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REMARKS ON THE FRACTIONAL BROWNIAN MOTION

Denis Feyel

Département de Maths., Université d'Evry-Val d'Essonne, Boulevard François Mitterand, 91025 Evry cedex, France, denis.feyel@univ-evry.fr

Arnaud de La Pradelle

Laboratoire d'analyse Fonctionnelle, Université Paris VI, 4 place Jussieu, 75052 Paris, France, adelapradelle@free.fr

Abstract. We study the fBm by use of convolution of the standard white noise with a certain distribution. This brings some simplifications and new results.

Key words: Hölder continuity, fractional Brownian motion, Skorohod integrals, Gaussian sobolev spaces, rough path, distributions.

AMS Subject classification (2000): 60G15, 60H05, 60H07.

Introduction.

There are many ways to tackle the fractional Brownian motion. In this paper, we use a convolution of a white noise by a distribution T. This distribution operates in principal value as explained in paragraphs I and II. In paragraph III anf IV is defined a Skorohod type integral with respect to the fBm. This allows to define vector valued rough paths which lead to rough paths in the sense of T.Lyons. In paragraph V, we indicate a regularization process of the fBm by convolution with some examples. We study convergence of Riemann sums in paragraph VI, this also leads to approximations by piecewise linear processes of fBm-Skorohod and fBm-Stratonovich type integrals. Paragraph VII is devoted to prove that every $\mathbb{R}^d$-valued fBm defined on $\mathbb{R}$ can be studied in this way.

This work is intended to simplify many previous papers (loc.cit.), and brings some new results.

I. The distribution T.

Consider the distribution-function

$$S(t) = \frac{|t|^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi\alpha/2)} = K(\alpha)|t|^{\alpha-1}$$

which is locally integrable for $\alpha > 0, \alpha \neq 1$. The distribution derivative

$$T(t) = \frac{|t|^{\alpha-2}\,\mathrm{Sign}(t)}{2\Gamma(\alpha-1)\cos(\pi\alpha/2)}$$

analytic with respect to $\alpha \in]1/2, 3/2[$, acts on the H'-Hölder continuous functions on $[a, b]$ with $\alpha + H' > 1$ in the following way

– $\langle T, \varphi\rangle = \displaystyle\int \varphi(t)T(t)\,dt$ if $0 \notin [a, b]$,

– p.v. $\int_a^b \varphi(t)T(t)\,dt = \varphi(0)[S(b) - S(a)] + \displaystyle\int_a^b [\varphi(t) - \varphi(0)]T(t)\,dt$ if $0 \in]a, b[$

– not defined if $0 = a$ or b.

Given t, the function $S_t(u) = S(u) - S(t-u)$ belongs to $L^2(du)$ for $1/2 < \alpha < 3/2$. The Fourier transform is worth

$$\widehat{S}_t(\xi) = \int S_t(u)\,\mathrm{e}^{\mathrm{i}u\xi}\,du = \frac{1 - \mathrm{e}^{\mathrm{i}t\xi}}{|\xi|^\alpha}$$

so that

$$\langle S_t, S_s\rangle_{L^2} = K(2\alpha)\left[|t|^{2\alpha-1} + |s|^{2\alpha-1} - |t-s|^{2\alpha-1}\right]$$

that is

$$\langle S_t, S_s\rangle_{L^2} = K(2\alpha)\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right] \tag{1}$$

with $H = \alpha - 1/2$ and $K(2\alpha) = [2\Gamma(2H+1)\sin(\pi H)]^{-1}$.

Note that the map $\alpha \to S_t(u)$ extends in a $L^2(du)$-valued holomorphic function in the band $1/2 < \mathrm{Re}\,\alpha < 3/2$. For $\alpha = 1$, it is worth $\pi^{-1}\,\mathrm{Log}\,|1 - t/u|$.

1 Theorem : *Let φ be a Banach-valued H'-Hölder continuous function defined on a segment $[a, b]$, with $H + H' > 1/2$. The convolution product*

$$f(u) = \text{p.v.} \int_a^b \varphi(t)T(u-t)\,dt \tag{2}$$

belongs to $L^2(du)$.

Proof: Note that f is defined for $u \neq a, b$. First, for u out of $[a, b]$ we get

$$|f(u)| \leq \|\varphi\|_\infty \int_a^b |T(u-t)|\, dt = \|\varphi\|_\infty |S(u-b) - S(u-a)|$$

Second, for $u \in]a, b[$, write $\|\varphi\|_{H'} = \mathrm{Sup}_{t \neq s} |\varphi(t) - \varphi(s)|/|t-s|^{H'}$, then

$$f(u) = \varphi(u)[-S(u-b) + S(u-a)] + \int_a^b [\varphi(t) - \varphi(u)] T(u-t)\, dt$$

$$|f(u)| \leq \|\varphi\|_\infty |S(u-b) - S(u-a)| + K(\alpha, H')\|\varphi\|_{H'} \left[|b-a|^{\alpha+H'-1}\right] 1_{]a,b[}(u)$$

with $K(\alpha, H') = K(\alpha)/(\alpha + H' - 1)$ (remark that $K(1, H') = 2/(\pi H')$). Finally we get

$$N_2(f) \leq \|\varphi\|_\infty N_2(S_b - S_a) + K(\alpha, H')\|\varphi\|_{H'} |b-a|^{\alpha+H'-1/2}$$
$$N_2(f) \leq [2K(2\alpha)]^{1/2} \|\varphi\|_\infty |b-a|^H + K(\alpha, H')\|\varphi\|_{H'} |b-a|^{\alpha+H'-1/2}$$

2 Remarks : 1°. If φ vanishes at a point of $[a, b]$, then $\|\varphi\|_\infty \leq \|\varphi\|_{H'} |b-a|^{H'}$, so that we get

$$N_2(f) \leq K'(H, H')\|\varphi\|_{H'} |b-a|^{H+H'} \tag{3}$$

2°. For $\alpha > 1$, T is locally integrable, so that the p.v. is unuseful.

II. The fBm.

Let B_t be an $\mathbb{R}^d$ valued Brownian motion for $t \in \mathbb{R}$. As $H = \alpha - 1/2 \in]0, 1[$, we have $S_t \in L^2(du)$ and we define

$$X_t^H = \int S_t(u)\, dB_u = K(\alpha) \int [|u|^{\alpha-1} - |t-u|^{\alpha-1}]\, dB_u \tag{4}$$

This is a centered Gaussian process with H'-Hölder continuous paths for $H' < H$ (Kolmogorov lemma).
The covariance matrix is given by formula (1)

$$\mathbb{E}(X_t^H X_s^H)_{ij} = \langle S_t, S_s \rangle_{L^2} \delta_{ij}$$

Let φ be a H'-Hölder continuous function on a segment $[a, b]$, we define

$$\int_a^b \varphi(t)\, dX_t^H = \int f(u)\, dB_u$$

where f is the convolution defined in formula (2). Then formula (3) writes

$$N_2\left(\int_a^b [\varphi(t)-\varphi(a)]\,dX_t^H\right) \leq K'(H,H')\|\varphi\|_{H'}|b-a|^{H+H'} \tag{5}$$

The process $Y_t = \int_0^t \varphi(t)\,dX_t^H$ centered Gaussian Banach valued process for $H \in]0,1[$ and $H+H' > 1/2$. Moreover the map $H \to \displaystyle\int_a^b \varphi(t)\,dX_t^H$ is analytic in $H \in]1/2-H',1[$.

Example: Let $\widetilde{X}^H$ an independant copy of X^H. If we take $\varphi(t) = \widetilde{X}_t^H$ as a L^2-valued H-Hölder continuous function, we get

$$\int_a^b [\widetilde{X}_t^H - \widetilde{X}_a^H] \otimes dX_t^H$$

By taking the coordinates, we can get the Lévy areas of the fBm for $H > 1/4$. Putting for the i-coordinate

$$\int_a^b [X_t^{H,i}(\omega) - X_a^{H,i}(\omega)]\,dX_t^{H,i}(\omega)] = [X_b^{H,i}(\omega) - X_a^{H,i}(\omega)]^2/2$$

we get finally ageometric rough path calculus for $H > 1/3$.

III. Using the Skorohod integral.

Recall (*cf.* [4,5]) that the Gaussian Sobolev space $\mathcal{D}^{1,2}(\Omega,\mu)$ constructed upon the Gaussian measure μ is the space of Wiener functionals $\Phi(\omega)$ such that $\nabla\Phi(\omega,\widetilde{\omega})$ belongs to $L^2(\mu\otimes\mu)$. The divergence is the transposed operator div : $L^2(\mu:\mu) \to L^2(\mu)$.

Let Φ be a H'-Hölder continuous $\mathcal{D}^{1,2}$ valued function. Consider the following integral

$$Z(\omega,\widetilde{\omega}) = \int_a^b \Phi_t(\omega) \otimes dX_t^H(\widetilde{\omega}) \tag{6}$$

then Z belongs to $\mathcal{D}^{1,2}\widehat{\otimes}W^1$ where W^1 is the first Wiener chaos and $\widehat{\otimes}$ is the Hilbert tensor product. We get by formula (5)

$$\|Z - \Phi_a \otimes (X_b^H - X_a^H)\|_{\mathcal{D}^{1,2}\widehat{\otimes}W^1} \leq K(H,H')\|\Phi\|_{H',\mathcal{D}^{1,2}}|b-a|^{H+H'}$$

Apply now the divergence operator, and define the fBm-Skorohod integral [1,6,7,8,13]

$$\int_a^b \Phi_t(\omega) \odot dX_t^H(\omega) = (\operatorname{div} Z)(\omega)$$

Hence we get by continuity of the divergence

$$\left\| \int_a^b [\Phi_t - \Phi_a] \odot dX_t^H \right\|_{L^2(d\omega)} \leq k' \|\Phi\|_{H,\mathcal{D}^{1,2}} |b-a|^{H+H'}$$

Example: suppose that Φ takes values into a non-homogneous Wiener chaos of degree k which is included in $\mathcal{D}^{1,2}$ with an equivalent norm as in L^2 [8]. For $H > 1/4$ it is straightforward to verify that we can define in this way

$$X_{ab,ij}^{H,(2)} = \int_a^b dX_{t,i}^H \odot \int_a^t dX_{s,j}^H, \qquad \text{and} \qquad X_{ab,ijk}^{H,(3)} = \int_a^b dX_{t,i}^H \odot X_{at,jk}^{H,(2)}$$

As $X_{ab}^{H,(2)}$ belongs to the second Wiener chaos, then $X_{ab}^{H,(3)}$ is well defined as an element of the third Wiener chaos for every $\{i,j,k\}$ (coordinate indices). Besides we have

$$\|X_{ab}^{H,(2)}\|_{L^2} \leq \text{Cst}\, |b-a|^{2H}, \qquad \|X_{ab}^{H,(3)}\|_{L^2} \leq \text{Cst}\, |b-a|^{3H}$$

It is easily seen that we get a vector valued rough path for $H > 1/4$, for example we have for every $c \in [a,b]$

$$X_{ab}^{H,(2)} - X_{ac}^{H,(2)} - X_{cb}^{H,(2)} = (X_b^H - X_c^H) \otimes (X_c^H - X_a^H)$$

Note the inversion of the tensor product, which does not matter.

We could recover $\int_a^b F(X_t^H) \odot dX_t^H$ thanks to the sewing lemma ([11,12]). Observe that we can also obtain pathwise rough paths in the sense of T.Lyons ([14]) for $H > 1/4$.

IV. Extending the classical calculus.

Let F be a polynomial. It is easily seen that $t \to F(X_t^H)$ is Hölder continuous, so that if $H > 1/2$ we get

$$\int_a^b F(X_t^H) \odot dX_t^H = \operatorname{div} \int_a^b F(X_t^H(\omega)) \otimes dX_(^H \widetilde{\omega})$$

where the second member is a Young integral. By computing it, we find

$$\int_a^b F(X_t^H(\omega))\,dX_t^H(\omega) - \int_a^b \nabla F(X_t^H)\mathbb{E}(X_t^H\,dX_t^H)$$

and finally

$$\int_a^b F(X_t^H)\odot dX_t^H = \int_a^b F(X_t^H(\omega))\,dX_t^H(\omega) - 2HK(2\alpha)\int_a^b \nabla F(X_t^H)t^{2H-1}\,dt$$

Then it is natural to put for $H > 1/4$

3 Definition : *For $H > 1/4$ and F a polynomial, we define*

$$\int_a^b F(X_t^H(\omega))\circ dX_t^H(\omega) = \int_a^b F(X_t^H)\odot dX_t^H + 2HK(2\alpha)\int_a^b \nabla F(X_t^H)t^{2H-1}\,dt$$

This formula can be read as an Ito formula for $H > 1/4$. Put

$$\widetilde{X}_{ab,ij}^{H,(2)} = \int_a^b dX_{t,i}^H \circ \int_a^t dX_{s,j}^H, \qquad \text{and} \qquad \widetilde{X}_{ab,ijk}^{H,(3)} = \int_a^b X_{tb,i}^H \circ dX_{t,j}^H \circ X_{at,k}^H$$

As for the Skorohod integral, we see that $\widetilde{X}_{ab}^{H,(2)}$ belongs to the second Wiener chaos, so that $\widetilde{X}_{ab}^{H,(3)}$ is well defined as an element of the third Wiener chaos for every $\{i,j,k\}$. Besides we have

$$\|\widetilde{X}_{ab}^{H,(2)}\|_{L^2} \le \mathrm{Cst}\,|b-a|^{2H}, \qquad \|\widetilde{X}_{ab}^{(H,3)}\|_{L^2} \le \mathrm{Cst}\,|b-a|^{3H}$$

Now, for $H > 1/2$ we get standard Young integrals, so that the rough path algebraic relations are satisfied. They also hold for $H > 1/4$ thanks to the analyticity with respect to H. Hence, we get another vector valued rough path for $H > 1/4$.

V. Approximations of the fBm.

1°. Let $\rho_n(t) \ge 0$ be a regularizing sequence, that is $\int \rho_n(t)\,dt = 1$ and ρ_n converges narrowly to the Dirac mass at 0. Put

$$S_n(u) = S * \rho_n(u), \qquad S_{t,n}(u) = S_t * \rho_n(u), \qquad T_{t,n}(u) = T * \rho_n(u)$$

Put

$$X_{t,n}^H = \int S_{t,n}\,dB_u, \qquad \int_a^b \varphi(t)\,dX_{t,n}^H = \int f_n(u)\,dB_u$$

with

$$f_n(u) = \int_a^b \varphi(t) T_{t,n}(u)\, dt$$

As n goes to infinity, $S_{n,t}$ converges to S_t in $L^2(du)$ so that $X^H_{t,n}$ converge to X^H_t in the first Wiener chaos.

We have $f_n = \varphi_{ab} * (T * \rho_n)$ where φ_{ab} is worth φ in $[a,b]$ and 0 elsewhere. It is easily seen that we have the associative relation $f_n = (\varphi_{ab} * T) * \rho_n$. As $\varphi_{ab} * T$ belongs to L^2, the convolution by ρ_n converges to f in L^2 when $n \to \infty$. It follows that

$$\int_a^b \varphi(t)\, dX^H_t = \operatorname*{Lim}_{n\to\infty} \int_a^b \varphi(t)\, dX^H_{t,n}$$

Example: take the 1-dimensional process B_u, and consider for $y > 0$ $\rho_y(t) = \dfrac{y}{\pi(t^2+y^2)}$. As well known $T_{t,y}(u) = \rho_y * T_t(u)$ is harmonic with respect to $(t,y) \in \mathbb{R} \times \mathbb{R}^+$. Hence we get a harmonic extension $X^H_{t,y}$ with respect to (t,y). When y goes to 0, $X^H_{t,y}$ converges to X^H_t. Note that $\displaystyle\int_a^b \varphi(t)\, dX^H_{t,y}$ is a usual integral with respect to a $\mathcal{C}^\infty$-function $X^H_{t,y}$ for every $y > 0$. It should be observed that $X^H_{t,y}$ is the real part of a holomorphic function of $t + \mathrm{i}y$ in the upper half-plane. This holomorphic function has been investigated in [15].

2°. (*cf.* [6,7]). For $\lambda > 0$ put

$$G^\alpha_\lambda(t) = \frac{2^{-\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty u^{\alpha/2-1}\, \mathrm{e}^{-\lambda u}\, h_u(t)\, du$$

where

$$h_u(t) = (2\pi u)^{-1/2} \exp(-t^2/2u)$$

Then

$$\widehat{G^\alpha_\lambda}(\xi) = 2^{-\alpha/2}(\lambda + \xi^2/2)^{-\alpha/2}$$

Put

$$Y^{H,\lambda}_t = \int G^\alpha_\lambda(t-u)\, dB_u$$

$$\mathrm{Cov}(Y^{H,\lambda}_t, Y^{H,\lambda}_s) = \int G^\alpha_\lambda(t-u) G^\alpha_\lambda(s-u)\, du = G^{2\alpha}_\lambda(t-s) = G^{2\alpha}_\lambda(h)$$

$$X^{H,\lambda}_t = Y^{H,\lambda}_t - Y^{H,\lambda}_0 = \int [G^\alpha_\lambda(u) - G^\alpha_\lambda(t-u)]\, dB_u$$

For $\alpha - 1/2 \in]0,1[$, we get by the Fourier transform

$$\mathrm{Cov}(Y^{H,\lambda}_t, Y^{H,\lambda}_s) = \frac{2^{-\alpha}}{2\pi} \int \frac{\cos h\xi}{(\lambda + \xi^2/2)^\alpha}\, d\xi$$

$$\mathbb{E}(X_t^{H,\lambda}, X_s^{H,\lambda}) = \frac{2^{-\alpha}}{2\pi} \int \frac{1 - \cos t\xi - \cos(s\xi) + \cos(t-s)\xi}{(\lambda + \xi^2/2)^\alpha} \, d\xi$$

As $\lambda \to 0$, $G_\lambda^\alpha(u) - G_\lambda^\alpha(t-u)$ converges towards $S_t(u)$ in $L^2(du)$, so that $X^{H,\lambda}$ converges towards X^H in $L^2(d\omega)$.

VI. Riemann sums associated with a partition.

Let $\Delta = \{a = t_0, t_1, \ldots, t_n = b\}$ be a partition of $[a, b]$ with mesh δ. Choose points $\tau_i \in [t_i, t_{i+1}]$, and consider the Riemann sum

$$Z = \sum_a^b \varphi(\tau_i)[X_{t_{i+1}}^H - X_i^H]$$

where φ is H'-Hölder continuous on $[a, b]$. Also consider the function

$$J(u) = \sum_a^b \varphi(\tau_i)[S_{t_{i+1}}(u) - S_i(u)]$$

4 Lemma : *For $H + H' > 1/2$, $J(u)$ converges to $\varphi_{ab} * T(u)$ in $L^2(du)$ as δ vanishes.*

Proof: We have

$$\varphi_{ab} * T(u) = \sum_a^b \text{p.v.} \int_{t_i}^{t_{i+1}} \varphi(t) T(u-t) \, dt$$

Hence the difference

$$D(u) = \varphi_{ab} * T(u) - J(u) = \sum_a^b \text{p.v.} \int_{t_i}^{t_{i+1}} [\varphi(t) - \varphi(\tau_i)] T(u-t) \, dt$$

For $u \notin [a, b]$ we have

$$|D(u)| \leq \|\varphi\|_{H'} |S_b(u) - S_a(u)| \delta^{H'}$$

For $u \in]a, b[$, we first look for $u \in]t_k, t_{k+1}[$. We have

$$D(u) = \left(\int_a^{t_k} + \int_{t_{k+1}}^b \right) [\varphi(t) - \varphi(\tau_i)] T(u-t) \, dt + \text{p.v.} \int_{t_k}^{t_{k+1}} [\varphi(t) - \varphi(\tau_i)] T(u-t) \, dt$$

$$|D(u)| \leq 2\|\varphi\|_{H'} \delta^{H'} [S_{t_k}(u) + S_{t_{k+1}}(u)] + |R(u)|$$

with

$$R(u) = [\varphi(u) - \varphi(\tau_i)][S_{t_{k+1}}(u) - S_{t_k}(u)] + \int_{t_k}^{t_{k+1}} [\varphi(t) - \varphi(u)]T(u-t)\,dt$$

$$|R(u)| \leq \|\varphi\|_{H'}\delta^{H'}[S_{t_k}(u) + S_{t_{k+1}}(u)] + \|\varphi\|_{H'}\delta^{\alpha+H'-1}/(\alpha+H'-1)$$

$$|D(u)| \leq 3\|\varphi\|_{H'}\delta^{H'}[S_{t_k}(u) + S_{t_{k+1}}(u)] + \|\varphi\|_{H'}\delta^{\alpha+H'-1}/(\alpha+H'-1)$$

We get

$$\int D(u)^2\,du = \left(\int_{-\infty}^{a} + \int_{b}^{\infty}\right) D(u)^2\,du + \sum_{a}^{b}\int_{t_k}^{t_{k+1}} D(u)^2\,du$$

$$\int D(u)^2\,du \leq K\|\varphi\|_{H'}^2\delta^{2H'}|b-a|^{2H} + K\|\varphi\|_{H'}^2\sum_{a}^{b}\delta^{2H'+2\alpha-1}$$

$$\int D(u)^2\,du \leq K\|\varphi\|_{H'}^2[\delta^{2H'}|b-a|^{2H} + \delta^{2H+2H'-1}|b-a|]$$

which converges to 0 as δ vanishes.

5 Remark : In fact this lemma also holds with a function $\tau_i(u)$ in place of τ_i.

Then we can claim

6 Theorem : *We have in $L^2(d\omega)$*

$$\int_a^b \varphi(t)dX_t^H(\omega) = \operatorname*{Lim}_{\delta\to 0}\sum_{a}^{b}\varphi(\tau_i)[X_{t_{i+1}}^H - X_{t_i}^H](\omega)$$

Application to the fBm integrals.

Applying this result to equation (6), we get

$$Z(\omega,\widetilde{\omega}) = \operatorname*{Lim}_{\delta\to 0}\sum_{a}^{b}\Phi_{\tau_i}(\omega)[X_{t_{i+1}}^H - X_{t_i}^H](\widetilde{\omega})$$

in the space $\mathcal{D}^{1,2}(d\omega)\widehat{\otimes}L^2(d\widetilde{\omega})$. By taking the divergence, we get

$$\int_a^b \Phi_t(\omega)\odot dX_t^H(\omega) = \operatorname*{Lim}_{\delta\to 0}\sum_{a}^{b}\Phi_{\tau_i}(\omega)[X_{t_{i+1}}^H - X_{t_i}^H](\omega) - \widetilde{E}[\nabla\Phi_{\tau_i}(\omega,\widetilde{\omega})[X_{t_{i+1}}^H(\widetilde{\omega}) - X_{t_i}^H(\widetilde{\omega})]$$

in $L^2(d\omega)$. When $\Phi_t = F(X_t^H)$, we have

$$\nabla\Phi_t(\omega,\widetilde{\omega}) = \nabla F(X_t^H(\omega))X_t^H(\widetilde{\omega})$$

$$\sum_a^b \nabla F(X^H_{\tau_i}(\omega))\widetilde{E}[X^H_{\tau_i}(\widetilde{\omega})[X^H_{t_{i+1}}(\widetilde{\omega}) - X^H_{t_i}(\widetilde{\omega})] =$$

$$= K(2\alpha)\sum_a^b \nabla F(X^H_{\tau_i}(\omega))[|t_{i+1}|^{2H} - |t_i|^{2H} - |t_{i+1} - \tau_i|^{2H} + |t_i - \tau_i|^{2H}]$$

$$= 2HK(2\alpha)\int_a^b \nabla F(X^H_t(\omega))t^{2H-1}\,dt - K(2\alpha)\sum_a^b \nabla F(X^H_{\tau_i}(\omega))[|t_{i+1} - \tau_i|^{2H} - |t_i - \tau_i|^{2H}]$$

7 Theorem : *Suppose that τ_i is the midpoint of $[t_i, t_{i+1}]$ then we have*

$$\int_a^b F(X^H_t)\odot dX^H_t = \operatorname*{Lim}_{\delta\to 0}\sum_a^b F(X^H_{\tau_i})[X^H_{t_{i+1}} - X^H_{t_i}] - 2HK(2\alpha)\int_a^b \nabla F(X^H_t)t^{2H-1}\,dt$$

$$\int_a^b F(X^H_t)\circ dX^H_t = \operatorname*{Lim}_{\delta\to 0}\sum_a^b F(X^H_{\tau_i})[X^H_{t_{i+1}} - X^H_{t_i}]$$

Hence the limite above exists in $L^2(d\omega)$.

Piecewise linear interpolation.

Let φ be a H'-Hölder continuous function, and let φ_n be the function deduced from φ by linear interpolation with vertices t_i of Δ_n. It is easily seen that the Hölder semi-norm satisfies $\|\varphi_n\|_{H'} \le \|\varphi\|_{H'}$ and that $\|\varphi - \varphi_n\|_{H''}$ converges to 0 for every $H'' < H'$. Hence $J(\varphi_n)$ is bounded in $L^2(du)$ and

$$N_2\left(\int_a^b [\varphi(t) - \varphi_n(t)]\,dX^H_t\right) \le k(H)\|\varphi - \varphi_n\|_{H''}|b-a|^{H+H'}$$

which tends to 0 when n tends to infinity.

Let $X^H_{t,n}$ be the linear interpolation of X^H_t with respect to Δ_n, and consider $\varphi_n = F(X^H_{t,n})$ for a polynomial F. As φ_n belongs to a non-hogoneous Wiener chaos of degree k, we have for $s \le t$

$$F(X^H_{t,n}) - F(X^H_{s,n}) = \int_0^1 \nabla F(X^H_{s,n} + \lambda[X^H_{t,n} - X^H_{s,n}])[X^H_{t,n} - X^H_{s,n}]\,d\lambda$$

$$N_2(F(X^H_{t,n}) - F(X^H_{s,n})) \le \operatorname*{Sup}_{\lambda\in[0,1]} N_4(\nabla F(X^H_{s,n} + \lambda[X^H_{t,n} - X^H_{s,n}]))N_4(X^H_{t,n} - X^H_{s,n})$$

By the Nelson inequalities for non-homogeneous chaos (*cf.* [8]), we get

$$N_2(F(X^H_{t,n}) - F(X^H_{s,n})) \le M'(a,b)N_2(X^H_{t,n} - X^H_{s,n}) \le M(a,b)|t-s|^H$$

where $M'(a,b)$ and $M(a,b)$ depends only on a and b.

8 Lemma : *Let $S_{t,n}$ be the interpolation of S_t relative to Δ_n. Then for any H'-Hölder continuous φ with $H+H'>1/2$, we have*

$$\varphi_{ab} * T = \operatorname*{Lim}_{\delta_n \to 0} \int_a^b \varphi_n(t)\, dS_{t,n}$$

Proof: First, if φ is real valued, there exists functions $u \to \tau_i(u) \in [t_i, t_{i+1}]$ such that the second member is worth

$$\int_a^b \varphi_n(t)\, dS_{t,n} = \sum_a^b \varphi(\tau_i(u))[S_{t_{i+1}} - S_{t_i}]$$

Applying lemma 4 and remark 5, we obtain

$$N_2\left(\varphi_{ab} * T - \int_a^b \varphi_n(t)\, dS_{t,n}\right) \leq K\|\varphi\|_{H'} \left[\delta^{2H'}|b-a|^{2H} + \delta^{2H+2H'-1}|b-a|\right]^{1/2}$$

Second, if φ is Banach valued, we get the same inequality thanks to the Hahn-Banach theorem.

9 Proposition : *We have*

$$\int_a^b \varphi(t)\, dX_t^H(\omega) = \operatorname*{Lim}_{\delta_n \to 0} \int_a^b \varphi_n(t)\, dX_{t,n}^H(\omega)$$

the limit being taken in $L^2(d\omega)$. Moreover if F is a polynomial,

$$\int_a^b F(X_t^H(\omega))\, dX_t^H(\widetilde{\omega}) = \operatorname*{Lim}_{\delta_n \to 0} \int_a^b F(X_{t,n}^H(\omega))\, dX_{t,n}^H(\widetilde{\omega})$$

in $L^2(d\omega \otimes d\widetilde{\omega})$.

Proof: The first assertion follows from the lemma, and the second one is a particular case in view of above.

10 Corollary :

$$\int_a^b F(X_t^H) \odot dX_t^H = \operatorname*{Lim}_{\delta_n \to 0} \int_a^b F(X_{t,n}^H)\, dX_{t,n}^H - 2HK(2\alpha) \int_a^b \nabla F(X_{t,n}^H) t^{2H-1}\, dt$$

Proof: A straightforward computation.

11 Corollary :

$$\int_a^b F(X_t^H) \circ dX_t^H = \lim_{\delta_n \to 0} \int_a^b F(X_{t,n}^H)\, dX_{t,n}^H$$

12 Remark : In particular, the $\odot$-type and $\circ$-type rough paths are obtained in this way (linear interpolation). Note that the $\circ$-type rough path was obtained by ([2]). See also [13].

VII. Retrieving the initial Brownian motion.

Now, let X_t^H be given a fBm, that is a continuous centered Gaussian process defined on $\mathbb{R}$, with covariance

$$\mathbb{E}(X_s^H X_t^H) = K(2\alpha)[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$$

We suppose that $H \in]0,1[$ and $\alpha = H + 1/2$. Does there exists a standard Brownian motion B_t such that formula (4) holds ?

First, let A_t be the $\mathbb{R}^d$-valued standard Brownian motion defined on $\mathbb{R}$, and let Y_t be the fBm defined above from A_t

$$Y_t = \int [S^\alpha(u) - S^\alpha(u-t)]\, dA_u$$

Then

$$A_t = -\int [S^{2-\alpha}(u) - S^{2-\alpha}(u-t)]\, dY_u \tag{7}$$

Indeed, (7) makes sense as $u \to |u|^{\alpha-2}[S^{2-\alpha}(u) - S^{2-\alpha}(u-t)]$ belongs to $L^2(du)$ as easily seen by Fourier transform. Put $\varphi(v) = S_t^{2-\alpha}(v)$, we get

$$\int [S^{2-\alpha}(u) - S^{2-\alpha}(u-t)]\, dY_u = \int (\varphi * T^\alpha)(v)\, dA_v = \int [T^\alpha * S^{2-\alpha}](v)\, dA_v$$

By Fourier transform, we check that $T^\alpha * S^{2-\alpha}](v) = -1_{[0,t](v)}$, so that we are done.

Let $\varphi(t)$ be a H-Hölder continuous function with compact support, we have seen that

$$\int_a^b \varphi(u)\, dY_u = \lim_{\delta \to 0} \sum_a^b \varphi(\tau_i)[Y_{u_{i+1}} - Y_{u_i}]$$

for every partition of $[a,b]$ with mesh $\delta \to 0$. It follows that

$$\int_a^b \varphi(u)\, dX_u^H = \operatorname*{Lim}_{\delta\to 0} \sum_a^b \varphi(\tau_i)[X_{u_{i+1}}^H - X_{u_i}^H]$$

exists since the covariances are the same. Now take

$$\varphi_R(u) = [S^{2-\alpha}(u) - S^{2-\alpha}(u-t)]1_{[-R,R]}(u)$$

As $R \to +\infty$, we get

$$A_t = -\operatorname*{Lim}_{R\to\infty} \int_R^R \varphi_R(u)\, dY_u$$

then

$$B_t = -\operatorname*{Lim}_{R\to\infty} \int_R^R \varphi_R(u)\, dX_u^H$$

exists. Then we obtain a continuous centered Gaussian process B_t on Ω, which is a Brownian motion, as easily checked by computing the covariance function. Conversely one retrieves X_t^H with the formula

$$X_t^H = \int [S^\alpha(u) - S^\alpha(u-t)]\, dB_u$$

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