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## A VARIATIONAL APPROACH FOR ALMOST PERIODIC SOLUTIONS IN RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

MOEZ AYACHI AND JÖEL BLOT

(communicated by R. L. Pouso)

*Abstract.* To study the a.p. (almost periodic) solutions of retarded functional differential equations in the form  $u''(t) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta + e(t)$ , we introduce variational formalisms to characterize the a.p. solutions as a critical points of functionals defined on Banach spaces of a.p. functions. We obtain an existence result of weak a.p. solutions and a result of density of the a.p. forcing termes  $e(\cdot)$  for which the equation possesses usual a.p. solutions.

### 1. Introduction

From a function  $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ , where  $\mathbb{E}$  is a finite-dimensional real Euclidean space, and from  $r \in (0, \infty)$  we consider the following (second order) retarded functional differential equation

$$u''(t) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta + e(t) \quad (1.1)$$

where  $D_j$ ,  $j = 1, 2$ , denotes the partial gradient and where  $e : \mathbb{R} \rightarrow \mathbb{E}$  is a forcing term.

We study the a.p. (almost periodic) solutions of (1.1) where  $e$  is an a.p. function.

A strong a.p. solution of (1.1) is a function  $u : \mathbb{R} \rightarrow \mathbb{E}$  which is twice differentiable (in ordinary sense) with  $u, u'$  and  $u''$  which are a.p. in the sense of Bohr [3, 6, 14]; the equality in (1.1) being satisfied for all  $t \in \mathbb{R}$ .

A weak a.p. solution of (1.1) is a function  $u : \mathbb{R} \rightarrow \mathbb{E}$  which is a.p. in the sense of Besicovitch [5, 18], which possesses a first-order and a second-order generalized derivative; the equality in (1.1) means that the difference between the two members has a quadratic mean value equal to zero.

For the ordinary differential equations, this kind of weak a.p. solutions was considered in [8]. For neutral delay differential equations, this kind of weak a.p. solutions is considered in [4].

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Our approach uses a variational method. The a.p. solutions (strong or weak) of (1.1) are characterized as critical points of functionals in the form

$$u \mapsto \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left( \frac{1}{2} |u'(t)|^2 + \int_{-t}^0 f(u(t), u(t+\theta)) d\theta + u(t) \cdot e(t) \right) dt$$

on Banach spaces of a.p. functions. And so (1.1) appears as an Euler-Lagrange equation.

Now we briefly describe the contents of the paper. After Section 2 devoted to precise our notations, in Section 3 we build a variational formalism to characterize the strong (also called usual) a.p. solutions of (1.1) (Theorem (3.3)), for which we can deduce a result on the structure of the set of strong a.p. solutions of (1.1) (Theorem (3.4)). In Section 4 we build a variational formalism to characterize the weak a.p. solutions of (1.1) (Theorem (4.5)), and to establish an existence result of weak a.p. solutions (Theorem (4.6)); we obtain also a result of the structure of the set of the weak a.p. solutions of (1.1).

In Section 5 we establish a result on the density of the a.p. forcing term for which (1.1) possesses a strong a.p. solutions (Theorem (5.3)); this result uses the weak a.p. solutions.

## 2. Notations

When  $\mathbb{X}$  is a Banach space,  $AP^0(\mathbb{X})$  denotes the space of the Bohr-a.p. functions from  $\mathbb{R}$  in  $\mathbb{X}$  [3, 6, 14]. It is a Banach space for the norm  $\|u\|_\infty := \sup \{|u(t)| : t \in \mathbb{R}\}$ . When  $u \in AP^0(\mathbb{X})$ , its mean value exists in  $\mathbb{X}$  :

$$\mathfrak{M}\{u\} = \mathfrak{M}_t\{u(t)\} := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt,$$

[3, 6, 14]. When  $k \in \mathbb{N}$ ,  $k \geq 1$ ,  $AP^k(\mathbb{X})$  denotes the space of the  $u \in \mathcal{C}^k(\mathbb{R}, \mathbb{X}) \cap AP^0(\mathbb{X})$  such that  $u^j = \frac{d^j u}{dt^j} \in AP^0(\mathbb{X})$  for all  $j = 1, \dots, k$ . It is a Banach space for the norm  $\|u\|_{\mathcal{C}^k} := \|u\|_\infty + \sum_{1 \leq j \leq k} \|u^j\|_\infty$ .

$B^1(\mathbb{X})$  denotes the completion of  $AP^0(\mathbb{X})$  with respect to the norm  $\|u\|_{B^1} := \mathfrak{M}\{|u|\}$ . It is a quotient space to transform the semi-norm  $u \mapsto \mathfrak{M}\{|u|\}$  into a norm. When  $\mathbb{X}$  is a Hilbert space,  $B^2(\mathbb{X})$  denotes the completion of  $AP^0(\mathbb{X})$  with respect to the norm  $\|u\|_{B^2} := \mathfrak{M}\{|u|^2\}^{\frac{1}{2}}$ . It is also a quotient space and it is a Hilbert space for the inner product  $(u|v)_{B^2} := \mathfrak{M}\{(u|v)_\mathbb{X}\}$ .

The generalized derivative of  $u \in B^2(\mathbb{X})$  (when it exists) is  $\nabla u \in B^2(\mathbb{X})$  such that  $\mathfrak{M}_t\left\{\left|\nabla u(t) - \frac{1}{\tau}(u(t+\tau) - u(t))\right|^2\right\} \rightarrow 0$  ( $\tau \rightarrow 0$ ) [8, 12]. We consider  $B^{1,2}(\mathbb{X}) := \{u \in B^2(\mathbb{X}) : \nabla u \in B^2(\mathbb{X})\}$  and  $B^{2,2}(\mathbb{X}) := \{u \in B^{1,2}(\mathbb{X}) : \nabla^2 u := \nabla(\nabla u) \in B^2(\mathbb{X})\}$ . They are Hilbert spaces for the respective norms

$$\|u\|_{B^{1,2}} := \left( \|u\|_{B^2}^2 + \|\nabla u\|_{B^2}^2 \right)^{\frac{1}{2}}, \quad \|u\|_{B^{2,2}} := \left( \|u\|_{B^{1,2}}^2 + \|\nabla^2 u\|_{B^2}^2 \right)^{\frac{1}{2}}.$$

When  $u : \mathbb{R} \rightarrow \mathbb{E}$  is a continuous function, it is usual, in the theory of retarded functional differential equations, to consider, for all  $t \in \mathbb{R}$ ,  $u_t \in \mathcal{C}^0([-r, 0], \mathbb{E})$  defined by  $u_t(\theta) := u(t + \theta)$  for all  $\theta \in [-r, 0]$ , [15].

When  $u \in L^2_{loc}(\mathbb{R}, \mathbb{E})$  (Lebesgue space), we denote by  $\tilde{u} : \mathbb{R} \rightarrow L^2_{loc}([-r, 0], \mathbb{E})$  the function defined by  $\tilde{u}(t)(\theta) := u(t + \theta)$ .

### 3. The Strong a.p. Solutions

We consider the following condition on  $f$ :

$$f \in \mathcal{C}^1(\mathbb{E} \times \mathbb{E}, \mathbb{R}). \tag{3.1}$$

LEMMA 3.1. *Under (3.1) we consider the mapping  $F_0 : \mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathbb{R}$  defined by  $F_0(x, \psi) := \int_{-r}^0 f(x, \psi(\theta))d\theta$ . Then  $F_0$  is of class  $\mathcal{C}^1$  on  $\mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E})$  and  $DF_0(x, \psi)(y, \xi) = \int_{-r}^0 D_1f(x, \psi).y d\theta + \int_{-r}^0 D_2f(x, \psi(\theta)).\xi(\theta)d\theta$ .*

*Proof.* The following Nemytskii operator build on  $f$ :

$$\mathcal{N}_f^0 : \mathcal{C}^0([-r, 0], \mathbb{E}) \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathcal{C}^0([-r, 0], \mathbb{E}),$$

$\mathcal{N}_f^0(\phi, \psi) := [\theta \mapsto f(\phi(\theta), \psi(\theta))]$ , is of class  $\mathcal{C}^1$  under (3.1), (see proposition 1 page 168, and proposition 2 page 170 in [1]).

The operator  $A^0 : \mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E}) \rightarrow \mathcal{C}^0([-r, 0], \mathbb{E}) \times \mathcal{C}^0([-r, 0], \mathbb{E})$  defined by  $A^0(x, \psi) = (x, \psi)$  where the vector  $x \in \mathbb{E}$  is considered as a (constant) continuous function, is a linear continuous and therefore  $A^0$  is of class  $\mathcal{C}^1$ . The operator  $I^0 : \mathcal{C}^0([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ ,  $I^0(w) := \int_{-r}^0 w(t)dt$ , is linear continuous and therefore it is of class  $\mathcal{C}^1$ .

Since  $F_0 := I^0 \circ \mathcal{N}_f^0 \circ A^0$ ,  $F_0$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -mappings.

By using the chaine rule, we have

$$DF_0(x, \psi).(y, \xi) = I_0(D_\cdot \mathcal{N}_f^0(A^0(x, \psi)).A^0(y, \xi))$$

We know that  $D_\cdot \mathcal{N}_f^0(A^0(x, \psi)).A^0(y, \xi) = [\theta \mapsto D_1f(x, \psi(\theta)).y + D_2f(x, \psi(\theta)).\xi(\theta)]$ , and so we obtain the announced formula.

LEMMA 3.2. *The operator  $S^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathbb{R})$ , defined by*

$$S^0(u) := \left[ t \mapsto \int_{-r}^0 f(u(t), u(t + \theta))d\theta \right],$$

*is of class  $\mathcal{C}^1$ , and*

$$DS^0(u)h = \left[ t \mapsto \int_{-r}^0 D_1f(u(t), u(t + \theta)).h(t)d\theta + \int_{-r}^0 D_2f(u(t), u(t + \theta)).h(t + \theta)d\theta \right].$$

*Proof.* The Nemytskii operator defined on the mapping  $F_0$  provided by Lemma (3.1),

$$\mathcal{N}_{F_0} : AP^0(\mathbb{E} \times \mathcal{C}^0([-r, 0], \mathbb{E})) \equiv AP^0(\mathbb{E}) \times AP^0(\mathcal{C}^0([-r, 0], \mathbb{E})) \rightarrow AP^0(\mathbb{R}),$$

defined by

$$\mathcal{N}_{F_0}(u, \phi) := \left[ t \mapsto F_0(u(t), \phi(t)) = \int_{-r}^0 f(u(t), \phi(t)(\theta)) d\theta \right]$$

is of class  $\mathcal{C}^1$ , since  $F_0$  is of class  $\mathcal{C}^1$ . ([9], Corollary 5.3).

We introduce the operator  $T^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathcal{C}^0([-r, 0], \mathbb{E}))$  by setting  $T^0(u) := [t \mapsto u_t]$ . Then  $T^0$  is linear,  $T^0$  is continuous since  $\|T^0(u)\|_\infty = \|u\|_\infty$ , and therefore  $T^0$  is of class  $\mathcal{C}^1$ .

Since  $S^0 = \mathcal{N}_{F_0} \circ (id, T^0)$ ,  $S^0$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -operators.

By using the chain rule we have  $DS^0(u).h = D\mathcal{N}_{F_0}((id, T^0)(u)).D(id, T^0)(h) = D\mathcal{N}_{F_0}(u, \tilde{u}).(h, \tilde{h})$ , and by using Lemma (3.1) we obtain

$$\begin{aligned} (DS^0(u).h)(t) &= \int_{-r}^0 D_1 f(u(t), \tilde{u}(t)(\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), \tilde{u}(t)(\theta)).\tilde{h}(t)(\theta) d\theta \\ &= \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \end{aligned}$$

**THEOREM 3.3.** *Under (3.1) the functional  $J_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$ , defined by*

$$J_0(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |u'(t)|^2 + \int_{-r}^0 f(u(t), u(t+\theta)) d\theta + u(t).e(t) \right\},$$

is of class  $\mathcal{C}^1$ , and when  $u \in AP^1(\mathbb{E})$  we have  $DJ_0(u) = 0$  if and only if  $u$  is a strong solution of (1.1)

*Proof.* We consider the functional  $Q_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$  defined by

$$Q_0(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |u'(t)|^2 \right\}.$$

The mapping  $q : \mathbb{E} \rightarrow \mathbb{R}$ ,  $q(x) := \frac{1}{2} |x|^2 = \frac{1}{2} x.x$ , is of class  $\mathcal{C}^1$ , therefore the Nemytskii operator  $\mathcal{N}_q^0 : AP^0(\mathbb{E}) \rightarrow AP^0(\mathbb{R})$ ,  $\mathcal{N}_q^0(\varphi) := [t \mapsto \frac{1}{2} |\varphi(t)|^2]$ , is also of class  $\mathcal{C}^1$ , [7]. The operator  $\frac{d}{dt} : AP^1(\mathbb{E}) \rightarrow AP^0(\mathbb{E})$ ,  $\frac{d}{dt}(u) := u'$ , is linear continuous, therefore it is of class  $\mathcal{C}^1$ . The functional  $\mathfrak{M}^0 : AP^0(\mathbb{R}) \rightarrow \mathbb{R}$ , defined by  $\mathfrak{M}^0(\varphi) := \mathfrak{M}_t \{ \varphi(t) \}$ , is linear continuous, therefore it is of class  $\mathcal{C}^1$ . Since  $Q_0 = \mathfrak{M}^0 \circ \mathcal{N}_q^0 \circ \frac{d}{dt}$ ,  $Q_0$  is of class  $\mathcal{C}^1$  as composition of  $\mathcal{C}^1$ -mappings, and by using the chain rule we have

$$DQ_0(u).h = \mathfrak{M}_t \{ u'(t).h'(t) \} \quad (3.2)$$

We consider the functional  $\Phi_0 : AP^1(\mathbb{E}) \rightarrow \mathbb{R}$  defined by

$$\Phi_0(u) := \mathfrak{M}_t \left\{ \int_{-r}^0 f(u(t), u(t+\theta)) d\theta \right\}.$$

We consider the operator  $in_0 : AP^1(\mathbb{E}) \rightarrow AP^0(\mathbb{E})$ ,  $in_0(u) := u$ , which is linear continuous, and consequently  $in_0$  is of class  $\mathcal{C}^1$ .

We note that we have  $\Phi_0$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -mappings. By using Lemma (3.2) we obtain

$$D\Phi_0(u).h = \mathfrak{M}_t \left\{ \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} \quad (3.3)$$

Now we want to improve this last formula.

Since  $(t, \theta) \mapsto D_2 f(u(t), u(t+\theta)).h(t+\theta)$  is continuous on  $\mathbb{R} \times [-r, 0]$ , it is Lebesgue-integrable and by using the Fubini theorem [2], we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T \left( \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right) dt \\ = \int_{-r}^0 \left( \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) dt \right) d\theta \end{aligned} \quad (3.4)$$

We set  $g_T(\theta) := \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) dt$ . We know that, for all  $\theta \in [-r, 0]$ ,

$$\lim_{T \rightarrow \infty} g_T(\theta) = \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \}$$

since  $t \mapsto D_2 f(u(t), u(t+\theta)).h(t+\theta)$  bellongs to  $AP^0(\mathbb{R})$ .

Furthermore, since  $u, h \in AP^0(\mathbb{E})$ ,  $\overline{u(\mathbb{R})}$  and  $\overline{h(\mathbb{R})}$  are compact, [3, 6, 14], and since the mapping  $(x, y, z) \mapsto D_2 f(x, y).z$  is continuous on the compact  $\overline{u(\mathbb{R})} \times \overline{u(\mathbb{R})} \times \overline{h(\mathbb{R})}$ , it is bounded, and consequently we have :

$$\sup_{\theta \in [-r, 0]} \sup_{t \in \mathbb{R}} |D_2 f(u(t), u(t+\theta)).h(t+\theta)| := \sigma < \infty,$$

that implies  $|g_T(\theta)| \leq \sigma$  for all  $T > 0$ ,  $\theta \in [-r, 0]$ . And so the assumptions of the dominated convergence theorem of Lebesgue are fulfilled, [2], and by using it we obtain

$$\lim_{T \rightarrow \infty} \int_{-r}^0 g_T(\theta) d\theta = \int_{-r}^0 \lim_{T \rightarrow \infty} g_T(\theta) d\theta,$$

and so by using (3.4) we obtain

$$\begin{aligned} \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} d\theta \\ = \lim_{T \rightarrow \infty} \int_{-r}^0 \left( \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)).h(t+\theta) \right) d\theta \\ = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left( \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right) dt \end{aligned}$$

and so we have proven the following equality

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} = \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} d\theta \quad (3.5)$$

By using a similar reasoning we obtain

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t-\theta), u(t)).h(t) d\theta \right\} = \int_{-r}^0 \mathfrak{M}_t \{ D_2 f(u(t-\theta), u(t)).h(t) \} d\theta \quad (3.6)$$

Since the mean value is invariant by translation, [3, 6, 14], we have, for all  $\theta \in [-r, 0]$ , the following equality

$$\mathfrak{M}_t \{ D_2 f(u(t), u(t+\theta)).h(t+\theta) \} = \mathfrak{M}_t \{ D_2 f(u(t-\theta), u(t)).h(t) \}.$$

By using it with (3.5) and (3.6) we obtain

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\} = \mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t-\theta), u(t)).h(t) d\theta \right\}.$$

And by using this last equality in (3.3) we obtain

$$\begin{aligned} D\Phi_0(u).h &= \mathfrak{M}_t \left\{ \left( \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta \right).h(t) \right\} \end{aligned} \quad (3.7)$$

We consider the functional  $\Lambda_0 : AP^0(\mathbb{E}) \rightarrow \mathbb{R}$ , defined by  $\Lambda_0(u) := \mathfrak{M}_t \{ u(t).e(t) \}$ . Note that  $\Lambda_0$  is linear continuous and consequently it is of class  $\mathcal{C}^1$  and we have

$$D\Lambda_0(u).h = \mathfrak{M}_t \{ h(t).e(t) \}. \quad (3.8)$$

Since  $J_0 = Q_0 + \Phi_0 + \Lambda_0$ ,  $J_0$  is of class  $\mathcal{C}^1$  as a sum of three  $\mathcal{C}^1$ -functionals, and by using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} DJ_0(u).h &= \mathfrak{M}_t \{ u'(t).h'(t) + \left( \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \\ &\quad \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t) \right).h(t) \} \end{aligned} \quad (3.9)$$

for all  $u, h \in AP^1(\mathbb{E})$

We set  $p(t) := \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t)$ , and we have  $p \in AP^0(\mathbb{E})$ .

When  $DJ_0(u) = 0$  then by using (3.9) we have  $\mathfrak{M}_t \{ u'(t).h'(t) \} = -\mathfrak{M}_t \{ p(t).h(t) \}$  for all  $h \in AP^1(\mathbb{E})$  and by using the same reasoning that this one of the proof of Theorem 1 in [7] we obtain that  $u \in AP^2(\mathbb{E})$  and  $u''(t) = p(t)$ , that is exactly (1.1).

Conversely, if  $u$  is a strong a.p. solution of (1.1), then we have  $u'' = p$  and so, for all  $h \in AP^1(\mathbb{E})$ , we have  $DJ_0(u).h = \mathfrak{M} \{ u'.h' + p.h \} = \mathfrak{M} \left\{ \frac{d}{dt}(u'.h) \right\} = 0$

**THEOREM 3.4.** *Under (3.1), if we additionally assume that  $f$  is convex function, then the set of the strong a.p. solutions of (1.1) is a convex subset of  $AP^2(\mathbb{E})$ .*

*Proof.* When  $f$  is convex, it is easy to verify that  $J_0$  is convex,  $DJ_0 = 0$  is equivalent to  $J_0 = \inf J_0(AP^1(\mathbb{E}))$ , [10], and  $\{u \in AP^1(\mathbb{E}) : J_0 = \inf J_0(AP^1(\mathbb{E}))\}$  is convex. And so  $\{u \in AP^1(\mathbb{E}) : DJ_0 = 0\}$  is convex, and we obtain the conclusion by using Theorem (3.3).

A consequence of Theorem (3.4) is the following one: when  $e = 0$ , if (1.1) possesses a non-constant  $T_1$ -periodic solution  $u_1$  and a non-constant  $T_2$ -periodic solution  $u_2$  with  $T_1/T_2 \notin \mathbb{Q}$  the  $\frac{1}{2}u_1 + \frac{1}{2}u_2$  is a non-periodic a.p. solution of (1.1) since it is a convex combination of a.p. solution.

#### 4. The Weak a.p. solutions

We begin this section by giving a precise definition of the notion of weak a.p. solution of (1.1). A weak a.p. solution of (1.1) is a function  $u \in B^{2,2}(\mathbb{E})$  such that

$$\nabla^2 u = \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t),$$

this equality holding in  $B^2(\mathbb{E})$ .

We begin by establishing two lemmas which contain general properties of the Besicovitch a.p. functions.

LEMMA 4.1. *Let  $u \in B^2(\mathbb{E})$ . Then the following equalities hold*

$$\begin{aligned} \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t+\theta)|^2 d\theta \right\} &= \int_{-r}^0 \mathfrak{M}_t \left\{ |u(t+\theta)|^2 \right\} d\theta \\ &= r \mathfrak{M}_t \left\{ |u(t)|^2 \right\} \end{aligned}$$

*Proof.* Since  $\mathfrak{M}_t \left\{ |u(t)|^2 \right\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt$  exists in  $\mathbb{R}_+$ , we have

$$M := \sup_{T \geq 1} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt < \infty.$$

For all  $\theta \in [-r, 0]$  we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt &= \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds \\ &\leq \frac{1}{2T} \int_{-T-r}^{T-r} |u(t)|^2 dt \\ &\leq \frac{1}{2T} \int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &= \frac{1.2(T+r)}{2T} \cdot \frac{1}{2(T+r)} \int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &\leq \left(1 + \frac{r}{T}\right) \cdot M \leq (1+r) \cdot M =: M_1, \end{aligned}$$



and so we have proven

$$\exists M_1 > 0, \forall \theta \in [-r, 0], \forall T \geq 1, \frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt \leq M_1 < \infty \quad (4.1)$$

For all  $T \geq 1$  we define  $\Phi_T : [-r, 0] \rightarrow \mathbb{R}$  by setting  $\Phi_T(\theta) := \frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt$ . Since  $\Phi_T(\theta) = \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds$  we see that  $\Phi_T$  is absolutely continuous on  $[-r, 0]$ , and consequently we have  $\Phi_T \in L^1([-r, 0], \mathbb{R})$ .

If  $[a, b]$  is segment in  $\mathbb{R}$ , for all  $\theta \in [-r, 0]$ , by using the Fubini theorem for the non negative measurable functions [2], we have

$$\begin{aligned} \int_{[a,b] \times [-r,0]} |u(t + \theta)|^2 dt d\theta &= \int_{[-r,0]} \left( \int_{[a,b]} |u(t + \theta)|^2 dt \right) d\theta \\ &= \int_{[-r,0]} \left( \int_{[a,b]+\theta} |u(s)|^2 ds \right) d\theta \\ &\leq \int_{[-r,0]} \left( \int_{[a,b]+[-r,0]} |u(s)|^2 ds \right) d\theta \\ &= r \int_{[a,b]+[-r,0]} |u(s)|^2 ds < \infty \end{aligned}$$

since  $|u|^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R}_+)$  and since  $[a, b] + [-r, 0]$  is compact. And so we have proven:

$$(t, \theta) \mapsto |u(t + \theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R}) \quad (4.2)$$

Then by using the Fubini theorem [2], for all  $T > 0$  we obtain

$$\frac{1}{2T} \int_{-T}^T \left( \int_{-r}^0 |u(t + \theta)|^2 d\theta \right) dt = \int_{-r}^0 \left( \frac{1}{2T} \int_{-T}^T |u(t + \theta)|^2 dt \right) d\theta \quad (4.3)$$

Since  $u \in B^2(\mathbb{E})$ , we have  $\lim_{T \rightarrow \infty} \Phi_T(\theta) = \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} = \mathfrak{M}_t \left\{ |u(t)|^2 \right\}$  since the mean value is invariant by translation, for all  $\theta \in [-r, 0]$ . The constant  $M_1$  is integrable on  $[-r, 0]$ . And so by using (4.1), we can apply the dominated convergence theorem of Lebesgue to obtain  $\int_{-r}^0 \lim_{T \rightarrow \infty} \Phi_T(\theta) d\theta = \lim_{T \rightarrow \infty} \int_{-r}^0 \Phi_T(\theta) d\theta$ , that implice by using (4.3) that  $\int_{-r}^0 \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} d\theta = \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta)|^2 d\theta \right\}$ . And since  $\mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} = \mathfrak{M}_t \left\{ |u(t)|^2 \right\}$  for all  $\theta$ , we have also

$$\int_{-r}^0 \mathfrak{M}_t \left\{ |u(t + \theta)|^2 \right\} d\theta = r \mathfrak{M}_t \left\{ |u(t)|^2 \right\}.$$

**LEMMA 4.2.** *If  $u \in B^2(\mathbb{E})$  then  $\tilde{u} \in B^2(L^2([-r, 0], \mathbb{E}))$  and we have*

$$\|\tilde{u}\|_{B^2(L^2([-r, 0], \mathbb{E}))} = \sqrt{r} \cdot \|u\|_{B^2(\mathbb{E})}$$

*Proof.* We fix  $u \in B^2(\mathbb{E})$ , and  $\varepsilon > 0$ . We can choose  $q_\varepsilon \in AP^0(\mathbb{E})$  such that  $\|u - q_\varepsilon\|_{B^2(\mathbb{E})} < \varepsilon$ .

Since  $L^2([-r, 0], \mathbb{E})$  is separable, there exists a countable subset  $D$  in  $L^2([-r, 0], \mathbb{E})$  which is dense, and consequently the set  $\{B(\varphi, \rho) : \varphi \in D, \rho \in \mathbb{Q} \cap (0, \infty)\}$  is a generator of the Borel  $\sigma$ -field of  $L^2([-r, 0], \mathbb{E})$ , where

$$B(\varphi, \rho) := \left\{ \psi \in L^2([-r, 0], \mathbb{E}) : \|\psi - \varphi\|_{L^2([-r, 0], \mathbb{E})} < \rho \right\}.$$

We arbitrarily fix  $\varphi \in D$  and  $\rho \in \mathbb{Q} \cap (0, \infty)$ , and we set

$$\alpha(t) := \int_{-r}^0 |u(t + \theta) - \varphi(\theta)|^2 d\theta.$$

By using the same reasoning that this one used to establish (4.2) we obtain that  $(t, \theta) \mapsto |u(t + \theta) - \varphi(\theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$  and consequently by using the Fubini theorem we know that  $\alpha \in L^1_{loc}(\mathbb{R}, \mathbb{R})$  and then we necessarily have  $\alpha$  measurable.

We note that  $t \in \tilde{u}^{-1}(B(\varphi, \rho))$  is equivalent to  $t \in \alpha^{-1}([0, \rho^2])$ . Since  $\alpha$  is measurable we have  $\alpha^{-1}([0, \rho^2]) \in \mathcal{B}(\mathbb{R})$  and consequently  $\tilde{u}^{-1}(B(\varphi, \rho)) \in \mathcal{B}(\mathbb{R})$ , and so we have proven:

$$\tilde{u} \text{ is measurable from } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ in } (L^2([-r, 0], \mathbb{E}), \mathcal{B}(L^2([-r, 0], \mathbb{E}))). \quad (4.4)$$

By using (4.2) we know that  $(t, \theta) \mapsto |u(t + \theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$  and consequently, by using the Fubini theorem we obtain that  $t \mapsto \int_{-r}^0 |u(t + \theta)|^2 d\theta = \|\tilde{u}(t)\|_{L^2([-r, 0], \mathbb{E})}^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ .

Therefore we have obtained, [2] :

$$\tilde{u} \in L^2_{loc}(\mathbb{R}, L^2([-r, 0], \mathbb{E})). \quad (4.5)$$

By using Lemma (4.1) with  $u - q_\varepsilon$  instead of  $u$ , we know that

$$\mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta) - q_\varepsilon(t + \theta)|^2 d\theta \right\}$$

exists and that we have

$$\begin{aligned} \mathfrak{M}_t \left\{ \|\tilde{u}(t) - \tilde{q}(t)\|_{L^2([-r, 0], \mathbb{E})} \right\} &= \mathfrak{M}_t \left\{ \int_{-r}^0 |u(t + \theta) - q_\varepsilon(t + \theta)|^2 d\theta \right\} \\ &= r \mathfrak{M}_t \left\{ |u(t) - q_\varepsilon(t)|^2 \right\} < r \varepsilon^2. \end{aligned}$$

Since  $\tilde{q}_\varepsilon \in AP^0(\mathcal{C}^0([-r, 0], \mathbb{E})) \subset AP^0(L^2([-r, 0], \mathbb{E}))$ , when  $\varepsilon \rightarrow 0$ , we obtain that  $\tilde{u} \in B^2(L^2([-r, 0], \mathbb{E}), \mathbb{E})$ .

The relation between the norms of  $u$  and  $\tilde{u}$  is a consequence of Lemma (4.1).

By modifying a function  $u \in B^2(\mathbb{E})$  on a bounded interval of  $\mathbb{R}$  we do not modify the (class of the) function  $u$ , and so we can ask to use  $\tilde{u}(t)$ , defined as the restriction of  $u$  on the interval  $[t - r, t]$ , possesses a meaning. Lemma (4.2) provides an answer

to this question, since if  $v \in B^2(\mathbb{E})$  is different of  $u$ , then we have  $\tilde{u} \neq \tilde{v}$ . And so the definition of  $\tilde{u}$  is consistent, and the notion of weak a.p. solution is also consistent.

Now we introduce the following condition on  $f$ :

$$\begin{cases} \text{There exists } a \in (0, \infty) \text{ and } b \in \mathbb{R} \text{ such that} \\ |Df(x, y)| \leq a(|x| + |y|) + b \text{ for all } x, y \in \mathbb{E}. \end{cases} \quad (4.6)$$

LEMMA 4.3. *Under (3.1) and (4.6), the operator  $S : B^2(\mathbb{E}) \rightarrow B^1(\mathbb{R})$  defined by  $S(u) := \left[ t \mapsto \int_{-r}^0 f(u(t), u(t + \theta)) d\theta \right]$  is of class  $\mathcal{C}^1$  and for all  $u, h \in B^2(\mathbb{E})$ , we have  $DS(u).h = \left[ t \mapsto \int_{-r}^0 D_1 f(u(t), u(t + \theta)).h(t) + D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right]$*

*Proof.* The Nemytskii operator build on  $f$ ,  $\mathcal{N}_f : L^2([-r, 0], \mathbb{E}) \times L^2([-r, 0], \mathbb{E}) \rightarrow L^1([-r, 0], \mathbb{R})$ ,  $\mathcal{N}_f(\varphi, \psi) := [\theta \mapsto f(\varphi(\theta), \psi(\theta))]$ , under (3.1) and (4.6) is of class  $\mathcal{C}^1$ , [11], and  $D\mathcal{N}_f(\varphi, \psi).(\xi, \zeta) = [\theta \mapsto Df(\varphi(\theta), \psi(\theta)).(\xi(\theta), \zeta(\theta))]$

The operator  $A : \mathbb{E} \times L^2([-r, 0], \mathbb{E}) \rightarrow (L^2([-r, 0], \mathbb{E}))^2$  defined by  $A(x, \psi) := (x, \psi)$ , where  $x$  is considered as a constant function, is linear continuous, therefore  $A$  is of class  $\mathcal{C}^1$  and  $DA(x, \psi) = A$ .

The functional  $I : L^1([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ ,  $I(w) := \int_{-r}^0 w(\theta) d\theta$ , is linear continuous, therefore  $I$  is of class  $\mathcal{C}^1$  and  $DI(w) = I$ .

We consider the mapping  $F : \mathbb{E} \times L^2([-r, 0], \mathbb{E}) \rightarrow \mathbb{R}$ , defined by  $F(x, \psi) := \int_{-r}^0 f(x, \psi(\theta)) d\theta$ .

We note that  $F = I \circ \mathcal{N}_f \circ A$ , and so  $F$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -mappings, and by using the chain rule we obtain, for all  $x, y \in \mathbb{E}$  and for all  $\psi, \xi \in L^2([-r, 0], \mathbb{E})$ , the following formula:

$$DF(x, \psi).(y, \xi) = \int_{-r}^0 (D_1 f(x, \psi(\theta)).y + D_2 f(x, \psi(\theta)).\xi(\theta)) d\theta.$$

Let  $(y, \xi) \in \mathbb{E} \times L^2([-r, 0], \mathbb{E})$  such that  $\|(y, \xi)\| \leq 1$ . Then we have

$$\begin{aligned} |DF(x, \psi).(y, \xi)| &\leq \int_{-r}^0 |Df(x, \psi(\theta))| \cdot |(y, \xi(\theta))| d\theta \\ &\leq \left( \int_{-r}^0 |Df(x, \psi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \cdot \left( \int_{-r}^0 |(y, \xi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \end{aligned}$$

by using the Cauchy-Schwarz-Buniakovski inequality.

We note that

$$\int_{-r}^0 |(y, \xi(\theta))|^2 d\theta = \int_{-r}^0 (|y|^2 + |\xi(\theta)|^2) d\theta = r|y|^2 + \int_{-r}^0 |\xi(\theta)|^2 d\theta \leq r_1 \cdot \|(y, \xi)\|^2 \leq r_1,$$

where  $r_1 := \max\{r, 1\}$ , and so we have:

$$\begin{aligned}
 |DF(x, \psi).(y, \xi)| &\leq \sqrt{r_1} \cdot \left( \int_{-r}^0 |Df(x, \psi(\theta))|^2 d\theta \right)^{\frac{1}{2}} \\
 &\leq \sqrt{r_1} \cdot \left( \int_{-r}^0 (a \cdot |x| + a \cdot |\psi(\theta)| + b)^2 d\theta \right)^{\frac{1}{2}} \\
 &= \sqrt{r_1} \cdot \|a \cdot |x| + a \cdot |\psi| + |b|\|_{L^2([-r,0],\mathbb{E})} \\
 &\leq \sqrt{r_1} \cdot \left( a \| |x| \|_{L^2([-r,0],\mathbb{R})} + a \| |\psi| \|_{L^2([-r,0],\mathbb{R})} + \| |b| \|_{L^2([-r,0],\mathbb{R})} \right).
 \end{aligned}$$

Since  $\| |x| \|_{L^2([-r,0],\mathbb{R})} = \sqrt{r} \cdot |x|$ ,  $\| |b| \|_{L^2([-r,0],\mathbb{R})} = \sqrt{r} \cdot |b|$  and  $\| |\psi| \|_{L^2([-r,0],\mathbb{R})} = \| \psi \|_{L^2([-r,0],\mathbb{E})}$ , we have

$$|DF(x, \psi).(y, \xi)| \leq a \cdot \sqrt{r_1} \cdot \sqrt{r} \left( |x| + \| \psi \|_{L^2([-r,0],\mathbb{E})} \right) + \sqrt{r_1} \cdot \sqrt{r} \cdot |b|.$$

We set  $a_1 := a \cdot \sqrt{r_1} \cdot \sqrt{r}$  and  $b_1 := \sqrt{r_1} \cdot \sqrt{r} |b|$  and so we obtain:

$$|DF(x, \psi)| \leq a_1 \cdot \left( |x| + \| \psi \|_{L^2([-r,0],\mathbb{E})} \right) + b_1.$$

And so the assumption of ([11], Theorem 2.6 page 14) are fulfilled and we can assert that  $\mathcal{N}_F : B^2(\mathbb{E}) \times B^2(L^2) \rightarrow B^1(\mathbb{R})$  is of class  $\mathcal{C}^1$  and that we have, for all  $u, h \in B^2(\mathbb{E})$  and for all  $V, K \in L^2([-r, 0], \mathbb{E})$ , the following formula

$$\begin{aligned}
 D_x \mathcal{N}_F(u, V).(h, K) &= [t \mapsto DF(u(t), V(t)).(h(t), K(t))] \\
 &= \int_{-r}^0 (D_1 f(u(t), V(t)(\theta)).h(t) + D_2 f(u(t), V(t)(\theta)).K(t)(\theta)) d\theta \quad (4.7)
 \end{aligned}$$

We consider the linear operator  $T : B^2(\mathbb{E}) \rightarrow B^2(L^2([-r, 0], \mathbb{E}))$  defined by  $T(u) := \tilde{u}$ . By using Lemma (4.2) we know that  $T$  is continuous, and therefore  $T$  is of class  $\mathcal{C}^1$  with  $DT(u) = T$ .

We note that we have  $S = \mathcal{N}_F \circ (id, T)$ , and so  $S$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -operators, and by using the chain rule and (4.7) we obtain the announced formula.

LEMMA 4.4. *Under (3.1) and (4.6), if  $u$  and  $h$  belong to  $B^2(\mathbb{E})$  then the following equality holds :*

$$\mathfrak{M}_t \left\{ \int_{-r}^0 D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right\} = \mathfrak{M}_t \left\{ \left( \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta \right) . h(t) \right\}$$

*Proof.* By using a reasoning similar to this one used to establish (4.2) we obtain that  $(t, \theta) \mapsto D_2 f(u(t), u(t + \theta)).h(t + \theta) \in L^1_{loc}(\mathbb{R} \times [-r, 0], \mathbb{R})$ . And so we can use the Fubini theorem to obtain

$$\begin{aligned}
 \frac{1}{2T} \int_{-T}^T \left( \int_{-r}^0 D_2 f(u(t), u(t + \theta)).h(t + \theta) d\theta \right) dt \\
 = \int_{-r}^0 \left( \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t + \theta)).h(t + \theta) dt \right) d\theta \quad (4.8)
 \end{aligned}$$

for all  $T \in (0, \infty)$ .

For all  $T \in [1, \infty)$  we introduce the function  $g_T : [-r, 0] \rightarrow \mathbb{R}$  defined by

$$g_T(\theta) := \frac{1}{2T} \int_{-T}^T D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta) dt$$

Ever using the Fubini theorem we know that the  $g_T$  are borelian.

Since  $t \mapsto D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta) \in B^1(\mathbb{R})$  we know that the mean value exists in  $\mathbb{R}$  and consequently we have

$$\lim_{T \rightarrow \infty} g_T(\theta) = \mathfrak{M}_t \{D_2 f(u(t), u(t+\theta)) \cdot h(t+\theta)\}$$

for all  $\theta \in [-r, 0]$ .

Since  $\mathfrak{M}_t \{|u(t)|^2\}$  exists in  $\mathbb{R}$ , we have  $\sup_{t \geq 1} \left( \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt \right) =: M < \infty$ .

For all  $\theta \in [-r, 0]$  and, for all  $T \geq 1+r$ , we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt &= \frac{1}{2T} \int_{-T+\theta}^{T+\theta} |u(s)|^2 ds \\ &\leq \frac{1}{2T} \int_{-T+\theta}^{T-\theta} |u(s)|^2 ds \\ &= \frac{2(T-\theta)}{2T} \cdot \frac{1}{2(T-\theta)} \cdot \int_{-(T-\theta)}^{T-\theta} |u(t)|^2 dt \\ &\leq (1+r) \cdot M =: M_0 \end{aligned}$$

And so we have proven the following assertion

$$\begin{cases} \text{There exists } M_0 \in (0, \infty) \text{ such that, for all} \\ \theta \in [-r, 0], \quad \sup_{T \geq 1+r} \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt \leq M_0. \end{cases} \quad (4.9)$$

Replacing  $u$  by  $h$  we similarly obtain the following assertion.

$$\begin{cases} \text{There exists } M_1 \in (0, \infty) \text{ such that, for all} \\ \theta \in [-r, 0], \quad \sup_{T \geq 1+r} \frac{1}{2T} \int_{-T}^T |h(t+\theta)|^2 dt \leq M_1. \end{cases} \quad (4.10)$$

By using the equivalence of the norms of  $\mathbb{R}^2$  and the usual inequality  $(A+B)^2 \leq 2(A^2+B^2)$  we obtain the existence of  $a_2 \in (0, \infty)$  such that

$$\begin{aligned} |D_2 f(u(t), u(t+\theta))|^2 &\leq \left( a_2 \left[ |u(t)|^2 + |u(t+\theta)|^2 \right]^{\frac{1}{2}} + b \right)^2 \\ &\leq 2 \cdot \left( a_2 |u(t)|^2 + a_2 |u(t+\theta)|^2 + b^2 \right) \end{aligned}$$

that implies

$$\begin{aligned} \left( \frac{1}{2T} \int_{-T}^T |D_2 f(u(t), u(t+\theta))|^2 dt \right)^{\frac{1}{2}} &\leq \sqrt{2} (a_2 \cdot \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt \\ &\quad + a_2 \cdot \frac{1}{2T} \int_{-T}^T |u(t+\theta)|^2 dt + b^2)^{\frac{1}{2}} \\ &\leq \sqrt{2} (a_2 M_0 + a_2 M_0 + b^2)^{\frac{1}{2}}. \end{aligned}$$

Then by setting  $\gamma := \sqrt{2} (2a_2M_0 + b^2)^{\frac{1}{2}} M_1^{\frac{1}{2}}$  we have proven the following assertion

$$\left( \frac{1}{2T} \int_{-T}^T |D_2f(u(t), u(t + \theta))|^2 dt \right)^{\frac{1}{2}} \cdot \left( \frac{1}{2T} \int_{-T}^T |h(t + \theta)|^2 dt \right)^{\frac{1}{2}} \leq \gamma \quad (4.11)$$

By using the Cauchy-Schwarz-Buniakovski inequality and (4.11) we obtain, for all  $T \geq 1 + r$  and for  $\theta \in [-r, 0]$ ,

$$|g_T(\theta)| \leq \frac{1}{2T} \int_{-T}^T |D_2f(u(t), u(t + \theta))| \cdot |h(t + \theta)| dt \leq \sigma$$

Since the Lebesgue measure of  $[-r, 0]$  is finite, the constant  $\sigma$  is Lebesgue integrable in  $[-r, 0]$ , and consequently the assumptions of the Lebesgue Dominated Convergence theorem are fulfilled and we can say :

$$\int_{-r}^0 \lim_{T \rightarrow \infty} g_T(\theta) d\theta = \lim_{T \rightarrow \infty} \int_{-r}^0 g_T(\theta) d\theta,$$

and we can conclude as in the proof of (3.5), (3.6), (3.7).

**THEOREM 4.5.** *We assume (3.1) and (4.6) fulfilled. Then the functional  $J : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$ , defined by*

$$J(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |\nabla u(t)|^2 + \int_{-r}^0 f(u(t), u(t + \theta)) d\theta + u(t) \cdot e(t) \right\},$$

*is of class  $\mathcal{C}^1$ . And when  $u \in B^{1,2}(\mathbb{E})$ , we have  $DJ(u).h = 0$  if and only if  $u$  is a weak a.p. solution of (1.1).*

*Proof.* We consider the functional  $Q : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$  defined by

$$Q(u) := \mathfrak{M}_t \left\{ \frac{1}{2} |\nabla u(t)|^2 \right\}.$$

We set  $q(x) := \frac{1}{2} |x|^2$ ;  $q : \mathbb{E} \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$ -function since  $\mathbb{E}$  is euclidean. Since  $Dq(x) = x$ ,  $q$  satisfies the condition of ([11], Theorem 2.6 page 14) to ensure that the Nemytskii operator  $\mathcal{N}_q : B^2(\mathbb{E}) \rightarrow B^1(\mathbb{R})$  is of class  $\mathcal{C}^1$  and  $D\mathcal{N}_q(v).h = [t \mapsto v(t).h(t)]$  for all  $v, h \in B^2(\mathbb{E})$ . Since the derivation operator  $\nabla : B^{1,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$  is linear continuous, it is of class  $\mathcal{C}^1$  and since the operator  $\mathfrak{M} : B^1(\mathbb{R}) \rightarrow \mathbb{R}$ ,  $\mathfrak{M}(v) := \mathfrak{M}_t \{v(t)\}$  is also linear continuous, it is of class  $\mathcal{C}^1$ . And so  $Q := \mathfrak{M} \circ \mathcal{N}_q \circ \nabla$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -mappings. Moreover by using the chain rule we have

$$DQ(u).h = \mathfrak{M}_t \{ \nabla u(t) \cdot \nabla h(t) \} \quad (4.12)$$

for all  $u, h \in B^{1,2}(\mathbb{E})$ .

We consider the functional  $\Phi : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$  defined by

$$\Phi(u) := \mathfrak{M}_t \left\{ \int_{-r}^0 f(u(t), u(t + \theta)) d\theta \right\}.$$

We note that the injection  $in : B^{1,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$ ,  $in(u) := u$ , is linear continuous and consequently it is of class  $\mathcal{C}^1$ . We note that  $\Phi = \mathfrak{M} \circ S \circ in$ , and by using Lemma (4.3) we know that  $S$  is of class  $\mathcal{C}^1$ . And so  $\Phi$  is of class  $\mathcal{C}^1$  as a composition of  $\mathcal{C}^1$ -mapping. Ever using Lemma (4.3) and the chain rule we obtain the following formula

$$\begin{aligned} D\Phi(u).h &= \mathfrak{M}_t \left\{ \int_{-r}^0 D_1 f(u(t), u(t+\theta)).h(t) d\theta \right. \\ &\quad \left. + \int_{-r}^0 D_2 f(u(t), u(t+\theta)).h(t+\theta) d\theta \right\}, \end{aligned}$$

and by using Lemma (4.4) we obtain

$$\begin{aligned} D\Phi(u).h &= \mathfrak{M}_t \left\{ \left( \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta \right).h(t) \right\}. \end{aligned} \tag{4.13}$$

We consider the linear functional  $\Lambda : B^{1,2}(\mathbb{E}) \rightarrow \mathbb{R}$  defined by

$$\Lambda(u) := \mathfrak{M}_t \{u(t).e(t)\},$$

and the linear functional  $L : B^2(\mathbb{E}) \rightarrow \mathbb{R}$  defined by  $L(u) := \mathfrak{M}_t \{u(t).e(t)\} = (u|e)_{B^2(\mathbb{E})}$ . Since  $L$  is continuous (by using the Cauchy-Schwarz-Buniakovski inequality),  $\Lambda := L \circ in$  is also continuous as a composition of continuous mappings, and consequently  $\Lambda$  is of class  $\mathcal{C}^1$ . Moreover, since  $D\Lambda(u) = 1$  we obtain the following formula

$$D\Lambda(u).h = \mathfrak{M}_t \{h(t).e(t)\}, \text{ for all } u, h \in B^{1,2}(\mathbb{E}). \tag{4.14}$$

We note that  $J = Q + \Phi + \Lambda$ , and so  $J$  is of class  $\mathcal{C}^1$  as a sum of  $\mathcal{C}^1$ -functionals. Moreover, by using (4.12), (4.13), (4.14), we obtain

$$\begin{aligned} DJ(u).h &= \mathfrak{M}_t \left\{ \nabla u(t). \nabla h(t) + \left( \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 D_2 f(u(t-\theta), u(t)) d\theta + e(t) \right).h(t) \right\} \end{aligned} \tag{4.15}$$

for all  $u, h \in B^{1,2}(E)$ .

We set  $p(t) := \int_{-r}^0 [D_1 f(u(t), u(t+\theta)) + D_2 f(u(t-\theta), u(t))] d\theta + e(t)$  ( $\in B^2(\mathbb{E})$ ). And so the condition  $DJ(u) = 0$  can be written as  $\mathfrak{M}_t \{ \nabla u(t). \nabla h(t) \} = -\mathfrak{M}_t \{ p(t).h(t) \}$  for all  $h \in B^{1,2}(\mathbb{E})$ . And so by using [8], this last condition implies that  $\nabla u \in B^{1,2}(\mathbb{E})$ , i.e.  $u \in B^{2,2}(\mathbb{E})$ , and  $\nabla^2 u = p$  which exactly means that  $u$  is a weak a.p. solutions of (1.1).

Conversely, since  $\mathfrak{M} \{ \nabla v \} = 0$  for all  $v \in B^{1,2}(\mathbb{R})$ , we have  $0 = \mathfrak{M} \{ \nabla(\nabla u.h) \} = \mathfrak{M} \{ \nabla^2 u.h \} + \mathfrak{M} \{ \nabla u. \nabla h \} = \mathfrak{M} \{ p.h \} + \mathfrak{M} \{ \nabla u. \nabla h \}$  for all  $h \in B^{1,2}(\mathbb{E})$ , that implies  $DJ(u) = 0$ .

Now we introduce an assumption of convexity :

$$f \text{ is a convex function on } \mathbb{E} \times \mathbb{E} \tag{4.16}$$

and an assumption of coerciveness :

$$\begin{cases} \text{There exists } c \in (0, \infty) \text{ and } d \in \mathbb{R} \text{ such that} \\ f(x, y) \geq c|x|^2 + d \text{ for all } (x, y) \in \mathbb{E} \times \mathbb{E}. \end{cases} \tag{4.17}$$

**THEOREM 4.6.** *Under (3.1), (4.6), (4.16), (4.17), for all  $e \in B^2(\mathbb{E})$ , there exists  $u \in B^{2,2}(\mathbb{E})$  which is a weak a.p. solution of (1.1). Moreover the set of the weak a.p. solutions of (1.1) is a convex set.*

*Proof.* After Theorem (4.5) we know that the functional  $J$  is of class  $\mathcal{C}^1$  on  $B^{1,2}(\mathbb{E})$ . By using (4.16) we deduce that  $J$  is a convex functional. Then  $J$  is weakly lower semi-continuous on the Hilbert space  $B^{1,2}(\mathbb{E})$ , [16]. From (4.17) we deduce that, for all  $u \in B^{1,2}(\mathbb{E})$ , we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|\nabla u\|_{B^2(\mathbb{E})}^2 + c \|u\|_{B^2(\mathbb{E})}^2 - \|u\|_{B^2(\mathbb{E})} \cdot \|e\|_{B^2(\mathbb{E})} \\ &\geq c_1 \cdot \|u\|_{B^{1,2}(\mathbb{E})}^2 - \|e\|_{B^2(\mathbb{E})} \cdot \|u\|_{B^{1,2}(\mathbb{E})} \end{aligned}$$

where  $c_1 := \min\{\frac{1}{2}, c\} \in (0, \infty)$ . Consequently  $J$  is coercive, i.e.  $J(u) \rightarrow \infty$  when  $\|u\|_{B^{1,2}(\mathbb{E})}^2 \rightarrow \infty$ . Then, [10], we can assert that there exists  $u \in B^{1,2}(\mathbb{E})$  such that  $J(u) = \inf J(B^{1,2}(\mathbb{E}))$ , and since  $J$  is of class  $\mathcal{C}^1$  we have  $DJ(u) = 0$ , and then, by using Theorem (4.5), we know that  $u$  is a weak a.p. solution of (1.1).

Ever using Theorem (4.5), we know that the set of the weak a.p. solutions of (1.1) is equal to the following set:  $\{u \in B^{1,2}(\mathbb{E}) : DJ(u) = 0\}$ , and since  $J$  is convex this last it is equal to the set  $\{u \in B^{1,2}(\mathbb{E}) : J(u) = \inf J(B^{1,2}(\mathbb{E}))\}$ . Since  $J$  is convex this last set is a convex set. And so the set of the weak a.p. solutions of (1.1) is convex.

### 5. Density

**LEMMA 5.1.** *Under (3.1) and (4.16) we consider the operator  $\Gamma_1 : B^2(\mathbb{E}) \rightarrow B^2(E)$  defined by*

$$\Gamma_1(u) := \left[ t \mapsto \int_{-r}^0 D_1 f(u(t), u(t+\theta)) d\theta \right].$$

*Then  $\Gamma_1$  is continuous.*

*Proof.* Under (3.1) and (4.6) we know that we have  $|D_1 f(x, y)| \leq a(|x| + |y|) + b$  for all  $x, y \in \mathbb{E}$ . Then ([11], Theorem 2.5 page 9), the Nemytskii operator  $\mathcal{N}_{D_1 f} : L^2([-r, 0], E) \times L^2([-r, 0], E) \rightarrow L^2([-r, 0], E)$ ,  $\mathcal{N}_{D_1 f}(\varphi, \psi) := [\theta \mapsto D_1 f(\varphi(\theta))$ ,



$\psi(\theta)]$ , is continuous. We know that the operator  $A$ ,  $A(x, \psi) = (x, \psi)$ , used in the proof of Lemma (4.3), is continuous from  $\mathbb{E} \times L^2([-r, 0], E)$  in  $L^2([-r, 0], E) \times L^2([-r, 0], E)$ . The functional  $I$  used in the proof of Lemma (4.3) is continuous.

We define  $F_1 : \mathbb{E} \times L^2([-r, 0], E) \rightarrow \mathbb{R}$  by setting

$$F_1(x, \psi) := I \circ \mathcal{N}_{D_1 f} \circ A(x, \psi) = \int_{-r}^0 D_1 f(u(t), u(t + \theta)) d\theta.$$

Then  $F_1$  is continuous as a composition of continuous mappings.

For all  $x \in \mathbb{E}$  and  $\psi \in L^2([-r, 0], E)$  we have

$$\begin{aligned} |F_1(x, \psi)| &\leq \int_{-r}^0 |D_1 f(x, \psi(\theta))| d\theta \\ &\leq \int_{-r}^0 (a|x| + a|\psi(\theta)| + b) d\theta \\ &= r.a. |x| + a. \int_{-r}^0 |\psi(\theta)| d\theta + r.b \\ &\leq r.a. |x| + a. \sqrt{r} \|\psi\|_{L^2([-r, 0], E)} + r.b \\ &\leq a_3 \cdot \left( |x| + \|\psi\|_{L^2([-r, 0], E)} \right) + r.b, \end{aligned}$$

where  $a_3 := a. \max\{r, \sqrt{r}\}$ . And so the assumptions of ([18], Remark 2.7 page 54) are fulfilled that ensure that the Nemytskii operator

$$\mathcal{N}_{F_1} : B^2(\mathbb{E}) \times B^2(L^2([-r, 0], E)) \rightarrow B^2(\mathbb{E}),$$

$$\mathcal{N}_{F_1}(u, \xi) := \left[ t \mapsto F_1(u(t), \xi(t)) = \int_{-r}^0 D_1 f(u(t), \xi(t)(\theta)) d\theta \right]$$

is continuous.

We note that  $\Gamma_1 = \mathcal{N}_{F_1} \circ (id, T)$ , where  $T(u) = \tilde{u}$ , and so  $\Gamma_1$  is continuous as a composition of continuous mappings.

LEMMA 5.2. *Under (3.1) and (4.6) we consider the operator  $\Gamma_2 : B^2(\mathbb{E}) \rightarrow B^2(\mathbb{E})$  defined by*

$$\Gamma_2(u) := \left[ t \mapsto \int_{-r}^0 D_2 f(u(t - \theta), u(t)) d\theta \right].$$

Then  $\Gamma_2$  is continuous.

*Proof.* By a reasoning similar to this one used in Lemma (5.1), the Nemytskii operator  $\mathcal{N}_{D_2 f} : L^2([-r, 0], E) \times L^2([-r, 0], E) \rightarrow L^2([-r, 0], E)$ ,  $\mathcal{N}_{D_2 f}(\varphi, \psi) := [\theta \mapsto D_2 f(\varphi(\theta), \psi(\theta))]$ , is continuous.

We introduce the operator  $A_1 : L^2([-r, 0], E) \times \mathbb{E} \rightarrow L^2([-r, 0], E) \times L^2([-r, 0], E)$ ,  $A_1(\varphi, y) := [\theta(\varphi(\theta), y)]$ .  $A_1$  is linear continuous.

We consider also the functional  $I$  like in the proof of Lemma (5.1).

We define  $F_2 : L^2([-r, 0], E) \times \mathbb{E} \rightarrow \mathbb{R}$  by setting

$$F_2(\varphi, y) := I \circ \mathcal{N}_{D_2F} \circ A_1(\varphi, y) = \int_{-r}^0 D_2f(\varphi(\theta), y) d\theta.$$

And also  $F_2$  is continuous as composition of continuous functions. Like in the proof of Lemma (5.1) we establish that

$$|F_2(\varphi, y)| \leq a_3 \left( \|\varphi\|_{L^2([-r, 0], E)} + |y| \right) + rb.$$

And so by using ([18], Remark 2.7 page 54) we know that the Nemytskii operator

$$\begin{aligned} \mathcal{N}_{F_2} : B^2(L^2([-r, 0], E)) \times B^2(\mathbb{E}) &\rightarrow B^2(\mathbb{E}), \\ \mathcal{N}_{F_2}(\xi, u) &:= \left[ t \mapsto \int_{-r}^0 D_2f(\xi(t)(\theta), u(t)) d\theta \right], \end{aligned}$$

is continuous.

For all  $u \in B^2(\mathbb{E})$  and for all  $t \in \mathbb{R}$ , we denote by  $\tilde{u}(t) := [\theta \mapsto u(t - \theta)] \in L^2([-r, 0], E)$ . Proceeding like in Lemma (4.2) we can establish that  $\tilde{u} \in B^2(L^2([-r, 0], E))$  and that  $\|\tilde{u}\|_{B^2(L^2([-r, 0], E))} = \sqrt{r} \cdot \|u\|_{B^2(\mathbb{E})}$ . And so the operator

$$T_1 : B^2(\mathbb{E}) \rightarrow B^2(L^2([-r, 0], E)), \quad T_1(u) := \tilde{u},$$

is linear continuous.

We note that  $\Gamma_2 = \mathcal{N}_{D_2f} \circ (T_1, id)$  that permits us to say that  $\Gamma_2$  is continuous as a composition of continuous mappings.

**THEOREM 5.3.** *Under (3.1), (4.6), (4.16), (4.17), for all  $e \in AP^0(\mathbb{E})$ , and for all  $\varepsilon \in (0, \infty)$ , there exists  $e_\varepsilon \in AP^0(\mathbb{E})$  such that  $\|e - e_\varepsilon\|_{B^2(\mathbb{E})} \leq \varepsilon$  and such that there exists  $u_\varepsilon \in AP^2(\mathbb{E})$  wich is a strong a.p. solution of*

$$u_\varepsilon''(t) = \int_{-r}^0 D_1f(u(t), u(t + \theta)) d\theta + \int_{-r}^0 D_2f(u(t - \theta), u(t)) d\theta + e_\varepsilon(t).$$

*Proof.* We set  $\Gamma := \Gamma_1 + \Gamma_2$  where  $\Gamma_1$  comes from Lemma (5.1) and  $\Gamma_2$  comes from Lemma (5.2). We consider the operator  $\mathcal{S} : B^{2,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$ ,  $\mathcal{S}(u) := \nabla^2(u) - \Gamma(u)$ . The operator  $\nabla^2 : B^{2,2}(\mathbb{E}) \rightarrow B^2(\mathbb{E})$  is linear continuous and by using Lemma (5.1) and Lemma (5.2), we see that  $\mathcal{S}$  is continuous.

By using Theorem (4.6) we know that  $\mathcal{S}(B^{2,2}(\mathbb{E})) = B^2(\mathbb{E})$  and consequently we have  $AP^0(\mathbb{E}) \subset \mathcal{S}(B^{2,2}(\mathbb{E}))$ . Since  $AP^2$  is dense in  $B^{2,2}(\mathbb{E})$  and since  $\mathcal{S}$  is continuous, for all  $e \in AP^0(\mathbb{E})$  and for all  $\varepsilon \in (0, \infty)$ , we obtain that there exists  $u_\varepsilon \in AP^2$  such that  $\|\mathcal{S}(u_\varepsilon) - e\|_{B^2(\mathbb{E})} < \varepsilon$ .

By proceeding like in the proof of Lemma (3.2), we obtain that  $\Gamma_1(u_\varepsilon)$  and  $\Gamma_2(u_\varepsilon)$  belong to  $AP^0(\mathbb{E})$ . Since  $\nabla^2(u_\varepsilon) = u_\varepsilon''$ , [8], we obtain that  $\mathcal{S}(u_\varepsilon) \in AP^0(\mathbb{E})$ . We set  $e_\varepsilon := \mathcal{S}(u_\varepsilon)$ , and so  $e_\varepsilon$  satisfies the announced conditions.

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Moez Ayachi  
Laboratoire Marin Mersenne  
Université Paris 1 Panthéon-Sorbonne, centre P.M.F.  
90 rue de Tolbiac, 75634 Paris cedex 13,  
France  
e-mail: ayachi.moez@laposte.net

Jöel Blot  
Laboratoire Marin Mersenne  
Université Paris 1 Panthéon-Sorbonne, centre P.M.F.  
90 rue de Tolbiac, 75634 Paris cedex 13  
France  
e-mail: blot@univ-paris1.fr