

Effective H^∞ interpolation constrained by Hardy and Bergman norms

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Abstract

Given a finite set σ of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a holomorphic function f in \mathbb{D} which belongs to a class X , we are looking for a function g in another class Y (smaller than X) which minimizes the norm $\|g\|_Y$ among all functions g such that $g|_\sigma = f|_\sigma$. For $Y = H^\infty$, $X = H^p$ (the Hardy space) or $X = L_a^2$ (the Bergman space), and for the corresponding interpolation constant $c(\sigma, X, H^\infty)$, we show that $c(\sigma, X, H^\infty) \leq a\varphi_X(1 - \frac{1-r}{n})$ where $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . The upper bound is sharp over sets σ with given n and r .

Résumé

Etant donné un ensemble fini σ du disque unité $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ et une fonction f holomorphe dans \mathbb{D} appartenant à une certaine classe X , on cherche g dans une autre classe Y (plus petite que X) qui minimise la norme de g dans Y parmi toutes les fonctions g satisfaisant la condition $g|_\sigma = f|_\sigma$. On montre que dans le cas $Y = H^\infty$, la constante d'interpolation correspondante $c(\sigma, X, H^\infty)$ admet une majoration $c(\sigma, X, H^\infty) \leq a\varphi_X(1 - \frac{1-r}{n})$ où $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ et $\varphi_X(t)$ est la norme de la fonctionnelle d'évaluation $f \mapsto f(t)$, $0 \leq t < 1$, sur l'espace X . La majoration est exacte sur l'ensemble des σ avec n et r donné.

Introduction

(1) General framework. The problem considered is the following: given X and Y two Banach spaces of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $X \supset Y$, and a finite set $\sigma \subset \mathbb{D}$, to find the least norm interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case of f .

The classical interpolation problems- those of Nevanlinna-Pick and Carathéodory-Schur (on the one hand) and Carleson's free interpolation (on the other hand)- are of this nature. Two first are "individual", in the sens that one looks simply to compute the norms $\|f\|_{H^\infty|_\sigma}$ or $\|f\|_{H^\infty/z^n H^\infty}$ for a given f , whereas the third one is to compare the norms $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$ and

$$\inf(\|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma).$$

Here and everywhere below, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc \mathbb{D} endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. Looking at this comparison problem, say, in the form of computing/estimating the interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf\{\|g\|_Y : g|_\sigma = f|_\sigma\},$$

which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$, one can think, of course, on passing (after) to the limit- in the case of an infinite sequence $\{\lambda_j\}$ and its finite sections $\{\lambda_j\}_{j=1}^n$ - in order to obtain a Carleson type interpolation theorem $X|_\sigma = Y|_\sigma$. But not necessarily. In particular, even the classical Pick-Nevanlinna theorem (giving a necessary and sufficient

condition on a function a for the existence of $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and $f(\lambda) = a_\lambda$, $\lambda \in \sigma$), does not lead immediately to Carleson's criterion for $H^\infty_\sigma = l^\infty(\sigma)$. (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for $c(\sigma, X, Y)$ is of interest in its own. It is a kind of "effective interpolation" because we are looking for sharp estimations or a computation of $c(\sigma, X, Y)$ for a variety of norms $\|\cdot\|_X$, $\|\cdot\|_Y$. For this paper, the following partial case was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set $\sigma \subset \mathbb{D}$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? (H^2 being the standard Hardy space of the disc).

Here, we consider the case of H^∞ interpolation ($Y = H^\infty$) and the following scales of Banach spaces X :

(a) $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the standard Hardy spaces on the disc \mathbb{D} (see [N2] p.31-p.57) of all $f \in \text{Hol}(\mathbb{D})$ satisfying

$$\sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} |f(rz)|^p dm(z) \right)^{1/p} < \infty,$$

m being the Lebesgue normalized measure on \mathbb{T} .

(b) $X = l_a^2(1/\sqrt{k+1})$, the Bergman space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} < \infty.$$

An equivalent description of this space is:

$X = L_a^2$, the space of holomorphic functions such that

$$\int_{\mathbb{D}} |f(z)|^2 dx dy < \infty.$$

For spaces of type (a)&(b), we show

$$c_1 \varphi_X \left(1 - \frac{1-r}{n} \right) \leq \sup \{ c(\sigma, X, H^\infty) : \#\sigma \leq n, |\lambda| \leq r, \lambda \in \sigma \} \leq c_2 \varphi_X \left(1 - \frac{1-r}{n} \right),$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X .

In order to prove the right hand side inequality, we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{k=1}^n$ is the Malmquist basis (effectively constructible) of the space $K_B = H^2 \Theta B \overline{H^2}$, $B = \prod_{i=1}^n b_{\lambda_i}$ being the corresponding Blaschke product, $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ (see N. Nikolski, [N1] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3 p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important is a Bernstein-type inequality $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$ for a (rational) function f in the star-invariant subspace $H^p \cap B \overline{H^p}$ generated by a (finite) Blaschke product B , (K. Dyakonov [Dy]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need.

The lower bound problem is treated by using the "worst" interpolation n -tuple $\sigma = \sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). The "worst" interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We notice that spaces X of (a)&(b) satisfy the condition $X \circ b_\lambda \subset X$ which makes the problem of upper/lower bound easier.

(2) Principal results. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence in the unit disc, where every λ_s is repeated according its multiplicity m_s , $\sum_{s=1}^t m_s = n$ and $r = \max_{i=1..t} |\lambda_i|$. Let X, Y be Banach spaces of holomorphic functions continuously embedded into the space $Hol(\mathbb{D})$ of holomorphic functions on the unit disc \mathbb{D} . In what follows, we systematically use the following conditions for the spaces X and Y ,

$$(P_1) \quad Hol((1 + \epsilon)\mathbb{D}) \text{ is continuously embedded into } Y \text{ for every } \epsilon > 0,$$

$$(P_2) \quad Pol_+ \subset X \text{ and } Pol_+ \text{ is dense in } X,$$

where Pol_+ stands for the set of all complex polynomials p , $p(z) = \sum_{k=0}^N a_k z^k$,

$$(P_3) \quad [f \in X] \Rightarrow \left[z^n f \in X, \forall n \geq 0 \text{ and } \overline{\lim} \|z^n f\|^{\frac{1}{n}} \leq 1 \right],$$

$$(P_4) \quad [f \in X, \lambda \in \mathbb{D}, \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right].$$

We are interested in estimating the quantity

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i \}.$$

In order to simplify the notation, the condition

$$g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i$$

will also be written as

$$g|_\sigma = f|_\sigma.$$

Supposing X verifies property (P_4) and $Y \subset X$, the quantity $c(\sigma, X, Y)$ can be written as follows,

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_\sigma X \},$$

where B_σ is the Blaschke product

$$B_\sigma = \prod_{i=1..n} b_{\lambda_i},$$

corresponding to σ , $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$ being an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

The interesting case occurs when X is larger than Y , and the sens of the issue lies in comparing $\| \cdot \|_X$ and $\| \cdot \|_Y$ when Y interpolates X on the set σ . For example, we can wonder what happens when $X = H^p$, the classical Hardy spaces of the disc or $X = L^p_a$, the Bergman spaces, etc..., and when $Y = H^\infty$, but also $Y = W$ the Wiener algebra (of absolutely converging Fourier series) or $Y = B_{\infty,1}^0$, a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition). Here, H^p stands for the classical Hardy space of the disc (see below).

It is also important to understand what kind of interpolation we are going to study when bounding the constant $c(\sigma, X, Y)$. Namely, comparing with the Carleson free interpolation, we can say that the latter one deals with the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup \{ \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a) : a \in l^\infty(\sigma), \|a\|_{l^\infty} \leq 1 \}.$$

We also can add some more motivations to our problem:

(a) One of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c(\sigma, X, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X|_\sigma} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on f).

(b) There is a straight link between the constant $c(\sigma, X, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an H^∞ -calculus $\|f(A)\| \leq c \|f\|_\infty$, $f \in H^\infty$, for an arbitrary Banach space n -dimensional contraction A with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c(\sigma, H^\infty, W)$, so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on n -dimensional Banach spaces. An interesting case occurs for $f \in H^\infty$ such that $f|_\sigma = \frac{1}{z|_\sigma}$ (estimation of condition numbers and the norm of inverses of $n \times n$ matrices) or $f|_\sigma = \frac{1}{\lambda - z|_\sigma}$ (for estimation of the norm of the resolvent of an $n \times n$ matrix).

We start studying general Banach spaces X and Y and give some sufficient condition under which $C_{n,r}(X, Y) < \infty$, where

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r \}.$$

In particular, we prove the following fact.

Theorem 1.1.1 *Let X, Y be Banach spaces verifying properties (P_i) , $i = 1..4$. Then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and $r, 0 \leq r < 1$.

Next, we add the condition that X is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,r}(X, Y)$.

Theorem 1.2.1 *Let Y be a Banach space verifying property (P_1) and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in \text{Hol}((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $\text{Hol}((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, HY) \leq \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_1,2}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, k_{\lambda_t,2}, \dots, k_{\lambda_t,m_t-1},$$

and $k_{\lambda,i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$, $i \in \mathbb{N}$.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma} k_z\|_H,$$

where $P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_H e_k$ stands for the orthogonal projection of H onto $K_{B_\sigma}(H)$,

$$K_{B_\sigma}(H) = \text{span} (k_{\lambda_j,i} : 1 \leq i < m_j, j = 1, \dots, t).$$

After that, we specialize the upper bound obtained in Theorem 1.2 (ii) to the case $X = H^2$, the standard Hardy space of the disc, which can be equivalently defined as

$$H^2(\mathbb{D}) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\}.$$

Among other results, we get the following (see Proposition 2.0): for every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{\frac{1}{2}} \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{\frac{1}{2}} \leq 2 \sqrt{\frac{n}{1-r}}.$$

Next, we present a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ proving an estimate in the following form:

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}} \leq \sqrt{\frac{2n}{1-r}}.$$

It is shown (in Section 6) that this estimate is sharp (over n and r). This sharpness result is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). More precisely, we prove the following Theorem A, which gathers the results from Corollary 2.1 (for the upper bound) and Theorem 6.1.0 (for the lower bound, with $N = 1$).

Theorem A. *We have*

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq c(\sigma_{n,r}, H^2, H^\infty) \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}},$$

for all $n \geq 1$, $0 \leq r < 1$.

Then, we extend these results to the H^p spaces, as follows. Theorem B sums up Theorem 3.0 (for the upper bound) and Theorem 6.2.0 (for the lower bound).

Theorem B. *Let $1 \leq p \leq \infty$. Then*

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n,r}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_+$.

In particular, this gives yet another proof of the fact that $C_{n,r}(H^2, H^\infty) \leq a\sqrt{n}/\sqrt{1-r}$.

For the Bergman space $X = L_a^2$ we have the following Theorem C, which gathers Theorem 4.0 (for the upper bound, with $\alpha = -1/2$) and Theorem 6.1.0 (for the lower bound, with $N = 2$).

Theorem C. *We have*

$$\frac{1}{32} \frac{n}{1-r} \leq c(\sigma_{n,r}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq 6\sqrt{2} \frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$.

In Section 5 we compare the method used in Sections 1, 2, 3 and 4 with those resulting from the Carleson-free interpolation. Especially, we are interested in the cases of circular and radial sequences σ (see below). Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_\infty \leq 1} \inf(\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$, as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$)

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X, \quad \text{and} \quad \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem. 5.0 *Let X be a Banach space, $X \subset \text{Hol}(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of distinct points in the unit disc \mathbb{D} ,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the Carleson interpolation constant.

Theorem 5.0 tells us that, for σ with a “reasonable” interpolation constant $C_I(\sigma)$, the quantity $c(\sigma, X, H^\infty)$ behaves as $\max_i \|\varphi_{\lambda_i}\|$. However, for “tight” sequences σ , the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 5.0 is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ , Theorem 5.0 should give a better estimate than those of Theorem A and Theorem C.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? We answer this question for some particular sequences σ , see Exemples 5.2, 5.3 and 5.4.

1. Upper bounds for $c(\sigma, X, Y)$, as a kind of the Nevanlinna-Pick problem

1.1. General Banach spaces X and Y satisfying properties (P_i) , $i = 1 \dots 4$

The following theorem shows that if X and Y satisfy properties (P_i) for $i = 1 \dots 4$, then our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity $M_{n,r}$ which depends only on $n = \#\sigma$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$ (and of course on X and Y). In this generality, we cannot discuss the question of sharpness of the bounds obtained. First, we prove the following lemma.

Lemma. 1.1.0. *Under (P_2) , (P_3) and (P_4) , $B_\sigma X$ is a closed subspace of X and moreover,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\}.$$

Proof. Since $X \subset Hol(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and $f \mapsto f^{(k)}(\lambda)$, $k \in \mathbb{N}^*$, are continuous on $Hol(\mathbb{D})$, the subspace

$$M = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\},$$

is closed in X .

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties (P_2) and (P_3) imply that $h.X \subset X$, for all $h \in Hol((1 + \epsilon)\mathbb{D})$ with $\epsilon > 0$; we can write $h(z) = \sum_{k \geq 0} \widehat{h}(k)z^k$ with $|\widehat{h}(k)| \leq Cq^n$, $C > 0$ and $q < 1$. Then $\sum_{n \geq 0} \left\| \widehat{h}(k)z^k f \right\|_X < \infty$ for every $f \in X$. Since X is a Banach space we get $hf = \sum_{n \geq 0} \widehat{h}(k)z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies [f/b_\lambda = (1 - \bar{\lambda}z)f/(\lambda - z) \in X].$$

But this is obvious from (P_4) and the previous arguments. \square

Theorem. 1.1.1 *Let X, Y be Banach spaces verifying properties (P_i) , $i = 1..4$, then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and r , $0 \leq r < 1$.

Proof. For $k = 1..n$, we set $f_k(z) = 1/(1 - \bar{\lambda}_k z)$, and define the family $(e_k)_{k=1}^n$, (which is known as Malmquist basis, see [N1] p.117), by

$$e_1 = f_1 / \|f_1\|_2 \quad \text{and} \quad e_k = (\prod_{j=1..k-1} b_{\lambda_j}) f_k / \|f_k\|_2,$$

for $k = 2..n$, where $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$. Now, taking $f \in X$, we set

$$g = \sum_{k=1}^n \left(\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{e}_k(j)} \right) e_k,$$

where the series $\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{e}_k(j)}$ are absolutely convergent. Indeed,

$$\widehat{e}_k(j) = (2\pi i)^{-1} \int_{R\mathbb{T}} e_k(w) w^{-j-1} dw,$$

for all $j \geq 0$ and for all R , $1 < R < \frac{1}{r}$. For a subset A of \mathbb{C} and for a bounded function h on A , we define $\|h\|_A := \sup_{z \in A} |h(z)|$. As a result,

$$|\widehat{e}_k(j)| \leq (2\pi R^{j+1})^{-1} \|e_k\|_{R\mathbb{T}} \quad \text{and} \quad \sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi R)^{-1} \|e_k\|_{R\mathbb{T}} \sum_{j \geq 0} |\widehat{f}(j)| R^{-j} < \infty,$$

since $R > 1$ and f is holomorphic in \mathbb{D} .

Next, we observe that the map $\Phi : Hol(\mathbb{D}) \rightarrow Y \subset Hol(\mathbb{D})$ defined by

$$\Phi : f \mapsto \sum_{k=1}^n \left(\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{e}_k(j)} \right) e_k,$$

is well defined and has the following properties.

(a) $\Phi|_{H^2} = P_{B_\sigma}$ where P_{B_σ} is the orthogonal projection on the n -dimensional subspace of H^2 , K_{B_σ} defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \Theta B_\sigma H^2,$$

the last equality being a consequence of Lemma 1.2.0 of Section 1.2.

(b) Φ is continuous on $Hol(\mathbb{D})$ for the uniform convergence on compact sets of \mathbb{D} . Indeed, the point (a) is obvious since $(e_k)_{k=1}^n$ is an orthonormal basis of K_{B_σ} and

$$\sum_{j \geq 0} \widehat{f}(j) \overline{\widehat{e}_k(j)} = \langle f, e_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \widehat{h}(k) \overline{\widehat{g}(k)}$. In order to check point (b), let $(f_l)_{l \in \mathbb{N}}$ be a sequence of $Hol(\mathbb{D})$ converging to 0 uniformly on compact sets of \mathbb{D} . We need to see that $(\Phi(f_l))_{l \in \mathbb{N}}$ converges to 0, for which it is sufficient to show that $\lim_l \left| \sum_{j \geq 0} \widehat{f_l}(j) \overline{\widehat{e}_k(j)} \right| = 0$, for every $k = 1, 2, \dots, n$. Let $\rho \in]0, 1[$, then $\widehat{f_l}(j) = (2\pi)^{-1} \int_{\rho\mathbb{T}} f_l(w) w^{-j-1} dw$, for all $j, l \geq 0$. As a result,

$$\left| \sum_{j \geq 0} \widehat{f_l}(j) \overline{\widehat{e}_k(j)} \right| \leq \sum_{j \geq 0} \left| \widehat{f_l}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi\rho)^{-1} \|f_l\|_{\rho\mathbb{T}} \sum_{j \geq 0} |\widehat{e}_k(j)| \rho^{-j}.$$

Now if ρ is close enough to 1, it satisfies the inequality $1 \leq \rho^{-1} < r^{-1}$, which entails $\sum_{j \geq 0} |\widehat{e}_k(j)| \rho^{-j} < +\infty$ for each $k = 1..n$. The result follows.

Let

$$\Psi = Id|_X - \Phi|_X.$$

Using point (a), since $Pol_+ \subset H^2$ (Pol_+ standing for the set of all complex polynomials p , $p(z) = \sum_{k=0}^N a_k z^k$), we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma H^2$. Now, since $Pol_+ \subset Y$ and $Im(\Phi) \subset Y$, we deduce that

$$Im(\Psi|_{Pol_+}) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since $Y \subset X$. Now $\Psi(p) \in X$ and satisfies $(\Psi(p))|_\sigma = 0$ (that is to say $(\Psi(p))(\lambda) = 0, \forall \lambda \in \sigma$ (including multiplicities)) for all $p \in Pol_+$. Using Lemma 1.1.0, we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma X$. Now, Pol_+ being dense in X (property (P_2)), and Ψ being continuous on X , we can conclude that $Im(\Psi) \subset B_\sigma X$.

Now, we return to the proof of Theorem 1.1.1. Let $f \in X$ such that $\|f\|_X \leq 1$ and $g = \Phi(f)$. Since $Hol(r^{-1}\mathbb{D}) \subset Y$, we have $g \in Y$ and

$$f - g = \Psi(f) \in B_\sigma X.$$

Moreover,

$$\|g\|_Y \leq \sum_{k=1..n} |\langle f, e_k \rangle| \|e_k\|_Y.$$

In order to bound the right hand side, recall that for all $j \geq 0$ and for $R = 2/(r+1) \in]1, 1/r[$,

$$\sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi)^{-1} \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j.$$

Since the norm $f \mapsto \sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j$ is continuous on $Hol(\mathbb{D})$, and the inclusion $X \subset Hol(\mathbb{D})$ is also continuous, there exists $C_r > 0$ such that

$$\sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j \leq C_r \|f\|_X,$$

for every $f \in X$. On the other hand, $Hol(2(r+1)^{-1}\mathbb{D}) \subset Y$ (continuous inclusion again), and hence there exists $K_r > 0$ such that

$$\|e_k\|_Y \leq K_r \sup_{|z| < 2(r+1)^{-1}} |e_k(z)| = K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}}.$$

It is more or less clear that the right hand side of the last inequality can be bounded in terms of r and n only. Let us give a proof to this fact. It is clear that it suffices to estimate

$$\sup_{1 < |z| < 2(r+1)^{-1}} |e_k(z)|.$$

In order to bound this quantity, notice that

$$(1.1.0) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2},$$

for all $\lambda \in \mathbb{D}$ and all $z \in |\lambda|^{-1}\mathbb{D}$. Using the identity (1.1.0) for $\lambda = \lambda_j$, $1 \leq j \leq n$, and $z = \rho e^{it}$, $\rho = 2(1+r)^{-1}$, we get

$$|e_k(\rho e^{it})|^2 \leq \left(\prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left| \frac{1}{1 - \bar{\lambda}_k \rho e^{it}} \right|^2 \leq \left(\prod_{j=1}^{k-1} \left(1 + \frac{(\rho^2 - 1)(1 - |\lambda_j|^2)}{1 - |\lambda_j|^2 \rho^2} \right) \right) \left(\frac{1}{1 - |\lambda_k| \rho} \right)^2,$$

for all $k = 2..n$. Expressing ρ in terms of r , we obtain

$$\|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq \frac{1}{1 - \frac{2r}{r+1}} \sqrt{2 \left(\prod_{j=1..n-1} \left(1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right)} =: C_1(r, n),$$

and

$$\sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi)^{-1} C_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \|f\|_X \leq (2\pi)^{-1} C_r C_1(r, n) \|f\|_X.$$

On the other hand, since

$$\|e_k\|_Y \leq K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq K_r C_1(r, n),$$

we get

$$\|g\|_Y \leq \sum_{k=1}^n (2\pi)^{-1} C_r C_1(r, n) \|f\|_X K_r C_1(r, n) = (2\pi)^{-1} n C_r K_r (C_1(r, n))^2 \|f\|_X,$$

which proves that

$$c(\sigma, X, Y) \leq (2\pi)^{-1} n C_r K_r (C_1(r, n))^2$$

and completes the proof of Theorem 1.1.1. \square

1.2. The case where X is a Hilbert space

In the following theorem, we suppose that X is a Hilbert space and both X, Y satisfy properties (P_i) for $i = 1 \dots 4$. In this case, we obtain a better estimate for $c(\sigma, X, Y)$ than in Theorem 1.1.1 (see point (i) of Theorem 1.2.1). For the case $Y = H^\infty$, (point (ii) of Theorem 1.2.1), we can considerably improve this estimate. We omit an easy proof of the following lemma.

Lemma. 1.2.0. *Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence of \mathbb{D} where every λ_s is repeated according to its multiplicity m_s , $\sum_{s=1}^t m_s = n$. Let $(H, (\cdot)_H)$ be a Hilbert space continuously embedded into $Hol(\mathbb{D})$ and satisfying properties (P_i) for $i = 2, 3, 4$. Then*

$$K_{B_\sigma}(H) =: H \ominus_{B_\sigma} H = \text{span} (k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq m_j - 1),$$

where $k_{\lambda, i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$ and k_λ is the reproducing kernel of H at point λ for every $\lambda \in \mathbb{D}$, i.e. $k_\lambda \in H$ and $f(\lambda) = (f, k_\lambda)_H, \forall f \in H$.

Theorem. 1.2.1. *Let Y be a Banach space verifying property (P_1) and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in Hol((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $Hol((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, X, Y) \leq \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1, 0}, k_{\lambda_1, 1}, k_{\lambda_1, 2}, \dots, k_{\lambda_1, m_1 - 1}, k_{\lambda_2, 0}, k_{\lambda_2, 1}, k_{\lambda_2, 2}, \dots, k_{\lambda_2, m_2 - 1}, \dots, k_{\lambda_t, 0}, k_{\lambda_t, 1}, k_{\lambda_t, 2}, \dots, k_{\lambda_t, m_t - 1},$$

notation $k_{\lambda, i}$ is introduced in Lemma 1.2.0.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma} k_z\|_H,$$

where $P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_H e_k$ stands for the orthogonal projection of H onto $K_{B_\sigma}(H)$,

$$K_{B_\sigma}(H) = \text{span} (k_{\lambda_j, i} : 1 \leq i < m_j, j = 1, \dots, t).$$

Proof. i). Let $f \in X, \|f\|_X \leq 1$. Lemma 1.2.0 shows that

$$g = P_{B_\sigma} f = \sum_{k=1}^n (f, e_k)_H e_k$$

is the orthogonal projection of f onto subspace K_{B_σ} . Function g belongs to Y because all $k_{\lambda_j, i}$ are in $Hol((1 + \epsilon)\mathbb{D})$ for a convenient $\epsilon > 0$, and Y satisfies (P_1) .

On the other hand, $g - f \in B_\sigma H$ (again by Lemma 1.2.0). Moreover, using Cauchy-Schwarz inequality,

$$\|g\|_Y \leq \sum_{k=1}^n |(f, e_k)_H| \|e_k\|_Y \leq \left(\sum_{k=1}^n |(f, e_k)_H|^2 \right)^{1/2} \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2} \leq \|f\|_H \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2},$$

which proves i).

ii). If $Y = H^\infty$, then

$$|g(z)| = |(P_{B_\sigma} f, k_z)_H| = |(f, P_{B_\sigma} k_z)_H| \leq \|f\|_H \|P_{B_\sigma} k_z\|_H,$$

for all $z \in \mathbb{D}$, which proves ii). □

2. Upper bounds for $C_{n,r}(H^2, H^\infty)$

In this section, we specialize the estimate obtained in point (ii) of Theorem 1.2.1 for the case $X = H^2$, the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences σ (see Section 6). We also develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ giving more estimates for individual sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} .

Proposition. 2.0. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} we have*

$$(I_1) \quad c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

$$(I_2) \quad c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{1/2} = \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i \zeta)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)} \right|^{1/2}.$$

Proof. We prove (I₁). Applying point (ii) of Theorem 1.2.1 for $X = H^2$ and $Y = H^\infty$, and using

$$k_z(\zeta) = \frac{1}{1 - \bar{z}\zeta} \quad \text{and} \quad (P_{B_\sigma} k_z)(\zeta) = \frac{1 - \overline{B_\sigma(z)} B_\sigma(\zeta)}{1 - \bar{z}\zeta},$$

(see [N1] p.199), we obtain

$$\|P_{B_\sigma} k_z\|_{H^2} = \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

which gives the result.

We now prove (I₂), using (I₁). The map $\zeta \mapsto \|P_B(k_\zeta)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\}$, and hence the map

$$\zeta \mapsto \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

is a subharmonic function so

$$\sup_{|\zeta| < 1} \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2} \leq \sup_{|w|=1} \lim_{r \rightarrow 1} \left(\frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{1/2}.$$

Now apply Taylor's Formula of order 1 for points $w \in \mathbb{T}$ and $u = rw$, $0 < r < 1$. (It is applicable because B is holomorphic at every point of \mathbb{T}). We get

$$(B(u) - B(w))(u - w)^{-1} = B'(w) + o(1),$$

and since $|u - w| = 1 - |u|$,

$$|(B(u) - B(w))(u - w)^{-1}| = |B(u) - B(w)| (1 - |u|)^{-1} = |B'(w) + o(1)|.$$

Now,

$$|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,$$

$$(1 - |B(u)|)(1 - |u|)^{-1} \leq (1 - |u|)^{-1} |B(u) - B(w)| = |B'(w) + o(1)|,$$

and

$$\lim_{r \rightarrow 1} ((1 - |B(rw)|)(1 - |rw|)^{-1})^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Moreover,

$$B'(w) = - \sum_{i=1}^n (1 - |\lambda_i|^2) (1 - \bar{\lambda}_i w)^{-2} \prod_{j=1, j \neq i}^n b_{\lambda_j}(w),$$

for all $w \in \mathbb{T}$. This completes the proof since

$$\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}. \quad \square$$

Corollary. 2.1. *Let $n \geq 1$ and $r \in [0, 1[$. Then,*

$$C_{n,r}(H^2, H^\infty) \leq 2 (n(1-r)^{-1})^{\frac{1}{2}}.$$

Indeed, applying Proposition 2.0 we obtain

$$|B'(w)| \leq \left| \sum_{i=1..n} \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1+r}{1-r} \leq \frac{2n}{1-r}.$$

□

Now, we develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$.

Theorem. 2.2. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{1/2}$$

Proof. In order to simplify the notation, we set $B = B_\sigma$. Consider K_B , the n -dimensional subspace of H^2 defined by $K_B = (BH^2)^\perp = H^2 \ominus BH^2$.

Then the family $(e_k)_{k=1}^n$ introduced in the proof of Theorem 1.1.1, (known as Malmquist's basis), is an orthonormal basis of K_B , (see [N1], Malmquist-Walsh Lemma, p.116). Recall that

$$e_1 = f_1 / \|f_1\|_2 \quad \text{and} \quad e_k = (\prod_{j=1..k-1} b_{\lambda_j}) f_k / \|f_k\|_2,$$

for all $k = 2..n$, where f_k , defined in the proof of Theorem 1.1.1 is the reproducing kernel of H^2 associated to λ_k . Now, let $f \in H^2$ and

$$g = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k.$$

Function g belongs to H^∞ (it is a finite sum of H^∞ functions) and satisfies $g - f \in B$. Applying Cauchy-Schwarz inequality we get

$$|g(\zeta)| \leq \sum_{k=1}^n |(f, e_k)_{H^2}| |e_k(\zeta)| \leq \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{1/2} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{1/2},$$

for all $\zeta \in \mathbb{D}$. As a result, since f is an arbitrary H^2 function, we obtain

$$c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{1/2},$$

which completes the proof. \square

Corollary. 2.3. *For any sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2}.$$

Indeed,

$$\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{1/2}$$

and the result follows from Theorem 2.2. \square

Remark 2.4. As a result, we get once more the same estimate for $C_{n,r}(H^2, H^\infty)$ as in Corollary 2.1, with the constant $\sqrt{2}$ instead of 2: since $1 + |\lambda_j| \leq 2$ and $1 - |\lambda_j| \geq 1 - r$, applying Corollary 2.3, we get

$$C_{n,r}(H^2, H^\infty) \leq \sqrt{2} n^{1/2} (1 - r)^{-1/2}.$$

It is natural to wonder if it is possible to improve the bound $\sqrt{2} n^{1/2} (1 - r)^{-1/2}$. We return to this question in Section 5 below.

3. Upper bounds for $C_{n,r}(H^p, H^\infty)$, $p \geq 1$

In this section we extend Corollary 2.1 to all Hardy spaces H^p .

Theorem. 3.0. *Let $1 \leq p \leq \infty$. Then*

$$C_{n,r}(H^p, H^\infty) \leq A_p (n(1 - r)^{-1})^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p .

We first prove the following lemma.

Lemma. 3.1. *Let $n \geq 1$ and $0 \leq r < 1$. Then,*

$$C_{n,r}(H^1, H^\infty) \leq 2n(1 - r)^{-1}.$$

Proof. Let $f \in H^1$ such that $\|f\|_1 \leq 1$ and let

$$g = P_B f = \sum_{k=1..n} \langle f, e_k \rangle e_k,$$

where, as always, $(e_k)_{k=1}^n$ is the Malmquist basis corresponding to σ , and where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle f, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. That is to say that,

$$g(\zeta) = \sum_{k=1..n} \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\rangle,$$

for all $\zeta \in \mathbb{D}$, which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

As we saw in the proof of Theorem 2.3,

$$\|e_k\|_{H^\infty} \leq (1 + |\lambda_k|)^{1/2} (1 - |\lambda_k|)^{-1/2}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| \leq \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n (1 + |\lambda_k|) (1 - |\lambda_k|)^{-1} \leq 2n(1 - r)^{-1},$$

for all $\zeta \in \mathbb{D}$, which completes the proof. \square

Proof of Theorem 3.0. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, and $T : H^p \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma H^p\},$$

for every f . Then,

$$\|T\|_{H^p \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, H^p, H^\infty).$$

There exists $0 \leq \theta \leq 1$ such that $1/p = 1 - \theta$, and since (we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be]) $[H^1, H^\infty]_\theta = H^p$ (a topological identity: the spaces are the same and the norms are equivalent (up to constants depending on p only), see [J]), by a known interpolation Theorem (see [Tr], Theorem 1.9.3, p.59),

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (A_1 c(\sigma, H^1, H^\infty))^{1-\theta} (A_\infty c(\sigma, H^\infty, H^\infty))^\theta,$$

where A_1, A_∞ are numerical constants, and using both Lemma 3.1 and the fact that $c(\sigma, H^\infty, H^\infty) \leq 1$, we find

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (2A_1 n(1-r)^{-1})^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta (n(1-r)^{-1})^{\frac{1}{p}},$$

which completes the proof. \square

4. Upper bounds for $C_{n,r}(L_a^2, H^\infty)$

In this section, we generalize Corollary 2.2 to the case of spaces X which contain H^2 : $X = l_a^2((k+1)^\alpha)$, $\alpha \leq 0$, the Hardy weighted spaces of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 (k+1)^{2\alpha} < \infty.$$

It is also important to recall that

$$l_a^2((k+1)^\alpha) = L_a^2\left((1-|z|^2)^{-(2\alpha+1)} dA\right), \alpha < 0,$$

where $L_a^2\left((1 - |z|^2)^\beta dA\right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dA < \infty.$$

Notice also that $H^2 = l_a^2(1)$ and $L_a^2(\mathbb{D}) = l_a^2\left((k+1)^{-\frac{1}{2}}\right)$, where $L_a^2(\mathbb{D})$ stands for the Bergman space of the unit disc \mathbb{D} .

Theorem. 4.0. *Let $n \geq 1$, $r \in [0, 1[$, $\alpha \in [-1, 0]$ and $\beta \in]-1, 1]$. Then,*

$$C_{n,r} \left(l_a^2((k+1)^\alpha), H^\infty \right) \leq A (n(1-r)^{-1})^{\frac{1-2\alpha}{2}},$$

$$C_{n,r} \left(L_a^2\left((1 - |z|^2)^\beta dA\right), H^\infty \right) \leq A' (n(1-r)^{-1})^{\frac{\beta+2}{2}},$$

where $A = A(\alpha)$ and $A' = A'(\beta)$ are constants depending only on α and β , respectively.

In particular, for $\alpha = -1/2$ (or equivalently $\beta = 0$) we get

$$C_{n,r} \left(L_a^2, H^\infty \right) \leq 2\sqrt{3}n(1-r)^{-1}.$$

First, we prove a lemma, which is in fact, a partial case ($p = 2$) of the following K. Dyakonov's result [D] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): for every p , $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that

$$\|f'\|_{H^p} \leq c_p \|B'\|_\infty \|f\|_{H^p}$$

for all $f \in K_B$, where B is a finite Blaschke product and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. For the partial case considered in Lemma 4.1 below, our proof is different and the constant is slightly better. We notice that in general, Bernstein type inequalities have already been the subject of a lot of papers. Among others, Chapter 7 of P. Borwein and T. Erdélyi book's, see [BoEr], is devoted to such inequalities. This is also the case of A. Baranaov's work, see [B1], [B2] and [B3], and also of R. A. DeVore and G. G. Lorentz's book, see [DeLo].

In Proposition 4.1 which follows, the numerical constant $\frac{5}{2}$ is obviously not sharp but the asymptotic sharpness of $\frac{n}{1-r}$ is shown as $n \rightarrow \infty$ and $r \rightarrow 1$ in [Z2].

Proposition. 4.1. *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B =: H^2 \ominus BH^2$. Then for every $n \geq 2$ and $r \in [0, 1)$,*

$$\|f'\|_{H^2} \leq [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1} \|f\|_{H^2},$$

and in particular,

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2},$$

for all $n \geq 1$ and $r \in [0, 1)$.

Proof. Since $f \in K_B$, $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$. Noticing that,

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k,$$

for $k = 2..n$, we get

$$f' = (f, e_1)_{H^2} e'_1 + \sum_{k=2}^n (f, e_k)_{H^2} e'_k =$$

$$= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k,$$

which gives

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n \sum_{i=1}^{n-1} (f, e_k)_{H^2} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k \chi_{[1, k-1]}(i) + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k = \\ &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \end{aligned}$$

where $\chi_{[1, k-1]}$ is the characteristic function of $[1, k-1]$. Now,

$$\begin{aligned} \left\| (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 \right\|_{H^2} &\leq |(f, e_1)_{H^2}| \left\| \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} \right\|_{\infty} \|e_1\|_{H^2} \leq \\ &\leq \|f\|_{H^2} \|e_1\|_{H^2} \frac{1}{1-r} \|e_1\|_{H^2} \leq \|f\|_{H^2} \frac{1}{1-r}, \end{aligned}$$

using Cauchy-Schwarz inequality and the fact that e_1 is a vector of norm 1 in H^2 . By the same reason, we have

$$\begin{aligned} \left\| \sum_{k=2}^n \bar{\lambda}_k (f, e_k)_{H^2} \frac{1}{(1 - \bar{\lambda}_k z)} e_k \right\|_{H^2} &\leq \sum_{k=2}^n |(f, e_k)_{H^2}| \left\| \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} \right\|_{\infty} \|e_k\|_{H^2} \leq \\ &\leq \frac{1}{1-r} \sum_{k=2}^n |(f, e_k)_{H^2}| \leq \frac{1}{1-r} \left(\sum_{k=2}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \sqrt{n-2} \leq \frac{1}{1-r} \|f\|_{H^2} \sqrt{n-2}. \end{aligned}$$

Further,

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n e_k (f, e_k)_{H^2} \right\|_{H^2} &\leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \left\| \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} = \\ &= \left(\max_{1 \leq i \leq n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \leq \max_i \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \sum_{i=1}^{n-1} \|f\|_{H^2}. \end{aligned}$$

Now, using

$$\left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \bar{\lambda}_i z)(\lambda_i - z)} \right\|_{\infty} \leq \frac{1 + |\lambda_i|}{1 - |\lambda_i|} \leq \frac{1+r}{1-r},$$

we get

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} \leq (1+r) \frac{n-1}{1-r} \|f\|_{H^2}.$$

Finally,

$$\|f'\|_{H^2} \leq [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1} \|f\|_{H^2}.$$

In particular,

$$\|f\|_{H^2} \leq (2n-1 + \sqrt{n-2}) (1-r)^{-1} \|f\|_{H^2} \leq 5.2^{-1} n (1-r)^{-1} \|f\|_{H^2},$$

for all $n \geq 2$ and for every $f \in K_B$. (The case $n = 1$ is obvious because $\|f'\|_{H^2} \leq (1-r)^{-1} \|f\|_{H^2}$ for every f of the form $f = (1 - \bar{\lambda}z)^{-1}$, $\lambda \in \mathbb{D}$). \square

Corollary. 4.2. *Let σ be a sequence in \mathbb{D} . Then,*

$$c(\sigma, l_a^2((k+1)^{-1}), H^\infty) \leq 6\sqrt{2} (n(1-r)^{-1})^{3/2}.$$

Indeed, let $H = l_a^2((k+1)^{-N})$ and $B = B_\sigma$ the finite Blaschke product corresponding to σ . We recall that P_B is the orthogonal projection of H onto $K_B = K_B(H^2)$. We notice that $P_B : H \rightarrow H$ is a bounded operator and the adjoint $P_B^* : H^* \rightarrow H^*$ of P_B relatively to the Cauchy pairing $\langle \cdot, \cdot \rangle$ satisfies $P_B^* \varphi = P_B \varphi$, $\forall \varphi \in H^* \subset H^2$, where $H^* = l_a^2((k+1)^N)$ is the dual of H with respect to this pairing. If $f \in H$, then $|P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, P_B^* k_\zeta \rangle|$, where $k_\zeta = (1 - \bar{\zeta}z)^{-1}$ and

$$|P_B f(\zeta)| \leq \|f\|_H \|P_B^* k_\zeta\|_{H^*} \leq \|f\|_H K \left(\|P_B^* k_\zeta\|_{H^2} + \left\| (P_B^* k_\zeta)' \right\|_{H^2} \right),$$

where

$$K = \max \{1, \sup_{k \geq 1} (k+1)k^{-1}\} = 2$$

Since $P_B^* k_\zeta \in K_B$, Lemma 4.1 implies

$$|P_B f(\zeta)| \leq \|f\|_H K (\|P_B^* k_\zeta\|_{H^2} + 5 \cdot 2^{-1} (n(1-r)^{-1}) \|P_B^* k_\zeta\|_{H^2}) \leq A (n(1-r)^{-1})^{3/2} \|f\|_H,$$

where $A = \sqrt{2}K(1/2 + 5/2) = 6\sqrt{2}$, since $\|P_B^* k_\zeta\|_2 \leq \sqrt{2} (n(1-r)^{-1})^{1/2}$, and since we can suppose $n \geq 2$, (the case $n = 1$ being obvious). \square

Proof of Theorem 4.0. The case $\alpha = 0$ corresponds to $X = H^2$ and has already been studied in Section 1 (we can choose $A(0) = \sqrt{2}$). We now suppose $\alpha < 0$. Let $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ and $T : l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma l_a^2((k+1)^\alpha)\},$$

for every f . Then,

$$\|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, l_a^2((k+1)^\alpha), H^\infty).$$

Setting $\theta = -\alpha$ with $0 < \theta \leq 1$, we have (as in Theorem 3.0, we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$[l_a^2((k+1)^0), l_a^2((k+1)^{-1})]_{\theta, 2} = l_a^2\left(\left((k+1)^0\right)^{2\frac{1-\theta}{2}} \left((k+1)^{-1}\right)^{2\frac{\theta}{2}}\right) = l_a^2((k+1)^\alpha),$$

which entails, using Corollary 4.2 and (again) [Tr] Theorem 1.9.3 p.59,

$$\begin{aligned} \|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} &\leq (c(\sigma_{\lambda, n}, l_a^2((k+1)^0), H^\infty))^{1-\theta} (c(\sigma_{\lambda, n}, l_a^2((k+1)^{-1}), H^\infty))^\theta \leq \\ &\leq \left(A(0) (n(1-r)^{-1})^{\frac{1}{2}}\right)^{1-\theta} \left(A(1) (n(1-r)^{-1})^{\frac{3}{2}}\right)^\theta = A(0)^{1-\theta} A(1)^\theta (n(1-r)^{-1})^{\frac{1-\theta}{2} + \frac{3\theta}{2}}. \end{aligned}$$

It remains to use $\theta = -\alpha$ and set $A(\alpha) = A(0)^{1-\theta} A(1)^\theta$. In particular, for $\alpha = -1/2$ we get $(1-\theta)/2 + 3\theta/2 = 1$ and

$$A(-1/2) = A(0)^{(1-1/2)} A(1)^{1/2} = \sqrt{2}^{1/2} (6\sqrt{2})^{1/2} = 2\sqrt{3}.$$

\square

5. About the links with Carleson interpolation

In this section, we compare the method used in Sections 1, 2, 3 and 4 with those resulting from Carleson-type interpolation. Especially, we are interested in the case of circular sequences σ and radial sequences σ . Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_{l^\infty} \leq 1} \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$, as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$)

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X, \quad \text{and} \quad \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem. 5.0. *Let X be a Banach space, $X \subset \text{Hol}(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of distinct points in the unit disc \mathbb{D} ,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the interpolation constant.

Proof. Let $f \in X$. By definition of $C_I(\sigma)$, there exists $g \in H^\infty$ such that

$$f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n \quad \text{with} \quad \|g\|_\infty \leq C_I(\sigma) \max_i |f(\lambda_i)| \leq C_I(\sigma) \max_i \|\varphi_{\lambda_i}\| \|f\|_X.$$

Now, taking the supremum over all $f \in X$ such that $\|f\|_X \leq 1$, we get the right hand side inequality. The left hand side one is clear since if $g \in H^\infty$ satisfies $f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n$, then $\|g\|_\infty \geq |g(\lambda_i)| = |f(\lambda_i)| = |\varphi_{\lambda_i}(f)|, \quad \forall i = 1..n. \quad \square$

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? In what follows, we compare these quantities for three geometrically simple configurations: two-points sets σ , radial and circular sequences σ .

Example. 5.2. *Two points sets. Let $\sigma = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. Then,*

$$|b_{\lambda_1}(\lambda_2)|^{-1} \leq C_I(\sigma) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1},$$

and Theorem 5.0 implies

$$c(\sigma, X, H^\infty) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1} \max_{i=1,2} \|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives

$$c(\sigma, X, H^\infty) \leq \|\varphi_{\lambda_1}\| + \max_{|\lambda| \leq r} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

where $r = \max(|\lambda_1|, |\lambda_2|)$ and the functional $\varphi_{\lambda,1}$ is defined in the beginning of Section 5. The difference is that the first upper bound blows up when $\lambda_1 \rightarrow \lambda_2$, whereas the second one is still well-bounded.

Indeed, for an H^∞ -function f solving the interpolation $f(\lambda_1) = 1, f(\lambda_2) = -1$, we have

$$2 = |f(\lambda_1) - f(\lambda_2)| \leq 2 \|f\|_\infty |b_{\lambda_1}(\lambda_2)|,$$

and hence, $\|f\|_\infty \geq |b_{\lambda_1}(\lambda_2)|^{-1}$, which shows $C_I(\sigma) \geq |b_{\lambda_1}(\lambda_2)|^{-1}$.

On the other hand, setting

$$f = a_1 b_{\lambda_2}(\lambda_1)^{-1} b_{\lambda_2} + a_2 b_{\lambda_1}(\lambda_2)^{-1} b_{\lambda_1},$$

for arbitrary $a_1, a_2 \in \mathbb{C}$, we get $\|f\|_\infty \leq (|a_1| + |a_2|) |b_{\lambda_1}(\lambda_2)|^{-1} \leq 2 \max(|a_1|, |a_2|) |b_{\lambda_1}(\lambda_2)|^{-1}$. This implies $C_I(\sigma) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1}$.

For the second estimate stated in the example, taking $f \in X$ we set

$$g = f(\lambda_1) + [f(\lambda_2) - f(\lambda_1)](\lambda_2 - \lambda_1)^{-1}(z - \lambda_1),$$

and we get

$$\|g\|_\infty \leq |f(\lambda_1)| + |(f(\lambda_2) - f(\lambda_1))(\lambda_2 - \lambda_1)^{-1}|(1 + |\lambda_1|) \leq \|\varphi_{\lambda_1}\| + \max_{\lambda \in [\lambda_1, \lambda_2]} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

and the result follows. \square

Example. 5.3. *Circular sequences.* Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ for every i , and let $\alpha = \min_{i \neq j} |\lambda_i - \lambda_j| / (1 - r)$. Then, $\alpha^{-1} \leq C_I(\sigma) \leq 8e^{K'(1+K\alpha^{-3})}$, where $K, K' > 0$ are absolute constants. Therefore,

$$c(\sigma, X, H^\infty) \leq 8e^{K'(1+K\alpha^{-3})} \max_{|\lambda|=r} \|\varphi_\lambda\|$$

for every r -circular set σ (an estimate does not depending on n explicitly). In particular, there exists an increasing function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for any n uniformly distributed points $\lambda_1, \dots, \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2r \sin(\frac{\pi}{2n})$, we have

(1) $c(\sigma, H^2, H^\infty) \leq \varphi(n(1-r)r^{-1})(1-r)^{-\frac{1}{2}}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, H^2, H^\infty) \leq c(1-r)^{-\frac{1}{2}},$$

whereas our specific upper bound in Theorem A, (which is sharp over all n elements sequences σ), gives

$$c(\sigma, H^2, H^\infty) \leq c(1-r)^{-1}$$

only.

(2) $c(\sigma, L_a^2, H^\infty) \leq \varphi(n(1-r)r^{-1})(1-r)^{-1}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, L_a^2, H^\infty) \leq c(1-r)^{-1},$$

whereas our specific upper bound in Theorem 4.0, (which, again, is sharp over all n elements sequences σ), gives

$$c(\sigma, L_a^2, H^\infty) \leq c(1-r)^{-2}$$

only.

In order to explain the statements of this example, we observe first that the Carleson interpolation constant $C_I(\sigma)$, for r -circular sets σ , essentially depends on α only. Indeed, as is known, the separation constant

$$\Delta = \inf_{1 \leq j, k \leq n, j \neq k} |b_{\lambda_j}(\lambda_k)|,$$

is of the order of $\min(\alpha, 1)$, and the Carleson measure density for $\mu = \sum_{i=1}^n (1 - |\lambda_i|^2) \delta_{\lambda_i}$ also depends on α only. All together, $C_I(\sigma)$ is bounded if and only if α is separated from 0; see [N1] p.158 for the details of this reasoning. In fact, we can show that

$$\alpha(1 + \alpha r)^{-1} \leq \Delta \leq \alpha \quad \text{and} \quad \alpha^{-1} \leq C_I(\sigma) \leq e^{K'(1+K\alpha^{-3})},$$

(as claimed above), where $K, K' > 0$ are absolute constants, see [Z1] for details.

For a special case of n equidistant points on the circle $|z| = r$, $\lambda_j = re^{\frac{2i\pi j}{n}}$, $j = 1, 2, \dots, n$, one obtains $|\lambda_i - \lambda_{i+1}| = 2r \sin(\frac{\pi}{2n}) \geq \frac{2r}{n}$, and hence $\alpha \geq 2rn(1-r)^{-1}$. The above estimate for $C_I(\sigma)$ entails that we can take $\varphi(t) = 8e^{K'(1+Kt^3)}$ and then,

$$C_I(\sigma) \leq 8e^{K'(1+K\alpha^{-3})} \leq \varphi(n(1-r)r^{-1}).$$

Now, for the space H^2 , we have $\|\varphi_\lambda\| = (1 - |\lambda|^2)^{-1/2}$, and the upper estimate for $c(\sigma, H^2, H^\infty)$ follows. For the space L_a^2 , we have $\|\varphi_\lambda\| = (1 - |\lambda|^2)^{-1}$, and the same reasoning works for $c(\sigma, L_a^2, H^\infty)$. \square

Example. 5.4. Radial sequences. Now we consider geometric sequences on the radius of the unit disc \mathbb{D} , say on the radius $[0, 1)$. Let $0 < \rho < 1$, $p \in (0, \infty)$ and

$$\lambda_j = 1 - \rho^{j+p}, \quad j = 0, \dots, n,$$

so that the distances $1 - \lambda_j = \rho^j \rho^p$ form a geometric progression; the starting point is $\lambda_0 = 1 - \rho^p$. Let

$$r = \max_{0 \leq j \leq n} \lambda_j = \lambda_k = 1 - \rho^{n+p},$$

and $\delta = \delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, where $B_k = b_{\lambda_k}^{-1} B$. It is known that $\delta^{-1} \leq C_I(\sigma) \leq 8\delta^{-2}$, see ([N1], p 189). So, we need to know the asymptotic behaviour of $\delta = \delta(B)$ when $n \rightarrow \infty$, or $\rho \rightarrow 1$, or $\rho \rightarrow 0$, or $p \rightarrow \infty$, or $p \rightarrow 0$.

Claim. Let $\sigma = \{1 - \rho^{p+k}\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 5.0) is comparable with or better than the estimates from Corollary 2.2 (for $X = H^2$) and Theorem 4.0 (for $X = L_a^2$ and $X = L_a^2((1 - |z|^2)^\beta)$) for sufficiently small values of ρ (as $\rho \rightarrow 0$) and/or for a fixed ρ and $n \rightarrow \infty$. In all other cases, as for $p \rightarrow \infty$ (which means $\lambda_1 \rightarrow 1$), or $\rho \rightarrow 1$, or $n \rightarrow \infty$ and $\rho \rightarrow 1$, it is worse.

In order to justify that claim, we use the following upper bound for $\delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, assuming (for the notation convenience) the n is an even integer $n = 2k$ and computing $B_k(\lambda_k)$,

$$\begin{aligned} |B_k(\lambda_k)| &= \prod_{j=1}^{k-1} \frac{\lambda_k - \lambda_j}{1 - \lambda_j \lambda_k} \cdot \prod_{j=k+1}^{2k} \frac{\lambda_j - \lambda_k}{1 - \lambda_j \lambda_k} = \prod_{j=1}^{k-1} \frac{1 - \rho^{k-j}}{1 + \rho^{k-j} - \rho^{k+p}} \cdot \prod_{j=k+1}^{2k} \frac{1 - \rho^{j-k}}{1 + \rho^{j-k} - \rho^{j+p}} \\ &= \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k})} \leq \left(\prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \right)^2 \leq \\ &\leq \left(\prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^p)} \right)^2 =: A(n, \rho, p). \end{aligned}$$

For a lower bound, we proceed as in [N1] p.160 and get

$$\begin{aligned} |B_k(\lambda_k)| &= \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \cdot \prod_{s=1}^{n-k} \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k})} \geq \\ &\geq \left(\prod_{s=1}^n \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+n})} \right)^2 =: C(n, \rho, p) \end{aligned}$$

for every $k = 0, 1, \dots, n$. Hence,

$$C(n, \rho, p) \leq \delta(B) \leq A(n, \rho, p).$$

On the other hand, using Corollary 2.4 (for $X = H^2$)

$$\begin{aligned} c(\sigma, H^2, H^\infty) &\leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2} \leq \left(\sum_{j=1}^n \frac{2}{\rho^{j+p}} \right)^{1/2} = \\ &= \left(\frac{2}{\rho^{n+p}} \right)^{1/2} \left(\sum_{j=1}^n \rho^{n-j} \right)^{1/2} = \left(\frac{2}{1 - \rho} \right)^{1/2} \left(\frac{1 - \rho^n}{1 - \rho} \right)^{1/2} =: D(n, \rho, p). \end{aligned}$$

Now, we can compare the behaviour of $D(n, \rho, p)$ and $C_I(\sigma) \cdot \max_j \|\varphi_{\lambda_j}\|_{H^2}$ for varying parameters n, ρ, p .

5.4. (a) *Sparse sequences* σ ($\rho \rightarrow 0$, or at least $0 < \rho \leq \epsilon < 1$).

If $\rho \rightarrow 0$, one has $\lim_{\rho \rightarrow 0} C(n, \rho, p) = 1$, and hence $\overline{\lim}_{\rho \rightarrow 0} C_I(\sigma_{n, \rho, p}) \leq 8$. So, asymptotically, Theorem 5.0 implies

$$c(\sigma_{n, \rho, p}, H^2, H^\infty) \leq (8 + \epsilon) \sqrt{2} (1 - r)^{-\frac{1}{2}},$$

and Corollary 2.4 gives slightly better but comparable estimate,

$$c(\sigma_{n, \rho, p}, H^2, H^\infty) \leq (1 + \epsilon) \sqrt{2} (1 - r)^{-\frac{1}{2}}.$$

In our definition, if $p > 0$ is fixed and $\rho \rightarrow 0$ then $\lambda_1 = \lambda_1(\rho, p) \rightarrow 1$. In order to keep λ_1 at a fixed position we can set $p = p(\rho) = \frac{c}{\log(\frac{1}{\rho})}$. Then $\lambda_1 = 1 - \rho^p = 1 - e^{-c}$, $c > 0$. Still, $\lim_{\rho \rightarrow 0} C(n, \rho, p(\rho)) = 1$.

5.4. (b) *Condensed sequences* σ ($\rho \rightarrow 1$). In this case, $\lim_{\rho \rightarrow 1} D(n, \rho, p) = \sqrt{2} (1 - r)^{-\frac{1}{2}} \sqrt{n}$, and hence using Corollary 2.4 we cannot get better than the general estimate of Corollary 2.5, $c(\sigma, H^2, H^\infty) \leq (\sqrt{n} + \epsilon) \sqrt{2} (1 - r)^{-\frac{1}{2}}$. To the contrary, $A(n, \rho, p) \sim_{\rho \rightarrow 1} 2^{-\frac{n}{2}} \left(\frac{n}{2}\right)! (1 - \rho)^{\frac{n}{2}}$, and therefore $C_I(\sigma) \geq \delta^{-1} \geq (A(n, \rho, p))^{-1}$ which blows up as $\text{const} (1 - \rho)^{-n}$. So, as it can be predicted, in this case the Carleson interpolation is worse for our problem. Fixing $\lambda_1 = 1 - \rho^p$ at an arbitrary position ($p = -c \cdot \log(\rho)^{-1}$) will not change the conclusion.

5.4 (c) *Long sequences* ($n \rightarrow \infty$). With fixed ρ and p , let $n \rightarrow \infty$. Then, by Corollary 2.4,

$$c(\sigma, H^2, H^\infty) \leq \sqrt{2} (1 - r)^{-\frac{1}{2}} (1 - \rho)^{-\frac{1}{2}}.$$

(Observe that $1 - r = \rho^{n+p}$). In turn, Theorem 5.0 gives

$$c(\sigma, H^2, H^\infty) \leq 8\delta^{-2} (1 - r)^{-\frac{1}{2}}, \quad \overline{\lim}_{n \rightarrow \infty} \delta^{-2} \leq \left(\prod_{s=1}^{\infty} \frac{1 - \rho^s}{1 + \rho^s} \right)^{-4},$$

because $\lim_n C(n, \rho, p) = \lim_n A(n, \rho, p) = \left(\prod_{s=1}^n \frac{1 - \rho^s}{1 + \rho^s} \right)^{-4}$ for every ρ , $0 < \rho < 1$. Of course, the latter estimate is much worse than the former one, because $\prod_{s=1}^{\infty} \frac{1 + \rho^s}{1 - \rho^s} \sim \frac{\sqrt{1 - \rho}}{2\sqrt{\pi}} \exp\left(\frac{3\pi^2}{12} \frac{1}{1 - \rho}\right)$ as $\rho \rightarrow 1$. Indeed, setting $\varphi(\rho) = \prod_{s=1}^{\infty} (1 - \rho^s)^{-1}$ for all $\rho \in [0, 1[$, we have (see [Ne] p.22),

$$\varphi(\rho) = \sqrt{\frac{1 - \rho}{2\pi}} \exp\left(\frac{\pi^2}{12} \frac{1 + \rho}{1 - \rho}\right) [1 + O(1 - \rho)].$$

Now, setting $\psi(\rho) = \prod_{s=1}^{\infty} (1 + \rho^s)^{-1}$ we get $(\varphi\psi)(\rho) = \prod_{k \geq 1} (1 - \rho^{2k})^{-1} = \varphi(\rho^2)$ and,

$$\begin{aligned} \prod_{s=1}^{\infty} \frac{1 + \rho^s}{1 - \rho^s} &= \frac{\varphi(\rho)}{\psi(\rho)} = \varphi(\rho) \frac{\varphi(\rho)}{\varphi(\rho^2)} = \frac{(\varphi(\rho))^2}{\varphi(\rho^2)} = \\ &= \frac{1 - \rho}{2\pi} \exp\left(\frac{\pi^2}{6} \frac{1 + 1}{1 - \rho}\right) \sqrt{\frac{2\pi}{1 - \rho^2}} \exp\left(-\frac{\pi^2}{12} \frac{1 + 1}{(1 - \rho)(1 + 1)}\right) [1 + o(1)] = \\ &= \frac{\sqrt{1 - \rho}}{2\sqrt{\pi}} \exp\left(\frac{3\pi^2}{12} \frac{1}{1 - \rho}\right) [1 + o(1)], \quad \text{as } \rho \rightarrow 1. \quad \square \end{aligned}$$

6. Lower bounds for $C_{n, r}(X, H^\infty)$

6.1. The cases $X = H^2$ and $X = L_a^2$

Here, we consider the standard Hardy and Bergman spaces on the disc \mathbb{D} : $X = H^2 = l_a^2(1)$ and $X = L_a^2 = l_a^2((k+1)^{-1/2})$, and the problem of lower estimates for the one point special case $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$, (n times) $\lambda \in \mathbb{D}$. Recall the definition of our constrained interpolation constant for this case

$$c(\sigma_{n,\lambda}, H, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$. Our goal is to prove the sharpness of the upper estimate from Corollary 2.2 and Theorem 4.0 for the quantities $C_{n,r}(H^2, H^\infty)$ and $C_{n,r}(L_a^2, H^\infty)$.

Recall that the spaces $l_a^2((k+1)^\alpha)$ are defined in Section 4.

Theorem. 6.1.0 *Let $\alpha = 0$ or $\alpha = -\frac{1}{2}$. Then,*

$$c(\sigma_{\lambda,n}, l_a^2((k+1)^\alpha), H^\infty) \geq a_\alpha (n(1-|\lambda|)^{-1})^{\frac{1-2\alpha}{2}},$$

where $a_0 = 1/4\sqrt{2}$ and $a_{-1/2} = 1/32$. In particular,

$$a_\alpha (n(1-r)^{-1})^{\frac{1-2\alpha}{2}} \leq C_{n,r}(l_a^2((k+1)^\alpha), H^\infty) \leq A (n(1-r)^{-1})^{\frac{1-2\alpha}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A(\alpha)$ is a constant defined in Theorem 4.0.

In the proof, we use properties of reproducing kernel Hilbert space on the disc \mathbb{D} , see for example

[N2]. Let us recall some of them adapting the general setting to special cases $X = l_a^2((k+1)^\alpha)$. As

it is mentioned in Section 4,

$$l_a^2((k+1)^\alpha) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty \right\}.$$

The reproducing kernel of $l_a^2((k+1)^\alpha)$, by definition, is a $l_a^2((k+1)^\alpha)$ -valued function $\lambda \mapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2((k+1)^\alpha)$, where (\cdot, \cdot) means the scalar product $(h, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} (k+1)^{-2\alpha}$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k (k+1)^{2\alpha} (k+1)^{-2\alpha}$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} (k+1)^{2\alpha} \overline{\lambda}^k z^k, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$ ($\alpha = 0$), we get the Szegő kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

for the Bergman space $L_a^2(\mathbb{D}) = l_a^2((k+1)^{-1/2})$ ($\alpha = -1/2$) - the Bergman kernel $k_\lambda^{-1/2}(z) = (1 - \overline{\lambda}z)^{-2}$.

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel k of H^2 and $\varphi \in \{z^N : N = 1, 2\}$, the function $\varphi \circ k$ is also positive definit and the corresponding Hilbert space is

$$H_\varphi = \varphi(H^2) = l_a^2 \left((k+1)^{\frac{1-N}{2}} \right).$$

It satisfies the following property : for every $f \in H^2$, $\varphi \circ f \in \varphi(H^2)$ and $\|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2}^2)$ (see [N2] p.320).

We notice in particular that

$$H_z = H^2 \text{ and } H_{z^2} = L_a^2,$$

(a topological identity: the spaces are the same and the norms are equivalent). The above relation between the weighted spaces $l_a^2((k+1)^\alpha)$ and the spaces $\varphi(H^2) = H_\varphi$ leads to the following result.

Lemma 6.1.1 *Let $\varphi = z$ or $\varphi = z^2$, and $H_\varphi = \varphi(H^2)$ be the corresponding Hilbert space. Then, there exists a constant $a_\varphi > 0$ such that*

$$c(\sigma_{n,\lambda}, H_\varphi, H^\infty) \geq a_\varphi \varphi(n^{1/2}(1-|\lambda|)^{-1/2}).$$

Moreover, we can choose $a_z = 1/4\sqrt{2}$ and $a_{z^2} = 1/32$. In particular, we have

$$(4\sqrt{2})^{-1} n^{1/2}(1-r)^{-1/2} \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} n^{1/2}(1-r)^{-1/2},$$

$$32^{-1} n(1-r)^{-1} \leq C_{n,r}(L_a^2, H^\infty) \leq 6\sqrt{2} n(1-r)^{-1},$$

$\forall n \geq 1, \forall r \in [0, 1[.$

Proof. 1) We set

$$Q_n = \sum_{k=0}^{n-1} (1-|\lambda|^2)^{1/2} b_\lambda^k (1-\bar{\lambda}z)^{-1}, \quad H_n = \varphi \circ Q_n \quad \text{and} \quad \Psi = bH_n, \quad b > 0.$$

Then $\|Q_n\|_2^2 = n$, and hence by the above Aronszajn-deBranges inequality,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi(\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let $b > 0$ such that $b^2 \varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{n,\lambda}, H_\varphi, H^\infty) = c(\sigma_{n,\mu}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider G such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree n if $\varphi = z$ and $2n$ if $\varphi = z^2$) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}b_\lambda(z)} = (1-|\lambda|^2)^{-\frac{1}{2}} \left(1 + (1-\bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda}z^n \right) = \\ &= (1-r^2)^{-1/2} \left(1 + (1+r) \sum_{k=1}^{n-1} z^k + rz^n \right) =: (1-r^2)^{-1/2} \psi_1. \end{aligned}$$

Hence, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1-r^2)^{-\frac{1}{2}} \psi_1 \right)$ and

$$\varphi \circ \psi_1 = \psi_1^N(z), \quad N = 1, 2.$$

4) Next, we show that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} (2\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} \text{ if } N = 1 \\ 16^{-1} n(1-r)^{-1} \text{ if } N = 2 \end{cases},$$

where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd.

Indeed, setting $S_n = \sum_{j=0}^n z^j$, we have both for $N = 1$ and $N = 2$

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left(\left(1 + (1+r) \sum_{t=1}^{n-1} z^t + rz^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned} \sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left(\left(\frac{1-z^n}{1-z} \right)^N \right) = \\ &= \sum_{j=0}^m ((1-z)^{-N}) = \sum_{j=0}^m \left(\sum_{k \geq 0} C_{N+j-1}^j z^k \right) = \sum_{j=0}^m C_{N+j-1}^j = \\ &= \begin{cases} m+1 \text{ if } N = 1 \\ (m+1)(m+2)/2 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ (n+2)(n+4)/8 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ n^2/8 \text{ if } N = 2 \end{cases}. \end{aligned}$$

Finally, since $\sum^m(\psi) = b \sum^m(\varphi \circ \psi_1) = b(1-r^2)^{-N/2} \sum^m(\psi_1^N)$ we get

$$\sum_{j=0}^m (\psi) \geq \begin{cases} (2(1-r))^{-1/2} nb/2 \text{ if } N = 1 \\ (2(1-r))^{-1} n^2 b/8 \text{ if } N = 2 \end{cases},$$

with $b = \varphi(n) = \begin{cases} n^{-1/2} \text{ if } N = 1 \\ n^{-1} \text{ if } N = 2 \end{cases}$. This gives the result claimed.

5) Now, using point 4) and denoting $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel, we get

$$\begin{aligned} \|\Psi\|_{H^\infty/b_n^2 H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq 2^{-1} \|\psi \star F_n\|_\infty \geq 2^{-1} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \begin{cases} (4\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} \text{ if } N = 1 \\ 32^{-1} n(1-r)^{-1} \text{ if } N = 2 \end{cases}. \end{aligned}$$

□

Proof of Theorem 6.1.0. In order to prove the left hand side inequality, it suffices to apply Lemma 6.1.1 with $\varphi(z) = z^N$. Indeed, in this case $H_\varphi = l_a^2 \left((k+1)^{\frac{1-N}{2}} \right) = H_{z^N}$. The right hand side inequality is a straightforward consequence of Corollary 2.2 and Theorem 4.0.

□

6.2. The case $X = H^p$

Here we prove the sharpness (for even p) of the upper estimate found in Theorem 3.0.

Theorem. 6.2.0 Let $p \in 2\mathbb{Z}_+$, then

$$c(\sigma_{n,\lambda}, H^p, H^\infty) \geq 32^{-\frac{1}{p}} (n(1-|\lambda|)^{-1})^{\frac{1}{p}},$$

for every $\lambda \in \mathbb{D}$ and every integer $n \geq 1$, where $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$. Therefore,

$$32^{-\frac{1}{p}} (n(1-r)^{-1})^{\frac{1}{p}} \leq C_{n,r}(H^p, H^\infty) \leq A_p (n(1-r)^{-1})^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p (defined in Theorem 3.0).

We first prove the following lemma.

Lemma. 6.2.1 Let p, q such that $\frac{p}{q} \in \mathbb{Z}_+$, then $c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{q}{p}}$ for every sequence σ of \mathbb{D} .

Proof. Step 1. Recalling that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

we first prove that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \}.$$

Indeed, we clearly have the inequality

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} \leq c(\sigma, H^p, H^\infty),$$

and if the inequality were strict, that is to say

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} < \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

then we could write that there exists $\epsilon > 0$ such that for every $f = f_i \cdot f_o \in H^p$ (where f_i stands for the inner function corresponding to f and f_o to the outer one) with $\|f\|_p \leq 1$ (which also implies that $\|f_o\|_p \leq 1$, since $\|f_o\|_p = \|f\|_p$), there exists a function $g \in H^\infty$ verifying both $\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ and $g|_\sigma = f_o|_\sigma$. This entails that $f|_\sigma = (f_i g)|_\sigma$ and since $\|f_i g\|_\infty = \|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, we get that $c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, which is a contradiction and proves the equality of Step 1.

Step 2. Using the result of Step 1, we get that $\forall \epsilon > 0$ there exists an outer function $f_o \in H^q$ with $\|f_o\|_q \leq 1$ and such that

$$\inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f_o|_\sigma \} \geq c(\sigma, H^q, H^\infty) - \epsilon.$$

Now let $F = f_o^{\frac{q}{p}} \in H^p$, then $\|F\|_p^p = \|f_o\|_q^q \leq 1$. We suppose that there exists $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$ with

$$\|g\|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}}.$$

Then, since $g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i)^{\frac{q}{p}}$ for all $i = 1..n$, we have $g(\lambda_i)^{\frac{p}{q}} = f_o(\lambda_i)$ and $g^{\frac{p}{q}} \in H^\infty$ since $\frac{p}{q} \in \mathbb{Z}_+$. We also have

$$\left\| g^{\frac{p}{q}} \right\|_\infty = \|g\|_\infty^{\frac{p}{q}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

for all $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$, which gives

$$c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

and since that inequality is true for every $\epsilon > 0$, we get the result. \square

Proof of Theorem 6.2.0. We first prove the lower estimate for $c(\sigma_{n,\lambda}, H^p, H^\infty)$. Writing $p = 2(p/2)$, we apply Lemma 6.2.1 with $q = 2$ and this gives

$$c(\sigma_{n,\lambda}, H^p, H^\infty) \geq c(\sigma_{n,\lambda}, H^2, H^\infty)^{\frac{2}{p}} \geq 32^{-\frac{1}{p}} (n(1 - |\lambda|)^{-1})^{\frac{2}{p}}$$

for all integer $n \geq 1$. The last inequality is a consequence of Theorem 6.1.0. The right hand side inequality is proved in Theorem 3.0. \square

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