

Constrained Nevanlinna-Pick interpolation in Hardy and Bergman spaces

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Abstract

Given a finite set σ of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a holomorphic function f in \mathbb{D} which belongs to a class X , we are looking for a function g in another class Y (smaller than X) which minimizes the norm $\|g\|_Y$ among all functions g such that $g|_\sigma = f|_\sigma$. For $Y = H^\infty$, $X = H^p$ (the Hardy space) or $X = L_a^2$ (the Bergman space), and for the corresponding interpolation constant $c(\sigma, X, H^\infty)$, we show that $c(\sigma, X, H^\infty) \leq a\varphi_X \left(1 - \frac{1-r}{n}\right)$ where $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . The upper bound is sharp over sets σ with given n and r .

Introduction

(1) General framework. The problem considered is the following: given X and Y two Banach spaces of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $X \supset Y$, and a finite set $\sigma \subset \mathbb{D}$, to find the least norm interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case of f .

The classical interpolation problems- those of Nevanlinna-Pick and Carathéodory-Schur (on the one hand) and Carleson's free interpolation (on the other hand)- are of this nature. Two first are "individual", in the sens that one looks simply to compute the norms $\|f\|_{H^\infty_\sigma}$ or $\|f\|_{H^\infty/z^n H^\infty}$ for a given f , whereas the third one is to compare the norms $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$ and

$$\inf (\|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma).$$

Here and everywhere below, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc \mathbb{D} endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. Looking at this comparison problem, say, in the form of computing/estimating the interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \},$$

which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$, one can think, of course, on passing (after) to the limit- in the case of an infinite sequence $\{\lambda_j\}$ and its finite sections $\{\lambda_j\}_{j=1}^n$ - in order to obtain a Carleson type interpolation theorem $X|_\sigma = Y|_\sigma$. But not necessarily. In particular, even the classical Pick-Nevanlinna theorem (giving a necessary and sufficient condition on a function a for the existence of $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and $f(\lambda) = a_\lambda, \lambda \in \sigma$), does not lead immediately to Carleson's criterion for $H^\infty_\sigma = l^\infty(\sigma)$. (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for $c(\sigma, X, Y)$ is of interest in its own. For this paper, the following question was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set $\sigma \subset \mathbb{D}$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? (H^2 being the standard Hardy space of the disc).

Here, we consider the case of H^∞ interpolation ($Y = H^\infty$) and the following scales of Banach spaces X :

- (a) $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the standard Hardy spaces on the disc \mathbb{D} ,
(b) $X = l_a^2(1/\sqrt{k+1})$, the Bergman space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \frac{1}{(k+1)} < \infty.$$

An equivalent description of this space is:

$X = L_a^2$, the space of holomorphic functions such that

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dx dy < \infty.$$

For spaces of type (a)&(b), we show

$$c_1 \varphi_X \left(1 - \frac{1-r}{n} \right) \leq \sup \{ c(\sigma, X, H^\infty) : \#\sigma \leq n, |\lambda| \leq r, \lambda \in \sigma \} \leq c_2 \varphi_X \left(1 - \frac{1-r}{n} \right),$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X .

In order to prove the right hand side inequality, we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{k=1}^n$ is the explicitly known Malmquist basis of the space $K_B = H^2 \ominus BH^2$, $B = \prod_{i=1}^n b_{\lambda_i}$ being the corresponding Blaschke product, $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ (see N. Nikolski, [N1] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3 p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important is a Bernstein-type inequality $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$ for a (rational) function f in the star-invariant subspace $H^p \cap B\overline{H}_0^p$ generated by a (finite) Blaschke product B , (K. Dyakonov [Dy]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need.

The lower bound problem is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{\lambda, n} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We notice that spaces X of (a)&(b) satisfy the condition $X \circ b_\lambda \subset X$ which makes the problem of upper/lower bound easier.

(2) Principal results. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence in the unit disc, where every λ_s is repeated according its multiplicity m_s , $\sum_{s=1}^t m_s = n$ and $r = \max_{i=1..t} |\lambda_i|$. Let X, Y be Banach spaces of holomorphic functions continuously embedded into the space $Hol(\mathbb{D})$ of holomorphic functions in the unit disc \mathbb{D} . In what follows, we systematically use the following conditions for the spaces X and Y ,

$$(P_1) \quad Hol((1 + \epsilon)\mathbb{D}) \text{ is continuously embedded into } Y \text{ for every } \epsilon > 0,$$

$$(P_2) \quad Pol_+ \subset X \text{ and } Pol_+ \text{ is dense in } X,$$

where Pol_+ stands for the set of all complex polynomials p , $p(z) = \sum_{k=0}^N a_k z^k$,

$$(P_3) \quad [f \in X] \Rightarrow \left[z^n f \in X, \forall n \geq 0 \text{ and } \overline{\lim} \|z^n f\|^{\frac{1}{n}} \leq 1 \right],$$

$$(P_4) \quad [f \in X, \lambda \in \mathbb{D}, \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right].$$

We are interested in estimating the quantity

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i \}.$$

In order to simplify the notation, the condition

$$g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i$$

will also be written as

$$g|_\sigma = f|_\sigma.$$

Supposing X verifies property (P_4) and $Y \subset X$, the quantity $c(\sigma, X, Y)$ can be written as follows,

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_\sigma X \},$$

where B_σ is the Blaschke product

$$B_\sigma = \prod_{i=1..n} b_{\lambda_i},$$

corresponding to σ , $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$ being an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

The interesting case is obviously when X is larger than Y , and the sens of the issue lies in comparing $\| \cdot \|_X$ and $\| \cdot \|_Y$ when Y interpolates X on the set σ . For example, we can wonder what happens when $X = H^p$, the classical Hardy spaces of the disc or $X = L_a^p$, the Bergman spaces, etc..., and when $Y = H^\infty$, but also $Y = W$ the Wiener algebra (of absolutely converging Fourier series) or $Y = B_{\infty,1}^0$, a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition). Here, H^p stands for the classical Hardy space of the disc (see below).

It is also important to understand what kind of interpolation we are going to study when bounding the constant $c(\sigma, X, Y)$. Namely, comparing with the Carleson free interpolation, we can say that the latter one deals with the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup \{ \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a) : a \in l^\infty(\sigma), \|a\|_\infty \leq 1 \}.$$

We also can add some more motivations to our problem:

(a) One of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c(\sigma, X, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X|_\sigma} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on f).

(b) There is a straight link between the constant $c(\sigma, X, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an H^∞ -calculus $\|f(A)\| \leq c \|f\|_\infty$,

$f \in H^\infty$, for an arbitrary Banach space n -dimensional contraction A with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c(\sigma, H^\infty, W)$, so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on n -dimensional Banach spaces. An interesting case occurs for $f \in H^\infty$ such that $f|_\sigma = \frac{1}{z}|_\sigma$ (estimation of condition numbers and the norms inverses of $n \times n$ matrices) or $f|_\sigma = \frac{1}{\lambda - z}|_\sigma$ (for estimation of the norm of the resolvent of an $n \times n$ matrix).

This paper is devoted to upper/lower bounds for generalized Nevanlinna-Pick interpolation.

We start studying general Banach spaces X and Y and gives some sufficient conditions under which $C_{n,r}(X, Y) < \infty$, where

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r \}.$$

In particular, we prove the following basic fact.

Theorem 1.1.1 *Let X, Y be Banach spaces verifying properties (P_i) , $i = 1..4$, then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and $r, 0 \leq r < 1$.

Next, we add the condition that X is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,r}(X, Y)$.

Theorem 1.2.1 *Let Y be a Banach space verifying property (P_1) and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in \text{Hol}((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $\text{Hol}((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, X, Y) \leq \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_1,2}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, k_{\lambda_t,2}, \dots, k_{\lambda_t,m_t-1},$$

$k_{\lambda,i} = \left(\frac{d}{d\lambda} \right)^i k_\lambda$ and k_λ is the reproducing kernel of X at point λ for every $\lambda \in \mathbb{D}$.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma} k_z\|_H,$$

where $P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_H e_k$ stands for the orthogonal projection of H onto K_{B_σ} ,

$$K_{B_\sigma} = \text{span} (k_{\lambda_j, i} : 1 \leq i < m_j, j = 1, \dots, t).$$

After that, we specialize the upper bound obtained in **Theorem 1.2 (ii)** to the case $X = H^2$ and prove the following (see **Corollary 2.0** and **Proposition 2.1**). We get among other results that for every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{\frac{1}{2}} \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{\frac{1}{2}} \leq 2\sqrt{\frac{n}{1-r}}.$$

Next we present a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ proving an estimate in the following form:

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}} \leq \sqrt{\frac{2n}{1-r}}.$$

In particular, we get once more the same estimate for $c(\sigma, H^2, H^\infty)$, and hence for $C_{n,r}(H^2, H^\infty)$. Later on (see **Section 6**), we show that this estimate is sharp (over n and r). This lower bound problem is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{\lambda, n} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). More precisely, we prove the following **Theorem A**, which gathers the results from **Corollary 2.2** (for the upper bound) and from **Theorem 6.1.0** (for the lower bound, with $N = 1$).

Theorem A. *We have*

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq c(\sigma_{r,n}, H^2, H^\infty) \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}},$$

for all $n \geq 1$, $0 \leq r < 1$.

Then, we extend these results to the H^p spaces through **Theorem B** which sums up **Theorem 3.0** (for the upper bound) and **Theorem 6.2.0** (for the lower bound).

Theorem B. *Let $1 \leq p \leq \infty$. Then*

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{r,n}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_+$.

In particular, this gives yet another proof of the fact that $C_{n,r}(H^2, H^\infty) \leq a\sqrt{n}/\sqrt{1-r}$.

The same study applied to the Bergman space $X = L_a^2$ leads us to the following **Theorem C** which again gathers **Theorem 4.0** (for the upper bound, with $\alpha = 3/2$) and **Theorem 6.1.0** (for the lower bound, with $N = 2$).

Theorem C. *We have*

$$\frac{1}{32} \frac{n}{1-r} \leq c(\sigma_{r,n}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq 6\sqrt{2} \frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$.

Section 5 is devoted to compare the method used in **Sections 1, 2, 3 and 4** with those resulting from the Carleson-free interpolation. Especially, we are interested in the cases of circular

and radial sequences σ (see below). Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_{H^\infty} \leq 1} \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$,

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X,$$

as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$),

$$\varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem 5.0 *Let X be a Banach space, $X \subset Hol(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of distinct points in the unit disc \mathbb{D} ,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the Carleson interpolation constant.

Theorem 5.0 tells us that, for σ with a “reasonable” interpolation constant $C_I(\sigma)$, the quantity $c(\sigma, X, H^\infty)$ behaves as $\max_i \|\varphi_{\lambda_i}\|$. However, for “tight” sequences σ , the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of **Theorem 5.0** is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ , **Theorem 5.0** should give a better estimate than those of **Theorem A**, and of **Theorem C**.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? We answer this question for some particular sequences σ .

Example 5.2. *Two points sets. Let $\sigma = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. Then,*

$$\frac{1}{|b_{\lambda_1}(\lambda_2)|} \leq C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|},$$

and **Theorem 5.0** implies

$$c(\sigma, X, H^\infty) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|} \max_{i=1,2} \|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives (see **Section 5**)

$$c(\sigma, X, H^\infty) \leq \|\varphi_{\lambda_1}\| + \max_{|\lambda| \leq r} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

where $r = \max(|\lambda_1|, |\lambda_2|)$. The difference is that the first upper bound blows up when $\lambda_1 \rightarrow \lambda_2$, whereas the second one is still well-bounded.

Example 5.3. *Circular sequences. Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ for every i , and let $\alpha = \frac{\min_{i \neq j} |\lambda_i - \lambda_j|}{1-r}$. Then, $\frac{1}{\alpha} \leq C_I(\sigma) \leq 8e^{K'(1+\frac{K}{\alpha^3})}$, where $K, K' > 0$ are absolute constants. Therefore,*

$$c(\sigma, X, H^\infty) \leq 8e^{K'(1+\frac{K}{\alpha^3})} \max_{|\lambda|=r} \|\varphi_\lambda\|$$

for every r – circular set σ (an estimate does not depending on n explicitly). In particular, there exists an increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for any n uniformly distributed points $\lambda_1, \dots, \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2r \sin(\frac{\pi}{2n})$, we have

(1) $c(\sigma, H^2, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right) \frac{1}{(1-r)^{\frac{1}{2}}}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, H^2, H^\infty) \leq c \frac{1}{(1-r)^{\frac{1}{2}}},$$

whereas our specific upper bound in Theorem A, (which is sharp over all n elements sequences σ), gives

$$c(\sigma, H^2, H^\infty) \leq c \frac{1}{(1-r)}$$

only.

(2) $c(\sigma, L_a^2, H^\infty) \leq \varphi \left(\frac{n(1-r)}{r} \right) \frac{1}{(1-r)}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, L_a^2, H^\infty) \leq c \frac{1}{(1-r)},$$

whereas our specific upper bound in Theorem C, (which, again, is sharp over all n elements sequences σ), gives

$$c(\sigma, L_a^2, H^\infty) \leq c \frac{1}{(1-r)^2}$$

only.

We finally deal with a special case of radial sequences, in which we study *sparse sequences*, *condensed sequences*, and *long sequences*, and prove the following claim.

Example 5.4. *Radial sequences.*

Claim. Let $\sigma = \{1 - \rho^{p+k}\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 5.0) is comparable with or better than the estimates from Theorem A (for $X = H^2$) and Theorem C (for $X = L_a^2$) for sufficiently small values of ρ (as $\rho \rightarrow 0$) and/or for a fixed ρ and $n \rightarrow \infty$. In all other cases, as for $p \rightarrow \infty$ (which means $\lambda_1 \rightarrow 1$), or $\rho \rightarrow 1$, or $n \rightarrow \infty$ and $\rho \rightarrow 1$, it is worse.

1. Upper bounds for $c(\sigma, X, Y)$, as a kind of the Nevanlinna-Pick problem

1.1. General Banach spaces X and Y satisfying properties (P_i) , $i = 1 \dots 4$

The following theorem shows that if X and Y satisfy properties (P_i) for $i = 1 \dots 4$, then our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity $M_{n,r}$ which depends only on $n = \#\sigma$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$ (and of course on X and Y). In this generality, we cannot discuss the question of sharpness of the bounds obtained. First, we prove the following lemma.

Lemma. 1.1.0. *Under (P_2) , (P_3) and (P_4) , $B_\sigma X$ is a closed subspace of X and moreover,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\}.$$

Proof. Since $X \subset \text{Hol}(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and

$$f \mapsto f^{(k)}(\lambda), k = 1, 2, \dots,$$

are continuous on $\text{Hol}(\mathbb{D})$, the subspace

$$M = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\},$$

is closed in X .

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties (P_2) and (P_3) imply that $h.X \subset X$, for all $h \in \text{Hol}((1+\epsilon)\mathbb{D})$ with $\epsilon > 0$; we can write $h(z) = \sum_{k \geq 0} \widehat{h}(k) z^k$ with $|\widehat{h}(k)| \leq Cq^k$, $C > 0$ and $q < 1$. Then $\sum_{n \geq 0} \left\| \widehat{h}(k) z^k f \right\|_X < \infty$ for every $f \in X$. Since X is a Banach space we can conclude that $hf = \sum_{n \geq 0} \widehat{h}(k) z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies \left[\frac{f}{b_\lambda} = (1 - \bar{\lambda}z) \cdot \frac{f}{\lambda - z} \in X \right].$$

But this is obvious from (P_4) and the previous arguments. \square

Theorem. 1.1.1 *Let X, Y be Banach spaces verifying properties (P_i) , $i = 1 \dots 4$, then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and r , $0 \leq r < 1$.

Proof. For $k = 1 \dots n$, we set

$$f_k(z) = \frac{1}{1 - \bar{\lambda}_k z},$$

and define the family $(e_k)_{k=1}^n$, (which is known as Malmquist basis, see [N1] p.117), by

$$e_1 = (1 - |\lambda_1|^2)^{\frac{1}{2}} f_1,$$

and

$$e_k = (1 - |\lambda_k|^2)^{\frac{1}{2}} \left(\prod_{j=1}^{k-1} b_{\lambda_j} \right) f_k = \frac{f_k}{\|f_k\|_2} \prod_{j=1}^{k-1} b_{\lambda_j}$$

for $k = 2 \dots n$. Now, taking $f \in X$, we set

$$g = \sum_{k=1}^n \left(\sum_{j \geq 0} \hat{f}(j) \overline{\widehat{e}_k(j)} \right) e_k,$$

where the series

$$\sum_{j \geq 0} \hat{f}(j) \overline{\widehat{e}_k(j)}$$

are absolutely convergent. Indeed,

$$\widehat{e}_k(j) = \frac{1}{2\pi i} \int_{RT} \frac{e_k(w)}{w^{j+1}} dw,$$

for all $j \geq 0$ and for all $1 < R < \frac{1}{r}$. For a subset A of \mathbb{C} and for a bounded function h on A , we define

$$\|h\|_A := \sup_{z \in A} |h(z)|.$$

As a result,

$$|\widehat{e}_k(j)| \leq \frac{1}{2\pi} \frac{1}{R^{j+1}} \|e_k\|_{RT}.$$

So

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq \frac{\|e_k\|_{RT}}{2\pi R} \sum_{j \geq 0} |\hat{f}(j)| \left(\frac{1}{R} \right)^j < \infty,$$

since $R > 1$ and f is holomorphic in \mathbb{D} .

Next, we observe that the map

$$\Phi : Hol(\mathbb{D}) \rightarrow Y \subset Hol(\mathbb{D})$$

$$\Phi : f \mapsto \sum_{k=1}^n \left(\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

is well defined and has the following properties.

(a) $\Phi|_{H^2} = P_{B_\sigma}$ where P_{B_σ} is the orthogonal projection on the n -dimensional subspace of H^2 , K_{B_σ} defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \Theta B_\sigma H^2,$$

the last equality being a consequence of Lemma 1.2.0 of Section 1.2. Here, H^2 stands for the classical Hardy space $H^2(\mathbb{D})$ of the disc,

$$H^2(\mathbb{D}) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\},$$

or equivalently,

$$H^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the normalized Lebesgue measure on \mathbb{T} . See [N2] p.31-p.57 for more details on the Hardy spaces H^p , $1 \leq p \leq \infty$.

(b) Φ is continuous on $Hol(\mathbb{D})$ for the uniform convergence on compact sets of \mathbb{D} .

Indeed, the point (a) is obvious since $(e_k)_{k=1}^n$ is an orthonormal basis of K_{B_σ} and

$$\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} = \langle f, e_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. In order to check point (b), let $(f_l)_{l \in \mathbb{N}}$ be a sequence of $Hol(\mathbb{D})$ converging to 0 uniformly on compact sets of \mathbb{D} . We need to see that $(\Phi(f_l))_{l \in \mathbb{N}}$ converges to 0, for which it is sufficient to show that $\lim_l \left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| = 0$, for every $k = 1, 2, \dots, n$. Let $\rho \in]0, 1[$, then

$$\hat{f}_l(j) = \frac{1}{2\pi} \int_{\rho\mathbb{T}} \frac{f_l(w)}{w^{j+1}} dw,$$

for all $j, l \geq 0$. As a result,

$$\left| \hat{f}_l(j) \right| \leq \frac{1}{2\pi} \frac{1}{\rho^{j+1}} \|f_l\|_{\rho\mathbb{T}}.$$

So

$$\left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq \sum_{j \geq 0} \left| \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq \frac{\|f_l\|_{\rho\mathbb{T}}}{2\pi\rho} \sum_{j \geq 0} |\hat{e}_k(j)| \frac{1}{\rho^j}.$$

Now if ρ is close enough to 1, it satisfies the inequality $1 \leq \frac{1}{\rho} < \frac{1}{r}$, which entails

$$\sum_{j \geq 0} |\widehat{e}_k(j)| \frac{1}{\rho^j} < +\infty$$

for each $k = 1..n$. The result follows.

Let

$$\Psi = Id|_X - \Phi|_X.$$

Using point (a), since $Pol_+ \subset H^2$ (Pol_+ standing for the set of all complex polynomials p , $p(z) = \sum_{k=0}^N a_k z^k$), we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma H^2$. Now, since $Pol_+ \subset Y$ and $Im(\Phi) \subset Y$, we deduce that

$$Im(\Psi|_{Pol_+}) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since $Y \subset X$. Now $\Psi(p) \in X$ and satisfies $(\Psi(p))|_\sigma = 0$ (that is to say $(\Psi(p))(\lambda) = 0, \forall \lambda \in \sigma$ (including multiplicities)) for all $p \in Pol_+$. Using Lemma 1.1.0, we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma X$. Now, Pol_+ being dense in X (property (P_2)), and Ψ being continuous on X , we can conclude that $Im(\Psi) \subset B_\sigma X$.

Now, we return to the proof of Theorem 1.1.1. Let $f \in X$ such that $\|f\|_X \leq 1$ and $g = \Phi(f)$. Since $Hol(\frac{1}{r}\mathbb{D}) \subset Y$, we have

$$g = \Phi(f) \in Y$$

and

$$f - g = \Psi(f) \in B_\sigma X.$$

Moreover,

$$\|g\|_Y \leq \sum_{k=1..n} |\langle f, e_k \rangle| \|e_k\|_Y.$$

In order to bound the right hand side, recall that for all $j \geq 0$ and for $R = \frac{2}{r+1} \in]1, \frac{1}{r}[$,

$$\sum_{j \geq 0} |\widehat{f}(j) \overline{\widehat{e}_k(j)}| \leq \frac{\|e_k\|_{\frac{2}{r+1}\mathbb{T}}}{2\pi} \sum_{j \geq 0} |\widehat{f}(j)| \left(\frac{r+1}{2}\right)^j.$$

Since the norm $f \mapsto \sum_{j \geq 0} |\widehat{f}(j)| \left(\frac{r+1}{2}\right)^j$ is continuous on $Hol(\mathbb{D})$, and the inclusion $X \subset Hol(\mathbb{D})$ is also continuous, there exists $C_r > 0$ such that

$$\sum_{j \geq 0} |\widehat{f}(j)| \left(\frac{r+1}{2}\right)^j \leq C_r \|f\|_X,$$

for every $f \in X$. On the other hand,

$$Hol\left(\frac{2}{r+1}\mathbb{D}\right) \subset Y,$$

(continuous inclusion again), and hence there exists $K_r > 0$ such that

$$\|e_k\|_Y \leq K_r \sup_{|z| < \frac{2}{r+1}} |e_k(z)| = K_r \|e_k\|_{\frac{2}{r+1}\mathbb{T}}.$$

It is more or less clear that the right hand side of the last inequality can be bounded in terms of r and n only. Let us give a proof to this fact. It is clear that it suffices to estimate

$$\sup_{1 < |z| < \frac{2}{r+1}} |e_k(z)|.$$

In order to bound this quantity, notice that

$$(1.1.0) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2},$$

for all $\lambda \in \mathbb{D}$ and all $z \in \frac{1}{|\lambda|}\mathbb{D}$. Using the identity (1.1.0) for $\lambda = \lambda_j$, $1 \leq j \leq n$, and $z = \rho e^{it}$, $\rho = \frac{2}{r+1}$, we get

$$\begin{aligned} |e_k(\rho e^{it})|^2 &= (1 - |\lambda_k|^2) \left(\prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left| \frac{1}{1 - \bar{\lambda}_k \rho e^{it}} \right|^2, \\ |e_k(\rho e^{it})|^2 &\leq \left(\prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left(\frac{1}{1 - |\lambda_k| \rho} \right)^2, \end{aligned}$$

for all $k = 2..n$,

$$|e_k(\rho e^{it})|^2 \leq 2 \left(\prod_{j=1}^{k-1} \left(1 + \frac{(\rho^2 - 1)(1 - |\lambda_j|^2)}{1 - |\lambda_j|^2 \rho^2} \right) \right) \left(\frac{1}{1 - |\lambda_k| \rho} \right)^2.$$

Hence,

$$|e_k(\rho e^{it})|^2 \leq 2 \left(\prod_{j=1}^{k-1} \left(1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right) \left(\frac{1}{1 - \frac{2r}{r+1}} \right)^2.$$

Finally,

$$\begin{aligned} \|e_k\|_{\frac{2}{r+1}\mathbb{T}} &\leq \\ &\leq \frac{1}{1 - \frac{2r}{r+1}} \sqrt{2 \left(\prod_{j=1..n-1} \left(1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right)} =: C_1(r, n). \end{aligned}$$

and

$$\begin{aligned} \sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| &\leq \frac{C_r \|e_k\|_{\frac{2}{r+1}\mathbb{T}}}{2\pi} \|f\|_X \leq \\ &\leq \frac{C_r C_1(r, n)}{2\pi} \|f\|_X. \end{aligned}$$

On the other hand,

$$\|e_k\|_Y \leq K_r \|e_k\|_{\frac{2}{r+1}\mathbb{T}} \leq K_r C_1(r, n).$$

So

$$\|g\|_Y \leq \sum_{k=1}^n |\langle f, e_k \rangle| \|e_k\|_Y \leq$$

$$\leq \sum_{k=1}^n \frac{C_r C_1(r, n)}{2\pi} \|f\|_X K_r C_1(r, n) = \frac{n C_r K_r}{2\pi} (C_1(r, n))^2 \|f\|_X,$$

which proves that

$$c(\sigma, X, Y) \leq \frac{n C_r K_r}{2\pi} (C_1(r, n))^2$$

and completes the proof of Theorem 1.1.1. \square

1.2. The case where X is a Hilbert space

In the following theorem, we suppose that X is a Hilbert space and both X, Y satisfy properties (P_i) for $i = 1 \dots 4$. In this case, we obtain a better estimate for $c(\sigma, X, Y)$ than in **Theorem 1.1.1** (see point **(i)** of **Theorem 1.2.1**). For the case $Y = H^\infty$, (point **(ii)** of **Theorem 1.2.1**), we can considerably improve this estimate.

Lemma 1.2.0. *Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence of \mathbb{D} where every λ_s is repeated according to its multiplicity m_s , $\sum_{s=1}^t m_s = n$. Let $(H, (\cdot)_H)$ be a Hilbert space continuously embedded into $Hol(\mathbb{D})$ and satisfying properties (P_i) for $i = 2, 3, 4$. Then*

$$K_{B_\sigma} =: H \ominus B_\sigma H = \text{span} \left(k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq m_j - 1 \right),$$

where $k_{\lambda, i} = \left(\frac{d}{d\lambda} \right)^i k_\lambda$ and k_λ is the reproducing kernel of X at point λ for every $\lambda \in \mathbb{D}$.

Proof. First, we explain the notation. Namely, since $H \subset Hol(\mathbb{D})$ (with continuous inclusion), the function $\lambda \mapsto f(\lambda)$ is holomorphic and since $f(\lambda) = (f, k_\lambda)_H$ for every f , the function $\bar{\lambda} \mapsto k_\lambda$ is (weakly, and hence strongly) holomorphic. We have $f'(\lambda) = \left(f, \frac{d}{d\lambda} k_\lambda \right)_H$, and by induction,

$$f^{(i)}(\lambda) = \left(f, \left(\frac{d}{d\lambda} \right)^i k_\lambda \right)_H \text{ for every } i, i = 0, 1, \dots. \text{ Denote}$$

$$\left(\frac{d}{d\bar{\lambda}} \right)^i k_\lambda = k_{\lambda, i},$$

we know, (see Lemma 1.1.0), that

$$\begin{aligned} B_\sigma H &= \{f \in H : f^{(i)}(\lambda_j) = 0, \forall i, j, 1 \leq i < m_j, j = 1, \dots, t\} = \\ &= \{f \in H : (f, k_{\lambda_j, i})_H = 0, \forall i, j, 1 \leq i < m_j, j = 1, \dots, t\}. \end{aligned}$$

This means that

$$H \ominus B_\sigma H = \text{span} \left(k_{\lambda_j, i} : 1 \leq i < m_j, j = 1, \dots, t \right).$$

\square

Theorem 1.2.1. *Let Y be a Banach space verifying property (P_1) and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $\bar{k}_\lambda \in Hol((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{k}_\lambda \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $Hol((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, X, Y) \leq \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_1,2}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, k_{\lambda_t,2}, \dots, k_{\lambda_t,m_t-1},$$

notation $k_{\lambda,i}$ is introduced in Lemma 1.2.0.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma} k_z\|_H,$$

where $P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_H e_k$ stands for the orthogonal projection of H onto K_{B_σ} ,

$$K_{B_\sigma} = \text{span} (k_{\lambda_j,i} : 1 \leq i < m_j, j = 1, \dots, t).$$

Proof. i). Let $f \in X$, $\|f\|_X \leq 1$. Lemma 1.2.0 shows that

$$g = P_{B_\sigma} f = \sum_{k=1}^n (f, e_k)_H e_k$$

is the orthogonal projection of f onto subspace K_{B_σ} . Function g belongs to Y because all $k_{\lambda_j,i}$ are in $\text{Hol}((1 + \epsilon)\mathbb{D})$ for a convenient $\epsilon > 0$, and Y satisfies (P_1) .

On the other hand,

$$g - f \in B_\sigma H,$$

again by Lemma 1.2.0.

Moreover,

$$\|g\|_Y \leq \sum_{k=1}^n |(f, e_k)_H| \|e_k\|_Y,$$

and by Cauchy-Schwarz inequality,

$$\begin{aligned} \|g\|_Y &\leq \left(\sum_{k=1}^n |(f, e_k)_H|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}} \leq \\ &\leq \|f\|_H \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which proves i).

ii). If $Y = H^\infty$, then

$$|g(z)| = |(P_{B_\sigma} f, k_z)_H| = |(f, P_{B_\sigma} k_z)_H| \leq \|f\|_H \|P_{B_\sigma} k_z\|_H,$$

for all $z \in \mathbb{D}$, which proves ii). □

2. Upper bounds for $C_{n,r}(H^2, H^\infty)$

In this section, we specialize the estimate obtained in point (ii) of **Theorem 1.2.1** for the case $X = H^2$, the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences σ (see **Section 6**). We also develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ giving more estimates for individual sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} .

Corollary. 2.0. *Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in \mathbb{D} . Then,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{\frac{1}{2}}.$$

Indeed, applying point (ii) of Theorem 1.2.1 for $X = H^2$ and $Y = H^\infty$, and using

$$k_z(\zeta) = \frac{1}{1 - \bar{z}\zeta}$$

and

$$(P_{B_\sigma} k_z)(\zeta) = \frac{1 - \overline{B_\sigma(z)} B_\sigma(\zeta)}{1 - \bar{z}\zeta},$$

(see [N1] p.199), we obtain

$$\|P_{B_\sigma} k_z\|_{H^2} = \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{\frac{1}{2}},$$

which gives the result. □

Proposition. 2.1. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} we have*

$$c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{\frac{1}{2}} = \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i \zeta)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)} \right|^{\frac{1}{2}}.$$

Proof. We use Corollary 2.0. The map $\zeta \mapsto \|P_B(k_\zeta)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\}$, and hence the map

$$\zeta \mapsto \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{\frac{1}{2}},$$

is a subharmonic function so

$$\sup_{|\zeta|<1} \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{\frac{1}{2}} \leq \sup_{|w|=1} \lim_{r \rightarrow 1} \left(\frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{\frac{1}{2}}.$$

Now apply Taylor's Formula of order 1 for points $w \in \mathbb{T}$ and $u = rw$, $0 < r < 1$. (It is applicable because B is holomorphic at every point of \mathbb{T}). We get

$$\frac{B(u) - B(w)}{u - w} = B'(w) + o(1),$$

and since

$$|u - w| = 1 - |u|,$$

$$\left| \frac{B(u) - B(w)}{u - w} \right| = \frac{|B(u) - B(w)|}{1 - |u|} = |B'(w) + o(1)|.$$

Now,

$$|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,$$

$$\frac{1 - |B(u)|}{1 - |u|} \leq \frac{|B(u) - B(w)|}{1 - |u|} = |B'(w) + o(1)|,$$

and

$$\lim_{r \rightarrow 1} \left(\frac{1 - |B(rw)|}{1 - |rw|} \right)^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Since we have

$$B'(w) = - \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i w)^2} \prod_{j=1, j \neq i}^n b_{\lambda_j}(w),$$

for all $w \in \mathbb{T}$. This completes the proof since

$$\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}. \quad \square$$

Corollary. 2.2. *Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$. Then*

$$c(\sigma, H^2, H^\infty) \leq 2 \left(\frac{n}{1 - r} \right)^{\frac{1}{2}},$$

and hence,

$$C_{n,r}(H^2, H^\infty) \leq 2 \left(\frac{n}{1 - r} \right)^{\frac{1}{2}}.$$

Indeed, we apply Proposition 2.1 and observe that

$$|B'(w)| \leq \left| \sum_{i=1..n} \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1 + r}{1 - r} \leq \frac{2n}{1 - r}.$$

□

Now, we develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$.

Theorem. 2.3. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{\frac{1}{2}}$$

Proof. We give two proofs to this estimate. The first proof is shorter than the second one, but it contains an extra $\sqrt{2}$ factor.

First proof. Using Proposition 2.1., we obtain

$$c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{j=1}^n \frac{1 - |\lambda_j|^2}{(1 - \bar{\lambda}_j \zeta)^2} \frac{B_\sigma}{b_{\lambda_j}} \right|^{\frac{1}{2}} \leq$$

$$\leq \sqrt{2} \sup_{|\zeta|=1} \left(\sum_{i=1}^n \frac{1 - |\lambda_i|^2}{|1 - \bar{\lambda}_i \zeta|^2} \right)^{\frac{1}{2}} = \sqrt{2} \sup_{|\zeta|=1} \left(\sum_{i=1}^n \frac{1 - |\lambda_i|^2}{|\bar{\zeta} - \bar{\lambda}_i|^2} \right)^{\frac{1}{2}}.$$

Second proof. In order to simplify the notation, we set $B = B_\sigma$. Consider K_B , the n -dimensional subspace of H^2 defined by

$$K_B = (BH^2)^\perp = H^2 \ominus BH^2.$$

Then the family $(e_k)_{k=1}^n$ introduced in the proof of Theorem 1.1.1, (known as Malmquist's basis), is an orthonormal basis of K_B , (see [N1], Malmquist-Walsh Lemma, p.116). Recall that

$$e_1 = \frac{f_1}{\|f_1\|_2},$$

and

$$e_k = \frac{f_k}{\|f_k\|_2} \prod_{j=1}^{k-1} b_{\lambda_j},$$

for all $k = 2..n$, where

$$f_k = \frac{1}{1 - \bar{\lambda}_k z},$$

is the reproducing kernel of H^2 associated to λ_k . Now, let $f \in H^2$ and

$$g = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k.$$

Function g belongs to H^∞ because it is a finite sum of H^∞ functions. Moreover,

$$g(\lambda_i) = f(\lambda_i)$$

for all $i = 1..n$, counting with multiplicities. (Indeed, we can write $f = P_B f + g_1$ with $g_1 \in K_B^\perp = BH^2$). We have

$$|g(\zeta)| \leq \sum_{k=1}^n |(f, e_k)_{H^2}| |e_k(\zeta)|,$$

for all $\zeta \in \mathbb{D}$. And by Cauchy-Schwarz inequality,

$$|g(\zeta)| \leq \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{\frac{1}{2}},$$

$$\|g\|_\infty \leq \|f\|_2 \sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{\frac{1}{2}}.$$

Since f is an arbitrary H^2 function, we obtain

$$c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{\frac{1}{2}},$$

which completes the proof. \square

Corollary 2.4. For any sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in \mathbb{D} ,

$$c(\sigma, H^2, H^\infty) \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}}.$$

Indeed,

$$\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{\frac{1}{2}}$$

and the result follows from Theorem 2.3. \square

Remark 2.5. As a result, we get once more the same estimate for $C_{n,r}(H^2, H^\infty)$ as in Corollary 2.2, with the constant $\sqrt{2}$ instead of 2: since $1 + |\lambda_j| \leq 2$ and $1 - |\lambda_j| \geq 1 - r$, applying Corollary 2.4, we get

$$C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}.$$

It is natural to wonder if it is possible to improve the bound $\sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}$. We return to this question in **Chapter 5** below.

3. Upper bounds for $C_{n,r}(H^p, H^\infty)$, $p \geq 1$

The aim of this section is to extend **Corollary 2.2** to all Hardy spaces H^p . This is the subject of the following theorem.

Theorem 3.0. Let $1 \leq p \leq \infty$. Then

$$C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p .

We first prove the following lemma.

Lemma 3.1. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$, then

$$c(\sigma, H^1, H^\infty) \leq \frac{2n}{1-r}$$

and hence,

$$C_{n,r}(H^1, H^\infty) \leq \frac{2n}{1-r}.$$

Proof. Let $f \in H^1$ such that $\|f\|_1 \leq 1$ and let,

$$g = P_B f = \sum_{k=1..n} \langle f, e_k \rangle e_k,$$

where, as always, $(e_k)_{k=1}^n$ is the Malmquist basis corresponding to σ , and where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle f, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. That is to say that,

$$g(\zeta) = \sum_{k=1..n} \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\rangle,$$

for all $\zeta \in \mathbb{D}$, which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

Now, we recall that

$$e_k = \frac{(1 - |\lambda_k|^2)^{\frac{1}{2}}}{(1 - \overline{\lambda_k} z)} (\prod_{j=1}^{k-1} b_{\lambda_j}),$$

and, as we saw it in Theorem 2.3. (second proof),

$$\|e_k\|_{H^\infty} \leq \frac{(1 + |\lambda_k|)^{\frac{1}{2}}}{(1 - |\lambda_k|)^{\frac{1}{2}}}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| = \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n \frac{(1 + |\lambda_k|)}{(1 - |\lambda_k|)} \leq \frac{2n}{1-r},$$

for all $\zeta \in \mathbb{D}$, which completes the proof. \square

Proof of Theorem 3.0. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, and $T : H^p \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma H^p\},$$

for every f . Then,

$$\|T\|_{H^p \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, H^p, H^\infty).$$

There exists $0 \leq \theta \leq 1$ such that $\frac{1}{p} = 1 - \theta$, and since (we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be]) $[H^1, H^\infty]_\theta = H^p$ (a topological identity: the spaces are the same and the norms are equivalent (up to constants depending on p only), see [J]), by a known interpolation Theorem (see [Tr], Theorem 1.9.3, p.59),

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (A_1 c(\sigma, H^1, H^\infty))^{1-\theta} (A_\infty c(\sigma, H^\infty, H^\infty))^\theta,$$

where A_1, A_∞ are numerical constants, and using both Lemma 3.1 and the fact that $c(\sigma, H^\infty, H^\infty) \leq 1$, we find

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq \left(A_1 \frac{2n}{1-r} \right)^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

which completes the proof. \square

4. Upper bounds for $C_{n,r}(L_a^2, H^\infty)$

In this section, we generalize **Corollary 2.2** to the case of spaces X which contain H^2 : $X = l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, the Hardy weighted spaces of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \frac{1}{(k+1)^{2(\alpha-1)}} < \infty.$$

It is also important to recall that

$$l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right) = L_a^2\left((1-|z|^2)^{2\alpha-3} dA\right), \quad \alpha > 1,$$

where $L_a^2\left((1-|z|^2)^\beta dA\right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\beta dA < \infty.$$

Notice also that $H^2 = l_a^2(1)$ and $L_a^2(\mathbb{D}) = l_a^2\left(\frac{1}{(k+1)^{\frac{1}{2}}}\right)$, where $L_a^2(\mathbb{D})$ stands for the Bergman space of the unit disc \mathbb{D} .

Theorem. 4.0. *Let σ be a sequence in \mathbb{D} , $\alpha \in [1, 2]$ and $\beta \in]-1, 1]$. Then*

$$c\left(\sigma, l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}}.$$

Otherwise,

$$C_{n,r}\left(l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}},$$

$$C_{n,r}\left(L_a^2\left((1-|z|^2)^\beta dA\right), H^\infty\right) \leq A'\left(\frac{n}{1-r}\right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A(\alpha - 1)$ is a constant depending only on α and $A' = A'(\beta)$ is a constant depending only on β .

In particular, for $\alpha = \frac{3}{2}$ (or equivalently $\beta = 0$) we get

$$C_{n,r}\left(L_a^2, H^\infty\right) \leq 2\sqrt{3}\frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$.

First, we prove a following lemma. In fact, **Lemma 4.1** below is a partial case ($p = 2$) of the following K. Dyakonov's result [D] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): for every p , $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that

$$\left\| f' \right\|_{H^p} \leq c_p \left\| B' \right\|_\infty \|f\|_{H^p}$$

for all $f \in K_B$, where B is a finite Blaschke product (of order n) and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. For our partial case, our proof is different and the constant is slightly better. We notice that in general, Bernstein type inequalities have already been the subject of a lot of papers. Among others, Chapter 7 of P. Borwein and T. Erdélyi book's, see [BoEr], is devoted to such inequalities. This is also the case of A. Baranaov's work, see [B1], [B2] and [B3], and also of R. A. DeVore and G. G. Lorentz's book, see [DeLo].

Lemma. 4.1. Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B =: H^2 \ominus BH^2$. Then,

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}.$$

Proof. Since $f \in K_B$, $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$. Noticing that,

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k,$$

for $k = 2..n$, we get

$$\begin{aligned} f' &= (P_B f)' = (f, e_1)_{H^2} e'_1 + \sum_{k=2}^n (f, e_k)_{H^2} e'_k = \\ &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \end{aligned}$$

which gives

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n \sum_{i=1}^{n-1} (f, e_k)_{H^2} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k \chi_{[1, k-1]}(i) + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k = \\ &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \end{aligned}$$

where $\chi_{[1, k-1]}$ is the characteristic function of $[1, k-1]$. Now,

$$\begin{aligned} \left\| (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 \right\|_{H^2} &\leq |(f, e_1)_{H^2}| \left\| \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} \right\|_{\infty} \|e_1\|_{H^2} \leq \\ &\leq \|f\|_{H^2} \|e_1\|_{H^2} \frac{1}{1-r} \|e_1\|_{H^2} \leq \|f\|_{H^2} \frac{1}{1-r}, \end{aligned}$$

using both Cauchy-Schwarz inequality and the fact that e_1 is a vector of norm 1 in H^2 . By the same reason, we have

$$\begin{aligned} \left\| \sum_{k=2}^n \bar{\lambda}_k (f, e_k)_{H^2} \frac{1}{(1 - \bar{\lambda}_k z)} e_k \right\|_{H^2} &\leq \sum_{k=2}^n |(f, e_k)_{H^2}| \left\| \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} \right\|_{\infty} \|e_k\|_{H^2} \leq \\ &\leq \frac{1}{1-r} \sum_{k=2}^n |(f, e_k)_{H^2}| \leq \frac{1}{1-r} \left(\sum_{k=2}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \sqrt{n-2} \leq \frac{1}{1-r} \|f\|_{H^2} \sqrt{n-2}. \end{aligned}$$

Finally,

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} &\leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \left\| \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} = \\ &= \left(\max_{1 \leq i \leq n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \leq \max_i \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \sum_{i=1}^{n-1} \|f\|_{H^2}. \end{aligned}$$

Moreover, since

$$\left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \bar{\lambda}_i z)(\lambda_i - z)} \right\|_{\infty} \leq \frac{2}{1 - |\lambda_i|} \leq \frac{2}{1-r},$$

we get,

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} \leq \frac{2(n-1)}{1-r} \|f\|_{H^2}.$$

Finally,

$$\begin{aligned} \|f'\|_{H^2} &\leq \frac{1}{1-r} \|f\|_{H^2} + \frac{2(n-1)}{1-r} \|f\|_{H^2} + \frac{1}{1-r} \sqrt{n-2} \|f\|_{H^2} \leq \frac{(2n-1 + \sqrt{n-2})}{1-r} \|f\|_{H^2} \leq \\ &\leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}, \end{aligned}$$

for all $n \geq 2$ and for every $f \in K_B$. (The case $n = 1$ is obvious since $\|f'\|_{H^2} \leq \frac{1}{1-r} \|f\|_{H^2}$). \square

Corollary. 4.2. *Let σ a sequence in \mathbb{D} . Then,*

$$c \left(\sigma, l_a^2 \left(\frac{1}{k+1} \right), H^\infty \right) \leq 6\sqrt{2} \left(\frac{n}{1-r} \right)^{\frac{3}{2}}.$$

Indeed, if $f \in l_a^2 \left(\frac{1}{(k+1)^N} \right) = H$ then $|P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, P_B k_\zeta \rangle|$, where $\langle \cdot, \cdot \rangle$ means the Cauchy pairing and $k_\zeta = (1 - \bar{\zeta}z)^{-1}$. Denoting H^* the dual of H with respect to this pairing, $H^* = l_a^2((k+1)^N)$, we get

$$|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + \left\| (P_B k_\zeta)' \right\|_{H^2} \right),$$

where

$$K = \max \left\{ 1, \sup_{k \geq 1} \frac{k+1}{k} \right\} = 2$$

Since $P_B k_\zeta \in K_B$, Lemma 4.1 implies

$$\begin{aligned} |P_B f(\zeta)| &\leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + \left(\frac{5}{2} \frac{n}{1-r} \right) \|P_B k_\zeta\|_{H^2} \right) \leq \\ &\leq A \left(\frac{n}{1-r} \right)^{\frac{3}{2}} \|f\|_H, \end{aligned}$$

where $A = \sqrt{2}K \left(\frac{1}{2} + \frac{5}{2} \right) = 6\sqrt{2}$, since $\|P_B k_\zeta\|_2 \leq \frac{\sqrt{2n}}{\sqrt{1-r}}$, and since we can suppose $n \geq 2$, (the case $n = 1$ being obvious). \square

Proof of Theorem 4.0. The case $\alpha = 1$ corresponds to $X = H^2$ and has already been studied in Section 1 (we can choose $A(0) = \sqrt{2}$). We now suppose $\alpha > 1$. Let $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ and $T : l_A^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{ g \in H^\infty : f - g \in B_\sigma l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \right\},$$

for every f . Then,

$$\|T\|_{l_A^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty} = c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right).$$

We set such that $\alpha - 1 = \theta$ with $0 < \theta \leq 1$, and since (as in Theorem 3.0, we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$\begin{aligned} \left[l_a^2 \left(\frac{1}{(k+1)^0} \right), l_a^2 \left(\frac{1}{(k+1)^1} \right) \right]_{\theta,2} &= l_a^2 \left(\left(\frac{1}{(k+1)^0} \right)^{2\frac{1-\theta}{2}} \left(\frac{1}{(k+1)^1} \right)^{2\frac{\theta}{2}} \right) = \\ &= l_a^2 \left(\frac{1}{(k+1)^\theta} \right) = l_A^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), \end{aligned}$$

this gives, using Corollary 4.2 and (again) [Tr] Theorem 1.9.3 p.59,

$$\begin{aligned} &\| T \|_{l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty} \leq \\ &\leq \left(c \left(\sigma_{\lambda,n}, l_a^2 \left(\frac{1}{(k+1)^0} \right), H^\infty \right) \right)^{1-\theta} \left(c \left(\sigma_{\lambda,n}, l_a^2 \left(\frac{1}{(k+1)^1} \right), H^\infty \right) \right)^\theta \leq \\ &\leq \left(A(0) \left(\frac{n}{1-r} \right)^{\frac{1}{2}} \right)^{1-\theta} \left(A(1) \left(\frac{n}{1-r} \right)^{\frac{3}{2}} \right)^\theta = \\ &= A(0)^{1-\theta} A(1)^\theta \left(\frac{n}{1-r} \right)^{\frac{1-\theta}{2} + \frac{3\theta}{2}}. \end{aligned}$$

It remains to use $\theta = \alpha - 1$ and set $A(\alpha - 1) = A(0)^{1-\theta} A(1)^\theta$. In particular, for $\alpha = 3/2$ we get $\frac{1-\theta}{2} + \frac{3\theta}{2} = 1$ and

$$A \left(\frac{3}{2} \right) = A(0)^{(1-\frac{1}{2})} A(1)^{\frac{1}{2}} = \sqrt{2}^{\frac{1}{2}} (6\sqrt{2})^{\frac{1}{2}} = 2\sqrt{3}.$$

□

5. About the links with Carleson interpolation

In this section, we compare the method used in **Sections 1, 2, 3 and 4** with those resulting from Carleson-type interpolation. Especially, we are interested in the case of circular sequences σ and radial sequences σ . Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_\infty \leq 1} \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$,

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X,$$

as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$),

$$\varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem. 5.0. *Let X be a Banach space, $X \subset Hol(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of distinct points in the unit disc \mathbb{D} ,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the interpolation constant.

Proof. Let $f \in X$. By definition of $C_I(\sigma)$, there exist a $g \in H^\infty$ such that

$$f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n,$$

with

$$\begin{aligned} \|g\|_\infty &\leq C_I(\sigma) \max_i |f(\lambda_i)| \leq \\ &\leq C_I(\sigma) \max_i \|\varphi_{\lambda_i}\| \|f\|_X. \end{aligned}$$

Now, taking the supremum over all $f \in X$ such that $\|f\|_X \leq 1$, we get the right hand side inequality. The left hand side one is clear since if $g \in H^\infty$ satisfies $f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n$, then $\|g\|_\infty \geq |g(\lambda_i)| = |f(\lambda_i)| = |\varphi_{\lambda_i}(f)|$, $\forall i = 1..n$. \square

Comments 5.1.

Theorem 5.0 tells us that, for σ with a “reasonable” interpolation constant $C_I(\sigma)$, the quantity $c(\sigma, X, H^\infty)$ behaves as $\max_i \|\varphi_{\lambda_i}\|$. However, for “tight” sequences σ , the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of **Theorem 5.0** is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ , **Theorem 5.0** should give a better estimate than those of **Corollary 2.2**, and of **Theorem 4.0**.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristic r and n of σ ? In what follows we try to answer that question when σ is a r -circular sequence. In that case, we recall the definition of the constant α :

$$\alpha = \frac{\min_{i \neq j} |\lambda_i - \lambda_j|}{1-r} = \frac{ra}{1-r}.$$

Example 5.2. *Two points sets.* Let $\sigma = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. Then,

$$\frac{1}{|b_{\lambda_1}(\lambda_2)|} \leq C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|},$$

and *Theorem 5.0* implies

$$c(\sigma, X, H^\infty) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|} \max_{i=1,2} \|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives

$$c(\sigma, X, H^\infty) \leq \|\varphi_{\lambda_1}\| + \max_{|\lambda| \leq r} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

where $r = \max(|\lambda_1|, |\lambda_2|)$ and the functional $\varphi_{\lambda,1}$ is defined in 5. The difference is that the first upper bound blows up when $\lambda_1 \rightarrow \lambda_2$, whereas the second one is still well-bounded.

Indeed, for an H^∞ -function f solving the interpolation $f(\lambda_1) = 1$, $f(\lambda_2) = -1$, we have

$$2 = |f(\lambda_1) - f(\lambda_2)| \leq 2 \|f\|_\infty |b_{\lambda_1}(\lambda_2)|,$$

(indeed, the function $g = \frac{f(\lambda_1) - f}{b_{\lambda_1}}$ is holomorphic in \mathbb{D} and its H^∞ -norm on \mathbb{T} is equal to $\|f(\lambda_1) - f\|_\infty$, (which is less or equal than $2 \|f\|_\infty$), since the Blaschke factor b_{λ_1} has modulus 1 on the torus \mathbb{T}). Hence, $\|f\|_\infty \geq \frac{1}{|b_{\lambda_1}(\lambda_2)|}$, which shows $C_I(\sigma) \geq \frac{1}{|b_{\lambda_1}(\lambda_2)|}$.

On the other hand, setting

$$f = a_1 \frac{b_{\lambda_2}}{b_{\lambda_2}(\lambda_1)} + a_2 \frac{b_{\lambda_1}}{b_{\lambda_1}(\lambda_2)},$$

for arbitrary $a_1, a_2 \in \mathbb{C}$, we get $\|f\|_\infty \leq \frac{|a_1| + |a_2|}{|b_{\lambda_1}(\lambda_2)|} \leq \frac{\max(|a_1|, |a_2|)}{|b_{\lambda_1}(\lambda_2)|}$. This implies $C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|}$.

For the second estimate stated in the example, taking $f \in X$ we set

$$g = f(\lambda_1) + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} (z - \lambda_1),$$

and we get

$$\begin{aligned} \|g\|_\infty &\leq |f(\lambda_1)| + \left| \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} \right| (1 + |\lambda_1|) \leq \\ &\leq \|\varphi_{\lambda_1}\| + \max_{\lambda \in [\lambda_1, \lambda_2]} \|\varphi_{\lambda, 1}\| (1 + |\lambda_1|), \end{aligned}$$

and the result follows. \square

Example. 5.3. *Circular sequences.* Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ for every i , and let $\alpha = \frac{\min_{i \neq j} |\lambda_i - \lambda_j|}{1-r}$. Then, $\frac{1}{\alpha} \leq C_I(\sigma) \leq 8e^{K'(1+\frac{K}{\alpha^3})}$, where $K, K' > 0$ are absolute constants. Therefore,

$$c(\sigma, X, H^\infty) \leq \left(8e^{K'(1+\frac{K}{\alpha^3})}\right) \cdot \max_{|\lambda|=r} \|\varphi_\lambda\|$$

for every r -circular set σ (an estimate does not depend on n explicitly). In particular, there exists an increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for n uniformly distinct points $\lambda_1, \dots, \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2r \sin\left(\frac{\pi}{2n}\right)$, we have

(1) $c(\sigma, H^2, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right) \frac{1}{(1-r)^{\frac{1}{2}}}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, H^2, H^\infty) \leq c \frac{1}{(1-r)^{\frac{1}{2}}},$$

whereas our specific Corollary 2.2, (which is sharp over all n elements sequences σ), gives

$$c(\sigma, H^2, H^\infty) \leq c \frac{1}{(1-r)}$$

only.

(2) $c(\sigma, L_a^2, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right) \frac{1}{(1-r)}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, L_a^2, H^\infty) \leq c \frac{1}{(1-r)},$$

whereas our specific Theorem 4.0, (which, again, is sharp over all n elements sequences σ), gives

$$c(\sigma, L_a^2, H^\infty) \leq c \frac{1}{(1-r)^2}$$

only.

In order to explain the statements of this example, we observe first that the Carleson interpolation constant $C_I(\sigma)$, for r -circular sets σ , essentially depends on α only. Indeed, as is known, the separation constant

$$\Delta = \inf_{1 \leq j, k \leq n, j \neq k} |b_{\lambda_j}(\lambda_k)|,$$

is of the order of $\min(\alpha, 1)$, and the Carleson measure density for $\mu = \sum_{i=1}^n (1 - |\lambda_i|^2) \delta_{\lambda_i}$ also depends on α only. All together, $C_I(\sigma)$ is bounded if and only if α is separated from 0; see [N1] p.158 for the details of this reasoning. In fact, we can show that

$$\frac{\alpha}{1 + \alpha r} \leq \Delta \leq \alpha$$

and

$$\frac{1}{\alpha} \leq C_I(\sigma) \leq e^{K'(1+\frac{K}{\alpha^3})},$$

(as claimed as above), where $K, K' > 0$ are absolute constants, see Appendix 5.5 for details.

Now, checking point (1) for n equidistant points on the circle $|z| = r$, $\lambda_j = re^{\frac{2i\pi j}{n}}, j = 1, 2, \dots, n$, one obtains $|\lambda_i - \lambda_{i+1}| = 2r \sin\left(\frac{\pi}{2n}\right) \geq \frac{2r}{n}$, and hence $\alpha \geq \frac{2r}{n(1-r)}$. The above estimate for $C_I(\sigma)$ entails that we can take $\varphi(t) = 8e^{K'(1+Kt^3)}$ and then,

$$C_I(\sigma) \leq 8e^{K'(1+\frac{K}{\alpha^3})} \leq \varphi\left(\frac{n(1-r)}{r}\right).$$

Since, for the space H^2 , we have $\|\varphi_\lambda\| = (1 - |\lambda|^2)^{-\frac{1}{2}}$, the upper estimate for $c(\sigma, H^2, H^\infty)$ follows. Since for the space L_a^2 , we have $\|\varphi_\lambda\| = (1 - |\lambda|^2)$, the same reasoning works for $c(\sigma, L_a^2, H^\infty)$. \square

Example. 5.4. *Radial sequences.* Now we compare our two estimates of the interpolation constant $c(\sigma, X, H^\infty)$ (through the Carleson interpolation, and by the preceding general and specific methods) for special (geometric) sequences on the radius of the unit disc \mathbb{D} , say on the radius $[0, 1)$. Let $0 < \rho < 1$, $p \in (0, \infty)$ and

$$\lambda_j = 1 - \rho^{j+p}, \quad j = 0, \dots, n,$$

so that the distances $1 - \lambda_j = \rho^j \rho^p$ form a geometric progression; the starting point is $\lambda_0 = 1 - \rho^p$. Let

$$r = \max_{0 \leq j \leq n} \lambda_j = \lambda_k = 1 - \rho^{n+p},$$

and $\delta = \delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, where $B_k = \frac{B}{b\lambda_k}$. It is known that $\frac{1}{\delta} \leq C_I(\sigma) \leq \frac{8}{\delta^2}$. (The left hand side inequality is easy: if $f \in H^\infty$, $f(\lambda_k) = 1$, $f(\lambda_j) = 0$ for $j \neq k$, then $f = B_k g$ and $\|f\|_\infty = \|g\|_\infty \geq |g(\lambda_k)| = \frac{1}{|B_k(\lambda_k)|}$, and hence $C_I(\sigma) \geq \frac{1}{|B_k(\lambda_k)|}$ for every $k = 0, 1, 2, \dots, n$. The right hand side inequality is a theorem by P. Jones and S. Vinogradov, see ([N1], p 189). So, we need to know the asymptotic behaviour of $\delta = \delta(B)$ when $n \rightarrow \infty$, or $\rho \rightarrow 1$, or $\rho \rightarrow 0$, or $p \rightarrow \infty$, or $p \rightarrow 0$.

Claim. Let $\sigma = \{1 - \rho^{p+k}\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 5.0) is comparable with or better than the estimates from Corollary 2.2 (for $X = H^2$) and Theorem 4.0 (for $X = L_a^2$ and $X = L_a^2((1 - |z|^2)^\beta)$) for sufficiently small values of ρ (as $\rho \rightarrow 0$) and/or for a fixed ρ and $n \rightarrow \infty$. In all other cases, as for $p \rightarrow \infty$ (which means $\lambda_1 \rightarrow 1$), or $\rho \rightarrow 1$, or $n \rightarrow \infty$ and $\rho \rightarrow 1$, it is worse.

In order to justify that claim, we use the following upper bound for $\delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, assuming (for the notation convenience) the n is an even integer $n = 2k$ and computing $B_k(\lambda_k)$,

$$\begin{aligned} |B_k(\lambda_k)| &= \prod_{j=1}^{k-1} \frac{\lambda_k - \lambda_j}{1 - \lambda_j \lambda_k} \cdot \prod_{j=k+1}^{2k} \frac{\lambda_j - \lambda_k}{1 - \lambda_j \lambda_k} = \\ &= \prod_{j=1}^{k-1} \frac{1 - \rho^{k-j}}{1 + \rho^{k-j} - \rho^{k+p}} \cdot \prod_{j=k+1}^{2k} \frac{1 - \rho^{j-k}}{1 + \rho^{j-k} - \rho^{j+p}} = \\ &= \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \cdot \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k})} \leq \\ &\leq \left(\prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \right)^2 \leq \left(\prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^p)} \right)^2 =: A(n, \rho, p). \end{aligned}$$

For a lower bound, we proceed as in [N1] p.160 and get

$$\begin{aligned} |B_k(\lambda_k)| &= \prod_{s=1}^k \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k-s})} \cdot \prod_{s=1}^{n-k} \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+k})} \geq \\ &\geq \left(\prod_{s=1}^n \frac{1 - \rho^s}{1 + \rho^s (1 - \rho^{p+n})} \right)^2 =: C(n, \rho, p) \end{aligned}$$

for every $k = 0, 1, \dots, n$. Hence,

$$C(n, \rho, p) \leq \delta(B) \leq A(n, \rho, p).$$

On the other hand, using Corollary 2.4 (for $X = H^2$)

$$\begin{aligned} c(\sigma, H^2, H^\infty) &\leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n \frac{2}{\rho^{j+p}} \right)^{\frac{1}{2}} = \\ &= \left(\frac{2}{\rho^{n+p}} \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \rho^{n-j} \right)^{\frac{1}{2}} = \left(\frac{2}{1-r} \right)^{\frac{1}{2}} \left(\frac{1 - \rho^n}{1 - \rho} \right)^{\frac{1}{2}} =: D(n, \rho, p). \end{aligned}$$

Now, we can compare the behaviour of $D(n, \rho, p)$ and $C_I(\sigma). \max_j \|\varphi_{\lambda_j}\|_{H^2}$ for every parameter n, ρ, p .

5.4. (a) Sparse sequences σ ($\rho \rightarrow 0$, or at least $0 < \rho \leq \epsilon < 1$).

If $\rho \rightarrow 0$, one has $\lim_{\rho \rightarrow 0} C(n, \rho, p) = 1$, and hence $\lim_{\rho \rightarrow 0} C_I(\sigma_{n, \rho, p}) \leq 8$. So, asymptotically, Theorem 5.0 implies

$$c(\sigma_{n, \rho, p}, H^2, H^\infty) \leq (8 + \epsilon) \left(\frac{2}{1-r} \right)^{\frac{1}{2}},$$

and Corollary 2.4 gives slightly better but comparable estimate,

$$c(\sigma_{n, \rho, p}, H^2, H^\infty) \leq (1 + \epsilon) \left(\frac{2}{1-r} \right)^{\frac{1}{2}}.$$

In our definition, if $p > 0$ is fixed and $\rho \rightarrow 0$ then $\lambda_1 = \lambda_1(\rho, p) \rightarrow 1$. In order to keep λ_1 at a fixed position we can set $p = p(\rho) = \frac{c}{\log(\frac{1}{\rho})}$. Then $\lambda_1 = 1 - \rho^p = 1 - e^{-c}$, $c > 0$. Still, $\lim_{\rho \rightarrow 0} C(n, \rho, p(\rho)) = 1$.

5.4. (b) Condensed sequences σ ($\rho \rightarrow 1$). In this case, $\lim_{\rho \rightarrow 0} D(n, \rho, p) = \left(\frac{2}{1-r} \right)^{\frac{1}{2}} \sqrt{n+1}$, and hence using **Corollary 2.4** we cannot get better than the general estimate of **Corollary 2.5**, $c(\sigma, H^2, H^\infty) \leq (\sqrt{n+1} + \epsilon) \left(\frac{2}{1-r} \right)^{\frac{1}{2}}$. To the contrary, $A(n, \rho, p) \sim_{\rho \rightarrow 1} \frac{(\frac{n}{2})!}{2^{\frac{n}{2}}} (1 - \rho)^{\frac{n}{2}}$, and therefore $C_I(\sigma) \geq \delta^{-1} \geq (A(n, \rho, p))^{-1}$ which blows up as $\frac{\text{const}}{(1-\rho)^n}$. So, as it can be predicted, in this case the Carleson interpolation is worse for our problem. Fixing $\lambda_1 = 1 - \rho^p$ at an arbitrary position $\left(p = \frac{c}{\log(\frac{1}{\rho})} \right)$ will not change the conclusion.

5.4 (c) Long sequences ($n \rightarrow \infty$). With fixed ρ and p , let $n \rightarrow \infty$. Then, by **Corollary 2.4**,

$$c(\sigma, H^2, H^\infty) \leq \left(\frac{2}{1-r} \right)^{\frac{1}{2}} \left(\frac{1}{1-\rho} \right)^{\frac{1}{2}}.$$

(Observe, however, that is also not constant $1 - r = \rho^{n+p}$). In its turn, **Theorem 5.0** gives

$$c(\sigma, H^2, H^\infty) \leq \frac{8}{\delta^2} \frac{1}{(1-r)^{\frac{1}{2}}} \sim_{n \rightarrow \infty} \left(\prod_{s=1}^{\infty} \frac{1 - \rho^s}{1 + \rho^s} \right)^{-4} \frac{8}{(1-r)^{\frac{1}{2}}},$$

because $\lim_n C(n, \rho, p) = \lim_n A(n, \rho, p) = \left(\prod_{s=1}^n \frac{1-\rho^s}{1+\rho^s} \right)^{-4}$ for every ρ , $0 < \rho < 1$. Of course, the latter estimate is much worse than the former one, because $\prod_{s=1}^{\infty} \frac{1+\rho^s}{1-\rho^s} \sim \frac{\sqrt{1-\rho}}{2\sqrt{\pi}} \exp\left(\frac{3\pi^2}{12} \frac{1}{1-\rho}\right)$ as $\rho \rightarrow 1$. Indeed, setting $\varphi(\rho) = \prod_{s=1}^{\infty} \frac{1}{1-\rho^s}$ for all $\rho \in [0, 1[$, we have (see [Ne] p.22),

$$\varphi(\rho) = \sqrt{\frac{1-\rho}{2\pi}} \exp\left(\frac{\pi^2}{12} \frac{1+\rho}{1-\rho}\right) [1 + O(1-\rho)].$$

Now, setting $\psi(\rho) = \prod_{s=1}^{\infty} \frac{1}{1+\rho^s}$ we get $(\varphi\psi)(\rho) = \frac{1}{\prod_{k \geq 1} (1-\rho^{2k})} = \varphi(\rho^2)$ and,

$$\begin{aligned} \prod_{s=1}^{\infty} \frac{1+\rho^s}{1-\rho^s} &= \frac{\varphi(\rho)}{\psi(\rho)} = \varphi(\rho) \frac{\varphi(\rho)}{\varphi(\rho^2)} = \frac{(\varphi(\rho))^2}{\varphi(\rho^2)} = \\ &= \frac{1-\rho}{2\pi} \exp\left(\frac{\pi^2}{6} \frac{1+1}{1-\rho}\right) \sqrt{\frac{2\pi}{1-\rho^2}} \exp\left(-\frac{\pi^2}{12} \frac{1+1}{(1-\rho)(1+1)}\right) [1 + o(1)] = \\ &= \frac{\sqrt{1-\rho}}{2\sqrt{\pi}} \exp\left(\frac{3\pi^2}{12} \frac{1}{1-\rho}\right) [1 + o(1)], \text{ as } \rho \rightarrow 1. \end{aligned}$$

Appendix 5.5.

Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a r -circular sequence, $|\lambda_i| = r \forall i = 1 \dots n$, $0 \leq r < 1$; here we show the links between the constants $\Delta = \Delta(\sigma) = \inf_{i \neq j} |b_{\lambda_i}(\lambda_j)|$, and $\alpha = \frac{\min_{i \neq j} |\lambda_i - \lambda_j|}{1-r}$, and establish an estimate for the Carleson interpolation constant $C_I(\sigma)$.

Lemma 5.6. *In the above notation, we have*

$$\frac{\alpha}{1+\alpha r} \leq \Delta \leq \alpha.$$

Lemma 5.7. *In the above notation, we have the following estimate for the Carleson interpolation constant $C_I(\sigma)$: there exists numerical constants $K, K' > 0$ such that*

$$C_I(\sigma) \leq 8e^{K'(1+\frac{K}{\alpha^3})}.$$

See [N1] for the proofs of these two Lemmas.

6. Lower bounds for $C_{n,r}(X, H^\infty)$

6.1. The cases $X = H^2$ and $X = L_a^2$

Here, we consider the reproducing kernel Hilbert spaces on the disc \mathbb{D} : $X = H^2 = l_a^2(1)$ and $X = L_a^2 = l_a^2(1/\sqrt{k+1})$, and the problem of lower estimates for the one point special case $\sigma_{\lambda,n} = \{\lambda, \lambda, \dots, \lambda\}$, (n times) $\lambda \in \mathbb{D}$. Recall the definition of the semi-free interpolation constant

$$c(\sigma_{\lambda,n}, H, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$. In particular, our aim is to prove the sharpness of the upper estimate for the quantities

$$C_{n,r}(H^2, H^\infty) \text{ and } C_{n,r}(L_a^2, H^\infty)$$

in **Corollary 2.2** and **Theorem 4.0**.

Theorem. 6.1.0 *Let $N \in \{1, 2\}$ be an integer. Then,*

$$c \left(\sigma_{\lambda, n}, l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \geq a_N \left(\frac{n}{1-|\lambda|} \right)^{\frac{N}{2}},$$

where $a_1 = 1/4\sqrt{2}$ and $a_2 = 1/32$. In particular,

$$a_N \left(\frac{n}{1-r} \right)^{\frac{N}{2}} \leq C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A \left(\frac{N-1}{2} \right)$ is a constant defined in Theorem 4.0, and where spaces $l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right)$ are defined in Section 4.

(1) We first recall some properties of reproducing kernel Hilbert space on the disc \mathbb{D} , $X =$

$l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right)$. As it is mentioned in **Section 4**,

$$l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{(k+1)^{2(\alpha-1)}} < \infty \right\}.$$

The reproducing kernel of $l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right)$, by definition, is a $l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right)$ -valued function $\lambda \mapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right)$, where (\cdot, \cdot) means the scalar product $(h, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} \frac{1}{(k+1)^{2(\alpha-1)}}$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k (k+1)^{2(\alpha-1)} \frac{1}{(k+1)^{2(\alpha-1)}}$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} (k+1)^{2(\alpha-1)} \overline{\lambda}^k z^k, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$ ($\alpha = 1$), we get the Szegő kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

for the Bergman space $L_a^2(\mathbb{D}) = l_a^2 \left(\frac{1}{(k+1)^{\frac{1}{2}}} \right)$ ($\alpha = \frac{3}{2}$) - the Bergman kernel $k_\lambda^{3/2}(z) = (1 - \overline{\lambda}z)^{-2}$.

(2) Reproducing kernel Hilbert spaces containing H^2 . We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel k of H^2 and $\varphi \in \{z^N : N = 1, 2\}$, the function $\varphi \circ k$ is also positive definit and the corresponding RKHS

$$H(\varphi \circ k) =: \varphi(H(k)) = \varphi(H^2) = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right)$$

satisfies the following. For every $f \in H(k)$ we have $\varphi \circ f \in \varphi(H(k))$ and $\|\varphi \circ f\|_{\varphi(H(k))}^2 \leq \varphi(\|f\|_{H(k)}^2)$ (see [N2] p.320).

We notice in particular that

$$H_z = H^2 \text{ and } H_{z^2} = L_a^2,$$

(a topological identity: the spaces are the same and the norms are equivalent). The link between spaces of type $l_a^2\left(\frac{1}{(k+1)\frac{N-1}{2}}\right)$ and of type $\varphi(H^2) = H_\varphi$ being established, we give the following result.

Lemma 6.1.1 *Let $\varphi \in \{z^N : N = 1, 2\}$, and $H_\varphi = \varphi(H^2)$ be the reproducing kernel Hilbert space corresponding to the kernel $\varphi\left(\frac{1}{1-\lambda z}\right)$. Then, there exists a constant a_N depending on φ such that*

$$c(\sigma_{\lambda,n}, H_\varphi, H^\infty) \geq a_N \varphi\left(\sqrt{\frac{n}{1-|\lambda|}}\right).$$

Moreover, we can choose $a_1 = 1/4\sqrt{2}$ and $a_2 = 1/32$. In particular, we have

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}, \text{ and}$$

$$\frac{1}{32} \frac{n}{1-r} \leq C_{n,r}(L_a^2, H^\infty) \leq 6\sqrt{2} \frac{n}{1-r}$$

$\forall n \geq 1, \forall r \in [0, 1[$.

Proof. 1) We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z}, \quad H_n = \varphi \circ Q_n,$$

$$\Psi = bH_n.$$

Then $\|Q_n\|_2^2 = n$, and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with $\varphi \in \{1, z\}$ and $K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\lambda z}$, and noticing that $H(\varphi \circ K) = H_\varphi$,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi(\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let $b > 0$ such that $b^2 \varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{\lambda,n}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider G such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree n if $\varphi = z$ and $2n$ if $\varphi = z^2$) with positive coefficients. Note that

$$\begin{aligned}
Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = \\
&= (1 - |\lambda|^2)^{-\frac{1}{2}} \left(1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\
&= (1 - r^2)^{-1/2} \left(1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1.
\end{aligned}$$

Hence, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$ and

$$\varphi \circ \psi_1 = \psi_1^N(z), \quad N = 1, 2.$$

4) Next, we show that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} \frac{1}{2\sqrt{2}} \sqrt{\frac{n}{1-r}} \text{ if } N = 1 \\ \frac{1}{16} \frac{n}{1-r} \text{ if } N = 2 \end{cases},$$

where $m \geq 1$ is such that $2m = n$ if n is even and $2m - 1 = n$ if n is odd.

Indeed, setting

$$S_n = \sum_{j=0}^n z^j,$$

we have for every $N \in \{1, 2\}$

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left(\left(1 + (1 + r) \sum_{t=1}^{n-1} z^t + r z^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned}
\sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left(\left(\frac{1 - z^n}{1 - z} \right)^N \right) = \sum_{j=0}^m \left(\sum_{k=0}^N C_k^j \frac{1}{(1 - z)^j} \cdot \left(\frac{-z^n}{1 - z} \right)^{N-j} \right) = \\
&= \sum_{j=0}^m \left(\frac{1}{(1 - z)^N} \right) = \sum_{j=0}^m \left(\sum_{k \geq 0} C_{N+j-1}^j z^j \right) = \sum_{j=0}^m C_{N+j-1}^j = \\
&= \begin{cases} m + 1 \text{ if } N = 1 \\ (m + 1)(m + 2)/2 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ (n + 2)(n + 4)/8 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ n^2/8 \text{ if } N = 2 \end{cases}.
\end{aligned}$$

Finally, since $\sum^m (\psi) = b \sum^m (\varphi \circ \psi_1) = b(1 - r^2)^{-N/2} \sum^m (\psi_1^N)$ we get

$$\sum^m (\psi) \geq \begin{cases} (2(1 - r))^{-1/2} nb/2 \text{ if } N = 1 \\ (2(1 - r))^{-1} n^2 b/8 \text{ if } N = 2 \end{cases},$$

with $b = \varphi(n) = \begin{cases} n^{-1/2} \text{ if } N = 1 \\ n^{-1} \text{ if } N = 2 \end{cases}$ and obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel, we get

$$\begin{aligned} \|\Psi\|_{H^\infty/b_\lambda^n H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \begin{cases} \frac{1}{4\sqrt{2}} \sqrt{\frac{n}{1-r}} \text{ if } N = 1 \\ \frac{1}{32} \frac{n}{1-r} \text{ if } N = 2 \end{cases}. \end{aligned}$$

□

Proof of Theorem 6.1.0. In order to prove the left hand side inequality, it suffices to apply Lemma 6.1.1 with $\varphi(z) = z^N$. Indeed, in this case $H_\varphi = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right) = H_{z^N}$. The right hand side inequality is a straightforward consequence of Corollary 2.2 and Theorem 4.0.

□

6.2. The case $X = H^p$

The aim of this section is to prove the sharpness (for even p) of the upper estimate, found in **Theorem 3.0**, of the quantity $C_{n,r}(H^p, H^\infty)$. This is the subject of the following theorem.

Theorem. 6.2.0 *Let $p \in 2\mathbb{Z}_+$, then*

$$c(\sigma_{\lambda,n}, H^p, H^\infty) \geq \frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|\lambda|} \right)^{\frac{1}{p}},$$

for every $\lambda \in \mathbb{D}$ and every integer $n \geq 1$, where $\sigma_{\lambda,n} = \{\lambda, \lambda, \dots, \lambda\}$ and hence

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-r} \right)^{\frac{1}{p}} \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p which is defined in Theorem 3.0.

We first prove the following lemma.

Lemma. 6.2.1 *Let p, q such that $\frac{p}{q} \in \mathbb{Z}_+$, then $c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{q}{p}}$ for every sequence σ of \mathbb{D} .*

Proof. Step 1. Recalling that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

we first prove that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \}.$$

Indeed, we clearly have the inequality

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} \leq c(\sigma, H^p, H^\infty),$$

and if the inequality were strict, that is to say

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} < \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

then we could write that there exists $\epsilon > 0$ such that for every $f = f_i \cdot f_o \in H^p$ (where f_i stands for the inner function corresponding to f and f_o to the outer one) with $\|f\|_p \leq 1$ (which also implies that $\|f_o\|_p \leq 1$, since $\|f_o\|_p = \|f\|_p$), there exists a function $g \in H^\infty$ verifying both $\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ and $g|_\sigma = f_o|_\sigma$. This entails that $f|_\sigma = (f_i g)|_\sigma$ and since $\|f_i g\|_\infty =$

$\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, we get that $c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, which is a contradiction and proves the equality of Step 1.

Step 2. Using the result of Step 1, we get that $\forall \epsilon > 0$ there exists an outer function $f_o \in H^q$ with $\|f_o\|_p \leq 1$ and such that

$$\inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f_o|_\sigma \} \geq c(\sigma, H^q, H^\infty) - \epsilon.$$

Now let $F = f_o^{\frac{q}{p}} \in H^p$, then $\|F\|_p^p = \|f_o\|_q^q \leq 1$. We suppose that there exists $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$ with

$$\|g\|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}}.$$

Then, since $g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i)^{\frac{q}{p}}$ for all $i = 1..n$, we have $g(\lambda_i)^{\frac{p}{q}} = f_o(\lambda_i)$ and $g^{\frac{p}{q}} \in H^\infty$ since $\frac{p}{q} \in \mathbb{Z}_+$. We also have

$$\left\| g^{\frac{p}{q}} \right\|_\infty = \|g\|_\infty^{\frac{p}{q}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

for all $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$, which gives

$$c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

and since that inequality is true for every $\epsilon > 0$, we get the result. \square

Proof of Theorem 6.2.0. We first prove the left hand side inequality. Writing $p = 2 \cdot \frac{p}{2}$, we apply Lemma 6.2.1 with $q = 2$ and this gives

$$c(\sigma_{\lambda,n}, H^p, H^\infty) \geq c(\sigma_{\lambda,n}, H^2, H^\infty)^{\frac{2}{p}} \geq \frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1 - |\lambda|} \right)^{\frac{2}{p}}$$

for all integer $n \geq 1$. The last inequality being a consequence of Theorem 2.1.2. The right hand side inequality is proved in Theorem 3.0. \square

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