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A VARIATIONAL APPROACH FOR ALMOST PERIODIC SOLUTIONS IN RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

MOEZ AYACHI AND JÖEL BLOT

(communicated by R. L. Pouso)

Abstract. To study the a.p. (almost periodic) solutions of retarded functional differential equations in the form $u''(t) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta + e(t)$, we introduce variational formalisms to characterize the a.p. solutions as a critical points of functionals defined on Banach spaces of a.p. functions. We obtain an existence result of weak a.p. solutions and a result of density of the a.p. forcing termes $e(\cdot)$ for which the equation possesses usual a.p. solutions.

1. Introduction

From a function $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$, where \mathbb{E} is a finite-dimensional real Euclidean space, and from $r \in (0, \infty)$ we consider the following (second order) retarded functional differential equation

$$u''(t) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta + e(t) \quad (1.1)$$

where D_j , $j = 1, 2$, denotes the partial gradient and where $e : \mathbb{R} \rightarrow \mathbb{E}$ is a forcing term.

We study the a.p. (almost periodic) solutions of (1.1) where e is an a.p. function.

A strong a.p. solution of (1.1) is a function $u : \mathbb{R} \rightarrow \mathbb{E}$ which is twice differentiable (in ordinary sense) with u, u' and u'' which are a.p. in the sense of Bohr [3, 6, 14]; the equality in (1.1) being satisfied for all $t \in \mathbb{R}$.

A weak a.p. solution of (1.1) is a function $u : \mathbb{R} \rightarrow \mathbb{E}$ which is a.p. in the sense of Besicovitch [5, 18], which possesses a first-order and a second-order generalized derivative; the equality in (1.1) means that the difference between the two members has a quadratic mean value equal to zero.

For the ordinary differential equations, this kind of weak a.p. solutions was considered in [8]. For neutral delay differential equations, this kind of weak a.p. solutions is considered in [4].

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Our approach uses a variational method. The a.p. solutions (strong or weak) of (1.1) are characterized as critical points of functionals in the form

$$u \mapsto \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\frac{1}{2} |u'(t)|^2 + \int_{-t}^0 f(u(t), u(t+\theta)) d\theta + u(t) \cdot e(t) \right) dt$$

on Banach spaces of a.p. functions. And so (1.1) appears as an Euler-Lagrange equation.

Now we briefly describe the contents of the paper. After Section 2 devoted to precise our notations, in Section 3 we build a variational formalism to characterize the strong (also called usual) a.p. solutions of (1.1) (Theorem (3.3)), for which we can deduce a result on the structure of the set of strong a.p. solutions of (1.1) (Theorem (3.4)). In Section 4 we build a variational formalism to characterize the weak a.p. solutions of (1.1) (Theorem (4.5)), and to establish an existence result of weak a.p. solutions (Theorem (4.6)); we obtain also a result of the structure of the set of the weak a.p. solutions of (1.1).

In Section 5 we establish a result on the density of the a.p. forcing term for which (1.1) possesses a strong a.p. solutions (Theorem (5.3)); this result uses the weak a.p. solutions.

2. Notations

When \mathbb{X} is a Banach space, $AP^0(\mathbb{X})$ denotes the space of the Bohr-a.p. functions from \mathbb{R} in \mathbb{X} [3, 6, 14]. It is a Banach space for the norm $\|u\|_\infty := \sup \{|u(t)| : t \in \mathbb{R}\}$. When $u \in AP^0(\mathbb{X})$, its mean value exists in \mathbb{X} :

$$\mathfrak{M}\{u\} = \mathfrak{M}_t\{u(t)\} := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt,$$

[3, 6, 14]. When $k \in \mathbb{N}$, $k \geq 1$, $AP^k(\mathbb{X})$ denotes the space of the $u \in \mathcal{C}^k(\mathbb{R}, \mathbb{X}) \cap AP^0(\mathbb{X})$ such that $u^j = \frac{d^j u}{dt^j} \in AP^0(\mathbb{X})$ for all $j = 1, \dots, k$. It is a Banach space for the norm $\|u\|_{\mathcal{C}^k} := \|u\|_\infty + \sum_{1 \leq j \leq k} \|u^j\|_\infty$.

$B^1(\mathbb{X})$ denotes the completion of $AP^0(\mathbb{X})$ with respect to the norm $\|u\|_{B^1} := \mathfrak{M}\{|u|\}$. It is a quotient space to transform the semi-norm $u \mapsto \mathfrak{M}\{|u|\}$ into a norm. When \mathbb{X} is a Hilbert space, $B^2(\mathbb{X})$ denotes the completion of $AP^0(\mathbb{X})$ with respect to the norm $\|u\|_{B^2} := \mathfrak{M}\{|u|^2\}^{\frac{1}{2}}$. It is also a quotient space and it is a Hilbert space for the inner product $(u|v)_{B^2} := \mathfrak{M}\{(u|v)_\mathbb{X}\}$.

The generalized derivative of $u \in B^2(\mathbb{X})$ (when it exists) is $\nabla u \in B^2(\mathbb{X})$ such that $\mathfrak{M}_t\left\{\left|\nabla u(t) - \frac{1}{\tau}(u(t+\tau) - u(t))\right|^2\right\} \rightarrow 0$ ($\tau \rightarrow 0$) [8, 12]. We consider $B^{1,2}(\mathbb{X}) := \{u \in B^2(\mathbb{X}) : \nabla u \in B^2(\mathbb{X})\}$ and $B^{2,2}(\mathbb{X}) := \{u \in B^{1,2}(\mathbb{X}) : \nabla^2 u := \nabla(\nabla u) \in B^2(\mathbb{X})\}$. They are Hilbert spaces for the respective norms

$$\|u\|_{B^{1,2}} := \left(\|u\|_{B^2}^2 + \|\nabla u\|_{B^2}^2 \right)^{\frac{1}{2}}, \quad \|u\|_{B^{2,2}} := \left(\|u\|_{B^{1,2}}^2 + \|\nabla^2 u\|_{B^2}^2 \right)^{\frac{1}{2}}.$$

When $u : \mathbb{R} \rightarrow \mathbb{E}$ is a continuous function, it is usual, in the theory of retarded functional differential equations, to consider, for all $t \in \mathbb{R}$, $u_t \in \mathcal{C}^0([-r, 0], \mathbb{E})$ defined by $u_t(\theta) := u(t + \theta)$ for all $\theta \in [-r, 0]$, [15].

When $u \in L^2_{loc}(\mathbb{R}, \mathbb{E})$ (Lebesgue space), we denote by $\tilde{u} : \mathbb{R} \rightarrow L^2_{loc}([-r, 0], \mathbb{E})$ the function defined by $\tilde{u}(t)(\theta) := u(t + \theta)$.

3. The Strong a.p. Solutions

We consider the following condition on f :

$$f \in \mathcal{C}^1(\mathbb{E} \times \mathbb{E}, \mathbb{R}). \tag{3.1}$$

LEMMA 3.1. *Under (3.1) we consider the mapping $F_0 : \mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathbb{R}$ defined by $F_0(x, \psi) := \int_{-r}^0 f(x, \psi(\theta))d\theta$. Then F_0 is of class \mathcal{C}^1 on $\mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E})$ and $DF_0(x, \psi)(y, \xi) = \int_{-r}^0 D_1f(x, \psi).y d\theta + \int_{-r}^0 D_2f(x, \psi(\theta)).\xi(\theta)d\theta$.*

Proof. The following Nemytskii operator build on f :

$$\mathcal{N}_f^0 : \mathcal{C}^0([-r, 0], \mathbb{E}) \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathcal{C}^0([-r, 0], \mathbb{E}),$$

$\mathcal{N}_f^0(\phi, \psi) := [\theta \mapsto f(\phi(\theta), \psi(\theta))]$, is of class \mathcal{C}^1 under (3.1), (see proposition 1 page 168, and proposition 2 page 170 in [1]).

The operator $A^0 : \mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathcal{C}^0([-r, 0], \mathbb{E}) \times \mathcal{C}^0([-r, 0], \mathbb{E})$ defined by $A^0(x, \psi) = (x, \psi)$ where the vector $x \in \mathbb{E}$ is considered as a (constant) continuous function, is a linear continuous and therefore A^0 is of class \mathcal{C}^1 . The operator $I^0 : \mathcal{C}^0([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$, $I^0(w) := \int_{-r}^0 w(t)dt$, is linear continuous and therefore it is of class \mathcal{C}^1 .

Since $F_0 := I^0 \circ \mathcal{N}_f^0 \circ A^0$, F_0 is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -mappings.

By using the chaine rule, we have

$$DF_0(x, \psi).(y, \xi) = I_0(D\mathcal{N}_f^0(A^0(x, \psi)).A^0(y, \xi))$$

We know that $D\mathcal{N}_f^0(A^0(x, \psi)).A^0(y, \xi) = [\theta \mapsto D_1f(x, \psi(\theta)).y + D_2f(x, \psi(\theta)).\xi(\theta)]$, and so we obtain the announced formula.

LEMMA 3.2. *The operator $S^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathbb{R})$, defined by*

$$S^0(u) := \left[t \mapsto \int_{-r}^0 f(u(t), u(t + \theta))d\theta \right],$$

is of class \mathcal{C}^1 , and

$$DS^0(u)h = \left[t \mapsto \int_{-r}^0 D_1f(u(t), u(t + \theta)).h(t)d\theta + \int_{-r}^0 D_2f(u(t), u(t + \theta)).h(t + \theta)d\theta \right].$$

Proof. The Nemytskii operator defined on the mapping F_0 provided by Lemma (3.1),

$$\mathcal{N}_{F_0} : AP^0(\mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E})) \equiv AP^0(\mathbb{E}) \times AP^0(\mathcal{C}^0([-r, 0], \mathbb{E})) \rightarrow AP^0(\mathbb{R}),$$

defined by

$$\mathcal{N}_{F_0}(u, \phi) := \left[t \mapsto F_0(u(t), \phi(t)) = \int_{-r}^0 f(u(t), \phi(t)(\theta)) d\theta \right]$$

is of class \mathcal{C}^1 , since F_0 is of class \mathcal{C}^1 . ([9], Corollary 5.3).

We introduce the operator $T^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathcal{C}^0([-r, 0], \mathbb{E}))$ by setting $T^0(u) := [t \mapsto u_t]$. Then T^0 is linear, T^0 is continuous since $\|T^0(u)\|_\infty = \|u\|_\infty$, and therefore T^0 is of class \mathcal{C}^1 .

Since $S^0 = \mathcal{N}_{F_0} \circ (id, T^0)$, S^0 is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -operators.

By using the chain rule we have $DS^0(u).h = D\mathcal{N}_{F_0}((id, T^0)(u)).D(id, T^0)(h) = D\mathcal{N}_{F_0}(u, \tilde{u}).(h, \tilde{h})$, and by using Lemma (3.1) we obtain

$$\begin{aligned} (DS^0(u).h)(t) &= \int_{-r}^0 D_1 f(u(t), \tilde{u}(t)(\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), \tilde{u}(t)(\theta)).\tilde{h}(t)(\theta) d\theta \\ &= \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \end{aligned}$$

THEOREM 3.3. *Under (3.1) the functional $J_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$, defined by*

$$J_0(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |u'(t)|^2 + \int_{-r}^0 f(u(t), u(t+\theta)) d\theta + u(t).e(t) \right\},$$

is of class \mathcal{C}^1 , and when $u \in AP^1(\mathbb{E})$ we have $DJ_0(u) = 0$ if and only if u is a strong solution of (1.1)

Proof. We consider the functional $Q_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$ defined by

$$Q_0(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |u'(t)|^2 \right\}.$$

The mapping $q : \mathbb{E} \rightarrow \mathbb{R}$, $q(x) := \frac{1}{2} |x|^2 = \frac{1}{2} x.x$, is of class \mathcal{C}^1 , therefore the Nemytskii operator $\mathcal{N}_q^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathbb{R})$, $\mathcal{N}_q^0(\varphi) := [t \mapsto \frac{1}{2} |\varphi(t)|^2]$, is also of class \mathcal{C}^1 , [7]. The operator $\frac{d}{dt} : AP^1(\mathbb{E}) \rightarrow AP^0(\mathbb{E})$, $\frac{d}{dt}(u) := u'$, is linear continuous, therefore it is of class \mathcal{C}^1 . The functional $\mathfrak{M}^0 : AP^0(\mathbb{R}) \rightarrow \mathbb{R}$, defined by $\mathfrak{M}^0(\varphi) := \mathfrak{M}_t \{ \varphi(t) \}$, is linear continuous, therefore it is of class \mathcal{C}^1 . Since $Q_0 = \mathfrak{M}^0 \circ \mathcal{N}_q^0 \circ \frac{d}{dt}$, Q_0 is of class \mathcal{C}^1 as composition of \mathcal{C}^1 -mappings, and by using the chain rule we have

$$DQ_0(u).h = \mathfrak{M}_t \{ u'(t).h'(t) \} \tag{3.2}$$

We consider the functional $\Phi_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$ defined by

$$\Phi_0(u) := \mathfrak{M}_t \left\{ \int_{-r}^0 f(u(t), u(t+\theta)) d\theta \right\}.$$

We consider the operator $in_0 : AP^1(\mathbb{E}) \rightarrow AP^0(\mathbb{E})$, $in_0(u) := u$, which is linear continuous, and consequently in_0 is of class \mathcal{C}^1 .

We note that we have Φ_0 is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -mappings. By using Lemma (3.2) we obtain

$$D\Phi_0(u).h = \mathfrak{M}_t \left\{ \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} \quad (3.3)$$

Now we want to improve this last formula.

Since $(t, \theta) \mapsto D_2 f(u(t), u(t+\theta)).h(t+\theta)$ is continuous on $\mathbb{R} \times [-r, 0]$, it is Lebesgue-integrable and by using the Fubini theorem [2], we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T \left(\int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right) dt \\ = \int_{-r}^0 \left(\frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) dt \right) d\theta \end{aligned} \quad (3.4)$$

We set $g_T(\theta) := \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) dt$. We know that, for all $\theta \in [-r, 0]$,

$$\lim_{T \rightarrow \infty} g_T(\theta) = \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \}$$

since $t \mapsto D_2 f(u(t), u(t+\theta)).h(t+\theta)$ bellongs to $AP^0(\mathbb{R})$.

Furthermore, since $u, h \in AP^0(\mathbb{E})$, $\overline{u(\mathbb{R})}$ and $\overline{h(\mathbb{R})}$ are compact, [3, 6, 14], and since the mapping $(x, y, z) \mapsto D_2 f(x, y).z$ is continuous on the compact $\overline{u(\mathbb{R})} \times \overline{u(\mathbb{R})} \times \overline{h(\mathbb{R})}$, it is bounded, and consequently we have :

$$\sup_{\theta \in [-r, 0]} \sup_{t \in \mathbb{R}} |D_2 f(u(t), u(t+\theta)).h(t+\theta)| := \sigma < \infty,$$

that implies $|g_T(\theta)| \leq \sigma$ for all $T > 0$, $\theta \in [-r, 0]$. And so the assumptions of the dominated convergence theorem of Lebesgue are fulfilled, [2], and by using it we obtain

$$\lim_{T \rightarrow \infty} \int_{-r}^0 g_T(\theta) d\theta = \int_{-r}^0 \lim_{T \rightarrow \infty} g_T(\theta) d\theta,$$

and so by using (3.4) we obtain

$$\begin{aligned} \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} d\theta \\ = \lim_{T \rightarrow \infty} \int_{-r}^0 \left(\frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) \right) d\theta \\ = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right) dt \end{aligned}$$

and so we have proven the following equality

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} = \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} d\theta \quad (3.5)$$

By using a similar reasoning we obtain

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t-\theta), u(t)).h(t) d\theta \right\} = \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t-\theta), u(t)).h(t) \} d\theta \quad (3.6)$$

Since the mean value is invariant by translation, [3, 6, 14], we have, for all $\theta \in [-r, 0]$, the following equality

$$\mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} = \mathfrak{M}_t \{ D_2 f(u(t-\theta), u(t)).h(t) \}.$$

By using it with (3.5) and (3.6) we obtain

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} = \mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t-\theta), u(t)).h(t) d\theta \right\}.$$

And by using this last equality in (3.3) we obtain

$$\begin{aligned} D\Phi_0(u).h &= \mathfrak{M}_t \left\{ \left(\int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta \right).h(t) \right\} \end{aligned} \quad (3.7)$$

We consider the functional $\Lambda_0 : AP^0(\mathbb{E}) \rightarrow \mathbb{R}$, defined by $\Lambda_0(u) := \mathfrak{M}_t \{ u(t).e(t) \}$. Note that Λ_0 is linear continuous and consequently it is of class \mathcal{C}^1 and we have

$$D\Lambda_0(u).h = \mathfrak{M}_t \{ h(t).e(t) \}. \quad (3.8)$$

Since $J_0 = Q_0 + \Phi_0 + \Lambda_0$, J_0 is of class \mathcal{C}^1 as a sum of three \mathcal{C}^1 -functionals, and by using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} DJ_0(u).h &= \mathfrak{M}_t \{ u'(t).h'(t) + \left(\int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \\ &\quad \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t) \right).h(t) \} \end{aligned} \quad (3.9)$$

for all $u, h \in AP^1(\mathbb{E})$

We set $p(t) := \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t)$, and we have $p \in AP^0(\mathbb{E})$.

When $DJ_0(u) = 0$ then by using (3.9) we have $\mathfrak{M}_t \{ u'(t).h'(t) \} = -\mathfrak{M}_t \{ p(t).h(t) \}$ for all $h \in AP^1(\mathbb{E})$ and by using the same reasoning that this one of the proof of Theorem 1 in [7] we obtain that $u \in AP^2(\mathbb{E})$ and $u''(t) = p(t)$, that is exactly (1.1).

Conversely, if u is a strong a.p. solution of (1.1), then we have $u'' = p$ and so, for all $h \in AP^1(\mathbb{E})$, we have $DJ_0(u).h = \mathfrak{M} \{ u'.h' + p.h \} = \mathfrak{M} \left\{ \frac{d}{dt}(u'.h) \right\} = 0$

THEOREM 3.4. *Under (3.1), if we additionally assume that f is convex function, then the set of the strong a.p. solutions of (1.1) is a convex subset of $AP^2(\mathbb{E})$.*

Proof. When f is convex, it is easy to verify that J_0 is convex, $DJ_0 = 0$ is equivalent to $J_0 = \inf J_0(AP^1(\mathbb{E}))$, [10], and $\{u \in AP^1(\mathbb{E}) : J_0 = \inf J_0(AP^1(\mathbb{E}))\}$ is convex. And so $\{u \in AP^1(\mathbb{E}) : DJ_0 = 0\}$ is convex, and we obtain the conclusion by using Theorem (3.3).

A consequence of Theorem (3.4) is the following one: when $e = 0$, if (1.1) possesses a non-constant T_1 -periodic solution u_1 and a non-constant T_2 -periodic solution u_2 with $T_1/T_2 \notin \mathbb{Q}$ the $\frac{1}{2}u_1 + \frac{1}{2}u_2$ is a non-periodic a.p. solution of (1.1) since it is a convex combination of a.p. solution.

4. The Weak a.p. solutions

We begin this section by giving a precise definition of the notion of weak a.p. solution of (1.1). A weak a.p. solution of (1.1) is a function $u \in B^{2,2}(\mathbb{E})$ such that

$$\nabla^2 u = \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t),$$

this equality holding in $B^2(\mathbb{E})$.

We begin by establishing two lemmas which contain general properties of the Besicovitch a.p. functions.

LEMMA 4.1. *Let $u \in B^2(\mathbb{E})$. Then the following equalities hold*

$$\begin{aligned} \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t+\theta)|^2 d\theta \right\} &= \int_{-r}^0 \mathfrak{M}_t \left\{ |u(t+\theta)|^2 \right\} d\theta \\ &= r \mathfrak{M}_t \left\{ |u(t)|^2 \right\} \end{aligned}$$

Proof. Since $\mathfrak{M}_t \left\{ |u(t)|^2 \right\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt$ exists in \mathbb{R}_+ , we have

$$M := \sup_{T \geq 1} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt < \infty.$$

For all $\theta \in [-r, 0]$ we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt &= \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds \\ &\leq \frac{1}{2T} \int_{-T-r}^{T-r} |u(t)|^2 dt \\ &\leq \frac{1}{2T} \int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &= \frac{1.2(T+r)}{2T} \cdot \frac{1}{2(T+r)} \int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &\leq \left(1 + \frac{r}{T}\right) \cdot M \leq (1+r) \cdot M =: M_1, \end{aligned}$$

and so we have proven

$$\exists M_1 > 0, \forall \theta \in [-r, 0], \forall T \geq 1, \frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt \leq M_1 < \infty \quad (4.1)$$

For all $T \geq 1$ we define $\Phi_T : [-r, 0] \rightarrow \mathbb{R}$ by setting $\Phi_T(\theta) := \frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt$. Since $\Phi_T(\theta) = \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds$ we see that Φ_T is absolutely continuous on $[-r, 0]$, and consequently we have $\Phi_T \in L^1([-r, 0], \mathbb{R})$.

If $[a, b]$ is segment in \mathbb{R} , for all $\theta \in [-r, 0]$, by using the Fubini theorem for the non negative mesurable functions [2], we have

$$\begin{aligned} \int_{[a,b] \times [-r,0]} |u(t + \theta)|^2 dt d\theta &= \int_{[-r,0]} \left(\int_{[a,b]} |u(t + \theta)|^2 dt \right) d\theta \\ &= \int_{[-r,0]} \left(\int_{[a,b]+\theta} |u(s)|^2 ds \right) d\theta \\ &\leq \int_{[-r,0]} \left(\int_{[a,b]+[-r,0]} |u(s)|^2 ds \right) d\theta \\ &= r \int_{[a,b]+[-r,0]} |u(s)|^2 ds < \infty \end{aligned}$$

since $|u|^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R}_+)$ and since $[a, b] + [-r, 0]$ is compact. And so we have proven:

$$(t, \theta) \mapsto |u(t + \theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R}) \quad (4.2)$$

Then by using the Fubini theorem [2], for all $T > 0$ we obtain

$$\frac{1}{2T} \int_{-T}^T \left(\int_{-r}^0 |u(t + \theta)|^2 d\theta \right) dt = \int_{-r}^0 \left(\frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt \right) d\theta \quad (4.3)$$

Since $u \in B^2(\mathbb{E})$, we have $\lim_{T \rightarrow \infty} \Phi_T(\theta) = \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} = \mathfrak{M}_t \left\{ |u(t)|^2 \right\}$ since the mean value is invariant by translation, for all $\theta \in [-r, 0]$. The constant M_1 is integrable on $[-r, 0]$. And so by using (4.1), we can apply the dominated convergence theorem of Lebesgue to obtain $\int_{-r}^0 \lim_{T \rightarrow \infty} \Phi_T(\theta) d\theta = \lim_{T \rightarrow \infty} \int_{-r}^0 \Phi_T(\theta) d\theta$, that implice by using (4.3) that $\int_{-r}^0 \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} d\theta = \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta)|^2 d\theta \right\}$. And since $\mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} = \mathfrak{M}_t \left\{ |u(t)|^2 \right\}$ for all θ , we have also

$$\int_{-r}^0 \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} d\theta = r \mathfrak{M}_t \left\{ |u(t)|^2 \right\}.$$

LEMMA 4.2. *If $u \in B^2(\mathbb{E})$ then $\tilde{u} \in B^2(L^2([-r, 0], \mathbb{E}))$ and we have*

$$\|\tilde{u}\|_{B^2(L^2([-r,0], \mathbb{E}))} = \sqrt{r} \cdot \|u\|_{B^2(\mathbb{E})}$$

Proof. We fix $u \in B^2(\mathbb{E})$, and $\varepsilon > 0$. We can choose $q_\varepsilon \in AP^0(\mathbb{E})$ such that $\|u - q_\varepsilon\|_{B^2(\mathbb{E})} < \varepsilon$.

Since $L^2([-r, 0], \mathbb{E})$ is separable, there exists a countable subset D in $L^2([-r, 0], \mathbb{E})$ which is dense, and consequently the set $\{B(\varphi, \rho) : \varphi \in D, \rho \in \mathbb{Q} \cap (0, \infty)\}$ is a generator of the Borel σ -field of $L^2([-r, 0], \mathbb{E})$, where

$$B(\varphi, \rho) := \left\{ \psi \in L^2([-r, 0], \mathbb{E}) : \|\psi - \varphi\|_{L^2([-r, 0], \mathbb{E})} < \rho \right\}.$$

We arbitrarily fix $\varphi \in D$ and $\rho \in \mathbb{Q} \cap (0, \infty)$, and we set

$$\alpha(t) := \int_{-r}^0 |u(t + \theta) - \varphi(\theta)|^2 d\theta.$$

By using the same reasoning that this one used to establish (4.2) we obtain that $(t, \theta) \mapsto |u(t + \theta) - \varphi(\theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$ and consequently by using the Fubini theorem we know that $\alpha \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ and then we necessarily have α measurable.

We note that $t \in \tilde{u}^{-1}(B(\varphi, \rho))$ is equivalent to $t \in \alpha^{-1}([0, \rho^2])$. Since α is measurable we have $\alpha^{-1}([0, \rho^2]) \in \mathcal{B}(\mathbb{R})$ and consequently $\tilde{u}^{-1}(B(\varphi, \rho)) \in \mathcal{B}(\mathbb{R})$, and so we have proven:

$$\tilde{u} \text{ is measurable from } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ in } (L^2([-r, 0], \mathbb{E}), \mathcal{B}(L^2([-r, 0], \mathbb{E}))). \quad (4.4)$$

By using (4.2) we know that $(t, \theta) \mapsto |u(t + \theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$ and consequently, by using the Fubini theorem we obtain that $t \mapsto \int_{-r}^0 |u(t + \theta)|^2 d\theta = \|\tilde{u}(t)\|_{L^2([-r, 0], \mathbb{E})}^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R})$.

Therefore we have obtained, [2] :

$$\tilde{u} \in L^2_{loc}(\mathbb{R}, L^2([-r, 0], \mathbb{E})). \quad (4.5)$$

By using Lemma (4.1) with $u - q_\varepsilon$ instead of u , we know that

$$\mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta) - q_\varepsilon(t + \theta)|^2 d\theta \right\}$$

exists and that we have

$$\begin{aligned} \mathfrak{M}_t \left\{ \|\tilde{u}(t) - \tilde{q}_\varepsilon(t)\|_{L^2([-r, 0], \mathbb{E})} \right\} &= \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta) - q_\varepsilon(t + \theta)|^2 d\theta \right\} \\ &= r \cdot \mathfrak{M}_t \left\{ |u(t) - q_\varepsilon(t)|^2 \right\} < r \cdot \varepsilon^2. \end{aligned}$$

Since $\tilde{q}_\varepsilon \in AP^0(\mathcal{C}^0([-r, 0], \mathbb{E})) \subset AP^0(L^2([-r, 0], \mathbb{E}))$, when $\varepsilon \rightarrow 0$, we obtain that $\tilde{u} \in B^2(L^2([-r, 0], \mathbb{E}), \mathbb{E})$.

The relation between the norms of u and \tilde{u} is a consequence of Lemma (4.1).

By modifying a function $u \in B^2(\mathbb{E})$ on a bounded interval of \mathbb{R} we do not modify the (class of the) function u , and so we can ask to use $\tilde{u}(t)$, defined as the restriction of u on the interval $[t - r, t]$, possesses a meaning. Lemma (4.2) provides an answer

to this question, since if $v \in B^2(\mathbb{E})$ is different of u , then we have $\tilde{u} \neq \tilde{v}$. And so the definition of \tilde{u} is consistent, and the notion of weak a.p. solution is also consistent.

Now we introduce the following condition on f :

$$\begin{cases} \text{There exists } a \in (0, \infty) \text{ and } b \in \mathbb{R} \text{ such that} \\ |Df(x, y)| \leq a(|x| + |y|) + b \text{ for all } x, y \in \mathbb{E}. \end{cases} \quad (4.6)$$

LEMMA 4.3. *Under (3.1) and (4.6), the operator $S : B^2(\mathbb{E}) \rightarrow B^1(\mathbb{R})$ defined by $S(u) := \left[t \mapsto \int_{-r}^0 f(u(t), u(t + \theta)) d\theta \right]$ is of class \mathcal{C}^1 and for all $u, h \in B^2(\mathbb{E})$, we have $DS(u).h = \left[t \mapsto \int_{-r}^0 D_1 f(u(t), u(t + \theta)).h(t) + D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right]$*

Proof. The Nemytskii operator build on f , $\mathcal{N}_f : L^2([-r, 0], \mathbb{E}) \times L^2([-r, 0], \mathbb{E}) \rightarrow L^1([-r, 0], \mathbb{R})$, $\mathcal{N}_f(\varphi, \psi) := [\theta \mapsto f(\varphi(\theta), \psi(\theta))]$, under (3.1) and (4.6) is of class \mathcal{C}^1 , [11], and $D\mathcal{N}_f(\varphi, \psi).(\xi, \zeta) = [\theta \mapsto Df(\varphi(\theta), \psi(\theta)).(\xi(\theta), \zeta(\theta))]$

The operator $A : \mathbb{E} \times L^2([-r, 0], \mathbb{E}) \rightarrow (L^2([-r, 0], \mathbb{E}))^2$ defined by $A(x, \psi) := (x, \psi)$, where x is considered as a constant function, is linear continuous, therefore A is of class \mathcal{C}^1 and $DA(x, \psi) = A$.

The functional $I : L^1([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$, $I(w) := \int_{-r}^0 w(\theta) d\theta$, is linear continuous, therefore I is of class \mathcal{C}^1 and $DI(w) = I$.

We consider the mapping $F : \mathbb{E} \times L^2([-r, 0], \mathbb{E}) \rightarrow \mathbb{R}$, defined by $F(x, \psi) := \int_{-r}^0 f(x, \psi(\theta)) d\theta$.

We note that $F = I \circ \mathcal{N}_f \circ A$, and so F is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -mappings, and by using the chain rule we obtain, for all $x, y \in \mathbb{E}$ and for all $\psi, \xi \in L^2([-r, 0], \mathbb{E})$, the following formula:

$$DF(x, \psi).(y, \xi) = \int_{-r}^0 (D_1 f(x, \psi(\theta)).y + D_2 f(x, \psi(\theta)).\xi(\theta)) d\theta.$$

Let $(y, \xi) \in \mathbb{E} \times L^2([-r, 0], \mathbb{E})$ such that $\|(y, \xi)\| \leq 1$. Then we have

$$\begin{aligned} |DF(x, \psi).(y, \xi)| &\leq \int_{-r}^0 |Df(x, \psi(\theta))| \cdot |(y, \xi(\theta))| d\theta \\ &\leq \left(\int_{-r}^0 |Df(x, \psi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \cdot \left(\int_{-r}^0 |(y, \xi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \end{aligned}$$

by using the Cauchy-Schwarz-Buniakovski inequality.

We note that

$$\int_{-r}^0 |(y, \xi(\theta))|^2 d\theta = \int_{-r}^0 (|y|^2 + |\xi(\theta)|^2) d\theta = r|y|^2 + \int_{-r}^0 |\xi(\theta)|^2 d\theta \leq r_1 \cdot \|(y, \xi)\|^2 \leq r_1,$$

where $r_1 := \max\{r, 1\}$, and so we have:

$$\begin{aligned}
 |DF(x, \psi).(y, \xi)| &\leq \sqrt{r_1} \cdot \left(\int_{-r}^0 |Df(x, \psi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \\
 &\leq \sqrt{r_1} \cdot \left(\int_{-r}^0 (a \cdot |x| + a \cdot |\psi(\theta)| + b)^2 d\theta \right)^{\frac{1}{2}} \\
 &= \sqrt{r_1} \cdot \|a \cdot |x| + a \cdot |\psi| + |b|\|_{L^2([-r,0],\mathbb{E})} \\
 &\leq \sqrt{r_1} \cdot \left(a \| |x| \|_{L^2([-r,0],\mathbb{R})} + a \| |\psi| \|_{L^2([-r,0],\mathbb{R})} + \| |b| \|_{L^2([-r,0],\mathbb{R})} \right).
 \end{aligned}$$

Since $\| |x| \|_{L^2([-r,0],\mathbb{R})} = \sqrt{r} \cdot |x|$, $\| |b| \|_{L^2([-r,0],\mathbb{R})} = \sqrt{r} \cdot |b|$ and $\| |\psi| \|_{L^2([-r,0],\mathbb{R})} = \| \psi \|_{L^2([-r,0],\mathbb{E})}$, we have

$$|DF(x, \psi).(y, \xi)| \leq a \cdot \sqrt{r_1} \cdot \sqrt{r} \left(|x| + \| \psi \|_{L^2([-r,0],\mathbb{E})} \right) + \sqrt{r_1} \cdot \sqrt{r} \cdot |b|.$$

We set $a_1 := a \cdot \sqrt{r_1} \cdot \sqrt{r}$ and $b_1 := \sqrt{r_1} \cdot \sqrt{r} |b|$ and so we obtain:

$$|DF(x, \psi)| \leq a_1 \cdot \left(|x| + \| \psi \|_{L^2([-r,0],\mathbb{E})} \right) + b_1.$$

And so the assumption of ([11], Theorem 2.6 page 14) are fulfilled and we can assert that $\mathcal{N}_F : B^2(\mathbb{E}) \times B^2(L^2) \rightarrow B^1(\mathbb{R})$ is of class \mathcal{C}^1 and that we have, for all $u, h \in B^2(\mathbb{E})$ and for all $V, K \in L^2([-r, 0], \mathbb{E})$, the following formula

$$\begin{aligned}
 D_x \mathcal{N}_F(u, V).(h, K) &= [t \mapsto DF(u(t), V(t)).(h(t), K(t))] \\
 &= \int_{-r}^0 (D_1 f(u(t), V(t)(\theta)).h(t) + D_2 f(u(t), V(t)(\theta)).K(t)(\theta)) d\theta \quad (4.7)
 \end{aligned}$$

We consider the linear operator $T : B^2(\mathbb{E}) \rightarrow B^2(L^2([-r, 0], \mathbb{E}))$ defined by $T(u) := \tilde{u}$. By using Lemma (4.2) we know that T is continuous, and therefore T is of class \mathcal{C}^1 with $DT(u) = T$.

We note that we have $S = \mathcal{N}_F \circ (id, T)$, and so S is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -operators, and by using the chain rule and (4.7) we obtain the announced formula.

LEMMA 4.4. *Under (3.1) and (4.6), if u and h belong to $B^2(\mathbb{E})$ then the following equality holds :*

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right\} = \mathfrak{M}_t \left\{ \left(\int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta \right) . h(t) \right\}$$

Proof. By using a reasoning similar to this one used to establish (4.2) we obtain that $(t, \theta) \mapsto D_2 f(u(t), u(t + \theta)).h(t + \theta) \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$. And so we can use the Fubini theorem to obtain

$$\begin{aligned}
 \frac{1}{2T} \int_{-T}^T \left(\int_{-r}^0 D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right) dt \\
 = \int_{-r}^0 \left(\frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t + \theta)).h(t + \theta) dt \right) d\theta \quad (4.8)
 \end{aligned}$$

for all $T \in (0, \infty)$.

For all $T \in [1, \infty)$ we introduce the function $g_T : [-r, 0] \rightarrow \mathbb{R}$ defined by

$$g_T(\theta) := \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta) dt$$

Ever using the Fubini theorem we know that the g_T are borelian.

Since $t \mapsto D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta) \in B^1(\mathbb{R})$ we know that the mean value exists in \mathbb{R} and consequently we have

$$\lim_{T \rightarrow \infty} g_T(\theta) = \mathfrak{M}_t \{D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta)\}$$

for all $\theta \in [-r, 0]$.

Since $\mathfrak{M}_t \{|u(t)|^2\}$ exists in \mathbb{R} , we have $\sup_{t \geq 1} \left(\frac{1}{2T} \int_{-T}^T |u(t)|^2 dt \right) =: M < \infty$.

For all $\theta \in [-r, 0]$ and, for all $T \geq 1+r$, we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt &= \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds \\ &\leq \frac{1}{2T} \int_{-T+\theta}^{T-\theta} |u(s)|^2 ds \\ &= \frac{2(T-\theta)}{2T} \cdot \frac{1}{2(T-\theta)} \cdot \int_{-(T-\theta)}^{T-\theta} |u(t)|^2 dt \\ &\leq (1+r) \cdot M =: M_0 \end{aligned}$$

And so we have proven the following assertion

$$\begin{cases} \text{There exists } M_0 \in (0, \infty) \text{ such that, for all} \\ \theta \in [-r, 0], \quad \sup_{T \geq 1+r} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt \leq M_0. \end{cases} \quad (4.9)$$

Replacing u by h we similarly obtain the following assertion.

$$\begin{cases} \text{There exists } M_1 \in (0, \infty) \text{ such that, for all} \\ \theta \in [-r, 0], \quad \sup_{T \geq 1+r} \frac{1}{2T} \int_{-T}^T |h(t+\theta)|^2 dt \leq M_1. \end{cases} \quad (4.10)$$

By using the equivalence of the norms of \mathbb{R}^2 and the usual inequality $(A+B)^2 \leq 2(A^2+B^2)$ we obtain the existence of $a_2 \in (0, \infty)$ such that

$$\begin{aligned} |D_2 f(u(t), u(t+\theta))|^2 &\leq \left(a_2 \left[|u(t)|^2 + |u(t+\theta)|^2 \right]^{\frac{1}{2}} + b \right)^2 \\ &\leq 2 \cdot \left(a_2 |u(t)|^2 + a_2 |u(t+\theta)|^2 + b^2 \right) \end{aligned}$$

that implies

$$\begin{aligned} \left(\frac{1}{2T} \int_{-T}^T |D_2 f(u(t), u(t+\theta))|^2 dt \right)^{\frac{1}{2}} &\leq \sqrt{2} (a_2 \cdot \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt \\ &\quad + a_2 \cdot \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt + b^2)^{\frac{1}{2}} \\ &\leq \sqrt{2} (a_2 M_0 + a_2 M_0 + b^2)^{\frac{1}{2}}. \end{aligned}$$

Then by setting $\gamma := \sqrt{2} (2a_2M_0 + b^2)^{\frac{1}{2}} M_1^{\frac{1}{2}}$ we have proven the following assertion

$$\left(\frac{1}{2T} \int_{-T}^T |D_2f(u(t), u(t + \theta))|^2 dt \right)^{\frac{1}{2}} \cdot \left(\frac{1}{2T} \int_{-T}^T |h(t + \theta)|^2 dt \right)^{\frac{1}{2}} \leq \gamma \quad (4.11)$$

By using the Cauchy-Schwarz-Buniakovski inequality and (4.11) we obtain, for all $T \geq 1 + r$ and for $\theta \in [-r, 0]$,

$$|g_T(\theta)| \leq \frac{1}{2T} \int_{-T}^T |D_2f(u(t), u(t + \theta))| \cdot |h(t + \theta)| dt \leq \sigma$$

Since the Lebesgue measure of $[-r, 0]$ is finite, the constant σ is Lebesgue integrable in $[-r, 0]$, and consequently the assumptions of the Lebesgue Dominated Convergence theorem are fulfilled and we can say :

$$\int_{-r}^0 \lim_{T \rightarrow \infty} g_T(\theta) d\theta = \lim_{T \rightarrow \infty} \int_{-r}^0 g_T(\theta) d\theta,$$

and we can conclude as in the proof of (3.5), (3.6), (3.7).

THEOREM 4.5. *We assume (3.1) and (4.6) fulfilled. Then the functional $J : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$, defined by*

$$J(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |\nabla u(t)|^2 + \int_{-r}^0 f(u(t), u(t + \theta)) d\theta + u(t) \cdot e(t) \right\},$$

is of class \mathcal{C}^1 . And when $u \in B^{1,2}(\mathbb{E})$, we have $DJ(u).h = 0$ if and only if u is a weak a.p. solution of (1.1).

Proof. We consider the functional $Q : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$ defined by

$$Q(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |\nabla u(t)|^2 \right\}.$$

We set $q(x) := \frac{1}{2} |x|^2$; $q : \mathbb{E} \rightarrow \mathbb{R}$ is a \mathcal{C}^1 -function since \mathbb{E} is euclidean. Since $Dq(x) = x$, q satisfies the condition of ([11], Theorem 2.6 page 14) to ensure that the Nemytskii operator $\mathcal{N}_q : B^2(\mathbb{E}) \rightarrow B^1(\mathbb{R})$ is of class \mathcal{C}^1 and $D\mathcal{N}_q(v).h = [t \mapsto v(t).h(t)]$ for all $v, h \in B^2(\mathbb{E})$. Since the derivation operator $\nabla : B^{1,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$ is linear continuous, it is of class \mathcal{C}^1 and since the operator $\mathfrak{M} : B^1(\mathbb{R}) \rightarrow \mathbb{R}$, $\mathfrak{M}(v) := \mathfrak{M}_t \{v(t)\}$ is also linear continuous, it is of class \mathcal{C}^1 . And so $Q := \mathfrak{M} \circ \mathcal{N}_q \circ \nabla$ is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -mappings. Moreover by using the chain rule we have

$$DQ(u).h = \mathfrak{M}_t \{ \nabla u(t) \cdot \nabla h(t) \} \quad (4.12)$$

for all $u, h \in B^{1,2}(\mathbb{E})$.

We consider the functional $\Phi : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) := \mathfrak{M}_t \left\{ \int_{-r}^0 f(u(t), u(t + \theta)) d\theta \right\}.$$

We note that the injection $in : B^{1,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$, $in(u) := u$, is linear continuous and consequently it is of class \mathcal{C}^1 . We note that $\Phi = \mathfrak{M} \circ S \circ in$, and by using Lemma (4.3) we know that S is of class \mathcal{C}^1 . And so Φ is of class \mathcal{C}^1 as a composition of \mathcal{C}^1 -mapping. Ever using Lemma (4.3) and the chain rule we obtain the following formula

$$\begin{aligned} D\Phi(u).h &= \mathfrak{M}_t \left\{ \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta \right. \\ &\quad \left. + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\}, \end{aligned}$$

and by using Lemma (4.4) we obtain

$$\begin{aligned} D\Phi(u).h &= \mathfrak{M}_t \left\{ \left(\int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta \right).h(t) \right\}. \end{aligned} \tag{4.13}$$

We consider the linear functional $\Lambda : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$ defined by

$$\Lambda(u) := \mathfrak{M}_t \{u(t).e(t)\},$$

and the linear functional $L : B^2(\mathbb{E}) \rightarrow \mathbb{R}$ defined by $L(u) := \mathfrak{M}_t \{u(t).e(t)\} = (u|e)_{B^2(\mathbb{E})}$. Since L is continuous (by using the Cauchy-Schwarz-Buniakovski inequality), $\Lambda := L \circ in$ is also continuous as a composition of continuous mappings, and consequently Λ is of class \mathcal{C}^1 . Moreover, since $D\Lambda(u) = 1$ we obtain the following formula

$$D\Lambda(u).h = \mathfrak{M}_t \{h(t).e(t)\}, \text{ for all } u, h \in B^{1,2}(\mathbb{E}). \tag{4.14}$$

We note that $J = Q + \Phi + \Lambda$, and so J is of class \mathcal{C}^1 as a sum of \mathcal{C}^1 -functionals. Moreover, by using (4.12), (4.13), (4.14), we obtain

$$\begin{aligned} DJ(u).h &= \mathfrak{M}_t \left\{ \nabla u(t). \nabla h(t) + \left(\int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t) \right).h(t) \right\} \end{aligned} \tag{4.15}$$

for all $u, h \in B^{1,2}(E)$.

We set $p(t) := \int_{-r}^0 [D_1 f(u(t), u(t+\theta)) + D_2 f(u(t-\theta), u(t))] d\theta + e(t)$ ($\in B^2(\mathbb{E})$). And so the condition $DJ(u) = 0$ can be written as $\mathfrak{M}_t \{ \nabla u(t). \nabla h(t) \} = -\mathfrak{M}_t \{ p(t).h(t) \}$ for all $h \in B^{1,2}(\mathbb{E})$. And so by using [8], this last condition implies that $\nabla u \in B^{1,2}(\mathbb{E})$, i.e. $u \in B^{2,2}(\mathbb{E})$, and $\nabla^2 u = p$ which exactly means that u is a weak a.p. solutions of (1.1).

Conversely, since $\mathfrak{M} \{ \nabla v \} = 0$ for all $v \in B^{1,2}(\mathbb{R})$, we have $0 = \mathfrak{M} \{ \nabla(\nabla u.h) \} = \mathfrak{M} \{ \nabla^2 u.h \} + \mathfrak{M} \{ \nabla u. \nabla h \} = \mathfrak{M} \{ p.h \} + \mathfrak{M} \{ \nabla u. \nabla h \}$ for all $h \in B^{1,2}(\mathbb{E})$, that implies $DJ(u) = 0$.

Now we introduce an assumption of convexity :

$$f \text{ is a convex function on } \mathbb{E} \times \mathbb{E} \tag{4.16}$$

and an assumption of coerciveness :

$$\begin{cases} \text{There exists } c \in (0, \infty) \text{ and } d \in \mathbb{R} \text{ such that} \\ f(x, y) \geq c|x|^2 + d \text{ for all } (x, y) \in \mathbb{E} \times \mathbb{E}. \end{cases} \tag{4.17}$$

THEOREM 4.6. *Under (3.1), (4.6), (4.16), (4.17), for all $e \in B^2(\mathbb{E})$, there exists $u \in B^{2,2}(\mathbb{E})$ which is a weak a.p. solution of (1.1). Moreover the set of the weak a.p. solutions of (1.1) is a convex set.*

Proof. After Theorem (4.5) we know that the functional J is of class \mathcal{C}^1 on $B^{1,2}(\mathbb{E})$. By using (4.16) we deduce that J is a convex functional. Then J is weakly lower semi-continuous on the Hilbert space $B^{1,2}(\mathbb{E})$, [16]. From (4.17) we deduce that, for all $u \in B^{1,2}(\mathbb{E})$, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|\nabla u\|_{B^2(\mathbb{E})}^2 + c \|u\|_{B^2(\mathbb{E})}^2 - \|u\|_{B^2(\mathbb{E})} \cdot \|e\|_{B^2(\mathbb{E})} \\ &\geq c_1 \cdot \|u\|_{B^{1,2}(\mathbb{E})}^2 - \|e\|_{B^2(\mathbb{E})} \cdot \|u\|_{B^{1,2}(\mathbb{E})} \end{aligned}$$

where $c_1 := \min\{\frac{1}{2}, c\} \in (0, \infty)$. Consequently J is coercive, i.e. $J(u) \rightarrow \infty$ when $\|u\|_{B^{1,2}(\mathbb{E})}^2 \rightarrow \infty$. Then, [10], we can assert that there exists $u \in B^{1,2}(\mathbb{E})$ such that $J(u) = \inf J(B^{1,2}(\mathbb{E}))$, and since J is of class \mathcal{C}^1 we have $DJ(u) = 0$, and then, by using Theorem (4.5), we know that u is a weak a.p. solution of (1.1).

Ever using Theorem (4.5), we know that the set of the weak a.p. solutions of (1.1) is equal to the following set: $\{u \in B^{1,2}(\mathbb{E}) : DJ(u) = 0\}$, and since J is convex this last it is equal to the set $\{u \in B^{1,2}(\mathbb{E}) : J(u) = \inf J(B^{1,2}(\mathbb{E}))\}$. Since J is convex this last set is a convex set. And so the set of the weak a.p. solutions of (1.1) is convex.

5. Density

LEMMA 5.1. *Under (3.1) and (4.16) we consider the operator $\Gamma_1 : B^2(\mathbb{E}) \rightarrow B^2(E)$ defined by*

$$\Gamma_1(u) := \left[t \mapsto \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right].$$

Then Γ_1 is continuous.

Proof. Under (3.1) and (4.6) we know that we have $|D_1 f(x, y)| \leq a(|x| + |y|) + b$ for all $x, y \in \mathbb{E}$. Then ([11], Theorem 2.5 page 9), the Nemytskii operator $\mathcal{N}_{D_1 f} : L^2([-r, 0], E) \times L^2([-r, 0], E) \rightarrow L^2([-r, 0], E)$, $\mathcal{N}_{D_1 f}(\varphi, \psi) := [\theta \mapsto D_1 f(\varphi(\theta))$,

$\psi(\theta)]$, is continuous. We know that the operator A , $A(x, \psi) = (x, \psi)$, used in the proof of Lemma (4.3), is continuous from $\mathbb{E} \times L^2([-r, 0], E)$ in $L^2([-r, 0], E) \times L^2([-r, 0], E)$. The functional I used in the proof of Lemma (4.3) is continuous.

We define $F_1 : \mathbb{E} \times L^2([-r, 0], E) \rightarrow \mathbb{R}$ by setting

$$F_1(x, \psi) := I \circ \mathcal{N}_{D_1 f} \circ A(x, \psi) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta.$$

Then F_1 is continuous as a composition of continuous mappings.

For all $x \in \mathbb{E}$ and $\psi \in L^2([-r, 0], E)$ we have

$$\begin{aligned} |F_1(x, \psi)| &\leq \int_{-r}^0 |D_1 f(x, \psi(\theta))| d\theta \\ &\leq \int_{-r}^0 (a|x| + a|\psi(\theta)| + b) d\theta \\ &= r.a. |x| + a. \int_{-r}^0 |\psi(\theta)| d\theta + r.b \\ &\leq r.a. |x| + a. \sqrt{r} \|\psi\|_{L^2([-r, 0], E)} + r.b \\ &\leq a_3. \left(|x| + \|\psi\|_{L^2([-r, 0], E)} \right) + r.b, \end{aligned}$$

where $a_3 := a. \max\{r, \sqrt{r}\}$. And so the assumptions of ([18], Remark 2.7 page 54) are fulfilled that ensure that the Nemytskii operator

$$\mathcal{N}_{F_1} : B^2(\mathbb{E}) \times B^2(L^2([-r, 0], E)) \rightarrow B^2(\mathbb{E}),$$

$$\mathcal{N}_{F_1}(u, \xi) := \left[t \mapsto F_1(u(t), \xi(t)) = \int_{-r}^0 D_1 f(u(t), \xi(t)(\theta)) d\theta \right]$$

is continuous.

We note that $\Gamma_1 = \mathcal{N}_{F_1} \circ (id, T)$, where $T(u) = \tilde{u}$, and so Γ_1 is continuous as a composition of continuous mappings.

LEMMA 5.2. *Under (3.1) and (4.6) we consider the operator $\Gamma_2 : B^2(\mathbb{E}) \rightarrow B^2(\mathbb{E})$ defined by*

$$\Gamma_2(u) := \left[t \mapsto \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta \right].$$

Then Γ_2 is continuous.

Proof. By a reasoning similar to this one used in Lemma (5.1), the Nemytskii operator $\mathcal{N}_{D_2 f} : L^2([-r, 0], E) \times L^2([-r, 0], E) \rightarrow L^2([-r, 0], E)$, $\mathcal{N}_{D_2 f}(\varphi, \psi) := [\theta \mapsto D_2 f(\varphi(\theta), \psi(\theta))]$, is continuous.

We introduce the operator $A_1 : L^2([-r, 0], E) \times \mathbb{E} \rightarrow L^2([-r, 0], E) \times L^2([-r, 0], E)$, $A_1(\varphi, y) := [\theta(\varphi(\theta), y)]$. A_1 is linear continuous.

We consider also the functional I like in the proof of Lemma (5.1).

We define $F_2 : L^2([-r, 0], E) \times \mathbb{E} \rightarrow \mathbb{R}$ by setting

$$F_2(\varphi, y) := I \circ \mathcal{N}_{D_2F} \circ A_1(\varphi, y) = \int_{-r}^0 D_2f(\varphi(\theta), y) d\theta.$$

And also F_2 is continuous as composition of continuous functions. Like in the proof of Lemma (5.1) we establish that

$$|F_2(\varphi, y)| \leq a_3 \left(\|\varphi\|_{L^2([-r, 0], E)} + |y| \right) + rb.$$

And so by using ([18], Remark 2.7 page 54) we know that the Nemytskii operator

$$\begin{aligned} \mathcal{N}_{F_2} : B^2(L^2([-r, 0], E)) \times B^2(\mathbb{E}) &\rightarrow B^2(\mathbb{E}), \\ \mathcal{N}_{F_2}(\xi, u) &:= \left[t \mapsto \int_{-r}^0 D_2f(\xi(t)(\theta), u(t)) d\theta \right], \end{aligned}$$

is continuous.

For all $u \in B^2(\mathbb{E})$ and for all $t \in \mathbb{R}$, we denote by $\tilde{u}(t) := [\theta \mapsto u(t - \theta)] \in L^2([-r, 0], E)$. Proceeding like in Lemma (4.2) we can establish that $\tilde{u} \in B^2(L^2([-r, 0], E))$ and that $\|\tilde{u}\|_{B^2(L^2([-r, 0], E))} = \sqrt{r} \cdot \|u\|_{B^2(\mathbb{E})}$. And so the operator

$$T_1 : B^2(\mathbb{E}) \rightarrow B^2(L^2([-r, 0], E)), \quad T_1(u) := \tilde{u},$$

is linear continuous.

We note that $\Gamma_2 = \mathcal{N}_{D_2f} \circ (T_1, id)$ that permits us to say that Γ_2 is continuous as a composition of continuous mappings.

THEOREM 5.3. *Under (3.1), (4.6), (4.16), (4.17), for all $e \in AP^0(\mathbb{E})$, and for all $\varepsilon \in (0, \infty)$, there exists $e_\varepsilon \in AP^0(\mathbb{E})$ such that $\|e - e_\varepsilon\|_{B^2(\mathbb{E})} \leq \varepsilon$ and such that there exists $u_\varepsilon \in AP^2(\mathbb{E})$ which is a strong a.p. solution of*

$$u''_\varepsilon(t) = \int_{-r}^0 D_1f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2f(u(t - \theta), u(t)) d\theta + e_\varepsilon(t).$$

Proof. We set $\Gamma := \Gamma_1 + \Gamma_2$ where Γ_1 comes from Lemma (5.1) and Γ_2 comes from Lemma (5.2). We consider the operator $\mathcal{S} : B^{2,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$, $\mathcal{S}(u) := \nabla^2(u) - \Gamma(u)$. The operator $\nabla^2 : B^{2,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$ is linear continuous and by using Lemma (5.1) and Lemma (5.2), we see that \mathcal{S} is continuous.

By using Theorem (4.6) we know that $\mathcal{S}(B^{2,2}(\mathbb{E})) = B^2(\mathbb{E})$ and consequently we have $AP^0(\mathbb{E}) \subset \mathcal{S}(B^{2,2}(\mathbb{E}))$. Since AP^2 is dense in $B^{2,2}(\mathbb{E})$ and since \mathcal{S} is continuous, for all $e \in AP^0(\mathbb{E})$ and for all $\varepsilon \in (0, \infty)$, we obtain that there exists $u_\varepsilon \in AP^2$ such that $\|\mathcal{S}(u_\varepsilon) - e\|_{B^2(\mathbb{E})} < \varepsilon$.

By proceeding like in the proof of Lemma (3.2), we obtain that $\Gamma_1(u_\varepsilon)$ and $\Gamma_2(u_\varepsilon)$ belong to $AP^0(\mathbb{E})$. Since $\nabla^2(u_\varepsilon) = u''_\varepsilon$, [8], we obtain that $\mathcal{S}(u_\varepsilon) \in AP^0(\mathbb{E})$. We set $e_\varepsilon := \mathcal{S}(u_\varepsilon)$, and so e_ε satisfies the announced conditions.

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