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Numerical Analysis /*Analyse Numérique.*

A consistent intermediate wave speed for a well-balanced HLLC solver.

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Abstract. In this work, we present a HLLC scheme modification for application to nonhomogeneous shallow-water equations with pollutant transport. This new version is related to the definition of a consistent approximation of the intermediate wave speed. Numerical results are presented to illustrate the importance of such approximation to get appropriate pollutant concentration profiles.

Une vitesse intermédiaire consistante pour un schéma HLLC bien équilibré.

Résumé - *Nous présentons une modification du schéma HLLC pour application aux équations de Saint-Venant non homogènes avec transport de polluants. Cette nouvelle version du schéma est reliée à la définition d'une vitesse d'onde intermédiaire consistante. Un exemple est donné afin d'illuster comment une mauvaise approximation de la vitesse intermédiaire peut entraîner de mauvais résultats sur l'approximation de la concentration en polluant même avec un schéma bien équilibré.*

Version française abrégée - Nous proposons une extension du schéma HLLC pour les équations de Saint-Venant avec topographie et propagation polluants (voir [6]). Pour ce système le schéma HLLC peut s'écrire en fonction du flux numérique provenant du schéma HLL.

Pour le cas homogène, ce schéma est basé sur le fait que la troisième composante du flux peut s'écrire en fonction de la première composante du flux et de la concentration en polluant. La définition de la troisième composante du flux est

$$[F^{hllc}_{i+1/2}]_3 = [F^{hll}_{i+1/2}]_1 \varphi^*$$

La définition de φ^* est simplement la concentration du polluant à gauche ou à droite de l'intervalle $x = x_{i+1/2}$ en fonction du signe de la vitesse intermédiaire S^*.

Pour le cas non homogène nous considérons un schéma de type HLL écrit comme fonction du flux numérique qui dépend de la définition du terme de front (voir [4], [8]). Ceci va nous permettre d'obtenir une extension naturelle du schéma HLLC au cas non homogène à partir du cas homogène. Nous prouvons que le schéma proposé calcule exactement la solution stationnaire de l'eau au repos et qu'il est asymptotiquement bien équilibré (voir [4]) indépendamment de la définition de S^*. Nous proposons alors une modification de S^*, qui dépend de la topographie, qui vaut zéro au repos et qui évite des pics anormaux de polluants. En guise d'exemple, nous exhibons un test numérique sur lequel nous observons l'importance de cette modification. L'expression de S^* est donnée par (5).

1 Introduction

In this paper we study the extension of the HLLC solver to the Shallow Water Equations (SWE) with topography term, either in the 1D case with pollutant or equivalently in the 2D case.

The HLLC solver is an improvement of HLL solvers. HLL solver is low cost-computing since no characteristic decomposition is required. But, HLL solver considers only the maximum and the minimum eigenvalues of the system, hence the scheme presents a large diffusion related to the intermediate field. For 1D SWE with pollutant or 2D SWE, we have three characteristics fields. At contrary, HLLC solver takes into account the intermediate velocity, see e.g. [15].

In order to extend the HLLC solver to the non-homogeneous case, we study its well-balance properties. Also, HLLC solver needs to predict the intermediate wave speed at intercells, thus we propose a consistent definition.

The well-balanced properties are related to the stationary solutions of the system. In our case, we seek numerical schemes that preserves at least the solution of water at rest. This is the so-called C-property introduced by Bermúdez and Vázquez in [2]. In [9], Greenberg and Leroux introduce the concept of well-balanced numerical schemes : they define a numerical scheme that balances the different terms for a non-homogeneous hyperbolic equation. The well-balanced property for hyperbolic systems with source terms has been studied for kinetic schemes (see Perthame et al., [13]), relaxation solvers (see Bouchut, [3]) and a family of Q-schemes (see Chacón et al., [5]).

An other approach consist to rewrite the hyperbolic system with source term as a purely nonconservative term, by including a new equation. In that approach, because of the presence of nonconservative products, a new definition of weak solution must be introduced, depending on the choice of a family paths, see DalMaso et al. [7], Castro et al. [12], Gosse et al [6], LeFloch-Tzavaras [10]. Also the well-balanced properties of a numerical scheme can be related to its convergence. In [14], the authors prove an extension of Lax-Wendroff theorem which is related to the well-balance properties.

In [4], the authors introduce the concept of asymptotically well-balanced. A numerical scheme verifies this property if it preserves all regular stationary solutions of the non-homogeneous hyperbolic system over a set whose the measure of the complementary tends to zero when the space step Δx tends to zero. We will consider this property in the present paper.

In the present paper, we present a natural extension of the HLLC solver to the non-homogeneous SWE with pollutant. We show that, independently of the definition of the intermediate wave speed S^*, the scheme is asymptotically well-balanced. Anyway, we show that we need to define a consistent S^* depending on the source term that means an expression which vanish when water is at rest. To illustrate such phenomena, we present a numerical test which compare the same well-balanced scheme but with two different definitions of S^*, consistent or not.

This paper is organized as follows. In Section 2, we present the 1D SWE equations we consider. In Section 3, we present the well-knowned HLL and HLLC solvers in the homogeneous case. In Section 4, we propose the extension of HLLC solvers, based on given HLL solvers. We show some properties of the proposed HLLC solver and propose a consistent definition of the intermediate wave speed at interfaces S^*. In Section 5, we present a numerical test which show the good results obtained when a well-balanced HLLC solver with a consistent definition of S^* are considered and bad results if S^* is not consistent.

2 The equations

We denote the vectors of unknows by $W : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}^m$, and by $F : \mathbf{R}^m \to \mathbf{R}^m$ the flux function. The conservation law we consider is the following :

$$\partial_t W(x,t) + \partial_x F(W(x,t)) = G(x,W), \quad x \in [0,L], t \in [0,T]. \tag{1}$$

with : $W = (h,q,r)^T$, $h = h(x,t)$ is the height of the water column at instant t and position x, $q = hu$ is the discharge, u is the averaged horizontal velocity, r is the third unknown, $F(W) = (q, \dfrac{q^2}{h} + \dfrac{1}{2}gh^2, \dfrac{q\,r}{h})^T$, g is the gravity constant and $G(x,W) = (0, -ghz_b'(x), 0)^T$ is the source term, where z_b denotes the topography.

In the case where we consider the 1D SWE with pollutant, $r = h\varphi$ with φ the pollutant concentration. In the case where we consider the 2D SWE, we have : $\partial_t W \ + \ \operatorname{div} F(W) \ = 0$, with $W = (h, \vec{q})^T$, h the water depth, $\vec{q} = (q_1, q_2)$ the discharge, $\vec{q} = h\vec{u}$, $\vec{u}$ the depth-averaged velocity,

$F(W) = \big(G(W), H(W)\big)^T$ is the flux vector. Then, if we use the rotational invariance property of the equations, see [15], then the 2D SWE are reduced to a sum of 1D local Riemann problems, which are the same equations than the previous 1D SWE with pollutant.

3 The standard HLL and HLLC schemes for homogeneous case

Using a conservative finite volume method, the general structure is the following. We define the points $x_i = i\,\Delta x$ as a partition of $[0, L]$ with constant step Δx. For a given time $t = t^n$, we set $t^{n+1} = t^n + \Delta t$. The time step Δt is such that stability condition is satisfied. Then, if we denote $W_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W(x, t^n)dx$, a conservative three point finite volume scheme is :

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + \frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x} = 0.$$

where $F_{i+1/2}^n = F_{i+1/2}(W_i^n, W_{i+1}^n)$ is the numerical flux function.

The eigenvalues of the jacobian matrix of the flux are : $\lambda_1 = (u - c)$, $\lambda_2 = u$, $\lambda_3 = (u + c)$ with $c = \sqrt{g\,h}$. The HLL solver is defined by :

$$F_{i+1/2}^{hll} = \frac{S_R F(W_i) - S_L F(W_{i+1}) + S_L S_R (W_{i+1} - W_i)}{S_R - S_L}$$

where S_L and S_R are approximation of lower and upper bounds respectively, of the smallest and largest local speeds. We set : $S_L = \min\{\lambda_1(W_i), 0\}$ and $S_R = \max\{\lambda_3(W_{i+1}), 0\}$, but other definitions are possible, see [15].

The HLLC solver takes into account the intermediate eigenvalue $\lambda_2 = u$. A possible version of HLLC solvers (for the homogeneous SWE) is such that the flux function relates the flux corresponding to the passive scalar to the flux of the mass conservation equation, see [15]. Namely, we set : $[F]_3 = [F_1] \cdot \varphi$ (with $\varphi = r/h$). So, the first and second component of the flux are approximated by a given HLL solver. And the third component is approximated using the first component of the numerical flux of HLL solver as follows :

$$[F_{i+1/2}^{hllc}]_3 = [F_{i+1/2}^{hll}]_1 \cdot \varphi^*$$

where $\varphi^* = r_i/h_i$ if $S^* \geq 0$ and $\varphi^* = r_{i+1}/h_{i+1}$ if $S^* < 0$. The value of S^* must approximate the intermediate wave speed at intercell $x_{i+1/2}$. In [15], different definitions of S^* can be found. The following definition is a common choice :

$$S^* = \frac{S_L h_{i+1}(u_{i+1} - S_R) - S_R h_i (u_i - S_L)}{h_{i+1}(u_{i+1} - S_R) - h_i(u_i - S_L)} \tag{2}$$

4 Extension to the non-homogeneous case

In this section we present the extension of the HLLC solvers to the non-homogeneous case. It is based on two points : i) the structure of the scheme in conservative form ; ii) a consistent definition of S^*. In Subsection 4.1, we present the extension of the HLLC solver and a consistent definition of S^*. In Subsection 4.2, we summarize some possible choices for HLL solvers. In Subsection 4.3, we show the well-balanced properties of the proposed HLLC scheme.

4.1 HLLC scheme with source term

If we consider numerical schemes in conservative form, the structure is obtained by integrating (1) on the control volume $(x_{i-1/2}, x_{i+1/2})$. This gives :

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + \frac{F_{G,i+1/2}^n - F_{G,i-1/2}^n}{\Delta x} = G_i^n, \tag{3}$$

where G_i^n is a centered approximation of G at $x = x_i : G_i = (G_{i-1/2} + G_{i+1/2})/2$. If we consider a Godunov solver, $F^n_{G,i+1/2}$ is the approximation of $F(W(x_{i+1/2}, t^n))$ where W is the solution of the non-homogeneous Riemann problem.

Let us point out that we do not follow the standard approach which consists to consider the homogeneous flux then upwind the source term, see [2]. Here, $F^n_{G,i+1/2}$ depends on the source term G. Moreover, we notice that under structure (3), the extension of HLLC solver is simple and natural. If we consider a given HLL numerical flux, we can apply the same idea as for the homogeneous system : let us denote by F_G^{hll} the HLL numerical flux in the presence of source term G. Then, the first and second components of F_G^{hllc} are equal to F_G^{hll}, and we set :

$$[F_G^{hllc}]_3 = [F_G^{hll}]_1 \cdot \varphi^* \tag{4}$$

with $\varphi^* = r_i/h_i$ if $S^* > 0$ and $\varphi^* = r_{i+1}/h_{i+1}$ if $S^* < 0$.

Concerning the definition of S^*, we can consider (2). Nevertheless, we can easily prove that for water at rest solution, this definition leads to $S^* \neq 0$. Thus, we propose the following definition :

$$S^* = \frac{S_L q_{i+1} - S_R q_i - S_L S_R (h_{i+1} - h_i - \Delta x [\widetilde{A^{-}1}(W_{i+1/2}) G_{i+1/2}]_1)}{h_{i+1}(u_{i+1} - S_R) - h_i(u_i - S_L)}. \tag{5}$$

where $G_{i+1/2}$ is approximation of G at $x = x_{i+1/2}$. Concretely we set $[G_{i+1/2}]_l = 0$, for $l = 1$, $l = 3$ and

$$[G_{i+1/2}]_2 = -g \frac{h_i + h_{i+1}}{2} \frac{z_b(x_{i+1}) - z_b(x_i)}{\Delta x}.$$

By $\widetilde{A}^{-1}(W_{i+1/2})$ we denote an approximation of the inverse matrix of A at $x_{i+1/2}$, and if an eigenvalue of A vanish then the corresponding eigenvalue of $\widetilde{A}^{-1}$ is setted to zero.

4.2 Well-balanced HLL solvers

In this subsection, we present briefly the existing HLL scheme extended to the non-homogeneous case which respect the well-balance property. We rewrite them under the structure (3)-(4) we are interested in.

We can classify these HLL solvers for SWE with topography term in two types : *i*) well-balanced schemes for the water at rest solution only ; *ii*) well-balanced schemes for all stationary solutions.
i) The idea followed by many studies consists to use a scheme for the homogeneous case but evaluated at different states taking into account the topography term. That is,

$$F^{hll}_{G,i+1/2} = F^{hll}_{i+1/2}(W^-_{i+1/2}, W^+_{i+1/2}), \quad \text{verifying } (W^+_{i+1/2} - W^-_{i+1/2}) = 0 \text{ for water at rest.}$$

For example, Zhou et al. propose in [16] a linear approximation satisfying the previous property. LeVeque in [11] proposes a technique that build the states preserving a desired stationary solution. In the case of water at rest, LeVeque proposes to replace h by the water surface. Also, the hydrostatic reconstruction proposed by Audusse et al. in [1] can be applied. In this case, the water column is evaluated by adding and substituing two evaluations of the topography. Another technique consists to rewrite the SWE in function of the water surface and not in function of the water.
ii) A well-balanced HLL solver for all stationary solutions is presented in [8]. It is defined as follows :

$$\begin{aligned} F^{hll}_{G,i+1/2} \quad = \quad & \frac{F(W_i) + F(W_{i+1})}{2} - \frac{1}{2}\Big(\frac{S_R + S_L}{S_R - S_L}(F(W_{i+1}) - F(W_i) - \Delta x G) + \\ & + \frac{S_R S_L}{S_R - S_L}(W_{i+1} - W_i - \Delta x \widetilde{A^{-1}}(W_{i+1/2}) G_{i+1/2}) \end{aligned} \tag{6}$$

This scheme can be applied to any hyperbolic system with source term, hence to SWE with topography term. It is asymptotically well-balanced in the sense of [4]. That is, this scheme preserves any steady state up to second order.

4.3 Well-balanced and consistent HLLC solvers

We can use any of the previous versions of HLL solvers. Then, following the structure (3)-(4), we obtain the corresponding HLLC solver. Let us study their well-balanced properties. We have :

Theorem 4.1 *If the HLL solver is well-balanced, then the HLLC solver defined by* (4) *is well-balanced too, independently of the definition of* S^*.

The proof is straightforward since all stationary solutions verify $q = 0$ or φ constant.

Corollary 4.2 *a) Any HLLC solver builded from a HLL solver of type* i) *(see subsection* 4.2) *preserves the water at rest solution.*
b)Any HLLC solvers builded from HLL solver of type ii) *is well-balanced for any stationary solutions (in the sense of* [4]*). This remains true independently of the definition of* S^*.

As we wrote above, the well-balanced properties are independent of the definition of S^*. Nevertheless, we proposed in (5) a modification of the classical definition of S^* in order to take into account the topography term. Then, we have the following result :

Theorem 4.3 *The intermediate wave speed* S^* *defined by* (5) *is an approximation of* u *at third order in space for any stationary solution. Furthermore,* S^* *vanishes when water is at rest.*

The proof is based on the fact that any stationary solution of SWE verifies q constant. Moreover $(h_{i+1} - h_i - \Delta x[\widetilde{A^{-1}G_{i+1/2}}]_1) = \mathcal{O}(\Delta x^3)$ for any stationary solution and it vanishs for water at rest.

5 Numerical tests

Let us consider the 1D SWE with pollutant : $r = h\varphi$ with φ the pollutant concentration. Then, we show an influence of the definition of S^* on the pollutant concentration. We consider a domain of 4 meters long discretized with 50 points, a CFL condition equal to 0.9, the initial conditions $h(x, 0) = (18 - z_b(x))$, $q(x, 0) = h(x, 0)$. We consider the topography z_b and r defined by, Fig. 1 :

$$z_b(x) = \begin{cases} 1 & 1.5 < x < 2.5 \\ 0 & \text{otherwise} \end{cases} \qquad r(x) = \begin{cases} h(x) & 1.5 < x < 2.5 \\ 0 & \text{otherwise} \end{cases}$$

We set the final time to $T = 0.5$s. In Fig.1 a), we plot the computed pollutant concentration and the topography. We can observe the two different concentration values obtained with S^* defined by (2) (black dots) and (5) (squares) respectively. In the first case, S^* is not a consistent approximation of the velocity. In that case, we obtain a negative value near $x = 2.5$, see Fig. 1 b). At left of $x = 2.5$, S^* is positive and it is negative at $x = 2.5$. Thus, the pollutant is transported to the right until this point and transported to the left at $x = 2.5$ only. This produces the pollutant pick observed in Fig.1 a). At contrary, if we consider the consistent definition of S^* (5) then S^* is positive everywhere and the pollutant is transported smoothly, without any pick.

Furthermore, we can notice in Fig.1 b) that S^* is defined at intercells $x = x_{i+1/2}$, and the velocity is defined at points $x = x_i$. Then, we can observe that S^* defined by (5) is effectively a good approximation of the velocity at intermediate points.

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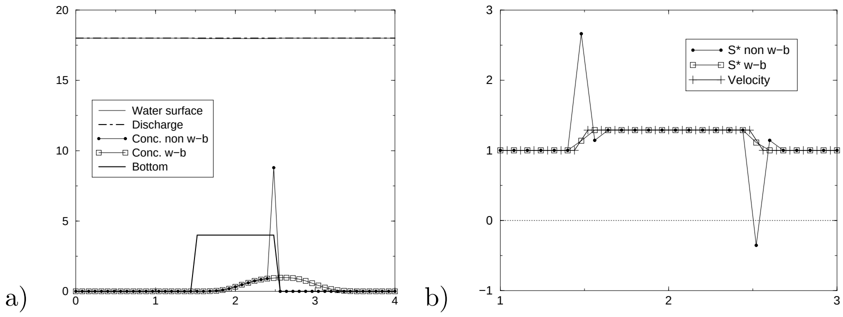

FIGURE 1 – a) Pollutant concentration and topography ; b) Value of S*

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