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# On the energy critical Schrödinger equation in $3D$ non-trapping domains

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## Abstract

We prove that the quintic Schrödinger equation with Dirichlet boundary conditions is locally well posed for  $H_0^1(\Omega)$  data on any smooth, non-trapping domain  $\Omega \subset \mathbb{R}^3$ . The key ingredient is a smoothing effect in  $L_x^5(L_t^2)$  for the linear equation. We also derive scattering results for the whole range of defocusing subquintic Schrödinger equations outside a star-shaped domain.

## 1 Introduction

The Cauchy problem for the semilinear Schrödinger equation in  $\mathbb{R}^3$  is by now relatively well-understood: after seminal results by Ginibre-Velo [10] in the energy class for energy subcritical equations, the issue of local well-posedness in the critical Sobolev spaces ( $\dot{H}^{\frac{3}{2}-\frac{2}{p-1}}$ ) was settled in [7]. Scattering for large time was proved in [10] for energy subcritical defocusing equations, while the energy critical (quintic) defocusing equation was only recently successfully tackled in [9]. The local well-posedness relies on Strichartz estimates, while scattering results combine these local results with suitable non concentration arguments based on Morawetz estimates. On domains, the same set of problems remains

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an elusive target, due to the difficulty in obtaining Strichartz estimates in such a setting. In [2], the authors proved Strichartz estimates with an half-derivative loss on non trapping domains: the non trapping assumption is crucial in order to rely on the local smoothing estimates. However, the loss resulted in well-posedness results for strictly less than cubic nonlinearities; this was later improved to cubic nonlinearities in [1] (combining local smoothing and semiclassical Strichartz near the boundary) and in [11] (on the exterior of a ball, through precised smoothing effects near the boundary). Recently there were two significant improvements, following different strategies:

- in [16], Luis Vega and the second author obtain an  $L_{t,x}^4$  Strichartz estimate which is scale invariant. However, one barely misses  $L_t^4(L^\infty(\Omega))$  control for  $H_0^1$  data, and therefore local wellposedness in the energy space was improved to all subcritical (less than quintic) nonlinearities, but combining this Strichartz estimate with local smoothing close to the boundary and the full set of Strichartz estimates in  $\mathbb{R}^3$  away from it. Scattering was also obtained for the cubic defocusing equation, but the lack of a good local wellposedness theory at the scale invariant level ( $\dot{H}^{\frac{1}{2}}$ ) led to a rather intricate incremental argument, from scattering in  $\dot{H}^{\frac{1}{4}}$  to scattering in  $H_0^1$ ;
- in [13], the first author proved the full set of Strichartz estimates (except for the endpoint) outside strictly convex obstacles, by following the strategy pioneered in [17] for the wave equation, and relying on the Melrose-Taylor parametrix. In the case of the Schrödinger equation, one obtains Strichartz estimates on a semiclassical time scale (taking advantage of a finite speed of propagation principle at this scale), and then upgrading to large time results from combining them with the smoothing effect (see [3] for a nice presentation of such an argument, already implicit in [19]). Therefore, one obtains the exact same local wellposedness theory as in the  $\mathbb{R}^3$  case, including the quintic nonlinearity, and scattering holds for all subquintic defocusing nonlinearities, taking advantage of the a priori estimates from [16].

In the present work, we aim at providing a local wellposedness theory for the quintic nonlinearity outside non trapping obstacles, a case which is not covered by [13]. From explicit computations with gallery modes ([12]), one knows that the full set of optimal Strichartz estimates does not hold for the Schrödinger equation on a domain whose boundary has at least one geodesically convex point; while this does not preclude a scale invariant Strichartz estimate with a loss (like the  $L_t^4(L_x^\infty)$  estimate in  $\mathbb{R}^3$  which is enough to solve the quintic NLS), it suggests to bypass the issue and use a different set of estimates, which we call smoothing estimates: in  $\mathbb{R}^3$ , these estimates may be stated as follows,

$$\|\exp(it\Delta)f\|_{L_t^4(L_x^2)} \lesssim \|f\|_{\dot{H}^{-\frac{1}{4}}}, \quad (1.1)$$

from which one can infer various estimates by using Sobolev in time and/or in space. Formally, (1.1) is an immediate consequence of the Stein-Tomas restriction theorem in  $\mathbb{R}^3$  (or, more accurately, its dual version, on the extension): let  $\tau > 0$  be a fixed radius, one sees  $\hat{f}(\xi)$  as a function on  $|\xi| = \sqrt{\tau}$ , and applies the extension estimate, with  $\delta$  the Dirac function and  $\mathcal{F}$  the space Fourier transform

$$\|\mathcal{F}^{-1}(\delta(\tau - |\xi|^2)\hat{f}(\xi))\|_{L_x^4} \lesssim \|\hat{f}(\xi)\|_{L^2(|\xi|=\sqrt{\tau})}.$$

Summing over  $\tau$  yields the  $L^2$  norm of  $f$  on the RHS, while on the left we use Plancherel in time and Minkowski to get (1.1). A similar estimate holds for the wave equation, replacing  $\sqrt{\tau} = |\xi|$  by  $\tau = \pm|\xi|$ , and usually goes under the denomination of square function (in time) estimates. In a compact setting (e.g. compact manifolds) a substitute for the Stein-Tomas theorem is provided by  $L^p$  eigenfunction estimates, or better yet, spectral cluster estimates. In the context of a compact manifold with boundaries, such spectral cluster estimates were recently obtained by Smith and Sogge in [18], and provided a key tool for solving the critical wave equation on domains, see [4, 6]. In this paper, we apply the same strategy to the Schrödinger equation:

- we derive an  $L^5(\Omega; L_t^2)$  smoothing estimate for spectrally localized data on compact manifolds with boundaries, from the spectral cluster  $L^5(\Omega)$  estimate; here  $I$  is a time interval whose size is such that  $|I||\sqrt{-\Delta_D}| \sim 1$ ;
- we decompose the solution to the linear Schrödinger equation on a non trapping domain into two main regions: close to the boundary, where we can view the region as embedded into a  $3D$  punctured torus, to which the previous semi-classical estimate may be applied, and then summed up using the local smoothing effect; and far away from the boundary where the  $\mathbb{R}^3$  estimates hold.
- Finally, we patch together all estimates to obtain an estimate which is valid on the whole exterior domain. Local wellposedness in the critical Sobolev space  $\dot{H}^{\frac{3}{2}-\frac{2}{p-1}}$  immediately follows for  $3+2/5 < p \leq 5$ , and together with the a priori estimates from [16], this implies scattering for the defocusing equation for  $3+2/5 < p < 5$ . The remaining range  $3 \leq p \leq 3+2/5$  is sufficiently close to 3 that, as alluded to in [16], a suitable modification of the arguments from [16] yields scattering as well, and we do not pursue this matter here.

**Remark 1.1.** *Clearly, such smoothing estimates are better suited to large values of  $p$ : the restriction  $3+2/5 < p$  for the critical wellposedness is directly linked to the exponent 5 in the spectral cluster estimates; in  $\mathbb{R}^3$ , where the correct (and optimal !) exponent is 4, one*

may solve down to  $p = 3$  by this method, while the Strichartz estimates allow to solve at scaling level all the way to the  $L^2$  critical value  $p = 1 + 4/3$ .

## 2 Statement of results

Let  $\Theta$  be a compact, non-trapping obstacle in  $\mathbb{R}^3$  and set  $\Omega = \mathbb{R}^3 \setminus \Theta$ . By  $\Delta_D$  we denote the Laplace operator with constant coefficients on  $\Omega$ . For  $s \in \mathbb{R}$ ,  $p, q \in [1, \infty]$  we denote by  $\dot{B}_p^{s,q}(\Omega) = \dot{B}_p^{s,q}$  the Besov spaces on  $\Omega$ , where the spectral localization in their definition is meant to be with respect to  $\Delta_D$ . We write  $L_x^p = L^p(\Omega)$  and  $\dot{H}^\sigma = \dot{B}_2^{\sigma,2}$  for the Lebesgue and Sobolev spaces on  $\Omega$ . It will be useful to introduce the Banach-valued Besov spaces  $\dot{B}_p^{s,q}(L_t^r)$ , and we refer to the Appendix for their definition. Whenever  $L_t^p$  is replaced by  $L_T^p$ , it is meant that the time integration is restricted to the interval  $(-T, T)$ .

We aim at studying wellposedness for the energy critical equation on  $\Omega \times \mathbb{R}$ , with Dirichlet boundary condition,

$$i\partial_t u + \Delta_D u = \pm |u|^4 u, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \quad (2.1)$$

and more generally

$$i\partial_t u + \Delta_D u = \pm |u|^{p-1} u, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \quad (2.2)$$

with  $p < 5$ .

**Theorem 2.1.** *(Well-posedness for the quintic Schrödinger equation) Let  $u_0 \in H_0^1(\Omega)$ . There exists a unique solution  $u \in C([-T, T], H_0^1(\Omega)) \cap \dot{B}_5^{1,2}(L_T^{\frac{20}{11}})$ , to the quintic nonlinear equation (2.1). Moreover, the solution is global in time and scatters in  $H_0^1$  if the data is small.*

The previous theorem extends to the following subcritical range:

**Theorem 2.2.** *Let  $3 + \frac{2}{5} < p < 5$ ,  $s_p = \frac{3}{2} - \frac{2}{p-1}$  and  $u_0 \in \dot{H}^{s_p}$ . There exists a unique solution  $u \in C([-T, T], \dot{H}^{s_p}) \cap \dot{B}_5^{s_p,2}(L_T^{\frac{20}{11}})$ , to the nonlinear equation (2.2). Moreover the solution is global in time and scatters in  $\dot{H}^{s_p}$  if the data is small.*

Finally, we consider the long time asymptotics for (2.2) in the defocusing case, namely the  $+$  sign on the left.

**Theorem 2.3.** *Assume the domain  $\Omega$  to be the exterior of a star-shaped compact obstacle (which implies  $\Omega$  is non trapping). Let  $3 + \frac{2}{5} < p < 5$ , and  $u_0 \in H_0^1(\Omega)$ . There exists a unique solution  $u \in C(\mathbb{R}, H_0^1(\Omega)) \cap \dot{B}_5^{s_p,2}(L_t^{\frac{20}{11}})$ , to the nonlinear equation (2.2). Moreover, there exists two scattering states  $u^\pm \in H_0^1(\Omega)$  such that*

$$\lim_{t \rightarrow \pm\infty} \|u(x, t) - e^{it\Delta_D} u^\pm\|_{H_0^1(\Omega)} = 0.$$

### 3 Smoothing type estimates

We start with definitions and notations. Let  $\psi(\xi^2) \in C_0^\infty(\mathbb{R} \setminus \{0\})$  and  $\psi_j(\xi^2) = \psi(2^{-2j}\xi^2)$ . On the domain  $\Omega$ , one has the spectral resolution of the Dirichlet Laplacian, and we may define smooth spectral projections  $\Delta_j = \psi_j(-\Delta_D)$  as continuous operators on  $L^2$ . Moreover, these operators are continuous on  $L^p$  for all  $p$ , and if  $f$  is Hilbert-valued and such that  $\| \|f\|_H \|_{L^p(\Omega)} < +\infty$ , then the operators  $\Delta_j$  are continuous as well on  $L^p(H)$ . We refer to [14] for an extensive discussion and references. We simply point out that if  $H = L_t^2$ , then  $\Delta_j$  is continuous on all  $L_x^p L_t^q$  by interpolation with the obvious  $L_t^p(L_x^p)$  bound and duality.

In this section we concentrate on estimates for the linear Schrödinger equation on  $\Omega \times \mathbb{R}$  with Dirichlet boundary conditions,

$$i\partial_t u_L + \Delta_D u_L = 0, \quad u_{L|\partial\Omega} = 0, \quad u_{L|t=0} = u_0 \quad (3.1)$$

**Theorem 3.1.** *The following local smoothing estimate holds for the homogeneous linear equation (3.1),*

$$\|\Delta_j u_L\|_{L_x^5 L_t^2} \lesssim 2^{-\frac{j}{10}} \|\Delta_j u_0\|_{L_x^2}. \quad (3.2)$$

Moreover, let  $2 \leq q \leq \infty$ , then

$$\|\Delta_j u_L\|_{L_x^5 L_t^q} \lesssim 2^{-j(\frac{2}{q} - \frac{9}{10})} \|\Delta_j u_0\|_{L_x^2}. \quad (3.3)$$

Consider now the inhomogeneous equation,

$$i\partial_t v + \Delta_D v = F, \quad v_{|\partial\Omega} = 0, \quad v_{|t=0} = 0. \quad (3.4)$$

From Theorem 3.1, we will obtain the following set of estimates:

**Theorem 3.2.** *Let  $2 \leq q < r \leq +\infty$ , then*

$$\|\Delta_j v\|_{C_t(L_x^2)} + 2^{j(\frac{2}{q} - \frac{9}{10})} \|\Delta_j v\|_{L_x^5 L_t^q} \lesssim 2^{-j(\frac{4}{r} - \frac{9}{5})} \|\Delta_j F\|_{L_x^{\frac{5}{4}} L_t^{r'}}, \quad (3.5)$$

with  $1/r + 1/r' = 1$ .

Combining the previous theorems with the results from [16], we finally state the set of estimates which will be used later for

$$i\partial_t u + \Delta_D u = F, \quad u_{|\partial\Omega} = 0, \quad v_{|t=0} = u_0. \quad (3.6)$$

**Theorem 3.3.** *Let  $2 < r \leq +\infty$ , then*

$$\|\Delta_j u\|_{C_t(L_x^2)} + 2^{\frac{j}{10}} \|\Delta_j u\|_{L_x^5 L_t^2} + 2^{-\frac{3}{4}j} \|\Delta_j u\|_{L_{t,x}^4} \lesssim \|\Delta_j u_0\|_{L_x^2} + 2^{-j(\frac{4}{r} - \frac{9}{5})} \|\Delta_j F\|_{L_x^{\frac{5}{4}} L_t^{r'}}, \quad (3.7)$$

with  $1/r + 1/r' = 1$ .

### 3.1 Proof of Theorem 3.1

Let  $\tilde{\psi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$  be such that  $\tilde{\psi} = 1$  on the support of  $\psi$ : hence, if  $\tilde{\Delta}_j$  denotes the corresponding localization operator,  $\tilde{\Delta}_j \Delta_j = \Delta_j$ . We now split the solution of the linear equation  $\Delta_j u_L(t, x) = \tilde{\Delta}_j \Delta_j u_L$  as a sum of two terms  $\tilde{\Delta}_j \chi \Delta_j u_L + \tilde{\Delta}_j (1 - \chi) \Delta_j u_L$ , where  $\chi \in C_0^\infty(\mathbb{R}^3)$  is compactly supported and it is equal to 1 near the boundary  $\partial\Omega$ .

#### 3.1.1 Far from the boundary: $\tilde{\Delta}_j (1 - \chi) \Delta_j u_L$

Set  $w_h(t, x) = (1 - \chi) \Delta_j e^{-it\Delta_D} u_0(x)$ . Then  $w_h$  satisfies

$$\begin{cases} i\partial_t w_h + \Delta_D w_h = -[\Delta_D, \chi] \Delta_j u_L, \\ w_h|_{t=0} = (1 - \chi) \Delta_j u_0. \end{cases} \quad (3.8)$$

Since  $\chi$  is equal to 1 near the boundary  $\partial\Omega$ , we can view the solution to (3.8) as the solution of a problem in the whole space  $\mathbb{R}^3$ . Consequently, the Duhamel formula writes

$$w_h(t, x) = e^{-it\Delta} (1 - \chi) \Delta_j u_0 - \int_0^t e^{-i(t-s)\Delta} [\Delta_D, \chi] \Delta_j u_L(s) ds, \quad (3.9)$$

where  $\Delta$  is the free Laplacian on  $\mathbb{R}^3$  and therefore the contribution of  $e^{-it\Delta} (1 - \chi) \Delta_j u_0$  satisfies the usual Strichartz estimates. We have thus reduced the problem to the study of the second term in the right hand-side of (3.9). Ideally, one would like to remove the time restriction  $s < t$  and use a variant of the Christ-Kiselev lemma. However, this would miss the endpoint case  $q = 2$ . Instead, we recall the following lemma:

**Lemma 3.1** (Staffilani-Tataru [19]). *Let  $x \in \mathbb{R}^n$ ,  $n \geq 3$  and let  $f(x, t)$  be compactly supported in space, such that  $f \in L_t^2(H^{-\frac{1}{2}})$ . Then the solution  $w$  to  $(i\partial_t + \Delta_x)w = f$  with  $w|_{t=0} = 0$ , is such that*

$$\|w\|_{L_t^2(L_x^{\frac{2n}{n-2}})} \lesssim \|f\|_{L_t^2(H^{-\frac{1}{2}})}. \quad (3.10)$$

In fact, one may shift regularity in (3.10) without difficulty. Now, the proof in [19] relies on a decomposition into traveling waves, to which homogeneous estimates are then applied. We can therefore use the  $L_x^4(L_t^2)$  smoothing estimate, Sobolev in space, and obtain, under the same hypothesis

$$\|w\|_{L_x^{\frac{5}{2}}(L_t^2)} \lesssim \|f\|_{L_t^2(H^{-\frac{1}{2}-\frac{1}{10}})}$$

We take  $f = -[\Delta_D, \chi] \Delta_j u_L \in L^2 H_{\text{comp}}^{-1/2-1/10}(\Omega)$  and

$$\|[\Delta_D, \chi] \Delta_j u_L\|_{L^2 H_{\text{comp}}^{-1/2-1/10}} \lesssim \|\Delta_j u_L\|_{L^2 \dot{H}^{1/2-1/10}(\Omega)} \lesssim \|\Delta_j u_0\|_{\dot{H}^{1/10}(\Omega)},$$

from which the smoothing estimates follow

$$\begin{aligned} \|(1 - \chi)\Delta_j u_L\|_{L^5(\mathbb{R}^3)L_t^2} &\lesssim \|(1 - \chi)\Delta_j u_0\|_{\dot{H}^{-\frac{1}{10}}(\mathbb{R}^3)} + \|[\Delta_D, \chi]\Delta_j u_L\|_{L^2 H_{\text{comp}}^{-1/2-1/10}} \\ &\lesssim \|\Delta_j u_0\|_{\dot{H}^{-\frac{1}{10}}(\Omega)}. \end{aligned} \quad (3.11)$$

We conclude using the continuity properties of  $\tilde{\Delta}_j$  which were recalled at the beginning of the section (see [14, Cor.2.5]). In fact, using (3.11), we get

$$\begin{aligned} \|\tilde{\Delta}_j(1 - \chi)\Delta_j u_L\|_{L_x^5 L_t^2} &\lesssim \|(1 - \chi)\Delta_j u_L\|_{L_x^5 L_t^2} \\ &\lesssim 2^{-\frac{j}{10}} \|\Delta_j u_0\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the spectral localization  $\Delta_j$  to estimate

$$\|\Delta_j u_0\|_{\dot{H}^\sigma(\Omega)} \simeq 2^{\sigma j} \|\Delta_j u_0\|_{L^2(\Omega)}.$$

### 3.1.2 Close to the boundary: $\tilde{\Delta}_j \chi \Delta_j u_L$

For  $l \in \mathbb{Z}$  let  $\varphi_l \in C_0^\infty((l - 1/2)\pi, (l + 1)\pi)$  equal to 1 on  $[l\pi, (l + 1/2)\pi]$ . We set  $v_j = \tilde{\Delta}_j \chi \Delta_j u_L$  and for  $l \in \mathbb{Z}$  we set  $v_{j,l} = \varphi_l(2^j t) v_j$ . We have

$$\begin{aligned} \|v_j\|_{L^5(\Omega)L^2(\mathbb{R})}^2 &= \left\| \sum_{l \in \mathbb{Z}} v_{j,l} \right\|_{L_x^5 L_t^2}^2 \simeq \left\| \left\| \sum_{l \in \mathbb{Z}} v_{j,l} \right\|_{L_t^2} \right\|_{L_x^{5/2}}^2 \\ &\lesssim \left\| \sum_{l \in \mathbb{Z}} \|v_{j,l}\|_{L_t^2}^2 \right\|_{L_x^{5/2}} \leq \sum_{l \in \mathbb{Z}} \|v_{j,l}\|_{L_x^5 L_t^2}^2, \end{aligned} \quad (3.12)$$

where for the first inequality we used the fact that the supports in time of  $\varphi_l$  are almost orthogonal. In order to estimate  $\|v_j\|_{L_x^5 L_t^2}^2$  it will be thus sufficient to estimate each  $\|v_{j,l}\|_{L_x^5 L_t^2}^2$ . The equation satisfied by  $\tilde{v}_{j,l} := \varphi_l(2^j t) \chi \Delta_j u_L$  is

$$i\partial_t \tilde{v}_{j,l} + \Delta_D \tilde{v}_{j,l} = -(\varphi_l(2^j t) [\Delta_D, \chi] \Delta_j u_L - i2^j \varphi_l'(2^j t) \chi \Delta_j u_L), \quad (3.13)$$

where we stress that  $\tilde{v}_{j,l}$  vanishes outside the time interval  $(2^{-j}(l - 1/2)\pi, 2^{-j}(l + 1)\pi)$ . We denote  $V_{j,l}$  the right hand side in (3.13) where we set

$$V_{j,l} := -\varphi_l(2^j t) [\Delta_D, \chi] \Delta_j u_L - i2^j \varphi_l'(2^j t) \chi \Delta_j u_L. \quad (3.14)$$

Let  $Q \subset \mathbb{R}^3$  be an open cube sufficiently large such that  $\partial\Omega$  is contained in the interior of  $Q$ . We denote by  $S$  the punctured torus obtained from removing the obstacle  $\Theta$  (recall that  $\Omega = \mathbb{R}^3 \setminus \Theta$ ) in the compact manifold obtained from  $Q$  with periodic boundary conditions on  $\partial Q$ . Notice that defined in this way  $S$  coincides with the Sinai billiard. Let also  $\Delta_S := \sum_{j=1}^3 \partial_j^2$  denote the Laplace operator on the compact domain  $S$ .



**Remark 3.1.** Notice that in a neighborhood of the boundary, the domains of  $\Delta_S$  and  $\Delta_D$  coincide, thus if  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$  is supported near  $\partial\Omega$  then

$$\Delta_S \tilde{\chi} = \Delta_D \tilde{\chi}.$$

Moreover, there exists  $\chi_1 \in C_0^\infty(\mathbb{R}^3)$  supported near the boundary and equal to 1 on the support of  $\tilde{\chi}$  such that

$$\chi_1 \tilde{\Delta}_j \tilde{\chi} = \chi_1 \tilde{\Delta}_j^S \tilde{\chi}. \quad (3.15)$$

On  $S$ , we may define a spectral localization operator using eigenvalues  $\lambda_k$  and eigenvectors  $e_k$  of  $\Delta_S$ : if  $f = \sum_k c_k e_k$ , then

$$\Delta_j^S f = \psi(2^{-2j} \Delta_S) f = \sum_k \psi(2^{-2j} \lambda_k^2) c_k e_k. \quad (3.16)$$

In what follows let  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$  be equal to 1 on the support of  $\chi$  and be supported in a neighborhood of  $\partial\Omega$  such that on its support the operator  $-\Delta_D$  coincide with  $-\Delta_S$ . From their respective definition,  $\tilde{v}_{j,l} = \tilde{\chi} v_{j,l}$ ,  $V_{j,l} = \tilde{\chi} V_{j,l}$ , consequently  $\tilde{v}_{j,l}$  will also solve the following equation on the compact domain  $S$

$$\begin{cases} i\partial_t \tilde{v}_{j,l} + \Delta_S \tilde{v}_{j,l} = V_{j,l}, \\ \tilde{v}_{j,l}|_{t < h(l-1/2)\pi} = 0, \quad \tilde{v}_{j,l}|_{t > h(l+1)\pi} = 0. \end{cases} \quad (3.17)$$

Writing the Duhamel formula for the last equation (3.17) on  $S$ , applying  $\tilde{\Delta}_j$  and using that  $v_{j,l} = \tilde{\Delta}_j \tilde{v}_{j,l}$ ,  $\tilde{\chi} \tilde{v}_{j,l} = \tilde{v}_{j,l}$  and  $\tilde{\Delta}_j \tilde{\chi} = \chi_1 \tilde{\Delta}_j^S \tilde{\chi} + (1 - \chi_1) \tilde{\Delta}_j \tilde{\chi}$  yields

$$\begin{aligned} v_{j,l}(t, x) &= \tilde{\Delta}_j \tilde{v}_{j,l} = \tilde{\Delta}_j \tilde{\chi} \tilde{v}_{j,l} \\ &= \chi_1 \tilde{\Delta}_j^S \tilde{\chi} \tilde{v}_{j,l} + (1 - \chi_1) \tilde{\Delta}_j \tilde{\chi} \tilde{v}_{j,l} = \chi_1 \tilde{\Delta}_j^S \tilde{v}_{j,l} + (1 - \chi_1) \tilde{\Delta}_j \tilde{v}_{j,l} \\ &= \chi_1 \int_{h(l-1/2)\pi}^t e^{-i(t-s)\Delta_S} \tilde{\Delta}_j^S V_{j,l}(s, x) ds \\ &\quad + (1 - \chi_1) \int_{h(l-1/2)\pi}^t \tilde{\Delta}_j e^{-i(t-s)\Delta_S} V_{j,l}(s, x) ds. \end{aligned} \quad (3.18)$$

Denote by  $v_{j,l,m}$  the first term in the second line of (3.18) and by  $v_{j,l,f}$  the second one. We deal with them separately. To estimate the  $L_x^5 L_t^2$  norm of the  $v_{j,l,f}$  we notice that it is supported far from the boundary therefor the estimates for the  $L_x^5 L_t^2$  norms will follow as shown in the previous section. Notice that since  $\tilde{v}_{j,l}$  solves also the equation (3.13) on  $\Omega$  hence we can replace  $\Delta_S$  by  $\Delta_D$  in the integral defining  $v_{j,l,f}$  and from Section 3.1.1 we deduce

$$\|(1 - \chi_1) \tilde{\Delta}_j e^{-i(t-s)\Delta_D} V_{j,l}\|_{L_x^5 L_t^2} \lesssim \|\tilde{\Delta}_j V_{j,l}\|_{\dot{H}^{-1/10}(\Omega)} \simeq 2^{-\frac{j}{10}} \|\tilde{\Delta}_j V_{j,l}\|_{L^2(\Omega)}. \quad (3.19)$$

We then apply the Minkowski inequality to deduce

$$\begin{aligned} & \|(1 - \chi_1) \int_{h(l-1/2)\pi}^t \tilde{\Delta}_j e^{-i(t-s)\Delta_D} V_{j,l}(s, x) ds\|_{L_x^5 L_t^2} \\ & \leq 2^{-j/2} \left( \int_{I_{j,l}} \|(1 - \chi_1) \tilde{\Delta}_j e^{-i(t-s)\Delta_D} V_{j,l}(s, \cdot)\|_{L^5(\Omega) L^2(I_{j,l})}^2 ds \right)^{1/2}, \end{aligned} \quad (3.20)$$

where we denoted  $I_{j,l} = [2^{-j}(l - 1/2)\pi, 2^{-j}(l + 1)\pi]$  and we used the Cauchy-Schwartz inequality. Using (3.19) we finally get

$$\|v_{j,l,f}\|_{L^5(\Omega) L^2(I_{j,l})} \leq 2^{-j(1/2+1/10)} \|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l}) L^2(\Omega)}. \quad (3.21)$$

To estimate the  $L_x^5 L_t^2$  norm of the main contribution  $v_{j,l,m}$  we need the following:

**Proposition 3.1.** *Let  $j \geq 0$ ,  $I_j = (-\pi 2^{-j}, \pi 2^{-j})$ ,  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$  be supported near  $\partial\Omega$  and  $V_0 \in L^2(\Omega)$ . Then there exists  $C > 0$  independent of  $j$  such that for the solution  $e^{-it\Delta_S} \tilde{\Delta}_j^S \tilde{\chi} V_0$  of the linear Schrödinger equation on  $S$  with initial data  $\tilde{\Delta}_j^S \tilde{\chi} V_0$  we have*

$$\|e^{-it\Delta_S} \tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^5(S) L^2(I_j)} \leq C 2^{-\frac{j}{10}} \|\tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^2(S)}. \quad (3.22)$$

We postpone the proof of Proposition 3.1.

Using the fact that  $v_{j,l}$  is supported in time in  $I_{j,l} = [2^{-j}(l - 1/2)\pi, 2^{-j}(l + 1)\pi]$ , the Minkowski inequality, Proposition 3.1 with  $\tilde{\chi} = 1$  on the support of  $\chi$  and with  $V_0 = V_{j,l}$ , and since  $\tilde{\chi}_1 v_{j,l,m} = v_{j,l,m}$  for any  $\tilde{\chi}_1 \in C^\infty(\mathbb{R}^3)$  with  $\tilde{\chi}_1 = 1$  on the support of  $\chi_1$ , we obtain

$$\begin{aligned} \|v_{j,l,m}\|_{L^5(\Omega) L^2(I_{j,l})} &= \|\tilde{\chi}_1 v_{j,l,m}\|_{L^5(\Omega) L^2(I_{j,l})} = \|v_{j,l,m}\|_{L^5(S) L^2(I_{j,l})} \\ &\leq \int_{2^{-j}(l-1)\pi}^{2^{-j}(l+1)\pi} \|e^{-i(t-s)\Delta_S} \tilde{\Delta}_j^S V_{j,l}(s, \cdot)\|_{L^5(S) L^2(I_{j,l})} ds \\ &\leq 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(S)} ds \\ &\leq 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\Delta}_j^S \tilde{\chi} V_{j,l}(s)\|_{L^2(S)} ds \\ &\leq 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\chi_1 \tilde{\Delta}_j V_{j,l}(s) + (1 - \chi_1) \tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(S)} ds \\ &\lesssim 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\Delta}_j V_{j,l}(s)\|_{L^2(\Omega)} ds \\ &\quad + 2^{-\frac{j}{10}} \int_{I_{j,l}} \|(1 - \chi_1) \tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(S)} ds \end{aligned} \quad (3.23)$$

where this time we wrote  $\tilde{\Delta}_j^S \tilde{\chi} = \chi_1 \tilde{\Delta}_j \tilde{\chi} + (1 - \chi_1) \tilde{\Delta}_j^S \tilde{\chi}$  and  $V_{j,l} = \tilde{\chi} V_{j,l}$ .

Using the Cauchy-Schwartz inequality in (3.23) yields

$$\begin{aligned} \|v_{j,l,m}\|_{L^5(\Omega)L^2(I_{j,l})} &\lesssim 2^{-j(1/2+1/10)} (\|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} \\ &\quad + \|(1-\chi_1)\tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(I_{j,l})L^2(S)}). \end{aligned} \quad (3.24)$$

We deal with the second term in (3.24). Using the precise form of  $V_{j,l}$  given in (3.14) together with the spectral localization  $\tilde{\Delta}_j^S$  we have

$$\begin{aligned} \|(1-\chi_1)\tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(I_{j,l})L^2(S)} &\lesssim (2^{j/2}\|\varphi_l(2^j t)[\Delta_D, \chi]\Delta_j u_L\|_{L^2(I_{j,l})H^{-1/2}(S)} \\ &\quad + 2^{j/2}\|\varphi_l'(2^j t)\chi\Delta_j u_L\|_{L^2(I_{j,l})H^{1/2}(S)}), \end{aligned} \quad (3.25)$$

and since the supports are localized near the boundary we can take the norms over  $\Omega$ . On the other hand, again thanks to the spectral cutoff  $\tilde{\Delta}_j$ , we obtain

$$\begin{aligned} \|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} &= \|\varphi_l(2^j t)\tilde{\Delta}_j[\Delta_D, \chi]\Delta_j u_L + i2^j \varphi_l'(2^j t)\tilde{\Delta}_j \chi \Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)} \\ &\lesssim 2^{j/2}\|\varphi_l(2^j t)\tilde{\Delta}_j[\Delta_D, \chi]\Delta_j u_L\|_{L^2(I_{j,l})H_0^{-1/2}(\Omega)} \\ &\quad + 2^{-j/2}\|2^j \varphi_l'(2^j t)\tilde{\Delta}_j \chi \Delta_j u_L\|_{L^2(I_{j,l})H_0^{1/2}(\Omega)}. \end{aligned} \quad (3.26)$$

Using the estimates (3.25) and (3.26) we obtain the same bounds for both terms in the right hand side of (3.24). Precisely, we have

$$\begin{aligned} \|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} + \|(1-\chi_1)\tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(I_{j,l})L^2(S)} \\ \lesssim 2^{j/2}\|\tilde{\varphi}_l(2^j t)\tilde{\Delta}_j \tilde{\chi} \Delta_j u_L\|_{L^2(I_{j,l})H_0^{1/2}(\Omega)}, \end{aligned} \quad (3.27)$$

where  $\tilde{\chi}$  and  $\tilde{\varphi}_l$  are equal to 1 on the supports of  $\chi$  and  $\varphi_l$ , respectively.

Let us recall the following local smoothing result on a non trapping domain:

**Lemma 3.2.** (*Burq, Gérard, Tzvetkov [2, Prop.2.7]*) *Assume that  $\Omega = \mathbb{R}^3 \setminus \Theta$ , where  $\Theta \neq \emptyset$  is a non-trapping obstacle. Then, for every  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ , one has*

$$\|\tilde{\chi} \Delta_j u_L\|_{L^2(\mathbb{R}, \dot{H}^{1/2}(\Omega))} \leq C \|\Delta_j u_0\|_{L^2(\Omega)}, \quad (3.28)$$

where, as usual,  $u_L(t, x) = e^{-it\Delta_D} u_0(x)$ .

Using (3.12), (3.21), (3.24), (3.27) and Lemma 3.2, we deduce

$$\begin{aligned} \|\tilde{\Delta}_j \chi \Delta_j u_L\|_{L^5(\Omega)L_t^2}^2 &\lesssim 2^{-2j(1/2+1/10)} \sum_{l \in \mathbb{Z}} \|\tilde{\Delta}_j V_{j,l}(s)\|_{L^2(I_{j,l})L^2(\Omega)}^2 \\ &\lesssim 2^{-\frac{2j}{10}} \sum_{l \in \mathbb{Z}} \|\tilde{\varphi}_l \tilde{\chi} \Delta_j u_L\|_{L^2(I_{j,l})\dot{H}^{1/2}(\Omega)}^2 \lesssim 2^{-\frac{2j}{10}} \|\tilde{\Delta}_j u_0\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.29)$$

### 3.1.3 End of the proof of Theorem 3.1

Until now we have prove Theorem 3.1 only for  $q = 2$ . We shall use the Gagliardo-Nirenberg inequality in order to deduce (3.3) for every  $q \geq 2$ . We have

$$\|\Delta_j u_L\|_{L_t^\infty} \lesssim \|\Delta_j u_L\|_{L_t^2}^{1/2} \|\Delta_j \partial_t u_L\|_{L_t^2}^{1/2}.$$

which gives, taking the  $L_x^5$  norms and using the Cauchy-Schwartz inequality

$$\|\Delta_j u_L\|_{L_x^5 L_t^\infty}^5 \lesssim \|\Delta_j u_L\|_{L_x^5 L_t^2}^{5/2} \|\Delta_j \partial_t u_L\|_{L_x^5 L_t^2}^{5/2}. \quad (3.30)$$

It remains to estimate  $\|\Delta_j \partial_t u_L\|_{L_x^5 L_t^\infty}$ : notice that since  $u_L = e^{-it\Delta_D} u_0$

$$\Delta_j \partial_t u_L = -i\Delta_D \Delta_j u_L = i2^{2j} \tilde{\Delta}_j u_L,$$

where  $\tilde{\Delta}_j$  is defined with  $\psi_1(x) = x\psi(x) \in C_0^\infty(\mathbb{R} \setminus \{0\})$ . Therefore

$$\|\Delta_j \partial_t u_L\|_{L_x^5 L_t^\infty} \leq C2^{j(2-1/10)} \|\tilde{\Delta}_j u_0\|_{L^2(\Omega)}, \quad (3.31)$$

consequently

$$\|\Delta_j \partial_t u_L\|_{L_x^5 L_t^q} \leq C2^{-j(2/q-9/10)} \|\Delta_j u_0\|_{L^2(\Omega)}$$

and Theorem 3.1 is proved.

## 3.2 Proof of Theorems 3.2 and 3.3

We recall a lemma due to Christ and Kiselev [8]. We state the corollary we will use, with only the time variable: we refer to [5] for a simple direct proof of all the different cases we use, with Banach-valued  $L_t^p(B)$  spaces or  $B(L_t^p)$ . Its use in the context of reversed norms  $L_x^q(L_t^p)$  goes back to [15] and it greatly simplifies obtaining inhomogeneous estimates from homogeneous ones.

**Lemma 3.3.** *(Christ and Kiselev [8]) Consider a bounded operator*

$$T : L^r(\mathbb{R}) \rightarrow L^q(\mathbb{R})$$

*given by a locally integrable kernel  $K(t, s)$ . Suppose that  $r < q$ . Then the restricted operator*

$$T_R f(t) = \int_{s < t} K(t, s) f(s) ds$$

*is bounded from  $L^r(\mathbb{R})$  to  $L^q(\mathbb{R})$  and*

$$\|T_R\|_{L^r(\mathbb{R}) \rightarrow L^q(\mathbb{R})} \leq C(1 - 2^{-(1/q-1/r)})^{-1} \|T\|_{L^r(\mathbb{R}) \rightarrow L^q(\mathbb{R})}.$$

From the lemma, the proof of the inhomogeneous set of estimates in Theorem 3.2 is routine from the homogeneous estimates in Theorem 3.1 and the Duhamel formula. Combining both homogeneous and inhomogeneous estimates yields Theorem 3.3.

### 3.3 Proof of Proposition 3.1

Let  $S$  denote the compact domain defined above. Recall  $(e_n)_n$  is the eigenbasis of  $L^2(S)$  consisting of eigenfunctions of  $-\Delta_S$  associated to the eigenvalues  $\lambda_n^2$ . Following [4], we define an abstract self adjoint operator on  $L^2(S)$  as follows

$$A_h(e_n) := -[h\lambda_n^2]e_n,$$

where  $[\lambda]$  is the integer part of  $\lambda$ . We first need to establish estimates for the linear Schrödinger equation on the compact domain  $S$  with spectrally localized initial data.

We now set  $h = 2^{-j}$  and state estimates on the evolution equation where  $h\Delta_D$  is replaced by  $A_h$ .

**Lemma 3.4.** *Let  $0 < h \leq 1$ ,  $q \geq 2$ ,  $I_h = (-\pi h, \pi h)$ ,  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$  be supported near  $\partial\Omega$  and  $V_0 \in L^2(\Omega)$ . There exists  $C > 0$  independent of  $h$  such that*

$$\|e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^5(S)L^q(I_h)} \leq Ch^{2/q-9/10}\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}. \quad (3.32)$$

We postpone the proof of Lemma 3.4 and proceed with the proof of Proposition 3.1. Denote by  $V_h(t, x) := e^{-it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0(x)$ , then

$$(ih\partial_t + A_h)V_h = (ih\partial_t + h\Delta_S)V_h + (A_h - h\Delta_S)V_h = (A_h - h\Delta_S)e^{-it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0.$$

Writing Duhamel formula for  $V_h$  yields

$$V_h(t, x) = e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S\tilde{\chi}V_0(x) - \frac{i}{h}\int_0^t e^{i\frac{t-s}{h}A_h}(A_h - h\Delta_S)e^{-is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0(x)ds. \quad (3.33)$$

Using (3.32) with  $q = 2$ , (3.33), the Minkowski inequality and boundedness of the operator

$$\|e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S\|_{L^2(S) \rightarrow L^5(S)L^2(I_h)} \lesssim 2^{-\frac{j}{10}} \sim h^{1/10}$$

(which follows from the proof of Lemma 3.4), we obtain

$$\begin{aligned} \|e^{-it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^5(S)L^2(I_h)} &\lesssim h^{\frac{1}{10}}\left(\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)} \right. \\ &\quad \left. + \frac{1}{h}\|(A_h + h\Delta_S)e^{-is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^1(-h\pi, h\pi)L^2(S)}\right), \end{aligned} \quad (3.34)$$

where to estimate the second term in the right hand side of (3.33) we used the fact that  $A_h$  commutes with the spectral localization  $\tilde{\Delta}_j^S$ . Changing variables  $s = h\tau$  in the second term in the right hand side of (3.34) yields

$$\begin{aligned} \frac{1}{h}\|(A_h + h\Delta_S)e^{-is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^1(-h\pi, h\pi)L^2(S)} &= \int_{-\pi}^{\pi} \|(A_h + h\Delta_S)e^{-i\tau h\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}d\tau \\ &\lesssim 2\pi\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}, \end{aligned} \quad (3.35)$$

where we used the fact that the operator  $(A_h + h\Delta_S)$  is bounded on  $L^2(S)$  and the mass conservation of the linear Schrödinger flow. It follows from (3.34) and (3.35) that

$$\|e^{-it\Delta_S} \tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^5(S)L^2(I_h)} \lesssim h^{1/10} \|\tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^2(S)},$$

which ends the proof of Proposition 3.1.

We now return to Lemma 3.4 for the rest of this section. Writing  $\tilde{\Delta}_j^S V_0 = \sum_n \tilde{\psi}(h^2 \lambda_n^2) V_{\lambda_n} e_n$ , we decompose (for  $0 < h \leq 1/4$ )

$$e^{i\frac{t}{h} A_h} \tilde{\Delta}_j^S V_0(t, x) = \sum_{k \in \mathbb{N}} e^{i\frac{t}{h} k} v_k(x)$$

with

$$v_k(x) = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1} \sum_{\lambda_n \in [\lambda, \lambda+1]} \tilde{\Psi}(h^2 \lambda_n^2) V_{\lambda_n} e_n = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1} \Pi_\lambda(\tilde{\Delta}_j^S V_0),$$

where  $\Pi_\lambda$  denotes the spectral projector  $\Pi_\lambda = 1_{\sqrt{-\Delta_S} \in [\lambda, \lambda+1]}$ . Let us estimate the  $L^5(S)L^q(I_h)$  norm of  $e^{i\frac{t}{h} A_h} \tilde{\Delta}_j^S V_0$ :

$$\begin{aligned} \|e^{i\frac{t}{h} A_h} \tilde{\Delta}_j^S V_0\|_{L^5(S)L^q(I_h)}^2 &\lesssim h^{2/q} \left\| \|e^{isA_h} \tilde{\Delta}_j^S V_0\|_{L_s^q(-\pi, \pi)}^2 \right\|_{L^{5/2}(S)} \\ &\lesssim h^{2/q} \left\| \|e^{isA_h} \tilde{\Delta}_j^S V_0\|_{H^{1/2-1/q}(s \in (-\pi, \pi))}^2 \right\|_{L^{5/2}(S)} \\ &\lesssim h^{2/q} \left\| \sum_{k \in \mathbb{N}} (1+k)^{2(\frac{1}{2}-\frac{1}{q})} \|e^{isk} v_k(x)\|_{L_s^2(-\pi, \pi)}^2 \right\|_{L^{5/2}(S)} \\ &\lesssim h^{2/q} \sum_{k \in \mathbb{N}} (1+k)^{1-2/q} \|e^{isk} v_k(x)\|_{L^5(S)L^2(-\pi, \pi)}^2 \\ &\lesssim h^{2/q} \sum_{k \in \mathbb{N}} (1+k)^{1-2/q} \|e^{isk} v_k(x)\|_{L^2(-\pi, \pi)L^5(S)}^2, \end{aligned}$$

where we used Sobolev injection in the time variable  $H^{1/2-1/q} \subset L^q$  and Plancherel in time. We recall a result of [18] of Smith and Sogge on the spectral projector  $\Pi_\lambda$ :

**Theorem 3.4.** (Smith and Sogge [18]) *Let  $S$  be a compact manifold of dimension 3, then*

$$\|\Pi_\lambda\|_{L^2(S) \rightarrow L^5(S)} \leq \lambda^{2/5}.$$

Using Theorem 3.4 we have

$$\begin{aligned} \|e^{i\frac{t}{h} A_h} \tilde{\Delta}_j^S V_0\|_{L^5(S)L^q(I_h)}^2 &\lesssim h^{2/q} \sum_{1/4h-1 \leq k \leq 4/h} (1+k)^{1-2/q+4/5} \|\tilde{\Delta}_j^S V_0\|_{L^2(S)}^2 \\ &\lesssim \sum_{hk \in [1/4, 4]} k^{1-4/q+4/5} \|\tilde{\Delta}_j^S V_0\|_{L^2(S)}^2 \\ &\lesssim \|\tilde{\Delta}_j^S V_0\|_{\dot{H}^{2/q-9/10}(S)}^2, \end{aligned}$$

since for  $hk > 4$  or  $h(k+1) < 1/4$  and  $\lambda_n \in [(k2^j)^{1/2}, ((k+1)2^j)^{1/2})$  we have  $\tilde{\Psi}(h^2\lambda_n^2) = 0$  and on the other hand for these values of  $k$  we have

$$k/\sqrt{2} \leq (k2^j)^{1/2} \leq \lambda_n \leq ((k+1)2^j)^{1/2} \leq \sqrt{2}(k+1), \quad h \leq 5(k+1)^{-1}.$$

This completes the proof of Lemma 3.4.

## 4 Local existence

In this section we prove Theorem 2.1.

**Definition 4.1.** Let  $u \in \mathcal{S}'(\mathbb{R} \times \Omega)$  and let  $\Delta_j = \psi(-2^{-2j}\Delta_D)$  be a spectral localization with respect to the Dirichlet Laplacian  $\Delta_D$  in the  $x$  variable, such that  $\sum_j \Delta_j = \text{Id}$  and let  $S_j = \sum_{k < j} \Delta_k$ . We introduce the "Banach valued" Besov space  $\dot{B}_p^{s,q}(L_t^r)$  as follows: we say that  $u \in \dot{B}_p^{s,q}(L_t^r)$  if

$$\left(2^{js} \|\Delta_j u\|_{L_x^p L_t^r}\right) \in l^q,$$

and  $\sum_j \Delta_j f$  converges to  $f$  in  $\mathcal{S}'$ . If  $L_t^r$  is replaced by  $L_T^r$ , the time integration is meant to be over  $(-T, T)$ . Moreover, when  $s < 0$ ,  $\Delta_j$  may be replaced by  $S_j$  in the norm and both norms are equivalent.

Applying Theorem 3.1 with  $q = 2, 5$  and taking  $s = 1$  in the definition above we obtain

$$u_L \in \dot{B}_5^{1+\frac{1}{10},2}(L_t^2) \cap \dot{B}_5^{\frac{1}{2},2}(L_t^5) \quad \text{and} \quad \partial_t u_L \in \dot{B}_5^{-\frac{3}{2},2}(L_t^5).$$

From this, by Gagliardo-Nirenberg in the time variable, one should have

$$u_L \in \dot{B}_5^{1,2}(L_t^{\frac{20}{9}}) \cap \dot{B}_5^{3/20,2}(L_t^{40}) \subset L_x^{20/3} L_t^{40},$$

and consequently

$$u_L^4 \in L_x^{5/3} L_t^{10} \quad \text{as well as} \quad |u_L|^4 u_L \in \dot{B}_{\frac{5}{4}}^{1,2}(L_t^{\frac{20}{11}})$$

which should be enough to iterate. However, our spaces are Banach valued Besov spaces (if one sees time as a parameter) and justifying Bernstein-like inequalities and Sobolev embedding is not entirely trivial (but doable, using the estimates from [14]). We choose an apparently complicated space in order to set up the fixed point, but the little gain in regularity from the smoothing estimate will turn out to be crucial for subcritical scattering.

**Remark 4.1.** By this choice, we only restrict the uniqueness class. It is likely that one may prove a better result, but there is no immediate benefit in the present setting, except proving additional estimates. We retained, however, the uniqueness class that would be provided by

the argument above in the Theorems' statements. Another remark is that one may dispense with the use of Lemma 3.1, miss the endpoint  $q = 2$  and still get the exact same nonlinear results, as there is room (due to the use of Sobolev embedding) in all mapping estimates. Moreover, as soon as we use an estimate with a (however small) gain in regularity, we do not need Lemma 4.5, as we could use a simpler embedding in a Besov space of negative regularity and play regularities against each other. In fact, in the same spirit as [15] one could replace the critical Sobolev norm by a Besov norm  $\dot{B}_2^{s_p, \infty}$ .

For  $T > 0$  let

$$X_T := \{u \mid u \in \dot{B}_5^{1+\frac{1}{10}, 2}(L_T^2) \cap \dot{B}_5^{\frac{1}{2}, 2}(L_T^5) \text{ and } \partial_t u \in \dot{B}_5^{-\frac{3}{2}, 2}(L_T^5)\}. \quad (4.1)$$

and for  $u \in X_T$  set  $F(u) := |u|^4 u$ .

**Proposition 4.1.** *Define a nonlinear map  $\phi$  as follows,*

$$\phi(u)(t) := \int_{s < t} e^{i(t-s)\Delta_D} F(u(s)) ds.$$

Then

$$\begin{aligned} \|\phi(u)\|_{C_T(\dot{H}_0^1)} + \|\phi(u)\|_{X_T} &\lesssim \|F(u)\|_{\dot{B}_{5/4}^{1,2}(L_T^{20/11})}, \\ \|\phi(u) - \phi(v)\|_{X_T} &\lesssim \|F(u) - F(v)\|_{\dot{B}_{5/4}^{1,2}(L_T^{20/11})}. \end{aligned}$$

The estimate for the inhomogeneous problem writes

$$\left\| \int e^{-is\Delta_D} F \right\|_{L_x^2} \leq C \|F\|_{\dot{B}_{5/4}^{0,2}(L_t^{20/11})},$$

Shifting the regularity to  $s = 1$  and using the Christ-Kiselev lemma provides both mapping estimates, as the difference estimate is identical to the first one (since we have an integer power). On the other hand,

$$\|F(u)\|_{\dot{B}_{5/4}^{1,2}(L_T^{20/11})} \lesssim \|u\|_{X_T}^5. \quad (4.2)$$

One may now set up the usual fixed point argument in  $X_T$  if  $T$  is sufficiently small.

We now consider local wellposedness for  $p < 5$ , e.g. Theorem 2.2. The critical Sobolev exponent w.r.t. scaling is  $s_p = 3/2 - 2/(p-1)$ . We aim at setting up a contraction argument in a small ball of

$$X_T := \{u \mid u \in \dot{B}_5^{s_p+\frac{1}{10}, 2}(L_T^2) \cap \dot{B}_4^{s_p-\frac{1}{4}, 2}(L_T^4) \text{ and } \partial_t u \in \dot{B}_4^{s_p-\frac{1}{4}-2, 2}(L_T^4)\}. \quad (4.3)$$

The important fact (if we were to ignore issues with Banach valued Besov spaces) would be that  $X_T \subset \dot{B}_5^{s_p, 2}(L_T^{20/9}) \cap L_x^{5(p-1)/3} L_T^{10(p-1)}$ .



**Remark 4.2.** *Some numerology is in order: if one were only to have the  $L_x^5 L_t^2$  smoothing estimate and use Sobolev (in time and in space), it would require  $5(p-1)/3 \geq 5$ , namely  $p \geq 4$ . However, we have the Strichartz estimate from [16], which allows  $5(p-1)/3 \geq 4$ , or  $p \geq 3 + 2/5$ .*

From the appendix, the nonlinear mapping verifies

$$\|F(u) - F(v)\|_{\dot{B}_{5/4}^{s_{p,2}}(L_T^{20/11})} \lesssim \|u - v\|_{X_T} (\|u\|_{X_T}^{p-1} + \|v\|_{X_T}^{p-1})$$

and existence and uniqueness follow by fixed point again.

We now deal with scattering in the same range of  $p$ : from [16], we have an a priori bound

$$\|S_j u\|_{L_t^4 L_x^4}^4 \lesssim \|u\|_{L_t^4 L_x^4}^4 \lesssim \|u_0\|_{L_x^2}^3 \sup_t \|u\|_{H_0^1} \leq M^{\frac{3}{2}} E^{\frac{1}{2}},$$

where  $M$  and  $E$  are the conserved charge and hamiltonian,

$$M = \int_{\Omega} |u|^2 dx \text{ and } E = \int_{\Omega} |\nabla u|^2 + \frac{2}{p+1} |u|^{p+1} dx.$$

Notice how this estimate is below the critical scaling  $s_p$ , as the RHS regularity is  $s = 1/4$ . From the energy a priori bound and Sobolev embedding, one has on the other hand

$$\|S_j u\|_{L_{t,x}^\infty} \lesssim 2^{\frac{j}{2}} \sup_t \|u\|_{H_0^1} \lesssim 2^{\frac{j}{2}} E^{\frac{1}{2}}.$$

Interpolating between the two bounds to get the right scaling yields,

$$\|S_j u\|_{L_{t,x}^q} \lesssim C(M, E) 2^{j(\frac{1}{2} - \frac{5-p}{3(p-1)})}, \quad (4.4)$$

where  $1/q = (5-p)/6(p-1)$ . In order to proceed with the usual scattering argument, we need to revisit the fixed point, or more precisely the nonlinear estimate on  $F(u)$ : indeed, if we wish to use (4.4), even at a power  $\varepsilon$ , we cannot afford to use the same regularity on both sides of the Duhamel formula. Fortunately, we have off diagonal inhomogeneous estimates, e.g.

$$\left\| \int e^{i(t-s)\Delta^D} F \right\|_{\dot{B}_5^{s_{p,2}}(L_t^{20/9}) \cap \dot{B}_4^{s_{p-3/4,2}}(L_t^4)} \leq C \|F(u)\|_{\dot{B}_{5/4}^{s_{p-1/10},2}(L_t^2)}.$$

In order to evaluate  $F(u)$ , one needs to place the  $S_j u$  factors in such a way that

$$\|(S_j u)^{p-1}\|_{L_x^{5/3} L_t^{20}} \lesssim 2^{\frac{j}{10}}.$$

However, we have from (4.4)

$$\|(\Delta_j u)^{p-1}\|_{L_{t,x}^{\frac{6}{5-p}}} \lesssim C(M, E) 2^{j(\frac{5p-13}{6})}, \quad (4.5)$$

and  $6/(5-p) > 5/3$ . As such, one may interpolate with

$$\|\Delta_j u\|_{L_x^4 L_t^4} \lesssim 2^{-j(s_p - \frac{1}{4})},$$

to get (after Sobolev embedding)

$$\|(\Delta_j u)^{p-1}\|_{L_x^{\frac{5}{3}} L_t^{20}} \lesssim 2^{\frac{j}{10}}.$$

Summing over low frequencies recovers the desired bound. Notice that scaling dictates the exponents (hence there is no need to compute explicitly the interpolation  $\theta$ ).

## Appendix

In order to perform the various product estimates, we need a couple of useful lemma. Observe that with the spectral localization one cannot take advantage of convolution of Fourier supports. As a first step and in order to avoid cumbersome notations, we only consider functions and Besov spaces which do not depend on time. We will then explain how to re-instate the time dependance in the nonlinear estimates.

It is worth noting at this stage, however, that both  $\Delta_j$  and  $S_j$  operators are well-defined on  $L_t^p L_x^q$  and  $L_x^q L_t^p$  for all the pairs  $(p, q)$  to be considered: this follows from [14] for the case  $L_t^p L_x^q$  where the time norm is harmless. In the case  $L_x^q L_t^2$ , the arguments from [14] apply as well (heat estimates are proved for data in  $L_x^p(H)$  where  $H$  is an abstract Hilbert space, and when  $H = L_t^2$ , the heat kernel is diagonal and therefore Gaussian as well). By interpolation and duality we recover all pairs  $(p, q)$ .

**Remark 4.3.** *In  $\mathbb{R}^n$ , one may perform product estimates in an easier way because of the convolution of Fourier supports. However, when dealing with non integer power-like nonlinearities, one cannot proceed so easily: the usual route is to use a characterization of Besov spaces via finite differences; here, because of the Banach valued Besov spaces, we perform a direct argument which is directly inspired by computations in [15], where the same sort of time-valued Besov spaces were unavoidable.*

**Lemma 4.1.** *Let  $f_j$  be such that  $S_j f_j = f_j$ , and  $\|f_j\|_{L^p} \lesssim 2^{-js} \eta_j$ , with  $s > 0$  and  $(\eta_j)_j \in l^q$ . Then  $g = \sum_j f_j \in \dot{B}_p^{s,q}$ .*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k < j} S_j f_j.$$

Now,

$$\|\Delta_k(\sum_{k < j} S_j f_j)\|_p \lesssim 2^{-ks} \sum_{k < j} 2^{-s(j-k)} \eta_j,$$

which by an  $l^1 - l^q$  convolution provides the result.

**Lemma 4.2.** *Let  $f_j$  be such that  $(I - S_j)f_j = f_j$ , and  $\|f_j\|_{L^p} \lesssim 2^{-js} \eta_j$ , with  $s < 0$  and  $(\eta_j)_j \in l^q$ . Then  $g = \sum_j f_j \in \dot{B}_p^{s,q}$ .*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k > j} (I - S_j) f_j.$$

Now,

$$\|\Delta_k(\sum_{k > j} (I - S_j) f_j)\|_p \lesssim 2^{-ks} \sum_{k > j} 2^{-s(j-k)} \eta_j,$$

which by an  $l^1 - l^q$  convolution provides the result.

**Lemma 4.3.** *Consider  $\alpha = 1$  or  $\alpha \geq 2$ ,  $f \in \dot{B}_p^{s,q}$  and  $g \in L^r$ , with  $0 < s < 1$ ,  $\frac{1}{m} = \frac{\alpha}{r} + \frac{1}{p}$ : let*

$$T_g^\alpha f = \sum_j (S_j g)^\alpha \Delta_j f.$$

Then

$$T_g^\alpha f \in \dot{B}_m^{s,q}.$$

We split the paraproduct  $T_g^\alpha f$ :

$$T_g^\alpha f = \sum_j S_j((S_j g)^\alpha \Delta_j f) + \sum_j (I - S_j)((S_j g)^\alpha \Delta_j f);$$

the first part is easily dealt with by Lemma 4.1. For the second one,  $K_g f$ , taking once again advantage of the spectral supports

$$\Delta_k K_g f = \Delta_k \sum_{j < k} (I - S_j)((S_j g)^\alpha \Delta_j f).$$

Notice the situation is close to the one in Lemma 4.2, but we don't have a negative regularity for summing. We therefore derive

$$\begin{aligned} \Delta_D K_g f &= \sum_{j < k} (I - S_j) \Delta_D((S_j g)^\alpha \Delta_j f) \\ &= \sum_{j < k} (I - S_j) (\Delta_D (S_j g)^\alpha \Delta_j f + (\Delta_D \Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1} \nabla S_j g \cdot \nabla \Delta_j f) \\ &= \sum_{j < k} (I - S_j) (\alpha \Delta_D S_j g (S_j g)^{\alpha-1} \Delta_j f + \alpha(\alpha-1) |\nabla S_j g|^2 (S_j g)^{\alpha-2} \Delta_j f \\ &\quad + (\Delta_D \Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1} \nabla S_j g \cdot \nabla \Delta_j f). \end{aligned}$$

The first two pieces are again easily dealt with with Lemma 4.2, and the resulting function is in  $\dot{B}_p^{s-2,q}$ . The remaining cross term is handled with some help from [14]:

$$\nabla \Delta_j f = \nabla \exp(4^{-j} \Delta_D) \tilde{\Delta}_j f,$$

where the new dyadic block  $\tilde{\Delta}_j$  is built on the function  $\tilde{\psi}(\xi) = \exp(|\xi|^2) \psi(\xi)$ . From the continuity properties of  $\sqrt{s} \nabla \exp(s \Delta_D)$  on  $L^p$ ,  $1 < p < +\infty$ , we immediatly deduce

$$\|\nabla \Delta_j f\|_p \lesssim 2^j \|\tilde{\Delta}_j f\|_p, \quad (4.6)$$

and taking advantage of  $s < 1$ , we can easily sum and conclude. This will be enough to deal with the critical case, but for differences of nonlinear power-like mappings, we need

**Lemma 4.4.** *Consider  $\alpha \geq 3$ ,  $f, g \in X = \dot{B}_p^{s,q} \cap L^r$ , with  $0 < s < 1$ ,  $\frac{1}{m} = \frac{\alpha-1}{r} + \frac{1}{p}$ . Then, if  $F(x) = |x|^{\alpha-1}x$  or  $F(x) = |x|^\alpha$ ,*

$$\|F(u) - F(v)\|_{\dot{B}_m^{s,q}} \lesssim \|u - v\|_X (\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1}).$$

In order to obtain a factor  $u - v$ , we write

$$F(u) - F(v) = (u - v) \int_0^1 F'(\theta u + (1 - \theta)v) d\theta. \quad (4.7)$$

In order to efficiently split this difference into two paraproducts involving  $u - v$  and  $F'(w)$  with  $w = \theta u + (1 - \theta)v$ , we need an estimate on  $F'(w)$ : write another telescopic series

$$\begin{aligned} F'(w) &= \sum_j F'(S_{j+1}w) - F'(S_j w) \\ &= \sum_j S_j (F'(S_{j+1}w) - F'(S_j w)) + \sum_j (I - S_j) (F'(S_{j+1}w) - F'(S_j w)) \\ &= S_1 + S_2. \end{aligned}$$

Exactly as before, the first sum  $S_1$  is easily disposed of with Lemma 4.1, as

$$|F'(S_{j+1}w) - F'(S_j w)| \lesssim |\Delta_j w| (|S_{j+1}w|^{\alpha-2} + |S_j w|^{\alpha-2}).$$

The second sum  $S_2$  requires the same trick as before; to avoid unnecessary cluttering, we set  $F(x) = x^\alpha$ , ignoring the sign issue (recall that  $\alpha \geq 3$ , hence  $F'''(x)$  is well-defined as a function): we apply  $\Delta_D$ , let  $\beta = \alpha - 1 \geq 2$

$$\begin{aligned} \Delta_D S_2 &= \sum_j (I - S_j) \Delta_D ((S_{j+1}w)^{\alpha-1} - (S_j w)^{\alpha-1}) \\ &= \sum_j (I - S_j) (\beta (S_{j+1}w)^{\beta-1} \Delta_D S_{j+1}w - \beta (S_j w)^{\beta-1} \Delta_D S_j w \\ &\quad + \beta(\beta - 1) (S_{j+1}w)^{\beta-2} (\nabla S_{j+1}w)^2 - \beta(\beta - 1) (S_j w)^{\beta-2} (\nabla S_j w)^2). \end{aligned}$$

We now apply Lemma 4.2 after inserting the right factors: we have four types of differences,

$$\begin{aligned} |((S_{j+1}w)^{\beta-1} - (S_jw)^{\beta-1})\Delta_D S_{j+1}w| &\lesssim C_\beta |\Delta_j w| |\Delta_D S_{j+1}| (|S_{j+1}w|^{\beta-2} + |S_jw|^{\beta-2}) \\ |(S_{j+1}w)^{\beta-1} \Delta_D \Delta_j w| &\leq |\Delta_D \Delta_j w| |S_{j+1}w|^{\beta-2} \\ |((S_{j+1}w)^{\beta-2} - (S_jw)^{\beta-2})(\nabla S_{j+1}w)^2| &\lesssim \tilde{C}_\beta |\Delta_j w|^{\beta-2} |\nabla S_{j+1}w|^2 \\ |(S_{j+1}w)^{\beta-2} ((\nabla S_j w)^2 - (\nabla S_{j+1}w)^2)| &\leq |\nabla \Delta_j w| (|\nabla S_j w| + |\nabla S_{j+1}w|) |S_{j+1}w|^{\beta-2} \end{aligned}$$

where on the third line we wrote the worst case, namely  $2 \leq \beta < 3$  (otherwise the power of  $\Delta_j w$  in the third bound will be replaced by  $|\Delta_j w|(|S_j w|^{\beta-3} + |S_{j+1}w|^{\beta-3})$ ).

By integrating, applying Hölder and using (4.6) to eliminate the  $\nabla$  operator, we obtain as an intermediary result

$$F'(w) \in \dot{B}_\lambda^{s,q}, \quad \text{with } \frac{1}{\lambda} = \frac{\alpha - 2}{r} + \frac{1}{p}.$$

We may now go back to the difference  $F(u) - F(v)$  as expressed in (4.7) and perform a simple paraproduct decomposition in two terms to which Lemma 4.3 may be applied. Observe that there is no difficulty in estimating  $F'(w)$  in  $L^{m/(\alpha-1)}$ , and that the integration in  $\theta$  is irrelevant. This completes the proof.

We now go back to the first nonlinear estimate, namely (4.2). We write a telescopic series for the product five factors  $u_1, u_2, u_3, u_4, u_5 \in X_T$ ,

$$u_1 u_2 u_3 u_4 u_5 = \sum_j S_{j+1} u_1 S_{j+1} u_2 S_{j+1} u_3 S_{j+1} u_4 S_{j+1} u_5 - S_j u_1 S_j u_2 S_j u_3 S_j u_4 S_j u_5$$

and we are reduced to studying five sums of the same type, of which the following is generic

$$S_1 = \sum_j \Delta_j u_1 S_j u_2 S_j u_3 S_j u_4 S_j u_5,$$

and we intend to apply Lemma 4.3, which is trivially extended to a product of several factors. In principle,

$$u_k \in \dot{B}_5^{1,2}(L_T^{\frac{20}{11}}) \cap L_x^{\frac{20}{3}} L_T^{40}$$

is enough, using the first space of the  $\Delta_j$  factor and the second one for all remaining  $S_j$  factors, except for the use of (4.6) in the proof. Consider, from  $u \in X_T$ ,

$$2^{\frac{11}{10}j} \|\Delta_j u\|_{L_x^5 L_T^2} + 2^{-\frac{3}{2}j} \|\partial_t \Delta_j u\|_{L_T^5 L_x^5} = \mu_j^0 \in l_j^2.$$

We will have, using [14],

$$2^{\frac{11}{10}j} \|\nabla \Delta_j u\|_{L_x^5 L_T^2} + 2^{-\frac{3}{2}j} \|\partial_t \nabla \Delta_j u\|_{L_T^5 L_x^5} = \mu_j^1 \in l_j^2, \quad \text{with } \|\mu^1\|_{l^2} \lesssim \|\mu^0\|_{l^2}.$$

By Gagliardo-Nirenberg in time, we have the correct estimate for  $\Delta_j u$ , for  $k = 0, 1$

$$2^{(1-k)j} \|\nabla^k \Delta_j u\|_{L_x^5 L_T^{\frac{20}{1}}} \lesssim \mu_j^k.$$

We proceed with the low frequencies by proving a suitable Sobolev embedding.

**Lemma 4.5.** *Let  $u \in \dot{B}_5^{\frac{1}{2},5}(L_T^5)$  and  $\partial_t u \in \dot{B}_5^{-\frac{3}{2},5}(L_T^5)$ . Then  $u \in L_x^{\frac{20}{3}} L_T^{40}$ .*

Let

$$2^{(\frac{1}{2}-k)j} \|\nabla^k \Delta_j u\|_{L_x^5 L_T^5} + 2^{-(k+\frac{3}{2})j} \|\partial_t \nabla^k \Delta_j u\|_{L_T^5 L_x^5} = \mu_j^k \in l_j^5,$$

notice we can easily switch time and space Lebesgue norms. Using Gagliardo-Nirenberg in time, we have

$$2^{(\frac{1}{6}-k)j} \|\nabla^k \Delta_j u\|_{L_x^5 L_T^{30}} \lesssim \mu_j^3 \in l_j^5. \quad (4.8)$$

Using now Gagliardo-Nirenberg in space, we also have

$$2^{-\frac{j}{10}} \|\Delta_j u\|_{L_x^\infty L_T^5} \lesssim 2^{-\frac{j}{10}} \|\Delta_j u\|_{L_T^5 L_x^\infty} \lesssim \mu_j^5$$

and the same thing for  $2^{-2j} \partial_t \Delta_j u$  (or with an additional  $2^j \nabla$ ). Now another Gagliardo-Nirenberg in time provides

$$2^{-(k+\frac{1}{2})j} \|\nabla^k \Delta_j u\|_{L_{T,x}^\infty} \lesssim \mu_j^6. \quad (4.9)$$

Now, we essentially take advantage of a discrete embedding between  $l^1$  and weighted  $l^\infty$  sequences:

$$\begin{aligned} |u| &\leq \sum_{j < J} |\Delta_j u| + \sum_{j \geq J} |\Delta_j u| \\ &\leq \sum_{j < J} 2^{\frac{j}{2}} \sup_j 2^{-\frac{j}{2}} |\Delta_j u| + \sum_{j \geq J} 2^{-\frac{j}{6}} \sup_j 2^{\frac{j}{6}} |\Delta_j u| \\ &\lesssim 2^{\frac{j}{2}} \sup_j 2^{-\frac{j}{2}} |\Delta_j u| + 2^{-\frac{j}{6}} \sup_j 2^{\frac{j}{6}} |\Delta_j u| \\ |u|^4 &\lesssim \sup_j 2^{-\frac{j}{2}} |\Delta_j u| \left( \sup_j 2^{\frac{j}{6}} |\Delta_j u| \right)^3 \\ \| |u|^4 \|_{L_x^{\frac{5}{3}} L_T^{10}} &\lesssim \| \sup_j 2^{-\frac{j}{2}} |\Delta_j u| \|_{L_{T,x}^\infty} \| \sup_j 2^{\frac{j}{6}} |\Delta_j u| \|_{L_x^5 L_T^{30}}^3 \\ \|u\|_{L_x^{\frac{20}{3}} L_T^{40}} &\lesssim \|u\|_{\dot{B}_\infty^{\frac{1}{2},\infty}(L_t^\infty)}^{\frac{1}{4}} \|u\|_{\dot{B}_5^{\frac{1}{2},5}(L_t^{30})}^{\frac{3}{4}} \end{aligned}$$

Notice that the estimate with a gradient is much easier: just interpolate between (4.8) and (4.9) with  $k = 1$  to obtain

$$2^{-j} \|\nabla \Delta_j u\|_{L_x^{\frac{20}{3}} L_T^{40}} \lesssim \mu_j^7,$$

which we can now sum over  $k < j$  to obtain control of  $S_j u$ .

The case  $p < 5$  is handled in a similar way, and we leave the details to the reader, sparing him the complete set of exponents (depending on  $p$ !) that would appear in the proof. For scaling reasons there is actually no need to perform the computation: the previous one on the critical case simply illustrates that we can sidestep issues related to the usual Littlewood-Paley theory by using direct arguments.

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