Propositional Mixed Logic: Its Syntax and Semantics

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ABSTRACT. *In this paper, we present a propositional logic (called mixed logic) containing disjoint copies of minimal, intuitionistic and classical logics. We prove a completeness theorem for this logic with respect to a Kripke semantics. We establish some relations between mixed logic and minimal, intuitionistic and classical logics. We present at the end a sequent calculus version for this logic.*

KEYWORDS: *propositional logic, minimal logic, intuitionistic logic, classical logic, mixed logic, completeness theorem, Kripke semantics.*

1. Introduction

Propositional intuitionistic and classical logics (abbreviated: PLI and PLC) are built by adding absurdity rules to propositional minimal logic (abbreviated PLM). The best known formalization consists to adding the intuitionistic absurdity rule (from the absurdity we can deduce all formulas) to PLM to obtain PLI, and to adding the classical absurdity rule (a non false formula is true) to PLM (or PLI) to obtain PLC. With this kind of formalism there are some problems.

– A classical formula does not contain any information on the smallest logical system in which it is derivable. To have this information, we must use the non effective decision algorithms of PLM and PLI. But with these algorithms we cannot know how many times we used the absurdity rules and on which formulas.

– A formula has several derivations and the formula does not contain informations to find its "better" derivation. For example, if one takes $A = (X \to Y) \vee (Y \to X)$, we can prove this formula using the classical absurdity rule on A (i.e. we prove $\neg\neg A$). And we can also prove it using the classical absurdity rule on the variable Y. Indeed, if Y is true, then we have (in PLM) $X \to Y$, and if Y is false, then we have (in PLI) $Y \to X$. The second derivation is nearer to the human reasoning. For this reason we want to call it "a good derivation" of the formula A.

– Each of these three logics has a semantics and a completeness theorem. For PLC it is the truth tables, for PLI it is the intuitionistic Kripke models and for PLM it is the minimal Kripke models. If we look closely at the proofs of the completeness theorems, a great resemblance is seen. Why not study all these logics at the same time? i.e. introduce a single semantics for these logics and only prove one completeness theorem in order to deduce the completeness of each system.

We propose in this paper a partial solution to these problems. We present a propositional logic (called mixed logic and abbreviated PML) containing three kinds of variables: minimal variables indexed by m, intuitionistic variables indexed by i and classical variables indexed by c. We restrict the absurdity rules to the formulas containing the corresponding variables. The main novelty of our system is that minimal, intuitionistic and classical logics appear as fragments. For instance a proof of an intuitionistic formula may use classical lemmas without any restriction. This approach is radically different from the one that consists in changing the rule of the game when we want to change logic. Here there is only one logic which, depending on its use, may appear classical, intuitionistic or minimal. We introduce for the system PML a Kripke semantics which is the superposition of minimal, intuitionistic and classical semantics. We show a completeness theorem which implies the completeness theorems of systems PLM, PLI and PLC. We deduce from this theorem a very significant result which is the following: "for a formula A to be derivable in a logic, it is necessary that the formula contains at least a variable which corresponds to this system". We were interested by labelling problems (we label variables by m, i or c) for classical formulas. We present decision algorithms for these problems and we formally define the concept of "good derivation" for a classical formula. We also present a sequent calculus version of this system. This presentation is coherent with what we already know on sequent calculus: classical logic comes from the possibility to put several formulas on the right.

This paper is an introduction to this domain and much questions remain open. For example, the standard proofs of cut-elimination are not adapted to our system. This comes primarily from impossibility of coding disjunction.

The idea to present only one system for different logics is not completely new. Indeed, J.Y. Girard presented in [GIR 93] a single sequent calculus (denoted LU) common to classical, intuitionistic and linear logics. The idea of Girard is to use a single variable set but different connectives which correspond to each fragment. Each formula is given with a polarity: positive, neutral and negative. For each connective

the rules depend on the polarity of the formulas. On the other hand the system LU has a cut-elimination theorem and then the sub-formula property.

Finally, let us mention that J.-L. Krivine and K. Nour introduced a second order mixed logic in order to type storage and control operators in λ-calculus (see [NOU 00]). The theoretical properties of this system are not difficult to prove because the only connectives are $\rightarrow$ and $\forall$. The presence of $\vee$ in system PML complicates our study.

2. The system PML

We present in this section the natural deduction version of propositional mixed logic.

DEFINITIONS 1. —

(1) We suppose that we have three disjoint countable sets of propositional variables: $\mathcal{V}_m = \{X_m, Y_m, Z_m, ...\}$ *the set of* minimal variables, $\mathcal{V}_i = \{X_i, Y_i, Z_i, ...\}$ *the set of* intuitionistic variables, $\mathcal{V}_c = \{X_c, Y_c, Z_c, ...\}$ *the set of* classical variables *and a special constant denoted* $\perp$.

(2) The formulas *are defined by induction. Each element of* $\mathcal{P} = \mathcal{V}_m \cup \mathcal{V}_i \cup \mathcal{V}_c \cup \{\perp\}$ *is a formula. And if A,B are formulas, then* $A \wedge B$, $A \vee B$ *and* $A \rightarrow B$ *are formulas. We denote* $\neg A = A \rightarrow \perp$.

(3) If A *is a formula, we denote by* $var(A)$ *the set of variables of* A. *A* classical formula *(resp. an* intuitionistic formula*) is a formula* A *such that* $var(A) \subseteq \mathcal{V}_c$ *(resp.* $var(A) \subseteq \mathcal{V}_i \cup \mathcal{V}_c$*). We allow the use of classical variables to build intuitionistic formulas because the intuitionistic absurdity rule is derivable in classical logic.*

(4) A simple sequent *is an expression of the form* $\Gamma \vdash A$ *where* $\Gamma \cup \{A\}$ *is a finite set of formulas. A derivation* $\mathcal{D}$ *may be constructed according to one of the rules below.*

$$(Ax)\ \frac{}{A \vdash A} \qquad (W)\ \frac{\Gamma \vdash A}{\Gamma, B \vdash A}$$

$$(\wedge_I)\ \frac{\Gamma_1 \vdash A_1 \quad \Gamma_2 \vdash A_2}{\Gamma_1, \Gamma_2 \vdash A_1 \wedge A_2} \qquad (\wedge_E)\ \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}$$

$$(\vee_I)\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \qquad (\vee_E)\ \frac{\Gamma_1 \vdash A_1 \vee A_2 \quad \Gamma_2, A_1 \vdash B \quad \Gamma_3, A_2 \vdash B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash B}$$

$$(\rightarrow_I)\ \frac{\Gamma, A_1 \vdash A_2}{\Gamma \vdash A_1 \rightarrow A_2} \qquad (\rightarrow_E)\ \frac{\Gamma_1 \vdash A_1 \rightarrow A_2 \quad \Gamma_2 \vdash A_1}{\Gamma_1, \Gamma_2 \vdash A_2}$$

$$(\perp_i)\ \frac{\Gamma \vdash \perp \quad A \text{ is an intuitionistic formula}}{\Gamma \vdash A}$$

$$(\perp_c)\ \frac{\Gamma \vdash \neg\neg A \quad A \text{ is a classical formula}}{\Gamma \vdash A}$$

The rules given above determine the natural deduction system, abbreviated PML. *If $\mathcal{D}$ is a derivation ending with a simple sequent $\Gamma \vdash A$, then we write $\Gamma \vdash_{pml} A$.*

EXAMPLE 2. —

a) $\vdash_{pml} X_c \vee \neg X_c$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X_c \vdash X_c}}{X_c \vdash X_c \vee \neg X_c} \quad \dfrac{}{\neg(X_c \vee \neg X_c) \vdash \neg(X_c \vee \neg X_c)}}{X_c, \neg(X_c \vee \neg X_c) \vdash \bot}}{\neg(X_c \vee \neg X_c) \vdash \neg X_c}}{\neg(X_c \vee \neg X_c) \vdash X_c \vee \neg X_c} \quad \dfrac{}{\neg(X_c \vee \neg X_c) \vdash \neg(X_c \vee \neg X_c)}}{\neg(X_c \vee \neg X_c) \vdash \bot}}{\vdash \neg\neg(X_c \vee \neg X_c)}}{\vdash X_c \vee \neg X_c}$$

b) $\vdash_{pml} (X_m \to X_c) \vee (X_c \to X_i)$.

$$\dfrac{\dfrac{\vdots}{\vdash X_c \vee \neg X_c} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X_c \vdash X_c}}{X_c, X_m \vdash X_c}}{X_c \vdash X_m \to X_c}}{X_c \vdash (X_m \to X_c) \vee (X_c \to X_i)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X_c \vdash X_c} \quad \dfrac{}{\neg X_c \vdash \neg X_c}}{X_c, \neg X_c \vdash \bot}}{X_c, \neg X_c \vdash X_i}}{\neg X_c \vdash X_c \to X_i}}{\neg X_c \vdash (X_m \to X_c) \vee (X_c \to X_i)}}{\vdash (X_m \to X_c) \vee (X_c \to X_i)}$$

c) $\vdash_{pml} (X_c \to X_m \vee X_i) \to (X_m \vee (X_c \to X_i))$ (left to the readers).

REMARK 3. — Note that the indices of variables used in the derivable formulas give some ideas on their derivations. For the formula $(X_m \to X_c) \vee (X_c \to X_i)$, the classical absurdity rule is used on the variable X_c and the intuitionistic absurdity rule is used on the variable X_i.

DEFINITION 4. — *Let A, F be formulas and $X \in \mathcal{P}$. The formula $A[F/X]$ represents the result of substitution of F to each occurrence of X.*

We have the following result.

THEOREM 5. — *Let $\Gamma \cup \{A, F\}$ be a set of formulas, X_m a minimal variable, X_i an intuitionistic variable, X_c a classical variable, F_i an intuitionistic formula, and F_c a classical formula. If $\Gamma \vdash_{pml} A$, then $\Gamma[F/X_m] \vdash_{pml} A[F/X_m]$, $\Gamma[F_i/X_i] \vdash_{pml} A[F_i/X_i]$ and $\Gamma[F_c/X_c] \vdash_{pml} A[F_c/X_c]$.*

PROOF. — By induction on the proof of $\Gamma \vdash_{pml} A$. ■

3. A semantics for PML

Now we are ready for a definition of Kripke semantics for PML.

DEFINITION 6. — *A* mixed Kripke model *is a triple $\mathcal{K} = (K, \leq, \Vdash)$, where $(K, \leq)$ is an inhabited, partially ordered set (poset), and $\Vdash$ a binary relation on $K \times \mathcal{P}$ such that:*

1) For all $\chi \in \mathcal{P}$, if $\alpha \Vdash \chi$ and $\beta \geq \alpha$, then $\beta \Vdash \chi$.

2) If $\alpha \Vdash \bot$, then, for all classical or intuitionistic variable X_s, $\alpha \Vdash X_s$.

3) If $\alpha \Vdash X_c$ and, $\alpha \not\Vdash \bot$, then for each $\beta \in K$: $\beta \Vdash X_c$.

The relation $\Vdash$ is then extended to logically compound formulas by the following clauses:

– $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$.

– $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$.

– $\alpha \Vdash A \rightarrow B$ iff for all $\beta \geq \alpha$, if $\beta \Vdash A$, then $\beta \Vdash B$.

LEMMA 7. — *For all formulas we have monotonicity: for all $\alpha, \beta \in K$ ($\alpha \Vdash A$ and $\beta \geq \alpha$ implies $\beta \Vdash A$).*

PROOF. — By formula induction. ■

DEFINITION 8. — *A formula A is* valid *in a mixed Kripke model $\mathcal{K} = (K, \leq, \Vdash)$ iff for all $\alpha \in K$, $\alpha \Vdash A$; notation $\mathcal{K} \Vdash A$. If Γ is a set of formulas, we say that $\Gamma \Vdash A$ iff in each mixed model $\mathcal{K}$ such that: if for all $B \in \Gamma$, $\mathcal{K} \Vdash B$, then also $\mathcal{K} \Vdash A$.*

REMARK 9. — To check if $\mathcal{K} \Vdash A$ it is enough to limit $\mathcal{K}$ to the variables of A.

We have the following lemmas.

LEMMA 10. — *Let A be an intuitionistic formula and $\mathcal{K}$ a mixed Kripke model. We have $\mathcal{K} \Vdash \bot \rightarrow A$.*

PROOF. — By induction on the complexity of A. ■

LEMMA 11. — *Let A be a classical formula and $\mathcal{K}$ a mixed Kripke model. We have $\mathcal{K} \Vdash \neg\neg A \rightarrow A$.*

PROOF. — We first prove, by induction, that if B is a classical formula, $\beta \in K$ and $\beta \Vdash B$, then, for each $\gamma \in K$, $\gamma \Vdash B$. Let $\alpha \in K$ such that $\alpha \Vdash \neg\neg A$. We may assume $\alpha \not\Vdash \bot$. Therefore $\alpha \not\Vdash \neg A$ and thus there is $\beta \geq \alpha$ such that $\beta \Vdash A$. We deduce $\alpha \Vdash A$. ■

We can deduce the soundness theorem for PML.

THEOREM 12. — *Let $\Gamma \cup \{A\}$ be a set of formulas. If $\Gamma \vdash_{pml} A$, then $\Gamma \Vdash A$.*

PROOF. — The proof is by induction on derivation of $\Gamma \vdash_{pml} A$ and we use Lemmas 10 and 11. ■

We present now a completeness proof for PML.

DEFINITION 13. — *A set of formulas Δ is said to be* saturated *iff: if $\Delta \vdash_{pml} C \vee D$, then $C \in \Delta$ or $D \in \Delta$.*

REMARK 14. — A saturated set of formulas Δ is closed by deduction. Indeed, if $\Delta \vdash_{pml} B$, then $\Delta \vdash_{pml} B \vee B$, thus $B \in \Delta$.

LEMMA 15. — *If $\Gamma \nvdash_{pml} A$, then there is a saturated set Γ_ω such that $\Gamma \subseteq \Gamma_\omega$ and $\Gamma_\omega \nvdash_{pml} A$.*

PROOF. — Same proof as the corresponding lemma in intuitionistic logic [DAV 01, DAL 94]. ■

DEFINITION 16. — *Let Γ_0 be any saturated set of formulas. Then we define $\mathcal{K} = (K, \subseteq, \Vdash)$ such that $K = \{\Delta \,/\, \Delta$ saturated sets and $\Gamma_0 \subseteq \Delta\}$, and, for each $\chi \in \mathcal{P}$: $\Delta \Vdash \chi$ iff $\chi \in \Delta$.*

LEMMA 17. — *$\mathcal{K}$ is a mixed Kripke model.*

PROOF. — We must prove the three needed conditions:

1) Trivial.

2) If $\Delta \Vdash \bot$, then $\Delta \vdash_{pml} \bot$, thus $\Delta \vdash_{pml} X_i$ and $\Delta \vdash_{pml} X_c$, i.e. $\Delta \Vdash X_i$ and $\Delta \Vdash X_c$.

3) Let $\Delta \Vdash X_c$, $\Delta \not\Vdash \bot$, and $\Delta' \not\Vdash \bot$. We have $\Gamma_0 \vdash_{pml} X_c \vee \neg X_c$, then $\Gamma_0 \vdash_{pml} X_c$ or $\Gamma_0 \vdash_{pml} \neg X_c$. Since $\Gamma_0 \subseteq \Delta$ and $\Gamma_0 \subseteq \Delta'$, we have $\Gamma_0 \Vdash X_c$ and $\Delta' \Vdash X_c$.

■

LEMMA 18. — *For all $\Delta \in K$ and each formula B, $\Delta \Vdash B$ iff $B \in \Delta$.*

PROOF. — By induction on the complexity of B. ■

THEOREM 19. — *Let $\Gamma \cup \{A\}$ be a set of formulas. If $\Gamma \Vdash A$, then $\Gamma \vdash_{pml} A$.*

PROOF. — Suppose $\Gamma \nvdash_{pml} A$, and let Γ_0 be a saturated extension of Γ such that $A \notin \Gamma_0$. By the last construction there is a mixed Kripke model $\mathcal{K} = (K, \subseteq, \Vdash)$ and $\alpha \in K$ such that for all B: $\alpha \Vdash B$ iff $B \in \Gamma_0$. In particular, $\alpha \Vdash B$ for $B \in \Gamma$ and $\alpha \not\Vdash A$. Hence $\Gamma \not\vdash A$. ■

We also have the following results.

THEOREM 20. —

1) The system PML *has the finite mixed Kripke model property.*

2) The system PML *is decidable.*

PROOF. — Same proof as the corresponding result in intuitionistic logic [DAV 01, DAL 94]. ■

4. Properties of PML

In this section we prove the principal result of the paper (Theorems 25 and 27): "To be derivable in the system using only classical (resp. intuitionistic, minimal) rules a mixed formula must contain at least a classical (resp. intuitionistic, minimal) variable". This result is easily shown if the system PML has some sub-formula property. However usually such a property is a direct consequence of the cut-elimination theorem which is difficult to show here because we cannot code the disjunctive formulas (indeed the formula $\neg(\neg A \wedge \neg B) \rightarrow A \vee B$ is not derivable) and eliminate the classical cuts.

DEFINITION 21. —

(1) An intuitionistic mixed Kripke model *(resp. a* minimal mixed Kripke model*) is a mixed Kripke model restricted on the formulas built on the set* $\mathcal{P}_{(i)} = \mathcal{V}_m \cup \mathcal{V}_i \cup \{\bot\}$ *(resp. the formulas built on the set* $\mathcal{P}_{(m)} = \mathcal{V}_m \cup \{\bot\}$*).*

(2) We write $\Gamma \vdash_{(i)} A$ *if* $\Gamma \vdash A$ *is derivable without using the rule* $(\bot_c)$ *and* $\Gamma \vdash_{(m)} A$ *if* $\Gamma \vdash A$ *is derivable without using the rules* $(\bot_i)$ *and* $(\bot_c)$.

We have the following results:

THEOREM 22. —

1) Let $\Gamma \cup \{A\}$ *be a set of formulas without classical variables.* $\Gamma \vdash_{(i)} A$ *iff for all intuitionistic mixed Kripke model* $\mathcal{K}$*:* $\mathcal{K} \Vdash \Gamma$ *implies* $\mathcal{K} \Vdash A$*.*

2) Let $\Gamma \cup \{A\}$ *be a set of formulas without classical and intuitionistic variables.* $\Gamma \vdash_{(m)} A$ *iff for all minimal mixed Kripke model* $\mathcal{K}$*:* $\mathcal{K} \Vdash \Gamma$ *implies* $\mathcal{K} \Vdash A$*.*

PROOF. — In the proof of Theorem 19, we use the derivation rules to prove Lemma 17. ∎

DEFINITION 23. — *For each mixed Kripke model* $\mathcal{K}$ *we define the intuitionistic (resp. the minimal) mixed Kripke model* $\mathcal{K}_{(i)}$ *(resp.* $\mathcal{K}_{(m)}$*) as being* $\mathcal{K}$ *restricted on the set* $\mathcal{P}_{(i)}$ *(resp.* $\mathcal{P}_{(m)}$*). By definition, it is clear that each intuitionistic mixed Kripke model (resp. minimal mixed Kripke model) can be seen as a* $\mathcal{K}_{(i)}$ *(resp. a* $\mathcal{K}_{(m)}$*) for a mixed Kripke model* $\mathcal{K}$*.*

LEMMA 24. —

1) Let A *be a formula without classical variables. We have* $\mathcal{K} \Vdash A$ *iff* $\mathcal{K}_{(i)} \Vdash A$*.*

2) Let A *be a formula without classical and intuitionistic variables. We have* $\mathcal{K} \Vdash A$ *iff* $\mathcal{K}_{(m)} \Vdash A$*.*

PROOF. — By induction on the complexity of A. ∎

The following theorem is now an easy corollary.

THEOREM 25. —

1) Let $\Gamma \cup \{A\}$ *be a set of formulas without classical variables. We have* $\Gamma \vdash_{pml} A$ *iff* $\Gamma \vdash_{(i)} A$*.*

2) Let $\Gamma \cup \{A\}$ be a set of formulas without classical and intuitionistic variables. We have $\Gamma \vdash_{pml} A$ iff $\Gamma \vdash_{(m)} A$.

PROOF. —

1) If $\Gamma \vdash_{pml} A$, then for all mixed Kripke model $\mathcal{K}$: $\mathcal{K} \Vdash \Gamma$ implies $\mathcal{K} \Vdash A$, thus, by Lemma 24, for all intuitionistic mixed Kripke model $\mathcal{K}_{(i)}$: $\mathcal{K}_{(i)} \Vdash \Gamma$ implies $\mathcal{K}_{(i)} \Vdash A$. Therefore, by Theorem 22, $\Gamma \vdash_{(i)} A$.

2) Same proof as 1).

■

DEFINITION 26. — *We write $\Gamma \vdash_{(i')} A$ if $\Gamma \vdash A$ is derivable without using the rule $(\bot_i)$.*

THEOREM 27. — *Let $\Gamma \cup \{A\}$ be a set of formulas without intuitionistic variables. $\Gamma \vdash_{pml} A$ iff $\Gamma \vdash_{(i')} A$.*

PROOF. — Same proof as Theorem 25. ■

The proof of Theorem 25 is not constructive. We will try to make a syntactical and constructive proof of this result (Corollary 37) but for a subsystem of PML.

DEFINITION 28. — *Let ${\mathcal{V}'}_m$ be a countable subset of $\mathcal{V}_m$, and $\mathtt{m}$ be a bijective mapping between $\mathcal{V}_i$ and $\mathcal{V}'_m$. For all formulas which do not contain classical variables the translation $^{\mathtt{m}}$ is defined inductively by: $\bot^{\mathtt{m}}{=}\bot$, ${X_m}^{\mathtt{m}} = X_m$, ${X_i}^{\mathtt{m}} = \neg\neg\mathtt{m}(X_i)$ and $(A \diamond B)^{\mathtt{m}} = A^{\mathtt{m}} \diamond B^{\mathtt{m}}$ if $\diamond \in \{\wedge, \vee, \rightarrow\}$.*

LEMMA 29. — *Let A be an intuitionistic formula. $\vdash_{(m)} \bot \rightarrow A^{\mathtt{m}}$.*

PROOF. — By induction on A. ■

THEOREM 30. — *Let $\Gamma \cup \{A\}$ be a set of formulas without classical variables. If $\Gamma \vdash_{(i)} A$, then $\Gamma^{\mathtt{m}} \vdash_{(m)} A^{\mathtt{m}}$.*

PROOF. — By induction on $\Gamma \vdash_{(i)} A$. ■

COROLLARY 31. — *Let $\Gamma \cup \{A\}$ be a set of formulas without classical and intuitionistic variables. We have $\Gamma \vdash_{(i)} A$ iff $\Gamma \vdash_{(m)} A$.*

PROOF. — By Theorem 30. ■

This method cannot be extended to get a syntactical proof of Theorem 25. We restrict our study to a subsystem of PML.

DEFINITION 32. — *We denote by* PML$^{\vee}$ *the system* PML *with this restriction on the rule $(\vee_E)$: if $A_1 \vee A_2$ is a classical formula, then B is also a classical formula. We denote $\Gamma \vdash^{\vee} A$, if A is derivable by Γ in* PML$^{\vee}$.

REMARK 33. — The following derivation cannot be done in the system PML$^{\vee}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash X_c \vee \neg X_c}\quad \dfrac{\dfrac{}{X_c \vdash X_c}\quad \dfrac{}{X_c \to X_m \vdash X_c \to X_m}}{X_c, X_c \to X_m \vdash X_m}\quad \dfrac{\dfrac{}{\neg X_c \vdash \neg X_c}\quad \dfrac{}{\neg X_c \to X_m \vdash \neg X_c \to X_m}}{\neg X_c, \neg X_c \to X_m \vdash X_m}}{X_c \to X_m, \neg X_c \to X_m \vdash X_m}}{X_c \to X_m \vdash (\neg X_c \to X_m) \to X_m}}{\vdash (X_c \to X_m) \to ((\neg X_c \to X_m) \to X_m)}$$

DEFINITION 34. — *Let $\mathcal{V}'_i$ be a countable subset of $\mathcal{V}_i$, and* i *be a bijective mapping between $\mathcal{V}_c$ and $\mathcal{V}'_i$. For all formulas of* PML *the translation* $^{\mathtt{i}}$ *is defined inductively by: $\bot^{\mathtt{i}}=\bot$, ${X_m}^{\mathtt{i}} = X_m$, ${X_i}^{\mathtt{i}} = X_i$, ${X_c}^{\mathtt{i}} = \neg\neg\mathtt{i}(X_c)$, $(A \diamond B)^{\mathtt{i}} = A^{\mathtt{i}} \diamond B^{\mathtt{i}}$ if $\diamond \in \{\wedge, \to\}$, and $(A \vee B)^{\mathtt{i}} = \neg\neg(A^{\mathtt{i}} \vee B^{\mathtt{i}})$.*

LEMMA 35. — *Let A be a classical formula. We have $\vdash_{(i)} \neg\neg A^{\mathtt{i}} \to A^{\mathtt{i}}$.*

PROOF. — By induction on A. ∎

THEOREM 36. — *Let $\Gamma \cup \{A\}$ be a set of formulas. If $\Gamma \vdash^{\vee} A$, then $\Gamma^{\mathtt{i}} \vdash_{(i)} A^{\mathtt{i}}$.*

PROOF. — By induction on $\Gamma \vdash^{\vee} A$. We use Lemma 35 for the rules $(\bot_c)$ and $(\vee_E)$. ∎

We can then deduce:

COROLLARY 37. —

1) Let $\Gamma \cup \{A\}$ be a set of formulas without classical variables. If $\Gamma \vdash^{\vee} A$, then $\Gamma \vdash_{(i)} A$.

2) Let $\Gamma \cup \{A\}$ be a set of formulas without classical and intuitionistic variables. If $\Gamma \vdash^{\vee} A$, then $\Gamma \vdash_{(m)} A$.

PROOF. — 1) by Theorem 36, and 2) by Corollary 31. ∎

5. Labels

We establish in this section relations between PML and minimal, intuitionistic and classical logics. If A is a derivable formula of ordinary propositional classical logic, we can label the propositional variables of A by m, i or c in order to obtain a derivable formula in PML. It is clear that such a labelling is not unique. We give in this section algorithms in order to give "minimal" labels of classical propositional formulas (Theorem 43) and classical propositional derivations (Theorem 48). We also define the notion of "good" derivation for a propositional classical formula (Definition 50).

DEFINITION 38. —

(1) Let $\mathcal{V} = \{X, Y, Z, ...\}$ be a countable set of propositional variables. We suppose that $\mathcal{V}_m$ (resp. $\mathcal{V}_i$, $\mathcal{V}_c$) are obtained by indexing the variables of $\mathcal{V}$. Using $\mathcal{V} \cup \{\bot\}$ we define, as usually, the minimal, intuitionistic, and classical logic denoted respectively by PLM, PLI *and* PLC. *We use as abbreviations $\vdash_m$, $\vdash_i$, $\vdash_c$ for derivability in* PLM, PLI, PLC *respectively. A formula built on $\mathcal{V} \cup \{\bot\}$ is called* ordinary formula.

(2) A label *is a function $l : \mathcal{V} \to \mathcal{P}$ such that $l(X) \in \{X_m, X_i, X_c\}$. A label l is extended to logical formulas by the following clauses: $l(\bot) = \bot$ and $l(A \diamond B) = l(A) \diamond l(B)$ if $\diamond \in \{\wedge, \vee, \to\}$.*

(3) We define on $\mathcal{V}_m \cup \mathcal{V}_i \cup \mathcal{V}_c$ a binary relation $<$ as follows: for all $X \in \mathcal{V}$, $X_m < X_i < X_c$. We define on labels a binary relation $<$ as follows: $l < l'$ iff (1) for all variable $X \in \mathcal{V}$, $l(X) \leq l'(X)$ and (2) there is a $X \in \mathcal{V}$ such that $l(X) < l'(X)$.

(4) Let l_m (resp. l_i, l_c) be the label defined by: for all $X \in \mathcal{V}$, $l_m(X) = X_m$ (resp. $l_i(X) = X_i$, $l_c(X) = X_c$).

The following result means that PML contains disjoint copies of systems PLM, PLI and PLC.

THEOREM 39. — *Let $\Gamma \cup \{A\}$ be a set of ordinary formulas. We have: $\Gamma \vdash_m A$ iff $l_m(\Gamma) \vdash_{(m)} l_m(A)$, $\Gamma \vdash_i A$ iff $l_i(\Gamma) \vdash_{(i)} l_i(A)$ and $\Gamma \vdash_c A$ iff $l_c(\Gamma) \vdash_{pml} l_c(A)$.*

PROOF. — Easy. ∎

DEFINITION 40. — *Let A be an ordinary formula such that $\vdash_c A$. A* label for A *is a label l such that $\vdash_{pml} l(A)$ and for every variable X which does not appear in A, $l(X) = X_m$.*

REMARK 41. — Let A be an ordinary formula such that $\vdash_c A$. By Theorem 39, l_c is a label for A.

DEFINITION 42. — *Let A be an ordinary formula such that $\vdash_c A$. A* minimal label for A *is a label l for A such that: if $l' \leq l$ is a label for A, then $l' = l$.*

THEOREM 43. — *Let A be an ordinary formula such that $\vdash_c A$. A has a minimal label.*

PROOF. — Since PML is decidable we try all possible labels for A. ∎

EXAMPLE 44. — Let b the label defined by: $b(X) = X_c$, $b(Y) = Y_i$, and for every $Z \neq X$ and Y, $b(Z) = Z_m$. It is easy to check that b is the unique minimal label for the ordinary formula $(Z \to X) \vee (X \to Y)$. The minimal label for an ordinary formula is not unique. Let $A = (X \to Y) \vee (Y \to X)$ and l, l' such that $l(X) = X_c$, $l(Y) = Y_i$, $l'(X) = X_i$ and $l'(Y) = Y_c$. It is easy to check that l and l' are two minimal labels for A but they are not comparable.

DEFINITION 45. — *Let $\mathcal{D}$ be a derivation in* PLC. *A* label for $\mathcal{D}$ *is a label l such that: (1) for every variable X which does not appear in $\mathcal{D}$, $l(X) = X_m$ and (2) by extending l on $\mathcal{D}$ we obtain a derivation in* PML. *A* minimal label for $\mathcal{D}$ *is a label l for $\mathcal{D}$ such that: if $l' \leq l$ is a label for $\mathcal{D}$, then $l' = l$.*

REMARK 46. — l_m (resp. l_i, l_c) is a label for all derivation in PLM (resp. PLI, PLC).

DEFINITION 47. — *Let $l_1, ..., l_n$ be labels. We define a new label $sup(l_1, ..., l_n)$ as follows: for every $X \in \mathcal{V}$, $sup(l_1, ..., l_n)(X) = sup(l_1(X), ..., l_n(X))$.*

THEOREM 48. — *Let $\mathcal{D}$ be a derivation in* PLC. *The derivation $\mathcal{D}$ has a unique minimal label.*

PROOF. — We define the unique minimal label $l_{\mathcal{D}}$ by induction on $\mathcal{D}$.

1) If $\mathcal{D}$ is (Ax), then $l_{\mathcal{D}} = l_m$.

2) If the last rule used in $\mathcal{D}$ is

- (W), $(\wedge_E)$, $(\vee_I)$, or $(\rightarrow_I)$, then $l_{\mathcal{D}} = l_{\mathcal{D}_1}$.
- $(\wedge_I)$, or $(\rightarrow_E)$, then $l_{\mathcal{D}} = sup(l_{\mathcal{D}_1}, l_{\mathcal{D}_2})$.
- $(\vee_E)$, then $l_{\mathcal{D}} = sup(l_{\mathcal{D}_1}, l_{\mathcal{D}_2}, l_{\mathcal{D}_3})$.
- $(\perp_i)$, then $l_{\mathcal{D}} = l \circ l_{\mathcal{D}_1}$, where

$$l(l_{\mathcal{D}_1}(X)) = \begin{cases} X_i & \text{if } X \in var(A) \text{ and } l_{\mathcal{D}_1}(X) \neq X_c \\ X_c & \text{if } X \in var(A) \text{ and } l_{\mathcal{D}_1}(X) = X_c \\ l_{\mathcal{D}_1}(X) & \text{otherwise} \end{cases}$$

- $(\perp_c)$, then $l_{\mathcal{D}} = l \circ l_{\mathcal{D}_1}$, where $l(l_{\mathcal{D}_1}(X)) = \begin{cases} X_c & \text{if } X \in var(A) \\ l_{\mathcal{D}_1}(X) & \text{otherwise} \end{cases}$

■

EXAMPLE 49. — It is easy to check that the label b of the Example 44 is the minimal label for the following derivation:

$$\dfrac{\dfrac{\vdots}{\vdash X \vee \neg X} \quad \dfrac{\dfrac{\dfrac{}{X \vdash X}}{X, Z \vdash X}}{X \vdash Z \rightarrow X}}{X \vdash (Z \rightarrow X) \vee (X \rightarrow Y)} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X} \quad \dfrac{}{\neg X \vdash \neg X}}{X, \neg X \vdash \perp}}{X, \neg X \vdash Y}}{\neg X \vdash X \rightarrow Y}}{\neg X \vdash (Z \rightarrow X) \vee (X \rightarrow Y)}$$
$$\vdash (Z \rightarrow X) \vee (X \rightarrow Y)$$

DEFINITION 50. — *Let A be an ordinary formula such that $\vdash_c A$. A* good derivation *for A is a derivation $\mathcal{D}$ of A in* PLC *such that $l_{\mathcal{D}}$ is a minimal label for A. Intuitively, a good derivation of a formula A is a derivation of A with minimal use of the absurdity rules.*

THEOREM 51. — *Let A be an ordinary formula such that $\vdash_c A$. The formula A has a good derivation.*

PROOF. — Let l_A be a minimal label of A. Since we can enumerate all derivable formulas, then we can find a derivation $\mathcal{D}$ ending with $l_A(A)$. The derivation obtained by erasing the indexes in the derivation $\mathcal{D}$ is a good derivation for A. ■

EXAMPLE 52. — The derivation of the Example 49 is a good derivation for the formula $(Z \rightarrow X) \vee (X \rightarrow Y)$.

6. Sequent calculus

We describe below a sequent calculus version of PML. This sequent calculus is non satisfactory because it does not satisfy the cut-elimination property (Theorem 61).

DEFINITION 53. — *In this section a sequent is of the form $\Gamma \vdash' A; \Delta$ where Γ (resp. Δ) is a finite set of formulas (resp. of classical formulas) and A is a formula. The rules of sequent calculus are the following:*

$$(Ax)\ \frac{}{A \vdash' A;} \qquad (Cut)\ \frac{\Gamma_1, A \vdash' B; \Delta_1 \quad \Gamma_2 \vdash' A; \Delta_2}{\Gamma_1, \Gamma_2 \vdash' B; \Delta_1, \Delta_2}$$

$$(S_r)\ \frac{\Gamma \vdash' A; \bot, \Delta}{\Gamma \vdash' A; \Delta} \qquad (S_l)\ \frac{\Gamma \vdash' A; A, \Delta}{\Gamma \vdash' A; \Delta}$$

$$(W_r)\ \frac{\Gamma \vdash' \bot; \Delta \quad A \text{ is an intuitionistic formula}}{\Gamma \vdash' A; \Delta}$$

$$(W_l)\ \frac{\Gamma \vdash' A; \Delta}{\Gamma, B \vdash' A; \Delta}$$

$$(W'_r)\ \frac{\Gamma \vdash' A; \Delta \quad B \text{ is a classical formula}}{\Gamma \vdash' A; B, \Delta}$$

$$(E)\ \frac{\Gamma \vdash' A; B, \Delta \quad A \text{ is a classical formula}}{\Gamma \vdash' B; A, \Delta}$$

$$(\wedge_r)\ \frac{\Gamma_1 \vdash' A_1; \Delta_1 \quad \Gamma_2 \vdash' A_2; \Delta_2}{\Gamma_1, \Gamma_2 \vdash' A_1 \wedge A_2; \Delta_1, \Delta_2} \qquad (\wedge_l)\ \frac{\Gamma, A_i \vdash' B; \Delta}{\Gamma, A_1 \wedge A_2 \vdash' B; \Delta}$$

$$(\vee_r)\ \frac{\Gamma \vdash' A_i; \Delta}{\Gamma \vdash' A_1 \vee A_2; \Delta} \qquad (\vee_l)\ \frac{\Gamma_1, A_1 \vdash' B; \Delta_1 \quad \Gamma_2, A_2 \vdash' B; \Delta_2}{\Gamma_1, \Gamma_2, A_1 \vee A_2 \vdash' B; \Delta_1, \Delta_2}$$

$$(\rightarrow_r)\ \frac{\Gamma, A_1 \vdash' A_2; \Delta}{\Gamma \vdash' A_1 \rightarrow A_2; \Delta} \qquad (\rightarrow_l)\ \frac{\Gamma_1 \vdash' A_1; \Delta_1 \quad \Gamma_2, A_2 \vdash' B; \Delta_2}{\Gamma_1, \Gamma_2, A_1 \rightarrow A_2 \vdash' B; \Delta_1, \Delta_2}$$

We write $\Gamma \vdash^{pml} A; \Delta$ if there is a derivation $\mathcal{D}$ ending with the sequent $\Gamma \vdash' A; \Delta$.

We wish to show $\Gamma \vdash^{pml} A;$ iff $\Gamma \vdash_{pml} A$.

LEMMA 54. —

1) If A is an intuitionistic formula, then $\vdash^{pml} \bot \rightarrow A;$.

2) If B is a classical formula, then $\vdash^{pml} \neg\neg B \rightarrow B;$.

PROOF. — 1) is easy. For 2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash' B;}}{B \vdash' B; \bot}}{B \vdash' \bot; B}}{\vdash' \neg B; B} \quad \dfrac{}{\bot \vdash' \bot;}}{\neg\neg B \vdash' \bot; B}}{\neg\neg B \vdash' B; \bot}}{\neg\neg B \vdash' B;}}{\vdash' \neg\neg B \to B;}$$

■

THEOREM 55. — *Let $\Gamma \cup \{A\}$ be a set of formulas. If $\Gamma \vdash_{pml} A$, then $\Gamma \vdash^{pml} A;$.*

PROOF. — By induction on the proof of $\Gamma \vdash_{pml} A$. We use the cut rule and Lemma 54. ■

LEMMA 56. — *If A, B are classical formulas, then $\vdash_{pml} [(\neg A \to A) \to A] \wedge [(\neg B \to A) \to (\neg A \to B)]$.*

PROOF. — Easy. ■

DEFINITION 57. — *Let $\neg\Delta$ indicate the negation of the formulas in Δ.*

THEOREM 58. — *Let Γ be a set of formulas, Δ a set of classical formulas, and A a formula. If $\Gamma \vdash^{pml} A; \Delta$, then $\Gamma, \neg\Delta \vdash_{pml} A$.*

PROOF. — By induction on the proof of $\Gamma \vdash^{pml} A; \Delta$. We use Lemma 56 for the rules (E) and (S_l). ■

We can then deduce:

COROLLARY 59. — *Let $\Gamma \cup \{A\}$ be a set of formulas. We have $\Gamma \vdash_{pml} A$ iff $\Gamma \vdash^{pml} A;$.*

PROOF. — We use Theorems 55 and 58. ■

REMARK 60. — The usual process to eliminate cuts in the sequent calculus is not valid for our system. For example, the elimination of cuts in the following derivation needs the use of several non classical formulas on the right.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X_c \vdash' X_c;}}{X_c \vdash' X_c; \bot}}{X_c \vdash' \bot; X_c}}{\vdash' \neg X_c; X_c}}{\vdash' X_c \vee \neg X_c; X_c}}{\vdash' X_c; X_c \vee \neg X_c}}{\vdash' X_c \vee \neg X_c; X_c \vee \neg X_c} \qquad \dfrac{\dfrac{\dfrac{}{X_c \vdash' X_c;} \quad \dfrac{}{X_m \vdash' X_m;}}{X_c, X_c \to X_m \vdash' X_m;} \quad \dfrac{\dfrac{}{\neg X_c \vdash' \neg X_c;} \quad \dfrac{}{X_m \vdash' X_m;}}{\neg X_c, \neg X_c \to X_m \vdash' X_m;}}{X_c \vee \neg X_c, X_c \to X_m, \neg X_c \to X_m \vdash' X_m;}$$

$$\dfrac{\vdash' X_c \vee \neg X_c; \qquad X_c \vee \neg X_c, X_c \to X_m, \neg X_c \to X_m \vdash' X_m;}{X_c \to X_m, \neg X_c \to X_m \vdash' X_m;}$$

THEOREM 61. — *The* PML *sequent calculus does not satisfy the cut-elimination (even weak) property.*

PROOF. — We prove that there is no normal derivation (i.e. without cuts) for the sequent $X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash^{pml} X_m;$. By using the following mixed Kripke model $\mathcal{K} = (K, \leq, \Vdash)$ where $K = \{\alpha, \beta\}$, $\alpha \leq \beta$, $\beta \Vdash X_m$, and $\beta \Vdash \bot$, we prove easily that $X_c \rightarrow X_m \not\vdash^{pml} X_m;$, $X_c \rightarrow X_m \not\vdash^{pml} \neg X_c;$, $\neg X_c \rightarrow X_m \not\vdash^{pml} X_c;$, $\neg X_c \rightarrow X_m \not\vdash^{pml} X_m;$, $\not\vdash^{pml} X_c;$, and $\not\vdash^{pml} \neg X_c;$. Let us take a minimal derivation of $X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash^{pml} X_m$; and look at the last used rule.

1) If it is the rule (W_l), then $X_c \rightarrow X_m \vdash^{pml} X_m$; or $\neg X_c \rightarrow X_m \vdash^{pml} X_m;$.

2) If it is the rule $(\rightarrow_l)$, then $\neg X_c \rightarrow X_m \vdash^{pml} X_c$; or $\vdash^{pml} X_c$; or $X_c \rightarrow X_m \vdash^{pml} \neg X_c$; or $\vdash^{pml} \neg X_c;$.

3) If it is the rule (S_r), then $X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash^{pml} X_m; \bot$. We again look at the last rule used.

- If it is the rule (W_l), then $X_c \rightarrow X_m \vdash^{pml} X_m; \bot$ or $\neg X_c \rightarrow X_m \vdash^{pml} X_m; \bot$.

- If it is the rule $(\rightarrow_l)$, then $\neg X_c \rightarrow X_m \vdash^{pml} X_c$; or $\neg X_c \rightarrow X_m \vdash^{pml} X_c; \bot$ or $\vdash^{pml} X_c$; or $\vdash^{pml} X_c; \bot$ or $X_c \rightarrow X_m \vdash^{pml} \neg X_c$; or $X_c \rightarrow X_m \vdash^{pml} \neg X_c; \bot$ or $\vdash^{pml} \neg X_c$; or $\vdash^{pml} \neg X_c; \bot$.

■

REMARK 62. — To get a normal derivation of the sequent $X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash' X_m;$, we need more flexible rules. For example:

– allowing the use of the logical rules each formula on the right;

– allowing several occurrences of the same non classical formula on the right.

Here is a derivation of sequent $X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash' X_m$ without using the cut rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X_c \vdash' X_c} \quad \dfrac{}{X_m \vdash' X_m}}{X_c, X_c \rightarrow X_m \vdash' X_m}}{X_c, X_c \rightarrow X_m \vdash' X_m, \bot}}{X_c \rightarrow X_m \vdash' X_m, \neg X_c} \quad \dfrac{}{X_m \vdash' X_m}}{X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash' X_m, X_m}}{X_c \rightarrow X_m, \neg X_c \rightarrow X_m \vdash' X_m}$$

OPEN QUESTION. — *"Is it possible to eliminate cuts in such a system?"*

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7. References

[DAL 94] VAN DALEN D., *Logic and structure*, Spriger-Verlag, Berlin Heidelberg, 1994.

[DAV 01] DAVID R., NOUR K., RAFFALLI C., *Introduction à la logique : théorie de la démonstration*, Dunod, Paris, 2001.

[GIR 93] GIRARD J.-Y., "On the unity of logic", *Annals of pure and applied logic*, vol. 59, 1993, p. 201–217.

[NOU 00] NOUR K., "Mixed logic and storage operators", *Archive for Mathematical Logic*, vol. 39, 2000, p. 261–280.