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Precise smoothing effect in the exterior of balls

Oana Ivanovici

Universite Paris-Sud, Orsay,

Mathematiques, Bat. 430, 91405 Orsay Cedex, France

oana.ivanovici@math.u-psud.fr

1 Introduction

We are interested in this article in investigating the smoothing effect properties of the solutions of the Schrödinger equation. Since the work by Craig, Kapeller and Strauss [13], Kato [16], Constantin and Saut [12] establishing the smoothing property, many works have dealt with the understanding of this effect. In particular the work by Doi [14] and Burq [5, 6] shows that it is closely related to the infinite speed of propagation for the solutions of Schrödinger equation. Roughly speaking, if one considers a wave packet with wave length λ , it is known that it propagates with speed λ and the wave will stay in any bounded domain only for a time of order $1/\lambda$. As a consequence, taking the L^2 in time norm will lead to an improvement of $1/\lambda^{1/2}$ with respect to taking an L^∞ norm, leading to a gain of $1/2$ derivatives. This heuristic argument can be transformed into a proof of the smoothing effect either by direct calculations (in the case of the free Schrödinger equation) or by means of resolvent estimates (see [3] for the case of a perturbation by a potential or [7] for the boundary value problem). In view of this simple heuristics, it is natural to ask whether one can refine (and improve) such smoothing type estimates if one considers smaller space domains (whose size will shrink as the wave length increases). A very natural context in which one can test this heuristics is the case of the exterior of a convex body (or more generally the exterior of several convex bodies), in which case natural candidates for the λ dependent domains are $\lambda^{-\alpha}$ neighborhoods of the boundary. This is the main aim of this paper. To keep the paper at a rather basic technical level, we choose to consider only balls, for which direct calculations (with Bessel functions) can be performed. Our first result reads as follows:

Theorem 1.1. *Let $\Omega = \mathbb{R}^3 \setminus B(0, 1)$, $T > 0$, $0 \leq \alpha < \frac{2}{3}$ and $\lambda \geq 1$. Let ψ and $\chi \in C_0^\infty(\mathbb{R}^*)$ be smooth functions with compact support, $\psi = 1$ near 1, $\chi = 1$ near 0. Set $\chi_\lambda(|x|) := \chi(\lambda^\alpha(|x| - 1))$, where x denotes the variable on Ω . Then one has*

- For $s \in [-1, 1]$ and $v(t) = \int_0^t e^{i(t-\tau)\Delta_D} \psi(\frac{-\Delta_D}{\lambda^2}) \chi_\lambda g d\tau$

$$\|\chi_\lambda \psi(\frac{-\Delta_D}{\lambda^2}) v\|_{L_T^2 H_D^{s+1}(\Omega)} \leq C \lambda^{-\frac{\alpha}{2}} \|\psi(\frac{-\Delta_D}{\lambda^2}) \chi_\lambda g\|_{L_T^2 H_D^s(\Omega)}, \quad (1.1)$$

- For $s \in [0, 1]$

$$\|\chi_\lambda e^{it\Delta_D} \psi(\frac{-\Delta_D}{\lambda^2}) u_0\|_{L_T^2 H_D^{s+\frac{1}{2}}(\Omega)} \leq C \lambda^{s-\frac{\alpha}{4}} \|\psi(\frac{-\Delta_D}{\lambda^2}) u_0\|_{L^2(\Omega)}. \quad (1.2)$$

Here the constants C do not depend on T , i.e. the estimates are global in time.

Using this result, we can deduce new Strichartz type estimates for the solution of the linear Schrödinger equation in the exterior, Ω , of a smooth bounded obstacle $\Theta \subset \mathbb{R}^3$,

$$(i\partial_t + \Delta_{D,N})u = 0, \quad u(0, x) = u_0(x), \quad u|_{\partial\Omega} = 0 \text{ or } \partial_n u|_{\partial\Omega} = 0, \quad (1.3)$$

that we denote respectively by $e^{it\Delta_D} u_0$ and $e^{it\Delta_N} u_0$.

Definition 1.2. Let $q, r \geq 2$, $(q, r, d) \neq (2, \infty, 2)$, $2 \leq p \leq \infty$. A pair (q, r) is called admissible in dimension d if q, r satisfy the scaling admissible condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \quad (1.4)$$

Theorem 1.3. (Strichartz estimates) Let $\Theta = B(0, 1) \subset \mathbb{R}^3$ and $\Omega = \mathbb{R}^3 \setminus \Theta$, $T > 0$ and (p, q) an admissible pair in dimension 3. Let $\epsilon > 0$ be an arbitrarily small constant. Then there exists a constant $C > 0$ such that, for all $u_0 \in H_{D,N}^{\frac{4}{5p}+\epsilon}(\Omega)$ the following holds

$$\|e^{it\Delta_{D,N}} u_0\|_{L^p([-T, T], L^q(\Omega))} \leq C \|u_0\|_{H_{D,N}^{\frac{4}{5p}+\epsilon}(\Omega)}. \quad (1.5)$$

Moreover, a similar result holds true for a class of trapping obstacles (Ikawa's example), i.e. for the case where Θ is a finite union of balls in \mathbb{R}^3 .

As a consequence of Theorem 1.3 we deduce new global well-posedness results for the non-linear Schrödinger equation in the exterior of several convex obstacles, improving previous results by Burq, Gerard and Tzvetkov [7]. Consider the nonlinear Schrödinger equation on Ω subject to Dirichlet boundary condition

$$(i\partial_t + \Delta_D)u = F(u) \quad \text{in } \mathbb{R} \times \Omega, \quad u(0, x) = u_0(x), \quad \text{on } \Omega, \quad u|_{\partial\Omega} = 0. \quad (1.6)$$

The nonlinear interaction F is supposed to be of the form $F = \partial V / \partial \bar{z}$, with $F(0) = 0$, where the "potential" V is real valued and satisfies $V(|z|) = V(z)$, $\forall z \in \mathbb{C}$. Moreover we suppose that V is of class C^3 , $|D_{z, \bar{z}}^k V(z)| \leq C_k (1 + |z|)^{4-k}$ for $k \in \{0, 1, 2, 3\}$ and that $V(z) \geq -(1 + |z|)^\beta$, for some $\beta < 2 + \frac{4}{d}$, $d = 3$ (the last assumption avoid blow-up in the focussing case).

Some phenomena in physics turn out to be modeled by exterior problems and one may expect rich dynamics under various boundary conditions. A first step in this direction is to establish well defined dynamics in the natural spaces determined by the conservation laws associated to (1.6). If $u(t, \cdot) \in H_0^1(\Omega) \cap H^2(\Omega)$ is a solution of (1.6) then it satisfies the conservation laws

$$\frac{d}{dt} \int_{\Omega} |u(t, x)|^2 dx = 0; \quad \frac{d}{dt} \left(\int_{\Omega} |\nabla u(t, x)|^2 dx + \int_{\Omega} V(u(t, x)) dx \right) = 0 \quad (1.7)$$

and therefore for a large class of potentials V the quantity $\|u(t, \cdot)\|_{H_0^1(\Omega)}$ remains finite along the trajectory starting from $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. This fact makes the study of (1.6) in the energy space $H_0^1(\Omega)$ of particular interest. It is also of interest to study (1.6) in $L^2(\Omega)$: the main issue in the analysis is that the regularities of H^1 and L^2 are a priori too poor to be achieved by the classical methods for establishing local existence and uniqueness for (1.6). We state the result concerning finite energy solutions, which will be a consequence of Theorem 1.3.

Theorem 1.4. (*Global existence theorem*) *Let $\Omega = \mathbb{R}^3 \setminus B(0, 1)$. For any $u_0 \in H_0^1(\Omega)$ the initial boundary value problem (1.6) has a unique global solution $u \in C(\mathbb{R}, H_0^1(\Omega))$ satisfying the conservation laws (1.7). Moreover, for any $T > 0$ the flow map $u_0 \rightarrow u$ is Lipschitz continuous from any bounded set of $H_0^1(\Omega)$ to $C(\mathbb{R}, H_0^1(\Omega))$.*

Remark 1.5. In [7], Strichartz type estimates with loss of $\frac{1}{p}$ derivative have been obtained for the Schrödinger equation in the exterior of a non-trapping obstacle $\Theta \subset \mathbb{R}^d$ which allowed to prove the same global existence result in dimension 3 provided $\|u_0\|_{H_0^1(\Omega)}$ is sufficiently small.

The Cauchy problem associated to (1.6) has been extensively studied in the case $\Omega = \mathbb{R}^d$, $d \geq 2$, by Bourgain [4], Cazenave [10], Sulem et Sulem [23], Ginibre and Velo [15], Kato [16] and the theory of existence of finite energy solutions to (1.6) for potentials V with polynomial growth has been much developed. Roughly speaking the argument for establishing finite energy solutions of (1.6) consists in combining H^1 local well-posedness with conservation laws (1.7) which provide a control on the H^1 norm.

The article is written as follows: in Section 2 we obtain bounds for the L^2 norms for the outgoing solution of the Helmholtz equation that will be used in Section 3 in order to prove Theorem 1.1. In Section 4 we use a strategy inspired from [22], [7] to handle the case when the solution is supported outside a neighborhood of $\partial\Omega$; in Section 5 we achieve the proof of Theorem 1.3. The last section is dedicated to the applications of these results; precisely, we give the proofs of Theorems 1.3 and 1.4 in the case where the obstacle consists of a union of balls. In the Appendix we recall some properties of the Hankel functions.

Acknowledgments: *This result is part of the author's PhD thesis in preparation at University Paris Sud, Orsay, under Nicolas Burq's direction.*

2 Precise smoothing effect

2.1 Preliminaries

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a smooth domain. For $s \geq 0$, $p \in [1, \infty]$ we denote by $W^{s,p}(\Omega)$ the Sobolev spaces on Ω . We write L^p and H^s instead of $W^{0,p}$ and $W^{s,2}$. By Δ_D (resp. $\Delta = \Delta_N$) we denote the Dirichlet Laplacian (resp. Neumann Laplacian) on Ω , with domain $H^2(\Omega) \cap H_0^1(\Omega)$ (resp. $\{u \in H^2(\Omega) | \partial_n u|_{\partial\Omega} = 0\}$). We next define the dual space of $H_0^1(\Omega)$, $H^{-1}(\Omega)$, which is a subspace of $D'(\Omega)$. We construct $H^{-s}(\Omega)$ via interpolation and due to [2, Cor.4.5.2] we have the duality between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$ for $s \in [0, 1]$.

Proposition 2.1. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a smooth domain. The following continuous embeddings hold:*

- $H_0^1(\Omega) \subset L^q(\Omega)$, $2 \leq q \leq \frac{2d}{d-2}$ ($p < \infty$ if $d = 2$),
- $H_D^s(\Omega) \subset L^q(\Omega)$, $\frac{1}{2} - \frac{1}{q} = \frac{s}{d}$, $s \in [0, 1)$,
- $H_D^{s+1}(\Omega) \subset W^{1,q}(\Omega)$, $\frac{1}{2} - \frac{1}{q} = \frac{s}{d}$, $s \in [0, 1)$,
- $W^{s,p}(\Omega) \subset L^\infty(\Omega)$, $s > \frac{d}{p}$, $p \geq 1$.

Proof. The proof follows from the Sobolev embeddings on \mathbb{R}^d and the use of extension operators. \square

Let $\psi, \chi \in C_0^\infty(\mathbb{R}^*)$ be smooth functions such that $\chi = 1$ in a neighborhood of 0 and for all $\tau \in \mathbb{R}_+$, $\sum_{k \geq 0} \psi(2^{-2k}\tau) = 1$. For $\lambda > 0$ let $\chi_\lambda(|x|) := \chi(\lambda^\alpha(|x| - 1))$, where x denotes variable on Ω . We introduce spectral localizations which commute with the linear evolution. Since the spectrum of $-\Delta_D$ is confined to the positive real axis it is convenient to introduce λ^2 as a spectral parameter. We will consider the linear problem (1.3) with initial data localized at frequency λ , $u|_{t=0} = \psi(-\frac{\Delta_D}{\lambda^2})u_0$.

Remark 2.2. In order to prove Theorem 1.3 it will be enough to prove (1.5) with the initial data of the form $\psi(-\frac{\Delta_D}{\lambda^2})u_0$, since then we have for some ϵ arbitrarily small

$$\begin{aligned} & \|e^{it\Delta_D} u_0\|_{L^p([-T,T], L^q(\Omega))} \approx \|(e^{it\Delta_D} \psi(-\frac{\Delta_D}{2^{2j}})u_0)_j\|_{L^p([-T,T], L^q(\Omega))l_j^2} \leq \quad (2.1) \\ & \leq \|(e^{it\Delta_D} \psi(-\frac{\Delta_D}{2^{2j}})u_0)_j\|_{l_j^2 L^p([-T,T], L^q(\Omega))} \leq \left(\sum_j 2^{2j(\frac{4}{5p} + \epsilon)} \|\psi(-\frac{\Delta_D}{2^{2j}})u_0\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \approx \|u_0\|_{H_D^{\frac{4}{5p} + \epsilon}(\Omega)}. \end{aligned}$$

2.2 Estimates in a small neighborhood of the boundary

In what follows let $d = 3$. We study first the outgoing solution to the equation

$$(\Delta_D + \lambda^2)w = \chi_\lambda f, \quad w|_{\partial\Omega} = 0. \quad (2.2)$$

In what follows we will establish high frequencies bounds for the L^2 norm of w in a small neighborhood $\Omega_\lambda = \{|x| \lesssim \lambda^{-\alpha}\}$ of size $\lambda^{-\alpha}$ of the boundary. We notice that a ray with transversal, equal-angle reflection spends in the neighborhood Ω_λ a time $\simeq \lambda^{-\alpha}$. If the ray is diffractive then the time spent in Ω_λ equals $\lambda^{-\frac{\alpha}{2}}$.

We analyze the outgoing solution of (2.2) outside the unit ball of \mathbb{R}^3 . The first step is to introduce polar coordinates and to write the expansion in spherical harmonics of the solution to

$$\begin{cases} (\Delta_D + \lambda^2)\tilde{w} = 0 & \text{on } \Omega = \{x \in \mathbb{R}^3 \mid |x| > 1\}, \\ \tilde{w}|_{\partial\Omega} = \tilde{f} & \text{on } \mathbb{S}^2, \\ r(\partial_r w - i\lambda w) \rightarrow_{r \rightarrow \infty} 0. \end{cases} \quad (2.3)$$

In this coordinates the Laplace operator on Ω writes

$$\Delta_D = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^2}, \quad (2.4)$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace operator on the sphere \mathbb{S}^2 . Thus the solution \tilde{w} of (2.3) satisfies

$$r^2\partial_r^2\tilde{w} + 2r\partial_r\tilde{w} + (\lambda^2r^2 + \Delta_{\mathbb{S}^2})\tilde{w} = 0, \quad r > 1. \quad (2.5)$$

In particular, if $\{e_j\}$ is an orthonormal basis of $L^2(\mathbb{S}^2)$ consisting of eigenfunctions of $\Delta_{\mathbb{S}^2}$, with eigenvalues $-\mu_j^2$ and if ω denotes the variable on the sphere \mathbb{S}^2 , we can write

$$\tilde{w}(r\omega) = \sum_j \tilde{w}_j(r)e_j(\omega), \quad r \geq 1, \quad (2.6)$$

where the functions $\tilde{w}_j(r)$ satisfy

$$r^2\tilde{w}_j''(r) + 2r\tilde{w}_j'(r) + (\lambda^2r^2 - \mu_j^2)\tilde{w}_j(r) = 0, \quad r > 1. \quad (2.7)$$

This is a modified Bessel equation, and the solution satisfying the radiation condition $r(\partial_r w - i\lambda w) \rightarrow_{r \rightarrow \infty} 0$ is of the form

$$\tilde{w}_j(r) = a_j r^{-\frac{1}{2}} H_{\nu_j}(\lambda r), \quad (2.8)$$

where $H_{\nu_j}(z)$ denote the Hankel function. Recall that the Hankel function is given by

$$H_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \frac{e^{i(z - \pi\nu/2 - \pi/4)}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-s} s^{\nu-1/2} \left(1 - \frac{s}{2iz}\right)^{\nu-1/2} ds, \quad (2.9)$$

where

$$\Gamma(\nu + 1/2) = \int_0^\infty e^{-s} s^{\nu-1/2} ds,$$

and it is valid for $Re\nu > \frac{1}{2}$ and $-\pi/2 < \arg z < \pi$. Also, in (2.8) ν_j is given by

$$\nu_j = (\mu_j^2 + \frac{1}{4})^{1/2} \quad (2.10)$$

and the coefficients a_j are determined by the boundary condition $\tilde{w}_j(1) = \langle \tilde{f}, e_j \rangle$, so

$$a_j = \frac{\langle \tilde{f}, e_j \rangle}{H_{\nu_j}(\lambda)}. \quad (2.11)$$

Let us introduce the self-adjoint operator

$$A = (-\Delta_{\mathbb{S}^2} + \frac{1}{4})^{1/2}, \quad Ae_j = \nu_j e_j. \quad (2.12)$$

Then the solution of (2.3) writes, formally,

$$\tilde{w}(r\omega) = r^{-1/2} \frac{H_A(\lambda r)}{H_A(\lambda)} \tilde{f}(\omega), \quad \omega \in \mathbb{S}^2. \quad (2.13)$$

Proposition 2.3. (see [24, Chp.3]) *The spectrum of A is $\text{spec}(A) = \{m + \frac{1}{2} | m \in \mathbb{N}\}$.*

Remark 2.4. (see [24, Chp.3]) The Hankel function $H_{m+1/2}(z)$ and Bessel functions of order $m + \frac{1}{2}$ are all elementary functions of z . We have

$$H_{m+1/2}(z) = (\frac{2z}{\pi})^{1/2} q_m(z), \quad q_m(z) = -i(-1)^m (\frac{1}{z} \frac{d}{dz})^m (\frac{e^{iz}}{z}). \quad (2.14)$$

We deduce that

$$r^{-1/2} \frac{H_{m+1/2}(\lambda r)}{H_{m+1/2}(\lambda)} = r^{-m-1} e^{i\lambda(r-1)} \frac{p_m(\lambda r)}{p_m(\lambda)}, \quad p_m(z) = i^{-m-1} \sum_{k=0}^m (i/2)^k \frac{(m+k)!}{k!(m-k)!} z^{m-k}. \quad (2.15)$$

We now look for a solution to (2.2). If we write

$$f(r\omega) = \sum_j f_j(r) e_j(\omega), \quad w(r\omega) = \sum_j w_j(r) e_j(\omega),$$

then the functions $w_j(r)$ satisfy

$$r^2 w_j''(r) + 2r w_j'(r) + (\lambda^2 r^2 - \mu_j^2) w_j(r) = r^2 f_j(r), \quad r > 1 \quad (2.16)$$

together with the vanishing condition at $r = 1$ and Sommerfield radiation condition when $r \rightarrow \infty$. Applying the variation of constants method together with the outgoing assumption and the formula (2.13), we obtain

$$w_j(r) = \int_0^\infty G_{\nu_j}(r, s, \lambda) \chi_\lambda(s) f_j(s) s^2 ds, \quad \nu_j = (\mu_j^2 + \frac{1}{4})^{1/2}, \quad (2.17)$$

where $G_\nu(r, s, \lambda)$ is the Green kernel for the differential operator

$$L_\nu = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right), \quad \nu = \left(\mu^2 + \frac{1}{4}\right)^{1/2}. \quad (2.18)$$

The fact that L_ν is self-adjoint implies that $G_\nu(r, s, \lambda) = G_\nu(s, r, \lambda)$ and from the radiation condition we obtain (for λ real)

$$G_\nu(r, s, \lambda) = \begin{cases} \frac{\pi}{2i}(rs)^{-1/2} \left(J_\nu(s\lambda) - \frac{J_\nu(\lambda)}{H_\nu(\lambda)} H_\nu(s\lambda) \right) H_\nu(r\lambda), & r \geq s, \\ \frac{\pi}{2i}(rs)^{-1/2} \left(J_\nu(r\lambda) - \frac{J_\nu(\lambda)}{H_\nu(\lambda)} H_\nu(r\lambda) \right) H_\nu(s\lambda), & r \leq s. \end{cases} \quad (2.19)$$

In what follows we will look for estimates of the L^2 norm of w_j on the interval $[1, 1 + \lambda^{-\alpha}]$. This problem has to be divided in several classes, according whether ν_j/λ is less than, nearly equal or greater than 1. We distinguish also the simple case of determining bounds when the argument is much larger than the order. Let us explain the meaning of this: in fact, applying the operator L_{ν_j} to $rw_j(r)$ instead of $w_j(r)$, we eliminate the term involving $w'_j(r)$ in (2.16). On the characteristic set we have

$$\rho_j^2 + \frac{\mu_j^2}{r^2} = \lambda^2, \quad \nu_j = \left(\mu_j^2 + \frac{1}{4}\right)^{1/2}, \quad (2.20)$$

where ρ_j denotes the dual variable of r . Let θ_j be defined by $\tan \theta_j = \frac{\rho_j}{\mu_j}$, then for λ big enough and r in a small neighborhood of 1 we can estimate

$$\tan^2 \theta_j + 1 \simeq \frac{\lambda^2}{\nu_j^2}.$$

- When the quotient $\frac{\nu_j}{\lambda}$ is smaller than a constant $1 - \epsilon_0$ where ϵ_0 is fixed, strictly positive, this corresponds to an angle θ_j between some fixed direction θ_0 and $\pi/2$, with $\tan \theta_0 = \epsilon_0$, and thus to a ray hitting the obstacle transversally. In this case we show that, since a unit speed bicharacteristic spends in the λ -depending neighborhood Ω_λ a time $\lambda^{-\alpha}$, we have the following

Proposition 2.5. *Let $\epsilon_0 > 0$ be fixed, small. Then there exists a constant $C = C(\epsilon_0)$ such that*

$$\|w_j\|_{L^2([1, 1 + \lambda^{-\alpha}])} \leq C\lambda^{-(1+\alpha)} \|f_j\|_{L^2([1, 1 + \lambda^{-\alpha}])}$$

uniformly for j such that $\{\nu_j/\lambda \leq 1 - \epsilon_0\}$.

- When the quotient $\frac{\nu_j}{\lambda}$ is close enough to 1 the angles θ_j become very small and this is the case of a diffractive ray, which spends in Ω_λ a time proportional to $\lambda^{-\alpha/2}$. In this case $\tan \theta_j \simeq \sqrt{1 - \frac{\nu_j^2}{\lambda^2}}$ and we show the following

Proposition 2.6. *If $1 - \frac{\nu_j}{\lambda} \simeq \lambda^{-\beta}$ for some $\beta > 0$ then*

$$\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\lambda^{-1-\alpha/2-(\alpha-\beta)}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])} \text{ if } \beta \leq \alpha$$

and

$$\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\lambda^{-(1+\alpha/2)}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])} \text{ if } \beta \geq \alpha.$$

- In the elliptic case $\frac{\lambda}{\nu_j} \ll 1$ or $\frac{\lambda}{\nu_j} \in [\epsilon_0, 1 - \epsilon_0]$ for some small $\epsilon_0 > 0$ there is nothing to do since away from the characteristic variety (2.20) we have nice bounds of the solution of the solution $w_j(r)$ of (2.16).

Proof. (of Proposition 2.5) The solution $w_j(r)$ given in (2.17) writes

$$\begin{aligned} w_j(r) = & \frac{\pi}{8i}r^{-1/2} \left(\int_0^r \chi_\lambda(s)f_j(s)s^{3/2}(\bar{H}_{\nu_j}(\lambda s) - \frac{\bar{H}_{\nu_j}(\lambda)}{H_{\nu_j}(\lambda)}H_{\nu_j}(\lambda s))dsH_{\nu_j}(\lambda r) + \right. \\ & \left. + \int_r^\infty \chi_\lambda(s)f_j(s)s^{3/2}H_{\nu_j}(\lambda s)ds(\bar{H}_{\nu_j}(\lambda r) - \frac{\bar{H}_{\nu_j}(\lambda)}{H_{\nu_j}(\lambda)}H_{\nu_j}(\lambda r)) \right) \end{aligned} \quad (2.21)$$

thus in order to obtain estimates for $\|w_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])}$ we have to determine bounds for $\|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])}$ since we have for some constant $C > 0$

$$\|w_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])} \leq \|f_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])}\|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])}^2. \quad (2.22)$$

We consider separately two regimes:

1. For $\frac{\nu_j}{\lambda} \ll 1$ we use (7.2) to obtain immediately

$$\|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])}^2 \lesssim \lambda^{-1-\alpha}. \quad (2.23)$$

2. For $0 < \epsilon_0 \leq c_j := \frac{\nu_j}{\lambda} \leq 1 - \epsilon_0$ for some fixed $\epsilon_0 > 0$ we use the expansions (7.3), (7.4): set

$$\cos \tau = \frac{c_j}{r}, \quad dr = c_j \frac{\sin \tau}{\cos^2 \tau} d\tau, \quad \tau \in [\arccos c_j, \arccos \frac{c_j}{1 + \lambda^{-\alpha}}] =: [\tau_{j,0}, \tau_{j,1}].$$

Then we have

$$\begin{aligned} \|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])}^2 &= \frac{2}{\pi\lambda} \int_{\tau_{j,0}}^{\tau_{j,1}} \frac{1}{\cos \tau} d\tau = \frac{1}{\pi\lambda} \ln \left(\frac{1 + \sin \tau}{1 - \sin \tau} \right) \Big|_{\tau_{j,0}}^{\tau_{j,1}} \\ &\simeq \frac{1}{\pi\lambda} \left((1 + \lambda^{-\alpha})^2 \frac{(1 + \sqrt{1 - c_j^2/(1 + \lambda^{-\alpha})^2})^2}{(1 + \sqrt{1 - c_j^2})^2} - 1 \right) \simeq \frac{1}{\pi\lambda} \frac{\lambda^{-\alpha}}{(1 - c_j^2 + \lambda^{-\alpha})^{1/2} + (1 - c_j^2)^{1/2}} \end{aligned} \quad (2.24)$$

and $1 - c_j^2 \in [2\epsilon_0 - \epsilon_0^2, 1 - \epsilon_0^2]$ with $0 < \epsilon_0 < 1$ fixed. Notice, however, that if one takes $c_j \simeq \lambda^{-\beta}$ for some $\beta > 0$, then $\tan \theta_j \simeq (1 - c_j^2)^{1/2} \simeq \lambda^{-\beta/2}$. If $\beta \leq \alpha$ we can estimate (2.24) by $\lambda^{-(1-\alpha/2-(\alpha-\beta))}$, otherwise we have the bound $\lambda^{-(1+\alpha/2)}$.

□

Proof. (of Proposition 2.6) Here we use Proposition 7.3. Let $1 - \nu_j/\lambda = \tau\lambda^{-\beta}$ for τ in a neighborhood of 1. For $s \in [1, 1 + \lambda^{-\alpha}]$ write $z(s)\nu_j = s\lambda$, thus $z(s) = \frac{s}{1 - \tau\lambda^{-\beta}}$. Since

$$\begin{aligned} \|w_j\|_{L^2([1, 1 + \lambda^{-\alpha}])} &\leq C \frac{\lambda^{-\alpha/2}}{|H_{\nu_j}(\lambda)|} \|f_j\|_{L^2([1, 1 + \lambda^{-\alpha}])} \|H_{\nu_j}(\lambda r)\|_{L^2([1, 1 + \lambda^{-\alpha}])} \times \\ &\quad \times \|J_{\nu_j}(\lambda s)Y_{\nu_j}(\lambda) - J_{\nu_j}(\lambda)Y_{\nu_j}(\lambda s)\|_{L^2([1, 1 + \lambda^{-\alpha}])}, \end{aligned} \quad (2.25)$$

we shall compute separately each factor in (2.25) (modulo small terms). We have

$$\|H_{\nu_j}(\lambda r)\|_{L^2([1, 1 + \lambda^{-\alpha}])}^2 = \int_1^{1 + \lambda^{-\alpha}} |J_{\nu_j}(\lambda r)|^2 + |Y_{\nu_j}(\lambda r)|^2 dr \simeq \quad (2.26)$$

$$\begin{aligned} \frac{2}{\pi} \nu_j^{-1} \int_1^{1 + \lambda^{-\alpha}} \frac{1}{(z^2(s) - 1)^{1/2} ds} &\simeq \frac{2}{\pi} \nu_j^{-1} \frac{\lambda^{-\alpha}}{\lambda^{-\alpha} + 2\tau\lambda^{-\beta}} \simeq \begin{cases} \lambda^{-1 - (\alpha - \beta)}, & \text{si } 0 \leq \beta \leq \alpha, \\ \lambda^{-1}, & \text{si } \beta > \alpha, \end{cases} \\ |H_{\nu_j}(\lambda)|^2 = |J_{\nu_j}(\lambda)|^2 + |Y_{\nu_j}(\lambda)|^2 &\simeq \frac{2}{\pi} \nu_j^{-1} \frac{1}{(z^2(1) - 1)^{1/2}} \simeq \lambda^{-1 + \beta/2}, \end{aligned} \quad (2.27)$$

while for the factor in the second line in (2.25) we have

$$\|J_{\nu_j}(\lambda s)Y_{\nu_j}(\lambda) - J_{\nu_j}(\lambda)Y_{\nu_j}(\lambda s)\|_{L^2([1, 1 + \lambda^{-\alpha}])} \leq \quad (2.28)$$

$$\frac{2}{\pi} \nu_j^{-1} \frac{1}{(z^2(1) - 1)^{1/4}} \left(\int_1^{1 + \lambda^{-\alpha}} \frac{1}{(z^2(s) - 1)^{1/2}} ds \right)^{1/2} \simeq \begin{cases} \lambda^{-1 - \alpha/2 + \beta/4}, & \text{si } 0 \leq \beta \leq \alpha, \\ \lambda^{-1 + \beta/4}, & \text{si } \beta > \alpha. \end{cases}$$

From (2.25), (2.26), (2.27), (2.28) we deduce

$$\|w_j\|_{L^2([1, 1 + \lambda^{-\alpha}])} \leq C \|f_j\|_{L^2([1, 1 + \lambda^{-\alpha}])} \times \begin{cases} \lambda^{-1 - \alpha/2 - (\alpha - \beta)}, & \text{si } 0 \leq \beta \leq \alpha, \\ \lambda^{-1 - \alpha/2}, & \text{si } \beta > \alpha. \end{cases} \quad (2.29)$$

□

Neumann: *As far as the Neumann problem is concerned, we must solve the problem*

$$\left(\partial_r^2 + \frac{2}{r} \partial_r + \left(\lambda^2 - \frac{\nu^2}{r^2} \right) \right) w_j(r) = \chi(\lambda^\alpha(r - 1)) f_j(r), \quad \frac{\partial w_j}{\partial r}(1) = 0, \quad (2.30)$$

which gives, after performing similar computations

$$\begin{aligned} w_j^N(r) &= \frac{\pi}{8i} r^{-1/2} H_{\nu_j}(\lambda r) \left(\frac{\bar{H}_{\nu_j}'^2(\lambda)}{|H_{\nu_j}'(\lambda)|^2} \int_1^\infty \chi_\lambda(s) f_j(s) s^{3/2} H_{\nu_j}(\lambda s) ds - \right. \\ &\quad \left. - \int_1^r f_j(s) s^{3/2} \bar{H}_{\nu_j}(\lambda s) ds \right) - \frac{\pi}{8i} r^{-1/2} \bar{H}_{\nu_j}(\lambda r) \int_r^\infty \chi_\lambda(s) f_j(s) s^{3/2} H_{\nu_j}(\lambda s) ds, \end{aligned} \quad (2.31)$$

Propositions 2.5, 2.6 hold true for the Neumann case too.

3 Proof of Theorem 1

3.1 Reduction to the Helmholtz equation

Proposition 3.1. *Consider the Helmholtz equation*

$$\begin{cases} (\Delta_D + \lambda^2)w = \chi_\lambda f & \text{on } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \quad (3.1)$$

with the Sommerfeld "radiation condition" (where $r = |x|$)

$$r(\partial_r w - i\lambda w) \rightarrow_{r \rightarrow \infty} 0. \quad (3.2)$$

In order to prove Theorem 1.1 it is enough to establish estimates for the L^2 norms of the Helmholtz equation (3.1).

Proof. Consider the inhomogeneous Schrödinger equation

$$(i\partial_t + \Delta_D)v(t, x) = \chi_\lambda g(t, x), \quad v|_{t=0} = 0. \quad (3.3)$$

We denote by v^\pm the solutions to the equations

$$(i\partial_t + \Delta_D)v^\pm(t, x) = \chi_\lambda g(t, x)\mathbf{1}_{\{\pm t > 0\}}, \quad v^\pm|_{t=0} = 0. \quad (3.4)$$

For $\epsilon > 0$ and $\pm t > 0$ we define $v_\epsilon^\pm = e^{\mp\epsilon t}v^\pm$ which satisfy

$$\begin{cases} (i\partial_t + \Delta_D \pm i\epsilon)v_\epsilon^\pm = \mathbf{1}_{\{\pm t > 0\}}e^{-\epsilon t}\chi_\lambda g(t, x), \\ v_\epsilon^\pm|_{t=0} = 0, \quad v_\epsilon^\pm|_{\partial\Omega} = 0. \end{cases} \quad (3.5)$$

After performing the Fourier transform \mathcal{F} with respect to the time variable t , the equation (3.5) becomes

$$\begin{cases} (\Delta_D - \tau \pm i\epsilon)\hat{v}_\epsilon^\pm(\tau, x) = \chi_\lambda \mathcal{F}(\mathbf{1}_{\{\pm t > 0\}}e^{-\epsilon t}\chi_\lambda g)(\tau, x), \\ \hat{v}_\epsilon^\pm|_{\partial\Omega} = 0, \end{cases} \quad (3.6)$$

where τ denotes the dual variable of t and we deduce

$$\chi_\lambda \hat{v}_\epsilon^\pm(\tau, x) = \chi_\lambda (\Delta_D - \tau \pm i\epsilon)^{-1} \chi_\lambda \mathcal{F}(\mathbf{1}_{\{\pm t > 0\}}e^{-\epsilon t}\chi_\lambda g)(\tau, x). \quad (3.7)$$

Since $-\Delta_D$ is a positive self-adjoint operator, the resolvent $(-\Delta_D - z)^{-1}$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$. Since the spectrum of $-\Delta_D$ is confined to the positive real axis it is convenient to introduce $\lambda^2 \in \mathbb{R}_+$ as a spectral parameter. Notice, however, that there are two manners to approach $\lambda^2 > 0$ in $\mathbb{C} \setminus \mathbb{R}_+$, choosing the positive imaginary part, which corresponds to considering $\lambda^2 + i\epsilon$, or the negative imaginary part, which corresponds to $\lambda^2 - i\epsilon$. The "physical" choice corresponds to the limiting absorption principal and consists of taking $\lambda^2 + i\epsilon$. In some sense, the limiting absorption principal allows to recover the "sense of time". If one replaces λ^2 by $\lambda^2 \pm i\epsilon$, $\epsilon > 0$, then $(\lambda^2 \pm i\epsilon)$ belongs to the resolvent of the Laplace operator on Ω with Dirichlet boundary conditions on $\partial\Omega$ and we can let ϵ tend to 0 in (3.7) (since the operator $\chi_\lambda(\Delta_D - \tau \pm i\epsilon)^{-1}\chi_\lambda$ has a limit as $\epsilon \rightarrow 0$) and so we can express the Fourier transform of the unique solutions v^\pm of (3.4) as

$$\chi_\lambda \hat{v}^\pm(\tau, x) = \lim_{\epsilon \rightarrow 0} \chi_\lambda (\Delta_D - \tau \pm i\epsilon)^{-1} \chi_\lambda \mathcal{F}(\mathbf{1}_{\{\pm t > 0\}}e^{-\epsilon t}\chi_\lambda g)(\tau, x). \quad (3.8)$$

Remark 3.2. Notice that here we used the fact that for $-\frac{\tau}{\lambda^2}$ away from a neighborhood of 1 we have the bounds

$$\|\chi_\lambda \hat{v}^+\|_{L^2} \leq \frac{C \|\chi_\lambda \hat{g}\|_{L^2}}{|\tau| + \lambda^2}.$$

We conclude using that if $w_\epsilon = \hat{v}_\epsilon^+$, $f_\epsilon = \mathcal{F}(1_{\{t>0\}} e^{-ct} \chi_\lambda g)$ the following holds

Proposition 3.3. ([24]) *As $\epsilon \rightarrow 0$, w_ϵ converges to the unique solution of (3.1)-(3.2), where $f = \lim_{\epsilon \rightarrow 0} f_\epsilon$.*

It remains to notice that if \mathcal{H} is any Hilbert space, than the Fourier transform defines an isometry on $L^2(\mathbb{R}, \mathcal{H})$. \square

3.2 Smoothing effect

In this section we prove Theorem 1.1:

- We prove (1.1) for $s = 0$. Let $\lambda > 0$, w and g be such that (2.2) holds. We multiply (2.2) by $\chi_\lambda \bar{w}$ and we integrate on Ω

$$\int \chi_\lambda |\nabla w|^2 dx = \lambda^2 \int \chi_\lambda |w|^2 dx - 2\lambda^\alpha \int \langle \nabla w, \nabla \chi(\lambda^\alpha(|x| - 1)) \rangle \bar{w} dx - \int \chi_\lambda^2 f \bar{w} dx, \quad (3.9)$$

thus, using that $0 \leq \chi_\lambda \leq 1$, $\chi_\lambda \leq \chi_\lambda^2$, $\nabla \chi(\lambda^\alpha(|x| - 1)) \lesssim \chi_\lambda$, we have for all $\delta > 0$

$$\int |\chi_\lambda \nabla w|^2 dx \lesssim \lambda^2 \int |\chi_\lambda w|^2 dx + \delta \int |\chi_\lambda f|^2 dx + \frac{1}{4\delta} \int |\chi_\lambda w|^2 dx + \lambda^\alpha \int \bar{w} |\nabla w| \chi_\lambda^2 dx. \quad (3.10)$$

Since for all $\delta_1 > 0$ one has

$$\int \bar{w} |\nabla w| \chi_\lambda^2 dx \leq \left(\int |\chi_\lambda \nabla w|^2 dx \right)^{1/2} \left(\int |\chi_\lambda w|^2 dx \right)^{1/2} \leq \delta_1 \|\chi_\lambda \nabla w\|_{L^2(\Omega)}^2 + \frac{1}{4\delta_1} \|\chi_\lambda w\|_{L^2(\Omega)}^2,$$

taking $\delta = \lambda^{-\alpha}$, $\delta_1 = \lambda^{-\alpha}/2$ together with $\int |\chi_\lambda w|^2 \lesssim \lambda^{-2-\alpha} \int |\chi_\lambda f|^2$ (according to the computations made in the preceding section) we deduce

$$\int |\chi_\lambda \nabla w|^2 dx \lesssim \lambda^{-\alpha} \int |\chi_\lambda f|^2 dx. \quad (3.11)$$

Thus we have obtained

$$\|\chi_\lambda w\|_{H_D^1(\Omega)} \lesssim \lambda^{-\frac{\alpha}{2}} \|\chi_\lambda f\|_{L^2(\Omega)}. \quad (3.12)$$

Dualizing (3.12) we find

$$\|\chi_\lambda w\|_{L^2(\Omega)} \lesssim \lambda^{-\frac{\alpha}{2}} \|\chi_\lambda f\|_{H_D^{-1}(\Omega)}, \quad (3.13)$$

which will yield (1.1) for $s = -1$. Now we prove (1.1) for $s = 1$. Let again w and f be such that (2.2) holds and let $\tilde{\chi}_\lambda$ be a smooth cutoff function equal to 1 on the support of χ_λ . Write

$$\|\chi_\lambda w\|_{H_D^2(\Omega)} \approx \|\chi_\lambda w\|_{H_D^1(\Omega)} + \|\Delta_D(\chi_\lambda w)\|_{L^2(\Omega)}. \quad (3.14)$$

Since $\|\chi_\lambda w\|_{H_D^1(\Omega)}$ can be estimated by means of (3.12), we only need to obtain bounds for $\|\Delta_D(\chi_\lambda w)\|_{L^2(\Omega)}$. We write

$$\Delta_D(\chi_\lambda w) = \chi_\lambda \Delta_D w + [\Delta_D, \chi_\lambda] \tilde{\chi}_\lambda w. \quad (3.15)$$

The commutator $[\Delta_D, \chi_\lambda]$ is bounded from $H_D^1(\Omega)$ to $L^2(\Omega)$ with norm less than $C\lambda^\alpha$ and we have chosen $\alpha < 1$ (C does not depend on λ), while the first term in the right hand side of (3.15) is in $H_D^1(\Omega)$ since $\Delta_D w = \chi_\lambda f - \lambda^2 w$ and satisfies (2.2) with $\chi_\lambda f$ replaced by $\Delta_D(\chi_\lambda f)$. Therefore, we can apply (3.12) in order to deduce the inequality (1.1) for the Fourier transforms in time of u and f which appear in the proposition. Since we have obtained the result for $s = -1$ and $s = 1$ we can use an interpolation argument to get it for $s \in [-1, 1]$.

- We turn to the proof of (1.2) for $s = 0$. If we denote by A_λ the operator which to a given $u_0 \in L^2(\Omega)$ associates $\chi_\lambda e^{it\Delta_D} \psi(\frac{-\Delta_D}{\lambda^2}) u_0$, we need to prove that A_λ is bounded from $L^2(\Omega)$ to $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ with the norm less than $C\lambda^{-\frac{\alpha}{4}}$ for some constant C independent of λ , which in turn is equivalent to the continuity of the adjoint operator,

$$A_\lambda^*(f) = \int_0^T \psi(\frac{-\Delta_D}{\lambda^2}) e^{-i\tau\Delta_D} \chi_\lambda f(\tau) d\tau, \quad (3.16)$$

from $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ to $L^2(\Omega)$ with the norm bounded by $C\lambda^{-\frac{\alpha}{4}}$, which is equivalent to showing that the operator $A_\lambda A_\lambda^*$ defined by

$$(A_\lambda A_\lambda^* f)(t) = \int_0^T \chi_\lambda e^{it\Delta_D} \psi^2(\frac{-\Delta_D}{\lambda^2}) e^{-i\tau\Delta_D} \chi_\lambda f(\tau) d\tau \quad (3.17)$$

is continuous from $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ to $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ and its norm is bounded from above by $C\lambda^{-\frac{\alpha}{2}}$. We write $(A_\lambda A_\lambda^* f)(t)$ as a sum

$$\begin{aligned} (A_\lambda A_\lambda^* f)(t) &= \int_0^t \chi_\lambda \psi(\frac{-\Delta_D}{\lambda^2}) e^{i(t-\tau)\Delta_D} \psi(\frac{-\Delta_D}{\lambda^2}) \chi_\lambda f(\tau) d\tau + \\ &+ \int_t^T \chi_\lambda \psi(\frac{-\Delta_D}{\lambda^2}) e^{i(t-\tau)\Delta_D} \psi(\frac{-\Delta_D}{\lambda^2}) \chi_\lambda f(\tau) d\tau. \end{aligned} \quad (3.18)$$

Hence, in order to conclude it is sufficient to apply (1.1) with $s = -\frac{1}{2}$ together with time inversion, since the second term on the right hand side of (3.18) will solve the same problem with initial data $u|_{t=T} = u_0$.

We prove now (1.2) for $s = 1$.

Lemma 3.4. *We have*

$$\|\Delta_D \left(\chi_\lambda e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0 \right)\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} \lesssim \lambda^{1-\frac{\alpha}{4}} \|\chi_\lambda \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0\|_{L^2(\Omega)}. \quad (3.19)$$

Corollary 3.5. *Lemma 3.4 yields*

$$\|\chi_\lambda e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0\|_{L_T^2 H_D^{\frac{3}{2}}(\Omega)} \lesssim \lambda^{1-\frac{\alpha}{4}} \|\chi_\lambda \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0\|_{L^2(\Omega)}. \quad (3.20)$$

Corollary 3.5 and an interpolation argument now yield for $\theta \in [0, 1]$

$$\|\chi_\lambda e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0\|_{L_T^2 H_D^{\frac{3\theta}{2}}(\Omega)} \lesssim \lambda^{\frac{3\theta}{2}-\frac{1}{2}-\frac{\alpha}{4}} \|\chi_\lambda \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0\|_{L^2(\Omega)}, \quad (3.21)$$

achieving the proof of Theorem 1.1.

Proof. (of Lemma 3.4) Write

$$\begin{aligned} (-\Delta_D + 1) \left(\chi_\lambda e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0 \right) &= \chi_\lambda (-\Delta_D + 1) \left(e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0 \right) - \\ &\quad - [\Delta_D, \chi_\lambda] \left(e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0 \right). \end{aligned} \quad (3.22)$$

– For the first term in the right hand side of (3.22) we show that the operator

$$B_\lambda := \left(H_D^1(\Omega) \ni u_0 \rightarrow \chi_\lambda (-\Delta_D + 1) \left(e^{it\Delta_D} \psi \left(\frac{-\Delta_D}{\lambda^2} \right) u_0 \right) \in L_T^2 H_D^{-\frac{1}{2}}(\Omega) \right) \quad (3.23)$$

is continuous and its norm from $H_D^1(\Omega)$ to $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ is bounded from above by $C\lambda^{-\frac{\alpha}{4}}$ for some constant C independent of λ , or equivalently that

$$\begin{aligned} \left(B_\lambda (-\Delta_D + 1)^{-1} B_\lambda^* f \right) (t) &= \int_0^T \chi_\lambda (-\Delta_D + 1) \psi^2 \left(\frac{-\Delta_D}{\lambda^2} \right) e^{i(t-\tau)\Delta_D} \chi_\lambda f(\tau) d\tau = \\ &= (-\Delta_D + 1) (A_\lambda A_\lambda^* f) (t) + [\Delta_D, \chi_\lambda] \int_0^T e^{i(t-\tau)\Delta_D} \psi^2 \left(\frac{-\Delta_D}{\lambda^2} \right) \chi_\lambda f(\tau) d\tau \end{aligned} \quad (3.24)$$

is bounded from $L_T^2 H_D^{\frac{1}{2}}(\Omega)$ to $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ by $C\lambda^{-\frac{\alpha}{2}}$. Here A_λ is the operator introduced in the proof of the case $s = 0$. For the first term in the right hand side of (3.24) we apply (1.1) with $s = \frac{1}{2}$ and we obtain a bound $C\lambda^{-\frac{\alpha}{2}}$, while for the second term we make use of (1.2) with $s = -\frac{1}{2}$ and of the fact that $[\Delta_D, \chi_\lambda]$ is bounded from $H_D^{\frac{1}{2}}(\Omega)$ to $H_D^{-\frac{1}{2}}(\Omega)$ with a norm bounded by $C\lambda^\alpha$ in order to find a bound from $L_T^2 H_D^{\frac{1}{2}}(\Omega)$ to $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ of at most $C\lambda^{\frac{\alpha}{2}-1} \leq C\lambda^{-\frac{\alpha}{2}}$.

– The second term in the right hand side of (3.22) gives

$$\begin{aligned} & \|[\Delta_D, \chi_\lambda] e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} \lesssim \\ & \lesssim \lambda^{2\alpha} \|\chi_\lambda'' e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} + 2\lambda^\alpha \|\chi_\lambda' \nabla \left(e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0 \right)\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)}. \end{aligned} \quad (3.25)$$

For evaluating the first term in the last sum we use again (1.2) with $s = 0$

$$\lambda^{2\alpha} \|\chi_\lambda'' e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} \lesssim \lambda^{2\alpha - \frac{\alpha}{4}} \|\tilde{\chi}_\lambda \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (3.26)$$

where $\tilde{\chi}_\lambda$ is a smooth cutoff function such that $\tilde{\chi}_\lambda$ is equal to 1 on the support of χ_λ and we conclude since $\alpha < 1$. For the second term we have

$$\begin{aligned} & \|\chi_\lambda' \nabla \left(e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0 \right)\|_{L^2(\Omega)}^2 = - \int \chi_\lambda'^2 (\Delta_D u) \bar{u} - 2\lambda^\alpha \operatorname{Re} \int \chi_\lambda' \chi_\lambda'' (\nabla u) \bar{u} \\ & \leq \lambda^2 \int \chi_\lambda'^2 \left| \left(e^{it\Delta_D} \tilde{\psi}\left(\frac{-\Delta_D}{\lambda^2}\right) u_0 \right) \bar{u} \right| + 2\lambda^{\alpha+1/2} \|\tilde{\chi}_\lambda\|_{L^2(\Omega)}^2 \lesssim \lambda^2 \|\tilde{\chi}_\lambda u\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.27)$$

where we have set $\tilde{\psi}(x) = x\psi(x)$. From (1.2) with $s = 0$ that we have already established, we find, since $\alpha < 1$,

$$\begin{aligned} & \lambda^\alpha \|\chi_\lambda' \nabla \left(e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0 \right)\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} \lesssim \lambda^\alpha \|\tilde{\chi}_\lambda u\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} \\ & \lesssim \lambda^{\frac{3\alpha}{4}} \|\tilde{\chi}_\lambda \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)} \lesssim \lambda^{1-\frac{\alpha}{4}} \|\tilde{\chi}_\lambda \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \end{aligned} \quad (3.28)$$

□

Neumann: All of the above results remain valid if we consider the Neumann Laplacian Δ_N : in fact let w and f be such that (2.2) holds. Using the same strategy as before we get:

$$\int_{\partial\Omega} \partial_n u (\chi_\lambda \bar{u}) d\sigma - \int_\Omega \chi_\lambda |\nabla w|^2 dx + \lambda^2 \int_\Omega \chi_\lambda |w|^2 dx - \lambda^\alpha \int_\Omega \langle \nabla w, \nabla \chi_\lambda \rangle \tilde{\chi}_\lambda \bar{w} dx = \int_\Omega \chi_\lambda^2 f \bar{w} dx. \quad (3.29)$$

Notice that, due to the Neumann boundary conditions, the first term in the left hand side vanishes, so the computations are almost the same as in the previous case. However we must pay a little attention to the spectrum of the Neumann Laplacian: for, we have to introduce a spectral cut-off $\phi \in C_0^\infty$ equal to 1 close to 0 and decompose

$$\begin{aligned} w &= \phi(-\Delta_N)w + (1 - \phi(-\Delta_N))w, & f &= \phi(-\Delta_N)f + (1 - \phi(-\Delta_N))f, \\ u_0 &= \phi(-\Delta_N)u_0 + (1 - \phi(-\Delta_N))u_0. \end{aligned} \quad (3.30)$$

We can then rewrite the above proof with w , f , u_0 replaced by $(1 - \phi(-\Delta_N))w$, $(1 - \phi(-\Delta_N))f$, $(1 - \phi(-\Delta_N))u_0$ in order to obtain similar estimates in Theorem 1.1. To deal with the contributions of the remaining terms we use the fact that in this situation the L^2 and H_N^k norms are equivalent.

4 Estimates away from the obstacle

In this section we obtain bounds away from the obstacle. The main idea is to construct a new function $v = \phi u$ which will solve a problem with a nonlinearity supported in a compact set; under these assumptions, it is proved that the free evolution satisfies the usual Strichartz bounds (see [22]). However, we have to take into account the fact that the neighborhood outside of which we will apply this result is of size $\lambda^{-\alpha}$ and thus we will "lose" α derivative; however, this will not pose any problem if α is chosen small enough, since it will be covered by the loss of derivatives near the boundary. After a change of variables we can assume that $\Omega = \{x_3 > 0\}$ and thus x_3 defines the distance to the boundary. We define $\phi \in C^\infty(\bar{\Omega})$ by $\phi(x)|_{x_3 \leq 1} = x_3$, $\phi(x)|_{x_3 \geq 2} = 1$.

Proposition 4.1. *We have*

$$\|(1 - \chi_\lambda)e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^2 L^{\frac{2d}{d-2}}(\Omega)} \lesssim \lambda^\alpha \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L^2(\Omega)}. \quad (4.1)$$

We set $v = \phi u$, where $u = e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0$ solves (1.3). Then v solves the equation

$$\begin{cases} (i\partial_t + \Delta_D)v = (\Delta_D\phi)u + 2\nabla\phi\nabla u + \frac{\partial v}{\partial n}|_{\partial\Omega} \otimes \delta_{\partial\Omega}^{(0)} - v|_{\partial\Omega} \otimes \delta_{\partial\Omega}^{(1)} = [\Delta_D, \phi]u, \\ v|_{t=0} = \phi u|_{t=0}, \quad v|_{\partial\Omega} = 0, \end{cases} \quad (4.2)$$

where δ is the Dirac measure on $\partial\Omega$. It can easily be seen that the last two terms vanish.

We have $[\Delta_D, \phi]u \in L_T^2 H_{comp}^{\frac{-1}{2}}(\Omega)$ and (see [7, Prop.2.7])

$$\|[\Delta_D, \phi]e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^2 H_D^{\frac{-1}{2}}(\Omega)} \lesssim \|e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L^2(\Omega)}. \quad (4.3)$$

The inhomogeneous part of the equation (4.2) satisfied by v has compact spatial support and therefore we can employ [22, Thm.3] (for $p = 2$) and [7, Prop.2.10] in order to obtain Strichartz estimates without losses for v

$$\|\phi e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^p L^q(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L^2(\Omega)}, \quad (4.4)$$

for every (p, q) d -admissible pair, which in turn yields

$$\begin{aligned} \lambda^{-\alpha} \|(1 - \chi_\lambda)e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^p L^q(\Omega)} &\lesssim \|(1 - \chi_\lambda)\phi e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^p L^q(\Omega)} \\ &\lesssim \|\phi e^{it\Delta_D}\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L_T^p L^q(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right)u_0\|_{L^2(\Omega)}. \end{aligned} \quad (4.5)$$

In particular, for $p = 2$ we get (4.1) from (4.5).

Neumann: *When dealing with the Neumann problem, this approach requires some adjustments, but the above result remains valid (with α slightly modified). Let $\epsilon > 0$ and consider $\phi|_{x_3 \leq 1}(x) = x_3^{1+\epsilon}$, $\phi|_{x_3 \geq 2}(x) = 1$. Then $v = \phi u$ solves the equation*

$$(i\partial_t + \Delta_D)v = [\Delta_D, \phi^{1+\epsilon}]u, \quad v|_{t=0} = \phi u|_{t=0}, \quad \frac{\partial v}{\partial \nu}|_{\partial\Omega} = (x_3^\epsilon u)|_{\partial\Omega} + \phi \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0. \quad (4.6)$$

It's easy to see that one can obtain similar estimates, with α replaced by $\alpha - \epsilon$; still, this makes no difference for our purpose, since ϵ can be chosen as small as we like.

5 Proof of Theorem 2

In this section we achieve the proof of Theorem 1.3. Taking $s = 1/2$ in (1.2) gives

$$\|\chi_\lambda e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^1(\Omega)} \lesssim \lambda^{\frac{1}{2} - \frac{\alpha}{4}} \|\chi_\lambda \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)} \quad (5.1)$$

from which we deduce

$$\|\chi_\lambda e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 L^{\frac{2d}{d-2}}(\Omega)} \lesssim \|\chi_\lambda e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^1(\Omega)} \lesssim \lambda^{\frac{1}{2} - \frac{\alpha}{4}} \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.2)$$

An energy argument yields

$$\|\chi_\lambda e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^\infty L^2(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.3)$$

Interpolation between (5.1) and (5.3) with weights $\frac{2}{p}$ and $1 - \frac{2}{p}$ respectively yields

$$\|\chi_\lambda e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^p L^q(\Omega)} \lesssim \lambda^{\frac{1}{p}(1 - \frac{\alpha}{2})} \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.4)$$

We have also obtained estimates away from the boundary

$$\|(1 - \chi_\lambda) e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^p L^q(\Omega)} \lesssim \lambda^\alpha \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.5)$$

If we take $\alpha = \frac{1}{p}(1 - \frac{\alpha}{2})$ we find $\alpha = \frac{2}{2p+1}$. Now, for $p = 2$ this gives $\alpha = \frac{2}{5} (\leq \frac{2}{3})$ and consequently we have

$$\|e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 L^{\frac{2d}{d-2}}(\Omega)} \lesssim \lambda^{\frac{2}{5}} \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.6)$$

Interpolation between (5.6) and

$$\|e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^\infty L^2(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}, \quad (5.7)$$

with weights $\frac{2}{p}$ and $1 - \frac{2}{p}$ yields, for every (p, q) admissible pair, $p \geq 2$

$$\|e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^p L^q(\Omega)} \lesssim \lambda^{\frac{4}{5p}} \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.8)$$

Neumann: The case of the Neumann conditions could be handled in the same way. However in (5.5) we have $\alpha - \epsilon$ instead of α so we find $\alpha = \frac{2}{5} + \frac{4\epsilon}{5}$ and for every (p, q) d -admissible pair and every $\epsilon > 0$ we have

$$\|e^{it\Delta_N} \psi\left(\frac{-\Delta_N}{\lambda^2}\right) u_0\|_{L_T^p L^q(\Omega)} \lesssim \lambda^{\frac{4}{5p} + \frac{8\epsilon}{5p}} \|\psi\left(\frac{-\Delta_N}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (5.9)$$

6 Applications

6.1 Proof of Theorem 1.4 for non-trapping obstacle

The proof of Theorem 1.4 relies on the contraction principle applied to the equivalent integral equation associated to (1.6) with Dirichlet boundary conditions

$$u(t) = e^{it\Delta_D} u_0 - i \int_0^t e^{i(t-\tau)\Delta_D} F(u(\tau)) d\tau. \quad (6.1)$$

The assumptions on F and on the potential V imply the following pointwise estimates:

$$|F(u)| \lesssim |u|(1+|u|^2), \quad |\nabla F(u)| \lesssim |\nabla u|(1+|u|^2), \quad |F(u) - F(v)| \lesssim |u-v|(1+|u|^2+|v|^2), \quad (6.2)$$

$$|\nabla(F(u) - F(v))| \lesssim |\nabla(u-v)|(1+|u|^2+|v|^2) + |u-v|(|\nabla u| + |\nabla v|)(1+|u|+|v|). \quad (6.3)$$

The aim is to show that for sufficiently small $T > 0$ we can solve the integral equation by a Picard iteration scheme in the Banach space $X_T := L_T^\infty H_0^1(\Omega) \cap L_T^p W^{\sigma,q}(\Omega)$, where $2 < p < \frac{12}{5}$ and $\sigma = 1 - \frac{4}{5p}$. We equip X_T with the norm

$$\|u\|_{X_T} := \|u\|_{L_T^\infty H_0^1(\Omega)} + \|(1 - \Delta_D)^{\frac{\sigma}{2}} u\|_{L_T^p L^q(\Omega)}.$$

Remark 6.1. For $2 < p < \frac{12}{5}$ one has the continuous embedding $W^{1-\frac{4}{5p},q}(\Omega) \subset L^\infty(\Omega)$ where (p, q) is an admissible pair, since in this case $1 - \frac{4}{5p} > \frac{3}{q}$ and Proposition 2.1 implies $u \in L_T^p L^\infty(\Omega)$ for any $u \in X_T$.

Define a nonlinear map Φ as follows

$$\Phi(u)(t) := e^{it\Delta_D} u_0 - i \int_0^t e^{i(t-\tau)\Delta_D} F(u(\tau)) d\tau. \quad (6.4)$$

We have to show that Φ is a contraction in a suitable ball $B(0, R)$ of X_T . Using the Minkowski integral inequality together with an energy argument we have, on the one hand

$$\begin{aligned} \|\Phi(u)\|_{L_T^\infty H_0^1(\Omega)} &\lesssim \|u_0\|_{H_0^1(\Omega)} + \|F(u)\|_{L_T^1 H_0^1(\Omega)} \lesssim \\ &(\|u_0\|_{H_0^1(\Omega)} + \|u\|_{L_T^1 H_0^1(\Omega)} \|u\|_{L_T^\infty H_0^1(\Omega)}^2 + T \|u\|_{L_T^\infty H_0^1(\Omega)} + T^{1-\frac{2}{p}} \|u\|_{L_T^\infty H_0^1(\Omega)} \|u\|_{L_T^p L^\infty(\Omega)}^2) \leq \\ &C \left(\|u_0\|_{H_0^1(\Omega)} + T \|u\|_{X_T} + (T^{1-\frac{2}{p}} + T) \|u\|_{X_T}^3 \right) \end{aligned} \quad (6.5)$$

and also

$$\begin{aligned} \|(1-\Delta_D)^{\frac{\sigma}{2}} \Phi(u)\|_{L_T^p L^q(\Omega)} &\lesssim (\|(1-\Delta_D)^{\frac{\sigma}{2}} u_0\|_{H_D^{\frac{4}{5p}}(\Omega)} + \int_0^T \|(1-\Delta_D)^{\frac{\sigma}{2}} (1+|u(s)|^2) u(s)\|_{H_D^{\frac{4}{5p}}(\Omega)} ds) \lesssim \\ &(\|u_0\|_{H_0^1(\Omega)} + \int_0^T (1+\|u(s)\|_{L^\infty}^2) \|u(s)\|_{H_0^1(\Omega)} ds) \leq C \left(\|u_0\|_{H_0^1(\Omega)} + T \|u\|_{X_T} + T^{1-\frac{2}{p}} \|u\|_{X_T}^3 \right). \end{aligned} \quad (6.6)$$

We chose $R > 4C\|u_0\|_{H_0^1(\Omega)}$ and T such that $4C\left(T + (T + T^{1-\frac{2}{p}})R^2\right) < 1$. In a similar way we obtain

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq C\|u - v\|_{X_T} \left(T + (T + T^{1-\frac{2}{p}})(1 + \|u\|_{X_T}^2 + \|v\|_{X_T}^2) \right). \quad (6.7)$$

Taking T eventually smaller such that $C\left(T + (T + T^{1-\frac{2}{p}})(1 + 2R^2)\right) < 1$, we deduce from (6.5), (6.6) and (6.7) that Φ is a contraction from $B(0, R) \subset X_T$ to $B(0, R)$. Therefore, if we consider the sequence $\{v_n\}_{n \in \mathbb{N}} \subset X_T$ such that $v_0 = u_0 \in B(0, R)$, $v_{n+1} = \Phi(v_n)$, then v_n converges in X_T to the unique solution in X_T of the integral equation

$$u(t) = e^{it\Delta_D}u_0 - i \int_0^t e^{i(t-\tau)\Delta_D} F(u(\tau)) d\tau \quad (6.8)$$

which yields the local well-posedness result. From (6.7) we obtain the Lipschitz property of the flow map. Using a standard approximation argument we can derive the conservation laws. Next due to (1.7), the assumption on V ($V(|u|^2) \geq 0$) and the Gagliardo-Nirenberg inequality we can extend the local solution to an arbitrary time interval by reiterating the local-posedness argument.

Neumann: When we consider the Neumann Laplacian we take X_T like before and we choose ϵ so small such that the embedding $W^{1-\frac{4}{5p}-\frac{8\epsilon}{5p}, q}(\Omega) \subset L^\infty(\Omega)$ still holds: for $\epsilon < \frac{12-5p}{16}$ we are led to the same conclusion.

6.2 Proof Theorem 1.3 for a class of trapping obstacles

In this part we prove Theorem 1.3 for a class of trapping obstacles. In this case, since there are trapped trajectories (e.g. any line minimizing the distance between two obstacles has an unbounded sojourn time), the plain smoothing effect $H^{\frac{1}{2}}$ does not hold. However, one can obtain a smoothing effect with a logarithmic loss (see [5, Thm1.7, Thm.4.2]).

Assumptions: We suppose here $\Theta = \cup_{i=1}^N \Theta_i$ is the disjoint union of a finite number of balls $\Theta_i = B_i(o_i, r_i)$ of radius $r_i > 0$ in \mathbb{R}^3 . We denote by k the minimum of the curvatures of the spheres $\mathbb{S}(r_i) = \partial\Theta_i$, that is $k = \min\{\frac{1}{\sqrt{r_i}}, i = \overline{1, N}\}$. Denote by L the minimum of the distances between two balls. Then, if $N > 2$, we assume that $kL > N$ ($L \geq l > 0$ if $N = 2$ for some strictly positive l). We keep the notations of the previous sections.

Let $\chi \in C_0^\infty((-1, 1))$, $\chi = 1$ close to 0 and for every i , set $\chi_i(x) := \chi(\frac{|x-o_i|}{r_i} - 1)$ and $\chi_{i,\lambda}(x) = \chi(\lambda^\alpha(\frac{|x-o_i|}{r_i} - 1))$ ($\chi_{i,\lambda}$ vanishes outside a neighborhood of size $\lambda^{-\alpha}$ of the ball Θ_i). Set $u = e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0$ and $v_i = \chi_i u$. Then v_i satisfies

$$(i\partial_t + \Delta_D)v_i = [\Delta_D, \chi_i]u, \quad v_i|_{t=0} = \chi_i\psi(\frac{-\Delta_D}{\lambda^2})u_0. \quad (6.9)$$

We introduce the Dirichlet Laplacian Δ_D^i acting on $\Omega_i := \mathbb{R}^3 \setminus \Theta_i$ and continue to denote Δ_D the Dirichlet Laplacian outside the obstacle Θ . Writing Duhamel's formula we get

$$v_i(t) = e^{it\Delta_D^i}\chi_i\psi(\frac{-\Delta_D}{\lambda^2})u_0 - i \int_0^t e^{it\Delta_D^i}\psi(\frac{-\Delta_D}{\lambda^2})e^{-it\tau\Delta_D}[\Delta_D, \chi_i](u(\tau))d\tau. \quad (6.10)$$

The solution v_i of (6.9) satisfies

$$\begin{aligned} \|\chi_{i,\lambda} v_i\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} &\leq \|\chi_{i,\lambda} e^{it\Delta_D^i} \chi_i \psi\left(\frac{-\Delta_D^i}{\lambda^2}\right) u_0\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} + \\ &\|\chi_{i,\lambda} \int_0^t e^{it\Delta_D^i} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) e^{-it\tau\Delta_D} [\Delta_D, \chi_i](u(\tau)) d\tau\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)}. \end{aligned} \quad (6.11)$$

From [5, Thm.4.2] we know that the operator

$$A_i^* f_i := \int_0^T \psi\left(\frac{-\Delta_D}{\lambda^2}\right) e^{-i\tau\Delta_D} \chi_i f_i(\tau) d\tau \quad (6.12)$$

is bounded from $L_T^2 H_D^{-\frac{1}{2}}(\Omega)$ to $H_D^{-\epsilon}(\Omega)$. Notice that in (6.12) we can introduce a cutoff function $\tilde{\chi}_i \in C_0^\infty$ equal to 1 on the support of χ_i without changing the integral modulo smoothing terms. We need the next lemma:

Lemma 6.2. *Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+^*)$ be a smooth function such that $\tilde{\psi}(\tau) = 1$ if $\psi(\tau) = 1$ and such that $\sum_{k \geq 0} \tilde{\psi}(2^{-2k}\tau) = 1$. Let $\tilde{\chi}_i \in C_0^\infty$ be equal to 1 on the support of χ_i . Then for $f \in L^2(\Omega)$*

$$\tilde{\psi}\left(\frac{-\Delta_D^i}{\lambda^2}\right) \tilde{\chi}_i \psi\left(\frac{-\Delta_D}{\lambda^2}\right) (\chi_i f) = \tilde{\chi}_i \psi\left(\frac{-\Delta_D}{\lambda^2}\right) (\chi_i f) + O_{L^2(\Omega)}(\lambda^{-\infty}) \chi_i f. \quad (6.13)$$

We postpone the proof of Lemma 6.2 for the end of this section.

End of the proof of Theorem 1.3: We introduce the operator

$$A_{i,\lambda} u_0(t, x) := \chi_{i,\lambda} e^{it\Delta_D^i} \tilde{\psi}\left(\frac{-\Delta_D^i}{\lambda^2}\right) u_0,$$

which is continuous from $L^2(\Omega_i)$ to $L_T^2 H_D^{\frac{1}{2}}(\Omega_i)$ with the norm bounded by $\lambda^{-\alpha/4}$. Indeed, since $\chi_{i,\lambda}$ vanishes outside a small neighborhood of Θ_i we can apply Theorem 1.1 in $\mathbb{R}^3 \setminus \Theta_i$. If we take $f_i = [\Delta_D, \chi_i]u$, then in view of Lemma 6.2 the last term in the right hand side of (6.11) writes $A_{i,\lambda} A_i^* f_i + O_{L^2(\Omega)}(\lambda^{-\infty})$.

If $\tilde{\chi}_i \in C_0^\infty$ equals 1 on the support of χ_i we can estimate

$$\|[\Delta_D, \chi_i]u\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} \lesssim \|[\Delta_D, \chi_i] \tilde{\chi}_i u\|_{L_T^2 H_D^{-\frac{1}{2}}(\Omega)} \lesssim \quad (6.14)$$

$$\|\tilde{\chi}_i e^{it\Delta_D} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} \lesssim \|\psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{H_D^\epsilon(\Omega)},$$

where in the last inequality we applied [5, Thm.1.7]. Hence (6.11) becomes

$$\|\chi_{i,\lambda} u\|_{L_T^2 H_D^{\frac{1}{2}}(\Omega)} \lesssim \lambda^{-\frac{\alpha}{4}} \|\chi_{i,\lambda} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L_T^2(\Omega)} + \lambda^{-\frac{\alpha}{4}+2\epsilon} \|\chi_{i,\lambda} \psi\left(\frac{-\Delta_D}{\lambda^2}\right) u_0\|_{L^2(\Omega)}. \quad (6.15)$$

Set $\chi_\lambda := \sum_{i=1}^N \chi_{i,\lambda}$. Since $\{\chi_{i,\lambda}\}$ have disjoint supports, (6.15) remains valid for $\chi_{i,\lambda}$ replaced by χ_λ . We have thus obtained a smoothing effect with a gain $\alpha/4 - 2\epsilon$ and by interpolation with the energy estimate we have

$$\|\chi_\lambda u\|_{L_T^p L^q(\Omega)} \lesssim \lambda^{\frac{1}{p}(1-\frac{\alpha}{2}+\frac{\epsilon}{4})} \|\Psi(-\frac{\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (6.16)$$

Away from $\partial\Theta$ we can use the same arguments as in Section 4 and find

$$\|(1-\chi_\lambda)u\|_{L_T^p L^q(\Omega)} \lesssim \lambda^\alpha \|\Psi(-\frac{\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (6.17)$$

Let $p = 2 + \epsilon$ (for $\epsilon > 0$ sufficiently small) and take $\frac{1}{2+\epsilon}(1 - \frac{\alpha}{2} + \frac{\epsilon}{4}) = \alpha$.

Proof. (of Lemma 6.2): We fix $\lambda = 2^{k_0}$, $k_0 \geq 1$. We write

$$\begin{aligned} \tilde{\chi}_i \psi(-2^{-2k_0} \Delta_D) &= \sum_{k \geq 0} \tilde{\psi}(-2^{-2k} \Delta_D^i) \tilde{\chi}_i \psi(-2^{-2k_0} \Delta_D) = \\ &= \tilde{\psi}(-2^{-2k_0} \Delta_D^i) \tilde{\chi}_i \psi(-2^{-2k_0} \Delta_D) + \sum_{k \neq k_0} \tilde{\psi}(-2^{-2k_0} (2^{-2(k-k_0)} \Delta_D^i)) \tilde{\chi}_i \psi(-2^{-2k_0} \Delta_D). \end{aligned} \quad (6.18)$$

Let $\tilde{B} \subset \mathbb{R}^3$ be a ball of sufficiently large radius such that $\cup_{i=1}^N \text{supp} \tilde{\chi}_i \subset \subset B$ and $\tilde{B}_i \subset \mathbb{R}^3$ be balls such that $\text{supp} \tilde{\chi}_i \subset \tilde{B}_i$. Suppose that $\tilde{\chi}_i$ are suitably chosen such that the distances between any two such balls \tilde{B}_i be strictly positive (this is always possible, eventually shrinking the supports of χ_i). If we denote $\tilde{\Delta}_D$, resp. $\tilde{\Delta}_D^i$ the Laplace operators in the bounded domains $\tilde{B} \cap \Omega$ and resp. $\tilde{B}_i \cap \Omega$, we notice that

$$\Delta_D(\chi_i f) = \tilde{\Delta}_D(\chi_i f) = \Delta_D^i(\chi_i f) = \tilde{\Delta}_D^i(\chi_i f), \quad \forall f \in L^2(\Omega). \quad (6.19)$$

Let ϕ_n , $n \geq 1$, resp. ϕ_m^i , $m \geq 1$ denote an orthonormal system of eigenfunctions of $\tilde{\Delta}_D$, resp. $\tilde{\Delta}_D^i$ associated to the eigenvalues λ_n^2 , resp. λ_m^{i2} , i.e. such that $-\tilde{\Delta}_D \phi_n = \lambda_n^2 \phi_n$, $-\tilde{\Delta}_D^i \phi_m^i = \lambda_m^{i2} \phi_m^i$ for $n, m \geq 1$. Consequently if $f_i \in L^2(\Omega)$ is supported in $\tilde{B}_i \cap \Omega$ we have

$$\psi(-2^{-2k_0} \Delta_D) f_i = \sum_{n \geq 1} \langle f_i, \phi_n \rangle \psi(2^{-2k_0} \lambda_n^2) \phi_n,$$

and if $g_i \in L^2(\Omega)$ is supported in $\tilde{B}_i \cap \Omega$ and we set $A_k = 2^{-2(k-k_0)}$, $k \geq 0$ than

$$\tilde{\psi}(-2^{-2k_0} A_k \Delta_D^i) g_i = \sum_{m \geq 1} \langle g_i, \phi_m^i \rangle \tilde{\psi}(2^{-2k_0} A_k \lambda_m^{i2}) \phi_m^i,$$

If the support of $\tilde{\psi}$ is sufficiently large we show that the contribution of the sum in the second line of (6.18) for $k \neq k_0$ is $O_{L^2(\Omega)}(\lambda^{-\infty})$. We consider separately the cases $k > k_0$, resp. $k < k_0$. Let first $k > k_0$: we distinguish two case, according to $2^{\epsilon k_0} \lesssim 2^k < 2^{k_0}$ for some $\epsilon \geq 1/4$ or $2^k \lesssim 2^{k_0/4}$.

- Let $2^k < 2^{k_0/4}$ and set $A_k = 2^{2(k_0-k)}$, then $A_k > 2^{k_0} = \lambda \simeq \lambda_n$.

$$\tilde{\psi}(-2^{-2k_0} A_k \Delta_D^i) \tilde{\chi}_i \psi(-2^{-2k_0} \Delta_D) \chi_i f = \quad (6.20)$$

$$\begin{aligned} & \sum_{n,m \geq 1} \tilde{\psi}(2^{-2k_0} A_k \lambda_m^{i2}) \psi(2^{-2k_0} \lambda_n^2) \langle \chi_i f, \phi_n \rangle \langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle \phi_m^i, \\ & \langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle = \frac{1}{\lambda_n^2} \langle -\tilde{\chi}_i \tilde{\Delta}_D \phi_n, \phi_m^i \rangle = \frac{1}{\lambda_n^2} \langle -\tilde{\Delta}_D(\tilde{\chi}_i \phi_n) + [\tilde{\Delta}_D, \tilde{\chi}_i] \phi_n, \phi_m^i \rangle. \end{aligned} \quad (6.21)$$

Since $-\tilde{\Delta}_D(\tilde{\chi}_i \phi_n) = -\tilde{\Delta}_D^i(\tilde{\chi}_i \phi_n)$ is self-adjoint, the first term in the last line writes

$$\frac{1}{\lambda_n^2} \langle \tilde{\chi}_i \phi_n, -\tilde{\Delta}_D^i \phi_m^i \rangle = \frac{\lambda_m^{i2}}{\lambda_n^2} \langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle,$$

and hence its contribution in the sum in (6.20) is

$$\frac{\lambda_m^{i2}}{\lambda_n^2} \tilde{\psi}(2^{-2k_0} A_k \lambda_m^{i2}) \psi(2^{-2k_0} \lambda_n^2) \langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle. \quad (6.22)$$

Since $\psi, \tilde{\psi}$ are compactly supported away from 0, the only nontrivial contribution in the sum (6.20) comes from indices n, m such that $\lambda_n^2 \simeq A_k \lambda_m^{i2} \simeq 2^{2k_0}$ and from (6.22) this will be $O_{L^2(\Omega)}(A_k^{-1}) = O_{L^2(\Omega)}(1/\lambda_n)$ which follows from the assumption $2^{\epsilon k_0} \lesssim 2^k$ for some $\epsilon \geq 1/4$. We estimate the last term in the right side of (6.21)

$$\frac{1}{\lambda_n^2} \langle [\tilde{\Delta}_D, \tilde{\chi}_i] \phi_n, \phi_m^i \rangle = O_{L^2(\Omega)}(\lambda_n^{-1}) = O_{L^2(\Omega)}(2^{-k_0}),$$

since on the support of $\psi, \lambda_n \simeq 2^{k_0}$. By iterating these arguments $M \geq 1$ times we deduce that the contribution of the sum (6.18) is $O_{L^2(\Omega)}(2^{-Mk_0})$ for every $M \geq 1$.

- Let now $2^{\epsilon k_0} \lesssim 2^k < 2^{k_0}$, $\epsilon \geq 1/4$: in this case a simple integration by part is useless since the "error" is a multiple of the number of integrations. If the support of $\tilde{\chi}_i$ is sufficiently small then (6.19) holds and we have

$$\langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle = \int (\tilde{\chi}_i - 1) \phi_n \bar{\phi}_m^i dx + \delta_{n=m}, \quad (6.23)$$

where $\delta_{n=m}$ is the Dirac distribution. In the sum (6.20) we see that the contribution from $n = m$ is zero, since the support of $\tilde{\psi}(A_k \cdot)$ and $\psi(\cdot)$ are disjoint. For first term in the right hand side of (6.23) we use an argument of N.Burq, P.Gérard and N.Tzvetkov [9, Lemma 2.6]. Let $\kappa \in \mathcal{S}(\mathbb{R})$ be a rapidly decreasing function such that $\kappa(0) = 1$. From a result of Sogge [19, Chp.5.1] we can write, on the support of $\tilde{\chi}$ where (6.19) holds

$$\kappa(\sqrt{-\tilde{\Delta}_D} - \lambda) f(x) = \lambda^{1/2} \int e^{i\lambda\varphi(x,y)} a(x,y,\lambda) f(y) dy + R_\lambda f(x),$$

where $a(x, y, \lambda) \in C_0^\infty$ is an asymptotic assumption in $1/\lambda$ and $-\varphi(x, y)$ is the geodesic distance between x and y , and where

$$\forall p, s \in \mathbb{N} : \quad \|R_\lambda f\|_{H^s(\text{supp } \tilde{\chi}_i)} \leq C_{p,s} \lambda^{-p} \|f\|_{L^2}.$$

We use (6.19) and $\kappa(\sqrt{-\Delta_D} - \lambda_n)\phi_n = \phi_n$, $\kappa(\sqrt{-\Delta_D^i} - \lambda_m^i)\phi_m^i = \phi_m^i$ in order to write $\langle (\tilde{\chi}_i - 1)\phi_n, \phi_m^i \rangle$ as an integral (modulo a remaining term, small)

$$\int_{x,y,z} e^{i\lambda_n^{1/2}\Phi(x,y,z)} a(x, y, \lambda_n) \bar{a}(x, z, \lambda_m^i) \phi_n(y) \bar{\phi}_m^i(z) dy dz dx,$$

with $\Phi(x, y, z) = \varphi(x, y) + \sqrt{\lambda_m^i/\lambda_n}\varphi(x, z)$. Since $|\nabla_x \varphi|$ is uniformly bounded from below and bounded from above together with all its derivatives, we obtain that there exist $c > 0$, $C_\beta > 0$ such that $|\nabla_x \Phi| \geq c$ and $\partial^\beta \Phi \leq C_\beta$. It remains to perform integrations by parts in the x variable as many times as we want, each such integration providing a gain of a power $\lambda_n^{-1/2}$.

For $k > k_0$ and $2^{k_0} \lesssim 2^{k/4}$, we write

$$\langle \tilde{\chi}_i \phi_n, \phi_m^i \rangle = \frac{1}{\lambda_m^{i2}} \langle -\tilde{\chi}_i \phi_n, \tilde{\Delta}_D^i \phi_m^i \rangle = \frac{1}{\lambda_m^{i2}} \langle -\tilde{\Delta}_D(\tilde{\chi}_i \phi_n) + [\tilde{\Delta}_D, \tilde{\chi}_i] \phi_n, \phi_m^i \rangle, \quad (6.24)$$

in which case, using the spectral localizations $\psi, \tilde{\psi}$, we gain a factor A_k from the first term in (6.24) and a factor $A_k(\lambda_m^i)^{-1}$ from the second one; iterating the argument as many times as we want we obtain contribution $O_{L^2(\Omega)}(A_k^M)$. In the last case $2^{\epsilon k} \lesssim 2^{k_0} < 2^k$, $\epsilon \geq 1/4$ we use again the arguments of [9, Lemma 2.6]. The proof is complete. \square

7 Appendix

7.1 Hankel functions

We will start by recalling some properties of Hankel functions that will be useful in determining the behavior of the solution to (2.2) (see [1]). The Hankel function of order ν is defined by

$$H_\nu(z) = \int_{-\infty}^{+\infty - i\pi} e^{z \sinh t - \nu t} dt. \quad (7.1)$$

For all values of ν , $\{H_\nu(z), \bar{H}_\nu(z)\}$ form a pair of linearly independent solutions of the Bessel's equation $z^2 y'' + z y' + (z^2 - \mu^2) y = 0$.

Proposition 7.1. *We have the following asymptotic expansions (see [1, Chp.9]):*

1. *If ν is fixed, bounded and $z = r\lambda \gg \nu > \frac{1}{2}$, $\frac{\nu}{z} \ll 1$ then*

$$H_\nu(z) \simeq \sqrt{\frac{2}{\pi z}} e^{i(\lambda r - \frac{\pi\nu}{2} - \frac{\pi}{4})} (1 + O(\lambda^{-1})), \quad \bar{H}_\nu(z) \simeq \sqrt{\frac{2}{\pi z}} e^{-i(\lambda r - \frac{\pi\nu}{2} - \frac{\pi}{4})} (1 + O(\lambda^{-1})); \quad (7.2)$$

2. If $z = r\lambda > \nu \gg 1$ and $\frac{\nu}{z} \in [\epsilon_0, 1 - \epsilon_1]$ for some small, fixed $\epsilon_0 > 0$, $\epsilon_1 > 0$, then writing $\frac{z}{\nu} = \frac{1}{\cos\beta}$ we have

$$H_\nu(z) = H_\nu\left(\frac{\nu}{\cos\beta}\right) \simeq \sqrt{\frac{2}{\pi\nu \tan\beta}} e^{i\lambda(\tan\beta - \beta) + \frac{i\pi}{4}} (1 + O(\nu^{-1})), \quad (7.3)$$

$$\bar{H}_\nu(z) = \bar{H}_\nu\left(\frac{\nu}{\cos\beta}\right) \simeq \sqrt{\frac{2}{\pi\nu \tan\beta}} e^{i\lambda(\tan\beta - \beta) + \frac{i\pi}{4}} (1 + O(\nu^{-1})); \quad (7.4)$$

3. If $z, \nu \gg 1$ are nearly equal we have the following formulas

(a) If $z - \nu = \tau\nu^{1/3}$ with fixed τ , bounded, then

$$J_\nu(\nu + \tau\nu^{1/3}) \simeq \frac{\sqrt[3]{2}}{\sqrt[3]{\nu}} Ai(-\sqrt[3]{2}\tau)(1 + O(\nu^{-1})), \quad (7.5)$$

$$Y_\nu(\nu + \tau\nu^{1/3}) \simeq -\frac{\sqrt[3]{2}}{\sqrt[3]{\nu}} (Bi(-\sqrt[3]{2}\tau)(1 + O(\nu^{-1}))), \quad (7.6)$$

where for $|\tau|$ large and $\xi = \frac{2}{3}\tau^{3/2}$ we have

$$Ai(-\tau) \simeq \frac{1}{\pi^{1/2}\tau^{1/4}} \sin(\xi + \frac{\pi}{4})(1 + O(\xi^{-1})), \quad Bi(-\tau) \simeq \frac{1}{\pi^{1/2}\tau^{1/4}} \cos(\xi + \frac{\pi}{4})(1 + O(\xi^{-1})); \quad (7.7)$$

(b) If $z = \nu$, then

$$J_\nu(\nu) \simeq \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)} \nu^{-1/3} (1 + O(\nu^{-1})), \quad (7.8)$$

$$Y_\nu(\nu) \simeq -\frac{2^{1/3}}{3^{1/6}\Gamma(2/3)} \nu^{-1/3} (1 + O(\nu^{-1})); \quad (7.9)$$

(c) If $|\nu - r\lambda| \leq C_0|r\lambda|$ then if $\nu z = r\lambda$ we have for $z < 1$ (resp. $z > 1$)

$$J_\nu(\nu z) \simeq \left(\frac{4\zeta}{1-z^2}\right)^{1/4} \left(\frac{Ai(\nu^{2/3}\zeta)}{\nu^{1/3}} + \frac{\exp(\frac{2}{3}\nu\zeta^{3/2})}{1 + \nu^{1/6}|\zeta|^{1/4}} O(1/\nu^{4/3})\right), \quad (7.10)$$

$$Y_\nu(\nu z) \simeq -\left(\frac{4\zeta}{1-z^2}\right)^{1/4} \left(\frac{Bi(\nu^{2/3}\zeta)}{\nu^{1/3}} + \frac{\exp(|\Re(\frac{2}{3}\nu\zeta^{3/2})|)}{1 + \nu^{1/6}|\zeta|^{1/4}} O(1/\nu^{4/3})\right), \quad (7.11)$$

where the function ζ is defined by

$$\frac{2}{3}\zeta^{3/2} = \int_z^1 \frac{\sqrt{1-t^2}}{t} dt = \ln[(1 + \sqrt{1-z^2})/z] - \sqrt{1-z^2}, \quad z \leq 1, \quad (7.12)$$

$$\frac{2}{3}(-\zeta)^{3/2} = \int_1^z \frac{\sqrt{t^2-1}}{t} dt = \sqrt{z^2-1} - \arccos(1/z), \quad z \geq 1, \quad (7.13)$$

and where for τ large and $\xi = \frac{2}{3}\tau^{3/2}$ we have

$$Ai(\tau) = \frac{1}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}}e^{-\xi}(1 + O(\xi^{-1})), \quad Bi(\tau) = \frac{1}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}}e^{\xi}(1 + O(\xi^{-1})). \quad (7.14)$$

Taking $\tau = \nu^{2/3}\zeta$, we compute $Ai(\nu^{2/3}\zeta)$ using (7.14), (7.7) with $\xi = \frac{2}{3}\zeta^{3/2}\nu$.

Here J_ν and Y_ν are the Bessel functions of the first kind and $H_\nu(z) = J_\nu(z) + iY_\nu(z)$.

Remark 7.2. We remark that ζ defined in (7.12), (7.13) is analytic in z , even at $z = 1$ and $d\zeta/dz < 0$ there; also, at $z = 1$, $\zeta = 0$ and $(1 - z^2)^{-1}\zeta = 2^{-2/3}$. The formulas (7.10), (7.11) are among the deepest and most important results in the theory of Bessel functions. These results do not appear in the treatise of Watson, having been established after Watson's second edition was published. See Olver [17] and [1]. In the paper we use a simpler form of these asymptotic expansions for which we give an idea of the proof inspired from [11].

Proposition 7.3. *For $z > 1$ close to 1 and $\nu \gg 1$ large enough we have*

$$J_\nu(\nu z) \simeq \frac{\sqrt{2}}{\nu^{1/3}} \left(\frac{\zeta}{z^2 - 1}\right)^{1/4} Ai(-\nu^{2/3}\zeta), \quad (7.15)$$

$$Y_\nu(\nu z) \simeq -\frac{\sqrt{2}}{\nu^{1/3}} \left(\frac{\zeta}{z^2 - 1}\right)^{1/4} Bi(-\nu^{2/3}\zeta), \quad (7.16)$$

uniformly in z , where $\frac{2}{3}\zeta^{3/2} = \sqrt{z^2 - 1} - \arccos(1/z)$ (see [17]).

Proof. With a suitable choice of contour we have

$$J_\nu(\nu z) = \frac{1}{2\pi} \int e^{i\nu\phi(t,z)} dt, \quad \phi(t, z) = z \sin t - t. \quad (7.17)$$

For $\nu \gg 1$ and $z > 1$ the saddle points are real, $\tilde{t} = \arccos(1/z)$ and the critical value is $\phi(t(z), z) = \sqrt{z^2 - 1} - \arccos(1/z)$. Now if $\zeta(z)$ is defined in terms of the exponent in the Debye expansion, it is analytic in z for z near 1 and we have

$$Ai(-\nu^{2/3}\zeta) = \frac{1}{2\pi} \int e^{-is\nu^{2/3}\zeta + is^3/3} ds = \frac{\nu^{1/3}}{2\pi} \int e^{i\nu(-t\zeta + t^3/3)} dt \quad (7.18)$$

with critical points $t^2 = \zeta$, thus we get (7.15) applying the stationary phase. We obtain the Debye approximations

$$J_\nu(\nu z) \simeq \left(\frac{2}{\pi\nu\sqrt{z^2 - 1}}\right)^{1/2} \cos[\nu\sqrt{z^2 - 1} - \nu \arccos(1/z) - \pi/4] \quad (7.19)$$

by replacing the Airy function by its asymptotic expansion, thus the proper condition for its validity is $\nu^{2/3}\zeta \gg 1$. For small $\nu^{2/3}\zeta$ we are in the regime $\nu(z - 1) = \tau\nu^{1/3}$ for which we have the estimations (7.5), (7.6). An extension of this result giving (7.10), (7.11) uses a result of Chester, Friedman and Ursell who showed that a similar reduction is possible

whenever two saddle points coalesce: if $\partial_t \phi(\tilde{t}, 1) = 0$ and $\partial_{tt}^2 \phi(\tilde{t}, 1) = 0$ but $\partial_{ttt}^3 \phi(\tilde{t}, 1) \neq 0$, then an integral of the form (7.17) has a uniform asymptotic expansion in terms of the Airy function and its derivative. Their method was to make a change of variables so that

$$\phi(t, z) = \zeta \tau - \tau^3/3, \quad (7.20)$$

which holds exactly and uniformly; it is not merely an approximation for z near 1. \square

References

- [1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965
- [2] J. Bergh, J. Löfstrom, *Interpolation Spaces*, Springer-Verlag, 1976
- [3] M. Ben Artzi, S. Kleinerman, *Decay and regularity for the Schrödinger equation*, J.Anal.Math., 58:25-37, 1992
- [4] J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations*, Colloquium Publications, American Mathematical Society, 46, 1999
- [5] N. Burq, *Smoothing effect for Schrödinger boundary value problems* Duke Math.Journal, no.2, vol.123, 2004
- [6] N. Burq, *Semi-classical Estimates for the Resolvent in Nontrapping Geometries*, Int.Math.Res.Not. 5, 2002
- [7] N. Burq, P. Gérard, N. Tzvetkov, *On nonlinear Schrödinger equations in exterior domains*, Ann.I.H.Poincaré, 21:295-318, 2004
- [8] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, American J.Math., 126: 569-605, 2004
- [9] N.Burq, P.Gérard, N.Tzvetkov, *Bilinear eigenfunctions estimates and the nonlinear Schrödinger equation on surfaces*, Invent.Math. 150: 187-223, 2005
- [10] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Textos de Métodos Matemáticos", 26, 19
- [11] C. Chester, B. Friedman, C. Ursell, *An extension of the method of steepest descent*, Proc.CambridgePhilos.Soc.,54:599-611, 1957
- [12] P. Constantin, J. C. Saut, *Local smoothing properties of dispersive equations*, J.Amer.Math.Soc., 1:413-439, 1988

- [13] W. Craig, T. Kappeler, W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm.Pures Appl.Math. 48, 8:769-860, 1995
- [14] S. I. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math.J., 82:679-706, 1996
- [15] J. Ginibre, G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation*, Ann.I.P., Anal.non.lin.,2:309-327, 1985 93
- [16] T. Kato, *On nonlinear Schrödinger equations*, Ann.Inst.H.Poincaré Phis.Théor.", 46:113-129, 1987
- [17] F. Olver, *Asymptotics and Special functions*, Academic Press, New York, 1974
- [18] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I-IV*, Academic Press, 1975
- [19] C.D.Sogge, *Fourier integrals in classical analysis*, Cambridge Univ.Press, Cambridge and New York, 1993
- [20] H. Smith, C. Sogge, *On the critical semilinear wave equation outside convex obstacles*, J.Amer.Math.Soc., 8:879-916, 1995
- [21] H. Smith, C. Sogge, *On Strichartz estimates for the Schrödinger operators in compact manifolds with boundary*, arxiv, 2006
- [22] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm.Part.Diff.Eq.,no. 27, vol.5-6, 1337-1372, 2002
- [23] C. Sulem, P. L. Sulem, *The Nonlinear Schrödinger Equation*, Springer, 1999
- [24] M. Taylor, *Partial Differential Equations I,II,III*, Springer, 1996
- [25] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ.Press 1944