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POLYNOMIAL SYSTEMS SUPPORTED ON CIRCUITS AND DESSINS D'ENFANTS

FREDERIC BIHAN

ABSTRACT. We study polynomial systems whose equations have as common support a set \mathcal{C} of $n + 2$ points in \mathbb{Z}^n called a circuit. We find a bound on the number of real solutions to such systems which depends on n , the dimension of the affine span of the minimal affinely dependent subset of \mathcal{C} , and the rank modulo 2 of \mathcal{C} . We prove that this bound is sharp by drawing so-called dessins d'enfant on the Riemann sphere. We also obtain that the maximal number of solutions with positive coordinates to systems supported on circuits in \mathbb{Z}^n is $n + 1$, which is very small comparatively to the bound given by the Khovanskii fewnomial theorem.

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The support of a multivariate polynomial is the set of exponent vectors of the monomials appearing in the polynomial. A polynomial system of n polynomial equations in n variables is supported on $\mathcal{A} \subset \mathbb{Z}^n$ if \mathcal{A} is the common support of each polynomial in the system. A theorem due to Kouchnirenko gives that the number of (simple) solutions in the complex torus $(\mathbb{C}^*)^n$ to a generic polynomial system supported on \mathcal{A} is the volume $v(\mathcal{A})$ of the convex hull of \mathcal{A} , normalized so that the unit cube $[0, 1]^n$ has volume $n!$.

Here, we consider generic polynomial systems with real coefficients, and are interested in their numbers of real solutions in the real torus $(\mathbb{R}^*)^n$. Contrary to the complex case, the number of real solutions depends on the coefficients of the system, and one of the most important question is to find a sharp upper bound (for related results, see [1, 5, 8, 9] for example). A trivial bound is given by $v(\mathcal{A})$, the number of complex solutions. Another bound due to Khovanskii depends only on the number n of variables and the cardinality of the support \mathcal{A} . The above Kouchnirenko result shows immediately that such a bound belongs to the “real world”, that is, cannot exist as a bound on the number of complex solutions. The Khovanskii bound is

$$2^n 2^{\binom{|\mathcal{A}|}{2}} \cdot (n + 1)^{|\mathcal{A}|},$$

and gives an easy way to construct supports \mathcal{A} for which the Kouchnirenko bound $v(\mathcal{A})$ is not sharp. The Khovanskii bound can be stated alternatively without the term 2^n as a bound on the number of *positive solutions*, which are solutions with positive coordinates.

Since we are interested only in the solutions in the real torus, we have the freedom to translate the support \mathcal{A} by an integral vector and also to choose a basis of the lattice \mathbb{Z}^n . A translation by an integral vector corresponds to multiplying each polynomial of the system by a monomial, while a change of basis for the lattice \mathbb{Z}^n corresponds to a monomial change of coordinates (with coefficients 1) for the torus. These operations do

not change the number of positive, real and complex solutions. In particular, we will always assume that $0 \in \mathcal{A}$. The *rank modulo 2* of \mathcal{A} is the rank of the reduction modulo 2 of the matrix obtained by putting in columns the non zero elements of \mathcal{A} . We will denote it by $rk(\bar{\mathcal{A}})$.

If $\mathcal{A} \subset \mathbb{Z}^n$ and $|\mathcal{A}| < n + 1$, then $v(\mathcal{A}) = 0$, so that supports with such numbers of elements are not interesting. The first case where $v(\mathcal{A})$ is different from 0 arises when $|\mathcal{A}| = n + 1$, and the convex hull of \mathcal{A} is an n -dimensional simplex. This case is easy. The number of positive solutions to such systems is at most 1, so that the number of real solutions is at most 2^n . In fact, the maximal number of real solutions is $2^{n-rk(\bar{\mathcal{A}})}$ ([10], see also Lemma 1.3).

The goal of this paper is to obtain similar results in the first non trivial case, when the support is a *circuit*. A circuit is a set $\mathcal{C} \subset \mathbb{Z}^n$ of $n + 2$ points which affinely span \mathbb{R}^n . It contains an unique minimal affinely dependent subset, and we denote by $m(\mathcal{C})$ the dimension of the affine span of this subset. If $m(\mathcal{C}) = n$, that is, if any proper subset of \mathcal{C} is affinely independent, then \mathcal{C} is called a *non degenerate circuit*. We have $m(\mathcal{C}) = 1$ for example when three points of \mathcal{C} lie on a same line. In [1], the authors consider circuits \mathcal{C} (and also more general supports called near circuits) such that the index of $\mathbb{Z}\mathcal{C}$ in \mathbb{Z}^n is odd. This corresponds to circuits $\mathcal{C} \subset \mathbb{Z}^n$ with $rk(\bar{\mathcal{C}}) = n$. They obtain the upper bound $2m(\mathcal{C}) + 1$ on the number of real solutions and also prove that this bound is sharp among systems supported on circuits $\mathcal{C} \subset \mathbb{Z}^n$ with $rk(\bar{\mathcal{C}}) = n$. In particular, this gives the sharp bound $2n + 1$ which can be attained only with non degenerate circuits. We generalize the results in [1] to arbitrary values of the rank modulo 2. The first result gives sharp bounds on the number of positive solutions. Obviously, multiplying by 2^n such a bound gives a bound on the number of real solutions, which is sharp among systems supported on circuits \mathcal{C} with $rk(\bar{\mathcal{C}}) = 0$.

Theorem A. *The number of positive solutions to a generic real polynomial system supported on a circuit $\mathcal{C} \subset \mathbb{Z}^n$ is at most*

$$m(\mathcal{C}) + 1.$$

Therefore, the number of real solutions to a generic real polynomial system supported on a circuit $\mathcal{C} \subset \mathbb{Z}^n$ is at most

$$2^n(m(\mathcal{C}) + 1).$$

Moreover, the first bound, and thus the second bound, is sharp. Namely, for any integer m such that $1 \leq m \leq n$, there exist a circuit $\mathcal{C} \subset \mathbb{Z}^n$ with $m(\mathcal{C}) = m$, and a system supported on \mathcal{C} which has $m + 1$ positive solutions.

As a consequence, the number of positive solutions to a generic system supported on a circuit in \mathbb{Z}^n is at most

$$n + 1$$

while its number of real solutions is at most

$$2^n \cdot (n + 1)$$

and these bounds are sharp and can be attained only with non degenerate circuits.

In particular, as in the case of supports forming a simplex, the Khovansky bound is far from being sharp among systems supported on circuits. Theorem A follows from Proposition 2.1 and Theorem 3.7.

Theorem B. *The number, N , of real solutions to a generic real polynomial system supported on a circuit $\mathcal{C} \subset \mathbb{Z}^n$ satisfies*

(1) *If $rk(\bar{\mathcal{C}}) \leq m(\mathcal{C})$, then*

$$N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (m(\mathcal{C}) + rk(\bar{\mathcal{C}}) + 1).$$

(2) *if $rk(\bar{\mathcal{C}}) \geq m(\mathcal{C})$, then*

$$N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (2m(\mathcal{C}) + 1).$$

Moreover, both these bounds are sharp. Namely, let n, m, R be integers such that $1 \leq m \leq n$ and $0 \leq R \leq n$. If $R \leq m$ (resp. $R \geq m$), there exist a circuit $\mathcal{C} \subset \mathbb{Z}^n$ with $m(\mathcal{C}) = m$, $rk(\bar{\mathcal{C}}) = R$, and a system supported on \mathcal{C} whose number of real solutions is the bound in (1) (resp. the bound in (2)).

The bounds in Theorem B look like the bound $2^{n-rk(\bar{\mathcal{A}})}$ when \mathcal{A} forms an n -dimensional simplex. However, there is an essential difference in that the sharp bounds in Theorem B do not provide the maximal number of real solutions to systems supported on a *given* circuit. Theorem B follows from Theorem 2.2 and Theorem 3.8.

We consider the eliminant defined in [1] of a system supported on a circuit $\mathcal{C} \subset \mathbb{Z}^n$. Assuming that $\mathcal{C} = \{0, \ell e_1, w_1, \dots, w_n\} \subset \mathbb{Z}^n$, this eliminant is a univariate polynomial of the form

$$f(x) = x^{\lambda_0} \prod_{i=1}^t (g_i(x))^{\lambda_i} - \prod_{i=t+1}^{\nu} (g_i(x))^{\lambda_i},$$

where each polynomial g_i has the form $g_i(x) = a_i + b_i x^\ell$ with a_i and b_i non zero real numbers. Here, the coefficients a_i and b_i come from the coefficients of the system while the other numbers are determined by \mathcal{C} . The number ν is equal to $m(\mathcal{C})$ if 0 and ℓe_1 belong to the minimal affinely dependent subset of \mathcal{C} . If $rk(\bar{\mathcal{C}}) \neq 0$, then the integer ℓ can be assumed to be odd. We choose to distinguish the case $rk(\bar{\mathcal{C}}) \neq 0$ from the case $rk(\bar{\mathcal{C}}) = 0$ for which the number of real solutions is given by the number of positive ones. When $rk(\bar{\mathcal{C}}) = n$, it is proved in [1] that the real solutions to the system are in bijection with the real roots of f via the projection onto the first coordinate axis. In general, the real solutions project onto the real roots of f which satisfy sign conditions involving products of polynomials g_i . Moreover, such a real root is the image of $2^{n-rk(\bar{\mathcal{C}})}$ solutions to the system, at least when ℓ is odd. The positive solutions project bijectively onto the positive roots of f at which all g_i are positive. We determine the above sign conditions, and prove the upper bounds in Theorem A and B using essentially Rolle's theorem.

Writing the eliminant as $f = P - Q$, we see that the number of real (and positive) solutions is closely related to the arrangement of the roots of P , Q and f , their multiplicities being determined by \mathcal{C} . We consider the rational function

$$\phi = f/Q = P/Q - 1 : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1.$$

The roots of P , Q and f are the inverse images of -1 , ∞ and 0 , respectively. These roots lie on the graph

$$\Gamma = \phi^{-1}(\mathbb{R}P^1) \subset \mathbb{C}P^1$$

and we see Γ as an example of so-called *dessin d'enfant*. We use then the observation made in [2, 6] (see also [7]) that polynomials P , Q and $f = P - Q$ with prescribed arrangement and multiplicities of their real roots can be constructed drawing so-called *real rational graphs* on $\mathbb{C}P^1$. The sharpness of the bounds in Theorem A and B is then proved by means of such graphs.

The paper is organized as follows. In Section 1, we define the eliminant f of a system and explain the relation between its real roots and the real (resp. positive) solutions to the system. In Section 2, we prove the upper bounds in Theorem A and B, while in Section 3 we introduce real rational graphs and achieve constructions proving the sharpness of these upper bounds.

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1. ELIMINATION

A real polynomial system supported on a circuit

$$\mathcal{C} = \{w_{-1}, w_0, \dots, w_n\} \subset \mathbb{Z}^n$$

is called *generic* if it has $v(\mathcal{C})$ solutions in $(\mathbb{C}^*)^n$. This forces each solution to be a simple solution. We are interested in the real solutions to such a system, by which we mean solutions in the real torus $(\mathbb{R}^*)^n$.

Translating \mathcal{C} by an integral vector and choosing a basis for the lattice \mathbb{Z}^n if necessary, we may assume without loss of generality that $w_{-1} = 0$, and $w_0 = \ell e_1$ for some positive integer ℓ . Perturbing slightly the system, and using Gaussian elimination, it becomes equivalent to a system of the form

$$(1.1) \quad S : \quad x^{w_i} = g_i(x_1), \quad i = 1, \dots, n,$$

where $g_i(x_1) = a_i + b_i x_1^{\ell}$ with a_i, b_i non zero real numbers for $i = 1, \dots, n$ and g_1, \dots, g_n have distinct roots. Reordering the vectors w_1, \dots, w_n if necessary, the affine primitive relation on $\{0, e_1, w_1, \dots, w_n\}$ can be written as

$$(1.2) \quad \lambda_0 e_1 + \sum_{i=1}^t \lambda_i w_i = \sum_{i=t+1}^{\nu} \lambda_i w_i,$$

where $1 \leq \nu \leq n$, $\lambda_0, \dots, \lambda_\nu$ are coprime integers, $\lambda_0 \geq 0$ and $\lambda_1, \dots, \lambda_\nu > 0$. Note that here we could have $t = 0$ or $t = \nu$ so that one of the two sums collapses to 0. The integer

$$\delta := \lambda_0 + \sum_{i=1}^t \lambda_i - \sum_{i=t+1}^{\nu} \lambda_i$$

is (up to sign) the coefficient of $w_{-1} = 0$ in (1.2) (the term δw_{-1} does not appear in (1.2) since it is 0). Multiplying (1.2) by ℓ gives an affine relation on \mathcal{C} . Set

$$(1.3) \quad m(\mathcal{C}) := \nu - \chi(\lambda_0 = 0) - \chi(\delta = 0)$$

where $\chi(Y)$ is the boolean truth value of Y : $\chi(Y) = 1$ if Y is true and $\chi(Y) = 0$ otherwise. Then $m(\mathcal{C}) + 2$ is the number of non zero coefficients in the primitive affine relation on \mathcal{C} , so that $m(\mathcal{C})$ is the dimension of the affine span of the minimal affinely dependent subset of \mathcal{C} . The case $m(\mathcal{C}) = n$ arises when \mathcal{C} is a non degenerate circuit, that is, when any proper subset of \mathcal{C} is affinely independent.

Define the *eliminant* of S to be the following univariate polynomial

$$(1.4) \quad f(x_1) = x_1^{\lambda_0} \prod_{i=1}^t (g_i(x_1))^{\lambda_i} - \prod_{i=t+1}^{\nu} (g_i(x_1))^{\lambda_i}.$$

From (1.1) and (1.2), it follows immediately that if $x = (x_1, \dots, x_n)$ is a solution to S then x_1 is a root of the eliminant f .

Write each vector $w_i = l_i e_1 + v_i$ with $v_i \in \mathbb{Z}^{n-1}$. We get the relations

$$(1.5) \quad \sum_{i=1}^t \lambda_i v_i = \sum_{i=t+1}^{\nu} \lambda_i v_i$$

and

$$(1.6) \quad \lambda_0 + \sum_{i=1}^t \lambda_i l_i = \sum_{i=t+1}^{\nu} \lambda_i l_i.$$

The fact that $\lambda_0, \dots, \lambda_\nu$ are coprime and (1.6) gives immediately that $\lambda_1, \dots, \lambda_\nu$ are also coprime, so that the relation (1.5) is in fact the primitive affine relation on $\{0, v_1, \dots, v_n\} \subset \mathbb{Z}^{n-1}$. In particular, at least one integer among $\lambda_1, \dots, \lambda_\nu$ should be odd.

Proposition 1.1 ([1]). *Assume that λ_j is odd with $j \in \{1, \dots, \nu\}$. Then $x = (x_1, \dots, x_n)$ is a real solution to the system S if and only if x_1 is a real root of f and (x_2, \dots, x_n) is a real solution to the system*

$$y^{v_i} = g_i(x_1)/x_1^{l_i}, \quad i \in \{1, \dots, n\} \setminus \{j\}.$$

For simplicity, we will often assume that λ_1 is odd and thus apply Proposition 1.1 with $j = 1$. Define

$$B := (v_1, v_2, \dots, v_n),$$

the $(n-1) \times n$ matrix whose columns are the vectors v_1, v_2, \dots, v_n . Let $\bar{B} \in M_{n-1, n}(\mathbb{Z}/2)$ be the the reduction modulo 2 of B and $rk(\bar{B})$ be the rank of \bar{B} .

Proposition 1.2. *The number of real solutions to S is equal to $2^{n-1-rk(\bar{B})}$ times the number of real roots r of f subjected to the sign conditions*

$$(g_1(r)/r^{l_1})^{\epsilon_1} \dots (g_n(r)/r^{l_n})^{\epsilon_n} > 0$$

for any $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ such that $\bar{\epsilon} \in \text{Ker } \bar{B}$.

It is worth noting that the number of sign conditions can be reduced to $n - rk(\bar{B})$ using a basis of $\text{Ker } \bar{B}$.

Before giving the proof, we present the following well-known lemma (see [10]).

Lemma 1.3. *Suppose that $0, u_1, \dots, u_n \in \mathbb{Z}^n$ form an n -dimensional simplex, and let $\bar{U} \in M_n(\mathbb{Z}/2)$ denote the matrix whose columns contains in this order the reductions modulo 2 of u_1, \dots, u_n . Then the number of real solutions in the torus to the system*

$$(1.7) \quad x^{u_i} = c_i, \quad i = 1, \dots, n,$$

where c_i is a non zero real number, is

- (1) 0 or $2^{n-rk(\bar{U})}$ if the volume of the simplex is even,
- (2) 1 if this volume is odd.

Moreover, we have $2^{n-rk(\bar{U})}$ real solutions in the first case if and only if

$$c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} > 0$$

for any $(\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}^n$ whose reduction modulo 2 belong to the kernel of \bar{U} .

Note that the case 2) is in fact redundant since if the volume of the simplex is odd then the kernel of \bar{U} is reduced to 0, hence $2^{n-rk(\bar{U})} = 1$, and the sign conditions are empty.

Proof. There exist two matrices $L, R \in GL_n(\mathbb{Z})$ and integers a_1, \dots, a_n such that LUR is the diagonal matrix $D = \text{diag}(a_1, \dots, a_n)$. Let $\tilde{u}_1, \dots, \tilde{u}_n$ be the columns vectors, in this order, of the matrix LU . The matrix L provides the multiplicative change of coordinates $\tilde{y}_i = \prod_{j=1}^n y_j^{(L^{-1})_{ji}}$, $i = 1, \dots, n$, of the complex torus. This change of coordinates transforms our system to the system $\tilde{y}^{\tilde{u}_i} = c_i$, $i = 1, \dots, n$, which has the same number of real (and complex) solutions. Multiplication on the right by R transforms this system to the system

$$(1.8) \quad \tilde{y}_i^{a_i} = d_i \quad \text{with} \quad d_i := \prod_{j=1}^n c_j^{R_{ji}}, \quad i = 1, \dots, n.$$

This system is equivalent to the previous one since $\tilde{y}_i^{a_i} = \prod_{j=1}^n (\tilde{y}^{\tilde{u}_j})^{R_{ji}}$ and $R \in GL_n(\mathbb{Z})$. In particular, the initial system (1.7) and the system (1.8) have the same number of real solutions. This number of real solutions is 0 if $d_i < 0$ for some even a_i , or $2^{n-rk(\bar{D})}$ if $d_i > 0$ for all even a_i . The conclusion is now obvious noting that $rk(\bar{D}) = rk(\bar{U})$ and that the reductions modulo 2 of the vector columns (R_{1i}, \dots, R_{ni}) for which a_i is even generate the kernel of \bar{U} . \square

Proof of Proposition 1.2 Assume that λ_1 odd. According to Proposition 1.1, we have to count, for any root r of f , the number of real solutions to

$$(1.9) \quad y^{v_i} = g_i(r)/r^{l_i}, \quad i = 2, \dots, n.$$

Set $c_i(r) := g_i(r)/r^{l_i}$, $i = 1, \dots, n$. Note that $0, v_2, \dots, v_n$ form an $(n-1)$ -dimensional simplex for otherwise there would be an additional affine relation on \mathcal{C} (so that the convex hull of \mathcal{C} would have dimension $< n$). Let $\bar{U} \in M_{n-1}(\mathbb{Z}/2)$ denote the matrix whose columns contains in this order the reductions modulo 2 of v_2, \dots, v_n . According to Lemma 1.3, the

number of solutions to (1.9) is either 0, or $2^{n-1-rk(\bar{U})}$. Moreover, it is $2^{n-1-rk(\bar{U})}$ if and only if

$$c_2(r)^{\epsilon_2} \cdots c_n(r)^{\epsilon_n} > 0$$

for any $(\epsilon_2, \dots, \epsilon_n)$ whose reduction modulo 2 belongs to the kernel of $\bar{U} = (\bar{v}_2, \dots, \bar{v}_n)$.

It follows from the relation (1.5) that the reduction modulo 2 of $\lambda := (\lambda_1, \dots, \lambda_\nu, 0, \dots, 0)$ belongs to the kernel of \bar{B} . Moreover, since λ_1 is odd, the kernel of \bar{B} is the direct sum of $(\mathbb{Z}/2) \cdot \bar{\lambda}$ and the kernel (in $\{0\} \times (\mathbb{Z}/2)^{n-1}$) of \bar{U} . This also implies that $rk(\bar{U}) = rk(\bar{B})$. To finish, it remains to see that $f(r) = 0 \Rightarrow c_1(r)^{\lambda_1} \cdots c_\nu(r)^{\lambda_\nu} > 0$. \square

Remark 1.4. *As explained at the end of the proof, the sign condition $c_1(r)^{\lambda_1} \cdots c_\nu(r)^{\lambda_\nu} > 0$ given by the affine relation on $\{0, v_1, \dots, v_n\}$ is in fact redundant in Proposition 1.2.*

A *positive* solution is a solution with positive coordinates.

Proposition 1.5. *The number of positive solutions to S is equal to the number of roots r of f subjected to the sign conditions*

$$r > 0, \quad g_1(r) > 0, \quad \dots, \quad g_n(r) > 0.$$

Proof. Assume that λ_1 is odd. According to Proposition 1.1 (x_1, \dots, x_n) is a positive solution to S if and only if x_1 is a positive root r of f and (x_2, \dots, x_n) is a positive solution to

$$(1.10) \quad y^{v_i} = g_i(r)/r^{l_i}, \quad i = 2, \dots, n.$$

As it is well-known, the previous system has a positive solution if and only if $g_i(r)/r^{l_i} > 0$ for $i = 2, \dots, n$, and in this case it has exactly one positive solution. To finish, it remains to note that $f(r) = 0$, $r > 0$ and $g_i(r) > 0$ for $i = 2, \dots, n$ implies that $g_1(r) > 0$. \square

2. UPPER BOUNDS

Let S be a system as in Section 1 and write the eliminant (1.4) as

$$f = P - Q,$$

where $P(x_1) = x_1^{\lambda_0} \prod_{i=1}^t (g_i(x_1))^{\lambda_i}$ and $Q(x_1) = \prod_{i=t+1}^\nu (g_i(x_1))^{\lambda_i}$. Consider the rational function

$$(2.1) \quad \phi := f/Q = P/Q - 1.$$

Recall that $m(\mathcal{C})$ is the dimension of the affine span of the minimal affinely dependent subset of \mathcal{C} .

Proposition 2.1. *The number of positive solutions to S is no more than $m(\mathcal{C}) + 1$.*

Proof. Let \mathcal{R}_+ denote the set of positive roots r of f satisfying $g_1(r) > 0, \dots, g_n(r) > 0$ and let N denote the number of elements of \mathcal{R}_+ . The number of positive solutions to S is equal to N by Proposition 1.5. Consider an open interval I formed by two consecutive elements of \mathcal{R}_+ . This interval contains no root of P and Q (since each g_i has simple real roots). As a consequence, the function ϕ is bounded on I . Moreover, ϕ takes the same value 0 on the endpoints of I . Therefore, by Rolle's theorem, the derivative ϕ' has at least

one real root in I . This gives $N - 1$ distinct positive roots of ϕ' , which are different from the roots of P and Q . The roots of ϕ' are the roots of $P'Q - PQ'$. As noticed in ([1], Proof of Proposition 4.3.), the quotient

$$(2.2) \quad \frac{P'(x)Q(x) - P(x)Q'(x)}{x^{\lambda_0-1} \prod_{i=1}^{\mu} (g_i(x))^{\lambda_i-1}},$$

or, if $\lambda_0 = 0$, the same quotient but with $\ell - 1$ in place of $\lambda_0 - 1$, is a polynomial H which can be written as $H(x) = h(x^\ell)$ where h is real polynomial of degree $m(\mathcal{C})$ and with non zero constant term. Thus the $N - 1$ positive roots of ϕ' different from the roots of P and Q should be roots of H . But H has at most $m(\mathcal{C})$ distinct positive roots, hence $N - 1 \leq m(\mathcal{C})$. \square

J. Maurice Rojas informed us that Proposition 2.1 can be obtained from [5] (Lemma 2, Section 3).

Define the matrix A as the matrix obtained by putting (in this order) the vectors w_0, \dots, w_n in columns

$$(2.3) \quad A := (w_0, \dots, w_n) \in M_{n,n+1}(\mathbb{Z})$$

We have by definition $rk(\bar{\mathcal{C}}) = rk(\bar{A})$, where \bar{A} is the reduction modulo 2 of A . Recall that $w_0 = \ell e_1$ and $w_i = l_i e_1 + v_i$ for $i = 1, \dots, n$, so that

$$(2.4) \quad A = \begin{pmatrix} \ell & l_1 & \dots & l_n \\ 0 & & & \\ \vdots & v_1 & \dots & v_n \\ 0 & & & \end{pmatrix}$$

The lower-right matrix (v_1, \dots, v_n) is the matrix B that we already defined.

Theorem 2.2. *Let N be the number of real solutions to S .*

(1) *If $rk(\bar{\mathcal{C}}) \leq m(\mathcal{C})$, then*

$$N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (m(\mathcal{C}) + rk(\bar{\mathcal{C}}) + 1).$$

(2) *if $rk(\bar{\mathcal{C}}) \geq m(\mathcal{C})$, then*

$$N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (2m(\mathcal{C}) + 1).$$

Proof. Consider first the case $rk(\bar{\mathcal{C}}) = 0$. This corresponds to the case $\mathcal{C} \subset 2\mathbb{Z}^n$. Setting $\tilde{w}_i = w_i/2$, a system supported on \mathcal{C} can be rewritten as a system supported on the circuit $\tilde{\mathcal{C}} = \{0, \tilde{w}_{-1}, \tilde{w}_0, \dots, \tilde{w}_n\}$ so that the number of real solutions (in the real torus) to the system supported on \mathcal{C} is equal to 2^n times the number of positive solutions to the system supported on $\tilde{\mathcal{C}}$. Obviously, we have $m(\tilde{\mathcal{C}}) = m(\mathcal{C})$. The bound $2^n(m(\mathcal{C}) + 1)$ for the number of real solutions to S follows then from Proposition 2.1.

Suppose now that $rk(\bar{\mathcal{C}}) \neq 0$, that is, \mathcal{C} is not contained in $2\mathbb{Z}^n$. Then, it is easy to show the existence of points $w_i, w_j \in \mathcal{C}$ such that $w_i - w_j \notin 2\mathbb{Z}^n$, and w_i belongs to the minimal affinely dependent subset of \mathcal{C} . Thus, reordering the points of \mathcal{C} so that w_i becomes w_{-1}

and w_j becomes w_0 if necessary, we can assume that the number ℓ is an odd number, and the coefficient δ is non zero. The fact that ℓ is odd gives that \bar{A} and \bar{B} have isomorphic kernels, so that $rk(\bar{\mathcal{C}}) = rk(\bar{A}) = rk(\bar{B}) + 1$.

Define

$$c(r) := (c_1(r), \dots, c_n(r)) \quad \text{with} \quad c_i(r) := g_i(r)/r^{\ell_i}, \quad i = 1, \dots, n.$$

Let $\pi_1, \dots, \pi_{n-rk(\bar{\mathcal{C}})+1}$ be vectors in \mathbb{Z}^n whose reductions modulo 2 form a basis of $\text{Ker } \bar{B}$. Then Proposition 1.2 gives that the number of real solutions to S is $2^{n-rk(\bar{\mathcal{C}})}$ times the number of real roots r of f subjected to the sign conditions

$$(2.5) \quad (c(r))^{\pi_i} > 0 \quad \text{for} \quad i = 1, \dots, n - rk(\bar{\mathcal{C}}) + 1.$$

Consider the $n - rk(\bar{\mathcal{C}}) + 1$ by n matrix Π whose rows are given by the exponent vectors of the sign conditions (2.5). Taking the Hermite normal form of the reduction modulo 2 of Π , we obtain another basis of $\text{Ker } \bar{B}$ producing equivalent sign conditions

$$(2.6) \quad \begin{aligned} c_{s_1}(r) \cdot \prod_{i>s_1, i \notin \mathcal{S}} (c_i(r))^* &> 0 \\ c_{s_2}(r) \cdot \prod_{i>s_2, i \notin \mathcal{S}} (c_i(r))^* &> 0 \\ &\vdots \\ c_{s_{n-rk(\bar{\mathcal{C}})+1}}(r) \cdot \prod_{i>s_{n-rk(\bar{\mathcal{C}})+1}, i \notin \mathcal{S}} (c_i(r))^* &> 0 \end{aligned}$$

where $1 \leq s_1 < s_2 < \dots < s_{n-rk(\bar{\mathcal{C}})+1} \leq n$, $\mathcal{S} = \{s_1, \dots, s_{n-rk(\bar{\mathcal{C}})+1}\}$ and the stars are exponents taking values 0 or 1.

It remains to show that the set \mathcal{R} of real roots r of f satisfying the sign conditions (2.6) has at most $m(\mathcal{C}) + 1 + rk(\bar{\mathcal{C}})$ elements if $rk(\bar{\mathcal{C}}) \leq m(\mathcal{C})$, and at most $2m(\mathcal{C}) + 1$ elements if $rk(\bar{\mathcal{C}}) \geq m(\mathcal{C})$.

First note that the integer ℓ being odd, each g_i has only one real root, that we denote by ρ_i . Moreover, the polynomial H defined in (2.2) has at most $m(\mathcal{C})$ real roots since $H(x_1) = h(x_1^\ell)$ with h polynomial of degree $m(\mathcal{C})$. Let I denote an open (bounded) interval formed by two consecutive elements of \mathcal{R} . We claim that I should contain at least one element among $0, \rho_1, \dots, \rho_\nu$ and the real roots of H . Indeed, if the function $\phi = f/Q$ is bounded on I , then I should contain a real critical point (with finite critical value) of ϕ by Rolle's theorem. The critical points of ϕ with finite critical values are contained in the set formed of 0 , the roots ρ_1, \dots, ρ_t of P and the roots of H (see the proof of Proposition 2.1). If ϕ is not bounded on I , then obviously I should contain a root of Q , that is, a root ρ_i for some $i = t + 1, \dots, \nu$.

There is at most one interval I containing 0 and at most $m(\mathcal{C})$ intervals I containing a real root of H . Let us concentrate now on the intervals I which do not contain 0 but contain a root ρ_i with $i = 1, \dots, \nu$. The number of such intervals is obviously no more than ν . We claim that this number of intervals is also no more than $rk(\bar{\mathcal{C}}) - 1$. Indeed,

such an interval I should contain a ρ_i with $i \in \{1, \dots, n\} \setminus \mathcal{S}$, for otherwise some sign condition in (2.6) would be violated at the endpoints of I . The claim follows then as the cardinality of $\{1, \dots, n\} \setminus \mathcal{S}$ is $rk(\bar{\mathcal{C}}) - 1$.

Therefore, the number of intervals I formed by two consecutive elements of \mathcal{R} is no more than $1 + m(\mathcal{C}) + rk(\bar{\mathcal{C}}) - 1 = m(\mathcal{C}) + rk(\bar{\mathcal{C}})$, and no more than $1 + m(\mathcal{C}) + \nu$. Obviously, the number of elements of \mathcal{R} is equal to the number of these intervals I added by one. The bound $m(\mathcal{C}) + rk(\bar{\mathcal{C}})$ on the number of intervals I gives thus

$$(2.7) \quad N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (m(\mathcal{C}) + rk(\bar{\mathcal{C}}) + 1).$$

Consider now the bound $1 + m(\mathcal{C}) + \nu$ on the number of intervals I . We see that if this bound is attained, then the polynomial H has $m(\mathcal{C})$ real roots, and each interval I contains exactly one element among the real roots of H , the roots ρ_1, \dots, ρ_ν , and 0. This forces each of these points to be a critical point of ϕ with even multiplicity (and unbounded critical values for the roots of Q). The $m(\mathcal{C})$ real roots of H are simple roots of H , hence critical points with multiplicity 2 of ϕ . The multiplicity of 0 with respect to ϕ is λ_0 , while the multiplicity of ρ_i , $i = 1, \dots, \nu$, with respect to ϕ is $\pm\lambda_i$. Consequently, if the number of intervals I is $1 + m(\mathcal{C}) + \nu$, then $\lambda_0, \lambda_1, \dots, \lambda_\nu$ are even: a contradiction since these numbers are coprime. Hence, the number of intervals I is no more than $m(\mathcal{C}) + \nu$. If λ_0 and δ are both non zero, then $\nu = m(\mathcal{C})$ so that the number of intervals I is no more than $2m(\mathcal{C})$. This gives the bound

$$(2.8) \quad N \leq 2^{n-rk(\bar{\mathcal{C}})} \cdot (2m(\mathcal{C}) + 1).$$

Recall that we have assumed at the beginning that $\delta \neq 0$. Hence, it remains to see what happens when $\lambda_0 = 0$. In this case, we have $m(\mathcal{C}) = \nu - 1$ and 0 cannot be a critical point of ϕ . Arguing as before, we obtain then that the number of intervals I is no more than $m(\mathcal{C}) + \nu$. Moreover, if this number is equal to $m(\mathcal{C}) + \nu$, then $\lambda_1, \dots, \lambda_\nu$ are even: a contradiction since $\lambda_1, \dots, \lambda_\nu$ are coprime. Hence, the number of intervals I is no more than $m(\mathcal{C}) + \nu - 1 = 2m(\mathcal{C})$, and we retrieve the bound (2.8). \square

3. REAL RATIONAL GRAPHS AND SHARPNESS OF BOUNDS

3.1. Real rational graphs. This subsection comes from [2] (see also [3, 6, 7]). Consider the function (2.1)

$$\phi = f/Q = P/Q - 1$$

as a rational function $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. The degree of ϕ coincides with that of f . Define

$$\Gamma := \phi^{-1}(\mathbb{R}P^1).$$

This is a real graph on $\mathbb{C}P^1$ (invariant with respect to the complex conjugation) and which contains $\mathbb{R}P^1$. Each vertex of Γ has even valency, and the multiplicity of a critical point with real critical value of ϕ is half its valency. The graph Γ contains the inverse images of -1 , ∞ and 0 which are the sets of roots of P , Q and f , respectively. Denote by the same letter p (resp. q and r) the points of Γ which are mapped to -1 (resp. ∞ and 0). Orient the real axis on the target space via the arrows $-1 \rightarrow \infty \rightarrow 0 \rightarrow -1$ (orientation given by the decreasing order in \mathbb{R}) and pull back this orientation by ϕ . The graph Γ becomes an oriented graph, with the orientation given by arrows $p \rightarrow q \rightarrow r \rightarrow p$.

Clearly, the arrangement of the real roots of f , P and Q together with their multiplicities can be extracted from the graph Γ . We encode this arrangement together with the multiplicities by what is called a root scheme.

Definition 3.1 ([2]). *A root scheme is a k -uple $((l_1, m_1), \dots, (l_k, m_k)) \in (\{p, q, r\} \times \mathbb{N})^k$. A root scheme is realizable by polynomials of degree d if there exist real polynomials P and Q such that $f = P - Q$ has degree d and if $\rho_1 < \dots < \rho_k$ are the real roots of f , P and Q , then $l_i = p$ (resp. q, r) if ρ_i is root of P (resp. Q, f) and m_i is the multiplicity of ρ_i .*

Conversely, suppose we are given a real graph $\Gamma \subset \mathbb{C}P^1$ together with a real continuous map $\varphi : \Gamma \rightarrow \mathbb{R}P^1$. Denote the inverse images of $-1, \infty$ and 0 by letters p, q and r , respectively, and orient Γ with the pull back by φ of the above orientation of $\mathbb{R}P^1$. This graph is called a *real rational graph* [2] if

- any vertex of Γ has even valence,
- any connected component of $\mathbb{C}P^1 \setminus \Gamma$ is homeomorphic to an open disk,

Then, for any connected component D of $\mathbb{C}P^1 \setminus \Gamma$, the map $\varphi|_{\partial D}$ is a covering of $\mathbb{R}P^1$ whose degree d_D is the number of letters p (resp. q, r) in ∂D . We define the *degree* of Γ to be half the sum of the degrees d_D over all connected components of $\mathbb{C}P^1 \setminus \Gamma$. Since φ is a real map, the degree of Γ is also the sum of the degrees d_D over all connected components D of $\mathbb{C}P^1 \setminus \Gamma$ contained in one connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$.

Proposition 3.2 ([2, 6]). *A root scheme is realizable by polynomials of degree d if and only if it can be extracted from a real rational graph of degree d on $\mathbb{C}P^1$.*

Let us explain how to prove the if part in Proposition 3.2 (see [2, 3, 6, 7]). For each connected component D of $\mathbb{C}P^1 \setminus \Gamma$, extend $\varphi|_{\partial D}$ to a, branched if $d_D > 1$, covering of one connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$, so that two adjacent connected components of $\mathbb{C}P^1 \setminus \Gamma$ project to different connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. Then, it is possible to glue continuously these maps in order to obtain a real branched covering $\varphi : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ of degree d . The map φ becomes a real rational map of degree d for the standard complex structure on the target space and its pull-back by φ on the source space. There exist then real polynomials P and Q such that $f = P - Q$ has degree d and $\varphi = f/Q = P/Q - 1$, so that the points p (resp. q, r) correspond to the roots of P (resp. Q, f) and $\Gamma = \varphi^{-1}(\mathbb{R}P^1)$.

As in [2], we will abbreviate a sequence S repeated u times in a root scheme by S^u . If $u = 0$, then S^u is the empty sequence.

3.2. Constructions. We are going to prove the existence of root schemes by constructing real rational graphs Γ on $\mathbb{C}P^1$. Since these graphs are real, only half of them will be drawn and the horizontal line will represent the real axis $\mathbb{R}P^1$.

Proposition 3.3. *For any even integer $n = 2k > 0$, the root scheme*

$$([(q, 2), (p, 2)]^k, (q, 1), (r, 1)^{2k+1}, (p, 1))$$

is realizable by polynomials of degree $n + 1$.

For any odd integer $n = 2k + 1 > 0$, the root scheme

$$([(q, 2), (p, 2)]^k, (q, 2), (p, 1)(r, 1)^{2k+2}, (p, 1))$$

is realizable by polynomials of degree $n + 1$.

Proof. According to Proposition 3.2, it suffices to construct a real rational graph Γ_{n+1} of degree $n + 1$ on $\mathbb{C}P^1$ from which the desired root scheme can be extracted. This is done in Figure 1 for $n = 2, 4$ and in Figure 2 for $n = 3, 5$. These figures provide the induction step $n \rightarrow n + 2$ for constructing suitable graphs Γ_{n+1} for any positive integer n . \square

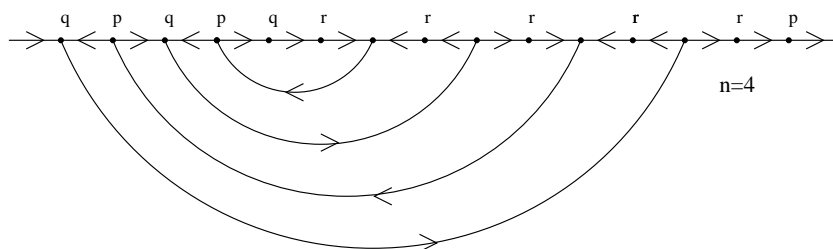
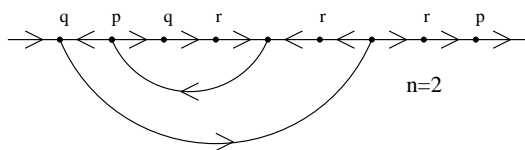


FIGURE 1.

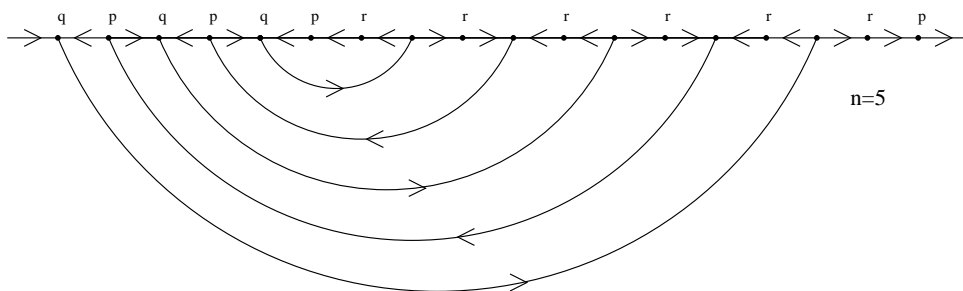
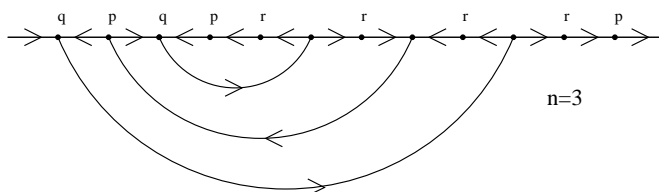


FIGURE 2.

Let n and R be integers such that $0 < R \leq n$. If $n - R$ is even, define I and J by

$$I := [(p, 2), (q, 2)]^{\frac{n-R}{2}}, (q, 3), (r, 1), (p, 2), (r, 1)^{n-R+1}$$

$$J := \begin{cases} (r, 1), (p, 1) & \text{if } R = 1 \\ (r, 1)^R, (q, 2), [(r, 1), (p, 4), (r, 1), (q, 4)]^{\frac{R}{2}-1}, (r, 1), (p, 3) & \text{if } R \text{ is even} \\ (r, 1)^R, (p, 2), [(r, 1), (q, 4), (r, 1), (p, 4)]^{\frac{R-3}{2}}, (q, 4), (r, 1), (p, 3) & \text{if } R \text{ is odd} \end{cases}$$

If $n - R$ is odd, define

$$I := [(p, 2), (q, 2)]^{\frac{n-R-1}{2}}, (p, 2), (q, 3), (r, 1), (p, 2), (r, 1)^{n-R+1}$$

$$J := \begin{cases} (r, 1), (q, 1) & \text{if } R = 1 \\ (r, 1)^R, (p, 2), [(r, 1), (q, 4), (r, 1), (p, 4)]^{\frac{R}{2}-1}, (r, 1), (q, 3) & \text{if } R \text{ is even} \\ (r, 1)^R, (q, 2), [(r, 1), (p, 4), (r, 1), (q, 4)]^{\frac{R-3}{2}}, (p, 4), (r, 1), (q, 3) & \text{if } R \text{ is odd} \end{cases}$$

Note that the definitions of J with $n - R$ odd and with $n - R$ even are obtained from each other by permuting p and q .

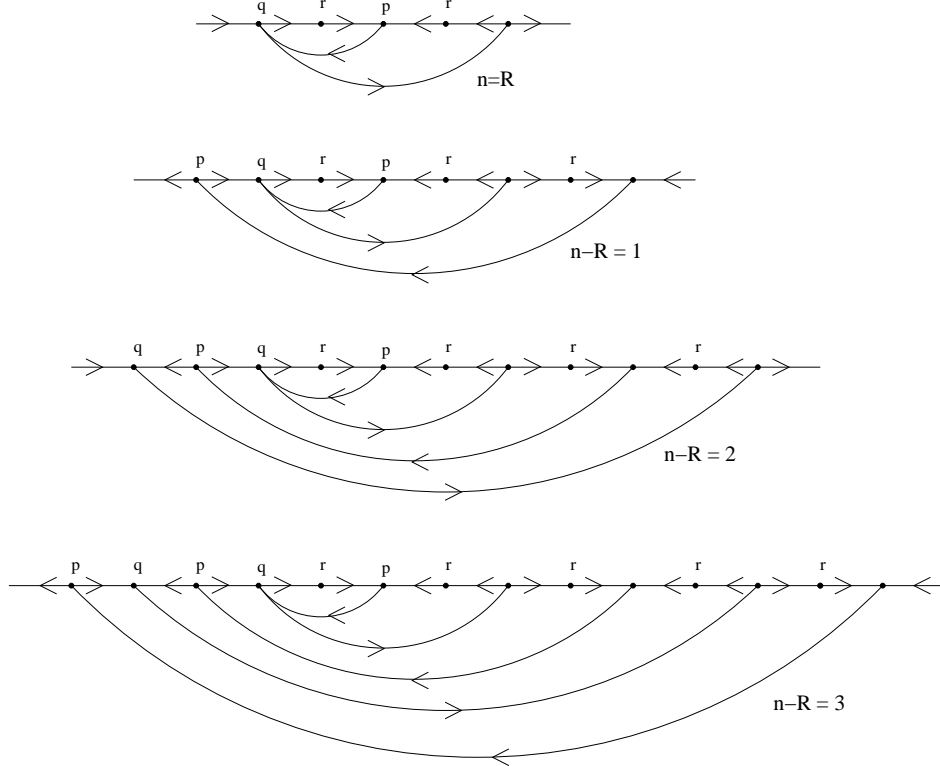


FIGURE 3: GRAPH Γ_I .

Proposition 3.4. *For any integers n and R such that $0 < R \leq n$, the root scheme (I, J) is realizable by polynomials of degree $n + R + 1$.*

Proof. First, we construct graphs Γ_I and Γ_J , which are not real rational graphs, but from which the sequences I and J can be extracted, respectively. Figure 3 shows Γ_I for $n - R = 0, 1, 2, 3$, and indicates how to construct Γ_I for any integer value of $n - R$ with $0 \leq n - R < n$. Similarly, Figure 4 shows Γ_J for $R = 1, 2, 3, 4$ and $n - R$ even, and indicates how to construct Γ_J for any integers n and R such that $0 < R \leq n$ and $n - R$ is even. The graph Γ_J for $n - R$ odd is obtained from the graph Γ_J with $n - R$ even and the same value of R by permuting p and q and reversing all the arrows. For any integers n and R such that $0 < R \leq n$ we can glue the corresponding Γ_I and Γ_J in order to obtain a real rational graph of degree $n + R + 1$ whose root scheme is (I, J) (see Figure 5 for $n = 4$ and $R = 3$). The result follows then from Proposition 3.2. \square

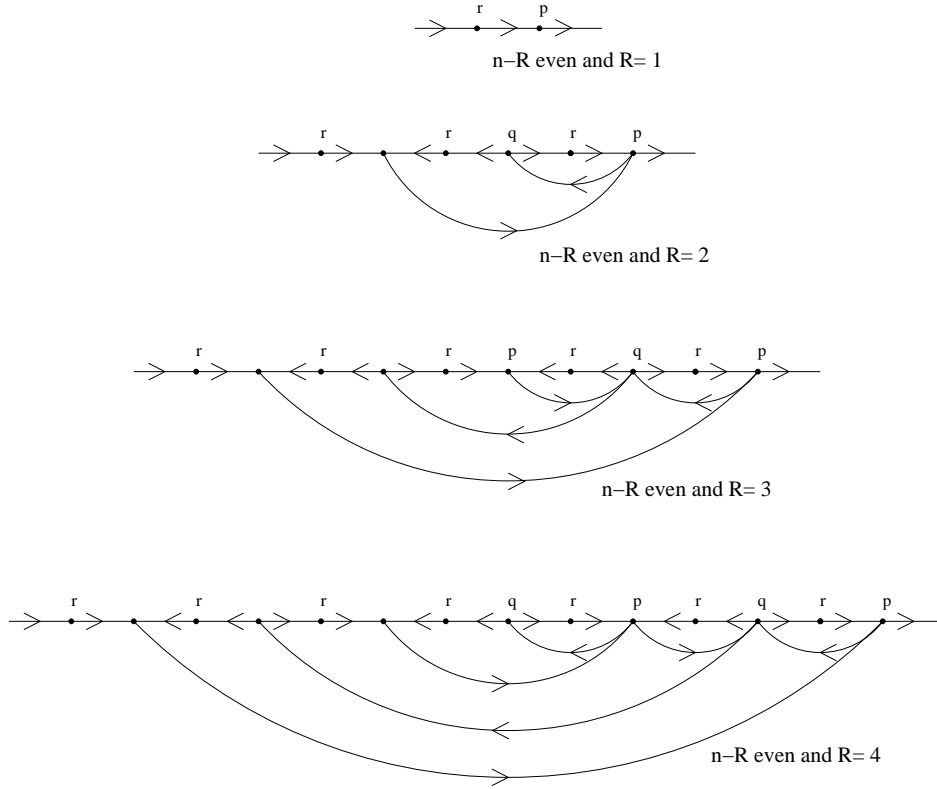


FIGURE 4: GRAPH Γ_J .

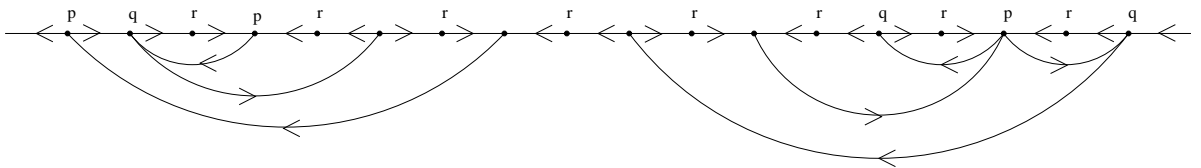


FIGURE 5: GLUING OF Γ_I AND Γ_J FOR $n = 4$, $R = 3$.

3.3. Sharpness of bounds.

Proposition 3.5. *For any positive integer n , there exist a non degenerate circuit $\mathcal{C} \subset \mathbb{Z}^n$ and a generic real polynomial system supported on \mathcal{C} whose number of positive solutions is equal to $n + 1$.*

Proof. We prove only the case n even since the case n odd is similar (the case $n = 1$ is trivial). So assume that $n = 2k > 0$ is even. According to Proposition 3.3, there exist polynomials P and Q such that $f = P - Q$ has degree $n + 1$ and the corresponding root scheme is

$$([(q, 2), (p, 2)]^k, (q, 1), (r, 1)^{2k+1}, (p, 1)).$$

Composing on the right $\phi = f/Q = P/Q - 1$ with a real automorphism of $\mathbb{C}P^1$ (a real rational map of degree 1), we can choose three values for the points in this root scheme. Reading the above scheme from the left to the right, we choose $p = 0$ for the last but one p , a positive value for the last q , and $p = \infty$ for the last p . This gives the following arrangement of the roots of P , Q and f (where we naturally omit the root at infinity)

$$q_1 < p_1 < \dots < q_k < p_k = 0 < q_{k+1} < r_1 < r_2 < \dots < r_{2k+1},$$

so that $f = P - Q$ with

$$P(x) = ax^2(x - p_1)^2 \dots (x - p_{k-1})^2, \quad Q(x) = b(x - q_{k+1})(x - q_1)^2 \dots (x - q_k)^2,$$

and $f = P - Q$ has $n + 1 = 2k + 1$ positive roots which are bigger than the roots of P and Q . Note that $ab > 0$ for otherwise the polynomial $f = P - Q$ could not have roots bigger than the roots of P and Q . Dividing f by a and setting $g_1(x) = x - p_1, \dots, g_{k-1}(x) = x - p_{k-1}, g_k(x) = x - q_1, \dots, g_{n-1}(x) = x - q_k$ and $g_n(x) = b(x - q_{k+1})/a$, we obtain a polynomial f of the form (1.4) with $\ell = 1$, the exponents are coprime, and such that \mathcal{R}_+ has $n + 1$ elements. Let us now give explicitly a non degenerate circuit $\mathcal{C} \subset \mathbb{Z}^n$ and then a system (1.1) whose eliminant is f . Take the circuit \mathcal{C} with $w_{-1} = 0, w_0 = e_1, w_1 = e_2, \dots, w_{n-1} = e_n$ and $w_n = 2(e_1 + e_2 + \dots + e_k - e_{k+1} - \dots - e_n)$. The primitive affine relation on $\{e_1, w_1, \dots, w_n\}$ is

$$2e_1 + 2(w_1 + \dots + w_{k-1}) = 2(w_k + \dots + w_{n-1}) + w_n$$

and has the desired coefficients. Hence $f(x_1)$ is the eliminant of the system $x^{w_i} = g_i(x_1)$, $i = 1, \dots, n$, given explicitly by

$$\begin{cases} x_2 = x_1 - p_1, \dots, x_k = x_1 - p_{k-1} \\ x_{k+1} = x_1 - q_1, \dots, x_n = x_1 - q_k \\ \left(\frac{x_1 \dots x_k}{x_{k+1} \dots x_n} \right)^2 = b(x_1 - q_{k+1})/a. \end{cases}$$

According to Proposition 1.5, this system has $n + 1$ positive solutions. \square

It may be interesting to note that the previous system has in fact only these $n + 1$ positive solutions as solutions in the complex torus. To see this, one can compute that the volume of the circuit is $n + 1$, and use Kouchnirenko theorem.

Proposition 3.6. *For any integers n and R such that $0 \leq R \leq n$, there exist a non degenerate circuit $\mathcal{C} \subset \mathbb{Z}^n$ with $\text{rk}(\mathcal{C}) = R$, and a generic real polynomial system supported on \mathcal{C} whose number of real solutions is equal to $2^{n-R}(n + R + 1)$.*

Proof. Note that for $R = 0$ this follows from Proposition 1.5 (see the proof of Theorem 2.2). We will give the proof only when n and R are even since the other cases are similar (the case $n = 1$ is trivial).

So let n and R be even integers such that $0 < R \leq n$. According to Proposition 3.4, there exist polynomials P and Q such that $f = P - Q$ has degree $n + R + 1$ and the corresponding root scheme is

$$\left([(p, 2), (q, 2)]^{\frac{n-R}{2}}, (q, 3), (r, 1), (p, 2), (r, 1)^{n+1}, (q, 2), [(r, 1), (p, 4), (r, 1), (q, 4)]^{\frac{R}{2}-1}, (r, 1), (p, 3) \right).$$

We may choose three values for the roots in this root scheme. Choose $p = 0$ for the p in the sequence $(q, 3), (r, 1), (p, 2)$, a negative value for the r in the same sequence, and the infinity value for the last p on the right in the root scheme. Then, the polynomials P and Q can be written as

$$P(x) = x^2 \prod_{i=1}^{\frac{n-R}{2}} (g_i(x))^2 \cdot \prod_{i=\frac{n-R}{2}+1}^{\frac{n}{2}-1} ((g_i(x))^4),$$

$$Q(x) = (g_{\frac{n}{2}}(x))^3 \cdot \prod_{i=\frac{n}{2}+1}^{n-\frac{R}{2}+1} (g_i(x))^2 \cdot \prod_{i=n-\frac{R}{2}+2}^n ((g_i(x))^4)$$

so that

- (\star) $f = P - Q$ has $n + R + 1$ (non zero) real roots, all but one are positive, and g_i is positive at the real roots of f for $i = 1, \dots, \frac{n-R}{2}$ and $i = \frac{n}{2}, \dots, n - \frac{R}{2}$.

Here the g_i have all, but one, the form, $g_i(x) = x - \rho_i$ with ρ_i a finite non zero root p if $i < \frac{n}{2}$ and a root q otherwise. The last one is of the form $c(x - \rho_i)$ where c is a positive real number due to the fact f has a real root bigger than the roots of P and Q (see the proof of Proposition 3.5). The polynomials g_i mentioned in (\star) correspond to the sequence $[(p, 2), (q, 2)]^{\frac{n-R}{2}}, (q, 3)$ on the left of the above root scheme.

Now, define $v_{n-\frac{R}{2}+1} = -(e_2 + \dots + e_{n-1}) - 3e_n$, $v_{\frac{n}{2}} = 2(e_2 + \dots + e_n)$, and

$$v_i = 2e_{i+1}, \quad i = 1, \dots, \frac{n-R}{2}$$

$$v_i = e_{i+1}, \quad i = \frac{n-R}{2} + 1, \dots, \frac{n}{2} - 1$$

$$v_i = -2e_i, \quad i = \frac{n}{2} + 1, \dots, n - \frac{R}{2}$$

$$v_i = -e_{i-1}, \quad i = n - \frac{R}{2} + 2, \dots, n$$

The primitive affine relation on $\{0, v_1, \dots, v_n\}$ is the desired one

$$2 \sum_{i=1}^{\frac{n-R}{2}} v_i + 4 \sum_{i=\frac{n-R}{2}+1}^{\frac{n}{2}-1} v_i = 3v_{\frac{n}{2}} + 2 \sum_{i=\frac{n}{2}+1}^{n-\frac{R}{2}+1} v_i + 4 \sum_{i=n-\frac{R}{2}+2}^n v_i,$$

and it's easy to see that the rank of the matrix $\bar{B} = (\bar{v}_1, \dots, \bar{v}_n)$ is $R - 1$. We want to find integers l_1, \dots, l_n verifying the relation (1.6) and such that the sign conditions in Proposition 1.2 are verified at each root r of f . Here, these sign conditions are given by the v_i which belong to $2\mathbb{Z}^n$

$$(3.1) \quad g_i(r)/r^{l_i} > 0 \quad , \quad i = 1, \dots, \frac{n-R}{2} \quad \text{and} \quad i = \frac{n}{2}, \dots, n - \frac{R}{2}$$

In view of property (\star) , we want to have l_i even for $i = 1, \dots, \frac{n-R}{2}$ and $i = \frac{n}{2}, \dots, n - \frac{R}{2}$ since one root of f is negative. In fact we can find integers l_i which are all even. Indeed, in our situation, the relation (1.6) is

$$(3.2) \quad 2 + 2 \sum_{i=1}^{\frac{n-R}{2}} l_i + 4 \sum_{i=\frac{n-R}{2}+1}^{\frac{n}{2}-1} l_i = 3l_{\frac{n}{2}} + 2 \sum_{i=\frac{n}{2}+1}^{n-\frac{R}{2}+1} l_i + 4 \sum_{i=n-\frac{R}{2}+2}^n l_i.$$

Replacing the first 2 on the left by 1, the existence of integers l_i verifying the resulting relation is due to the fact that 2, 4, 3 are coprime. Multiplying then by 2 the members of this modified relation gives then the existence of even integers l_i verifying (3.2).

So, let us choose even integers l_i verifying (3.2) so that the sign conditions (3.1) are satisfied. The proof is almost finished.

Consider the system

$$(3.3) \quad x^{w_i} = g_i(x_1) \quad , \quad i = 1, \dots, n,$$

where $w_i := l_i e_1 + v_i$. This system is equivalent to a (generic) system supported on the circuit $\mathcal{C} = \{w_{-1} = 0, w_0 = e_1, w_1, \dots, w_n\}$. The primitive affine relation on \mathcal{C} is

$$2w_0 + 2 \sum_{i=1}^{\frac{n-R}{2}} w_i + 4 \sum_{i=\frac{n-R}{2}+1}^{\frac{n}{2}-1} w_i = 3w_{\frac{n}{2}} + 2 \sum_{i=\frac{n}{2}+1}^{n-\frac{R}{2}+1} w_i + 4 \sum_{i=n-\frac{R}{2}+2}^n w_i,$$

so that f is the eliminant of (3.3). The rank modulo 2 of \mathcal{C} is $rk(\bar{\mathcal{C}}) = rk(\bar{B}) + 1 = R$. The eliminant f has $n + R + 1$ real roots and the sign conditions (3.1) are satisfied at each real root of f . Therefore, according to Proposition 1.2, the system (3.3) has $2^{n-R}(n + R + 1)$ real solutions. \square

The case $R = n$ in Proposition 3.6 has already been obtained in [1] by different methods.

Theorem 3.7. *For any integers n, m such that $1 \leq m \leq n$, there exist a circuit $\mathcal{C} \subset \mathbb{Z}^n$ with $m(\mathcal{C}) = m$ and a generic real polynomial system supported on \mathcal{C} whose number of positive solutions is $m + 1$*

Proof. By Proposition 3.5, there exists a circuit $\mathcal{C}' = \{0, w_0 = l e_1, w_1, \dots, w_m\} \subset \mathbb{Z}^m$ with $m(\mathcal{C}') = m$ and polynomials g_1, \dots, g_m such that the system $x^{w_i} = g_i(x_1)$, $i = 1, \dots, m$, has $m + 1$ positive solutions. By Proposition 1.5, this means that the eliminant f of this system has $m + 1$ positive roots at which g_1, \dots, g_m are positive. Define

$$w_i := e_i \quad , \quad i = m + 1, \dots, n,$$

and choose polynomials g_{m+1}, \dots, g_n (as in (1.1)) which are positive at the real roots of f . Then, the system $x^{w_i} = g_i(x_1)$, $i = 1, \dots, n$, is equivalent to a system supported on the circuit $\mathcal{C} = \{0, w_0 = \ell e_1, w_1, \dots, w_n\} \subset \mathbb{Z}^n$. The eliminant of this system is also f and we have $m(\mathcal{C}) = m(\mathcal{C}') = m$. Proposition 1.5 and our choice of g_{m+1}, \dots, g_n implies that this system has also $m + 1$ positive solutions. \square

Theorem 3.8. *For any integers n, m, R such that $0 \leq R \leq n$ and $1 \leq m \leq n$, there exist a circuit $\mathcal{C} \subset \mathbb{Z}^n$ with $m(\mathcal{C}) = m$, $rk(\bar{\mathcal{C}}) = R$, and a generic real polynomial system supported on \mathcal{C} whose number of real solutions, N , verifies:*

(1) if $R \leq m$, then

$$N = 2^{n-R} \cdot (m + R + 1).$$

(2) if $R \geq m$, then

$$N = 2^{n-R} \cdot (2m + 1).$$

Proof. The case $R = 0$ is an immediate consequence of Theorem 3.7. Suppose that $0 < R \leq m$. By Proposition 3.6, there exist a circuit $\mathcal{C}' = \{0, w_0 = \ell e_1, w_1, \dots, w_m\} \subset \mathbb{Z}^m$ with $m(\mathcal{C}') = m$, $rk(\bar{\mathcal{C}}') = R$, and polynomials g_1, \dots, g_m such that the system

$$(3.4) \quad x^{w_i} = g_i(x_1), \quad i = 1, \dots, m,$$

has $2^{m-R} \cdot (m + R + 1)$ real solutions. Here and in the rest of the proof g_i is a polynomial as in (1.1). Since $rk(\mathcal{C}') \neq 0$, we may assume that ℓ is odd. Define

$$w_i := 2e_i, \quad i = m + 1, \dots, n,$$

and choose polynomials g_{m+1}, \dots, g_n which are positive at the real roots of the eliminant f of (3.4). Then, the system

$$(3.5) \quad x^{w_i} = g_i(x_1), \quad i = 1, \dots, n,$$

is equivalent to a system supported on the circuit $\mathcal{C} = \{0, w_0 = \ell e_1, w_1, \dots, w_n\} \subset \mathbb{Z}^n$. We have $m(\mathcal{C}) = m(\mathcal{C}') = m$, $rk(\bar{\mathcal{C}}) = rk(\bar{\mathcal{C}}') = R$ and f is also the eliminant of (3.5). Moreover the sign conditions in Proposition 1.2 which correspond to (3.5) are obtained from those corresponding to (3.4) by adding $g_i(r) > 0$ for $i = m + 1, \dots, n$ (note that $l_{m+1} = \dots = l_n = 0$). But since (3.4) has $2^{m-R} \cdot (m + R + 1)$ real solutions, and ℓ is odd, it follows from Proposition 1.2 that the sign conditions corresponding to (3.4) are satisfied at $m + R + 1$ real roots of f . By our choice of g_{m+1}, \dots, g_n , the sign conditions corresponding to (3.5) are satisfied at the same $m + R + 1$ real roots of f . Hence, again by Proposition 1.2, the system (3.5) has $2^{n-R} \cdot (m + R + 1)$ real solutions.

Finally, suppose that $R > m$. The idea is exactly as before. By Proposition 3.6, there exist a circuit $\mathcal{C}' = \{0, w_0 = \ell e_1, w_1, \dots, w_m\} \subset \mathbb{Z}^m$ with $m(\mathcal{C}') = m$, $rk(\bar{\mathcal{C}}') = m$ and polynomials g_1, \dots, g_m such that the system (3.4) has $2m + 1$ real solutions. According to Proposition 1.2 (see also Remark 1.4), this means that the eliminant f of this system has $2m + 1$ real roots. Define

$$w_i := e_i, \quad i = m + 1, \dots, R, \quad w_i := 2e_i, \quad i = R + 1, \dots, n$$

and choose polynomials g_{R+1}, \dots, g_n which are positive at the roots of the eliminant f . Then, the system (3.5) is equivalent to a system supported on the circuit $\mathcal{C} = \{0, w_0 =$

$\ell e_1, w_1, \dots, w_n\} \subset \mathbb{Z}^n$. We have $m(\mathcal{C}) = m$, $rk(\bar{\mathcal{C}}) = R$ and (3.5) has $2^{n-R} \cdot (2m + 1)$ real solutions. \square

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