A continuum-tree-valued Markov process

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A CONTINUUM-TREE-VALUED MARKOV PROCESS[1]

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We present a construction of a Lévy continuum random tree (CRT) associated with a super-critical continuous state branching process using the so-called exploration process and a Girsanov theorem. We also extend the pruning procedure to this super-critical case. Let ψ be a critical branching mechanism. We set $\psi_\theta(\cdot) = \psi(\cdot + \theta) - \psi(\theta)$. Let $\Theta = (\theta_\infty, +\infty)$ or $\Theta = [\theta_\infty, +\infty)$ be the set of values of θ for which ψ_θ is a conservative branching mechanism. The pruning procedure allows to construct a decreasing Lévy-CRT-valued Markov process $(\mathcal{T}_\theta, \theta \in \Theta)$, such that $\mathcal{T}_\theta$ has branching mechanism ψ_θ. It is sub-critical if $\theta > 0$ and super-critical if $\theta < 0$. We then consider the explosion time A of the CRT: the smallest (negative) time θ for which the continuous state branching process (CB) associated with $\mathcal{T}_\theta$ has finite total mass (i.e., the length of the excursion of the exploration process that codes the CRT is finite). We describe the law of A as well as the distribution of the CRT just after this explosion time. The CRT just after explosion can be seen as a CRT conditioned not to be extinct which is pruned with an independent intensity related to A. We also study the evolution of the CRT-valued process after the explosion time. This extends results from Aldous and Pitman on Galton–Watson trees. For the particular case of the quadratic branching mechanism, we show that after explosion the total mass of the CB behaves like the inverse of a stable subordinator with index 1/2. This result is related to the size of the tagged fragment for the fragmentation of Aldous's CRT.

1. Introduction. Continuous state branching processes (CB in short) are nonnegative real valued Markov processes first introduced by Jirina [19] that satisfy a branching property: the process $(Z_t, t \geq 0)$ is a CB if its law when starting from $x + x'$ is equal to the law of the sum of two independent copies of Z starting respectively from x and x'. The law of such a process

is characterized by the so-called branching mechanism ψ via its Laplace functionals. The branching mechanism ψ of a CB is given by

$$\psi(\lambda) = \tilde{\alpha}\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell)[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\mathbf{1}_{\{\ell\leq 1\}}],$$

where $\tilde{\alpha} \in \mathbb{R}$, $\beta \geq 0$ and π is a Radon measure on $(0,+\infty)$ such that $\int_{(0,+\infty)}(1 \wedge \ell^2)\pi(d\ell) < +\infty$. The CB is said to be respectively sub-critical, critical, super-critical when $\psi'(0) > 0$, $\psi'(0) = 0$ or $\psi'(0) < 0$. We will write (sub)critical for critical or sub-critical. Notice that ψ is smooth and strictly convex if $\beta > 0$ or $\pi \neq 0$.

It is shown in [20] that all these CBs can be obtained as the limit of renormalized sequences of Galton–Watson processes. A genealogical tree is naturally associated with a Galton–Watson process and the question of existence of such a genealogical structure for CB arises naturally. This question has given birth to the theory of continuum random trees (CRT), first introduced in the pioneer work of Aldous [7–9]. A continuum random tree (called Lévy CRT) that codes the genealogy of a general (sub)critical branching process has been constructed in [22, 23] and studied further in [16]. The main tool of this approach is the so-called exploration process $(\rho_s, s \in \mathbb{R}^+)$, where ρ_s is a measure on $\mathbb{R}^+$, which codes for the CRT. For (sub)critical quadratic branching mechanism ($\pi = 0$), the measure ρ_s is just the Lebesgue measure over an interval $[0, H_s]$, and the so-called height process $(H_s, s \in \mathbb{R}^+)$ is a Brownian motion with drift reflected at 0. In [15], a CRT is built for super-critical quadratic branching mechanism using the Girsanov theorem for Brownian motion.

We propose here a construction for general super-critical Lévy tree, using the exploration process, based on ideas from [15]. We first build the super-critical tree up to a given level a. This tree can be coded by an exploration process, and its law is absolutely continuous with respect to the law of a (sub)critical Lévy tree, whose leaves above level a are removed. Moreover, this family of processes (indexed by parameter a) satisfies a compatibility property, and hence there exists a projective limit which can be seen as the law of the CRT associated with the super-critical CB. This construction enables us to use most of the results known for (sub)critical CRT. Notice that another construction of a Lévy CRT that does not make use of the exploration process has been proposed in [18] as the limit, for the Gromov–Hausdorff metric, of a sequence of discrete trees. This construction also holds in the super-critical case but is not easy to use to derive properties for super-critical CRT.

In a second time, we want to construct a "decreasing" tree-valued Markov process. To begin with, if ψ is (sub)critical, for $\theta > 0$ we can construct, via the pruning procedure of [5], from a Lévy CRT $\mathcal{T}$ associated with ψ, a sub-

tree $\mathcal{T}_\theta$ associated with the branching mechanism ψ_θ defined by

$$\forall \lambda \geq 0 \qquad \psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta).$$

By [1, 25], we can even construct a "decreasing" family of Lévy CRTs ($\mathcal{T}_\theta$, $\theta \geq 0$) such that $\mathcal{T}_\theta$ is associated with ψ_θ for every $\theta \geq 0$.

In this paper, we consider a critical branching mechanism ψ and denote by Θ the set of real numbers θ (including negative ones) for which ψ_θ is a well-defined conservative branching mechanism (see Section 5.3 for some examples). Notice that $\Theta = [\theta_\infty, +\infty)$ or $(\theta_\infty, +\infty)$ for some $\theta_\infty \in [-\infty, 0]$. We then extend the pruning procedure of [5] to super-critical branching mechanisms in order to define a Lévy CRT-valued process ($\mathcal{T}_\theta, \theta \in \Theta$) such that:

- for every $\theta \in \Theta$, the Lévy CRT $\mathcal{T}_\theta$ is associated with the branching mechanism ψ_θ;
- all the trees $\mathcal{T}_\theta$, $\theta \in \Theta$ have a common root;
- the tree-valued process ($\mathcal{T}_\theta, \theta \in \Theta$) is decreasing in the sense that for $\theta < \theta'$, $\mathcal{T}_{\theta'}$ is a sub-tree of $\mathcal{T}_\theta$.

Let ρ^θ be the exploration process that codes for $\mathcal{T}_\theta$. We denote by $\mathbf{N}^\psi$ the excursion measure of the process ($\rho^\theta, \theta \in \Theta$), that is under $\mathbf{N}^\psi$, each ρ^θ is the excursion of an exploration process associated with ψ_θ. Let σ_θ denote the length of this excursion. The quantity σ_θ corresponds also to the total mass of the CB associated with the tree $\mathcal{T}_\theta$. We say that the tree $\mathcal{T}_\theta$ is finite (under $\mathbf{N}^\psi$) if σ_θ is finite (or equivalently if the total mass of the associated CB is finite). By construction, we have that the trees $\mathcal{T}_\theta$ for $\theta \geq 0$ are associated with (sub)critical branching mechanisms and hence are a.e. finite. On the other hand, the trees $\mathcal{T}_\theta$ for negative θ are associated with super-critical branching mechanisms. We define the explosion time

$$A = \inf\{\theta \in \Theta, \sigma_\theta < +\infty\}.$$

For $\theta \in \Theta$, we define $\bar{\theta}$ as the unique nonnegative real number such that

$$(1) \qquad\qquad \psi(\bar{\theta}) = \psi(\theta)$$

(notice that $\bar{\theta} = \theta$ if $\theta \geq 0$). If $\theta_\infty \notin \Theta$, we set $\bar{\theta}_\infty = \lim_{\theta \downarrow \theta_\infty} \bar{\theta}$. We give the distribution of A under $\mathbf{N}^\psi$ (Theorem 6.5). In particular we have, for all $\theta \in [\theta_\infty, +\infty)$,

$$\mathbf{N}^\psi[A > \theta] = \bar{\theta} - \theta.$$

We also give the distribution of the trees after the explosion time ($\mathcal{T}_\theta, \theta \geq A$) (Theorem 6.7 and Corollary 8.2). Of particular interest is the distribution of the tree at its explosion time, $\mathcal{T}_A$.

The pruning procedure can been viewed, from a discrete point of view, as a percolation on a Galton–Watson tree. This idea has been used in [11]

(percolation on branches) and in [4] (percolation on nodes) to construct tree-valued Markov processes from a Galton–Watson tree. The CRT-valued Markov process constructed here can be viewed as the continuous analog of the discrete models of [11] and [4] (or maybe a mixture of both constructions). However, no link is actually pointed out between the discrete and the continuous frameworks.

In [11] and [4], another representation of the process up to the explosion time is also given in terms of the pruning of an infinite tree [a (sub)critical Galton–Watson tree conditioned on nonextinction]. In the same spirit, we also construct another tree-valued Markov process $(\mathcal{T}_\theta^*, \theta \geq 0)$ associated with a critical branching mechanism ψ. In the case of a.s. extinction (i.e., when $\int^{+\infty} \frac{dv}{\psi(v)} < +\infty$), $\mathcal{T}_0^*$ is distributed as $\mathcal{T}_0$ conditioned to survival. The tree $\mathcal{T}_0^*$ is constructed via a spinal decomposition along an infinite spine. Then we define the continuum-tree-valued Markov process $(\mathcal{T}_\theta^*, \theta \geq 0)$ again by a pruning procedure. Let $\theta \in (\theta_\infty, 0)$. We prove that under the excursion measure $\mathbf{N}^\psi$, given $A = \theta$, the process $(\mathcal{T}_{\theta+u}, u \geq 0)$ is distributed as the process $(\mathcal{T}_{\tilde\theta+u}^*, u \geq 0)$ (Theorem 8.1).

When the branching mechanism is quadratic, $\psi(\lambda) = \lambda^2/2$, some explicit computations can be carried out. Let σ_θ^* be the total mass of $\mathcal{T}_\theta^*$ and $\tau = (\tau_\theta, \theta \geq 0)$ be the first passage process of a standard Brownian motion, that is a stable subordinator with index $1/2$. We get (Proposition 9.1) that $(\sigma_\theta^*, \theta \geq 0)$ is distributed as $(1/\tau_\theta, \theta \geq 0)$ and that $(\sigma_{A+\theta}, \theta \geq 0)$ is distributed as $(1/(V + \tau_\theta), \theta \geq 0)$ for some random variable V independent of τ. Let us recall that the pruning procedure of the tree can be used to construct some fragmentation processes (see [1, 6, 25]) and the process $(\sigma_\theta, \theta \geq 0)$, conditionally on $\sigma_0 = 1$, represents then the evolution of a tagged fragment. We hence recover a well-known result of Aldous–Pitman [10]: conditionally on $\sigma_0 = 1$, $(\sigma_\theta, \theta \geq 0)$ is distributed as $(1/(1 + \tau_\theta), \theta \geq 0)$ (see Corollary 9.2).

The paper is organized as follows. In Section 2, we introduce an exponential martingale of a CB and give a Girsanov formula for CBs. We recall in Section 3 the construction of a (sub)critical Lévy CRT via the exploration process and some useful properties of this exploration process. Then we construct, in Section 4, the super-critical Lévy CRT via a Girsanov theorem involving the same martingale as in Section 2. We recall in Section 5 the pruning procedure for critical or sub-critical CRTs and extend this procedure to super-critical CRTs. We construct in Section 6 the tree-valued process $(\mathcal{T}_\theta, \theta \in \Theta)$, or more precisely the family of exploration processes $(\rho^\theta, \theta \in \Theta)$ which codes for it. We also give the law of the explosion time A and the law of the tree at this time. In Section 7, we construct an infinite tree and the corresponding pruned sub-trees $(\mathcal{T}_\theta^*, \theta \geq 0)$, which are given by a spinal representation using exploration processes. We prove in Section 8 that the process $(\mathcal{T}_{A+u}, u \geq 0)$ is distributed as the process $(\mathcal{T}_{U+u}^*, u \geq 0)$

where U is a positive random time independent of $(\mathcal{T}_\theta^*, \theta \geq 0)$. We finally make the explicit computations for the quadratic case in Section 9.

Notice that all the results in the following sections are stated using exploration processes which code for the CRT, instead of the CRT directly. An informal description of the links between the CRT and the exploration process is given at the end of Section 3.6.

2. Girsanov's formula for continuous branching process.

2.1. *Continuous branching process.* Let ψ be a branching mechanism of a CB: for $\lambda \geq 0$,

$$(2) \qquad \psi(\lambda) = \tilde{\alpha}\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell)[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\mathbf{1}_{\{\ell \leq 1\}}],$$

where $\tilde{\alpha} \in \mathbb{R}$, $\beta \geq 0$, and π is a Radon measure on $(0,+\infty)$ such that $\int_{(0,+\infty)} (1 \wedge \ell^2)\pi(d\ell) < +\infty$. We shall say that ψ has parameter $(\tilde{\alpha}, \beta, \pi)$.

We shall assume that $\beta \neq 0$ or $\pi \neq 0$. We have $\psi(0) = 0$ and $\psi'(0^+) = \tilde{\alpha} - \int_{(1,+\infty)} \ell\pi(d\ell) \in [-\infty,+\infty)$. In particular, we have $\psi'(0^+) = -\infty$ if and only if $\int_{(1,+\infty)} \ell\pi(d\ell) = +\infty$. We say that ψ is conservative if for all $\varepsilon > 0$

$$(3) \qquad \int_0^\varepsilon \frac{1}{|\psi(u)|}\,du = +\infty.$$

Notice that (3) is fulfilled if $\psi'(0+) > -\infty$, that is, if $\int_{(1,+\infty)} \ell\pi(d\ell) < +\infty$. If ψ is conservative, the CB associated with ψ does not explode in finite time a.s.

Let P_x^ψ be the law of a CB $Z = (Z_a, a \geq 0)$ started at $x \geq 0$ and with branching mechanism ψ, and let E_x^ψ be the corresponding expectation. The process Z is a Feller process and thus has a càd-làg version. Let $\mathcal{F} = (\mathcal{F}_a, a \geq 0)$ be the filtration generated by Z completed the usual way. For every $\lambda > 0$, for every $a \geq 0$, we have

$$(4) \qquad \mathrm{E}_x^\psi[\mathrm{e}^{-\lambda Z_a}] = \mathrm{e}^{-xu(a,\lambda)},$$

where function u is the unique nonnegative solution of

$$(5) \qquad u(a,\lambda) + \int_0^a \psi(u(s,\lambda))\,ds = \lambda, \qquad \lambda \geq 0, a \geq 0.$$

This equation is equivalent to

$$(6) \qquad \int_{u(a,\lambda)}^\lambda \frac{dr}{\psi(r)} = a, \qquad \lambda \geq 0, a \geq 0.$$

If (3) holds, then the process is conservative: a.s. for all $a \geq 0$, $Z_a < +\infty$.

Let q_0 be the largest root of $\psi(q) = 0$. Since $\psi(0) = 0$, we have $q_0 \geq 0$. If ψ is (sub)critical, since ψ is strictly convex, we get that $q_0 = 0$. If ψ is super-critical, if we denote by $q^* > 0$ the only real number such that $\psi'(q^*) = 0$, we have $q_0 > q^* > 0$. See Lemma 2.4 for the interpretation of q_0.

If f is a function defined on $[\gamma, +\infty)$, then for $\theta \geq \gamma$, we set for $\lambda \geq \gamma - \theta$

$$f_\theta(\lambda) = f(\theta + \lambda) - f(\theta).$$

If ν is a measure on $(0, +\infty)$, then for $q \in \mathbb{R}$, we set

(7) $$\nu^{(q)}(d\ell) = \mathrm{e}^{-q\ell}\nu(d\ell).$$

REMARK 2.1. If $\pi^{(q)}((1, +\infty)) < +\infty$ for some $q < 0$, then ψ given by (2) is well defined on $[q, +\infty)$ and, for $\theta \in [q, +\infty)$, ψ_θ is a branching mechanism with parameter $(\tilde{\alpha} + 2\beta\theta + \int_{(0,1]} \pi(d\ell)\ell(1 - \mathrm{e}^{-\theta\ell}), \beta, \pi^{(\theta)})$. Notice that for all $\theta > q$, ψ_θ is conservative. And, if the additional assumption

$$\int_{(1,+\infty)} \ell\pi^{(q)}(d\ell) = \int_{(1,+\infty)} \ell\mathrm{e}^{|q|\ell}\pi(d\ell) < +\infty$$

holds, then $|(\psi_q)'(0+)| < +\infty$ and ψ_q is conservative.

2.2. *Girsanov's formula.* Let $Z = (Z_a, a \geq 0)$ be a conservative CB with branching mechanism ψ given by (2) with $\beta \neq 0$ or $\pi \neq 0$, and let $(\mathcal{F}_a, a \geq 0)$ be its natural filtration. Let $q \in \mathbb{R}$ such that $q \geq 0$ or $q < 0$ and $\int_{(1,+\infty)} \ell\mathrm{e}^{|q|\ell}\pi(d\ell) < +\infty$. Then, thanks to Remark 2.1, $\psi(q)$ and ψ_q are well defined and ψ_q is conservative. Then we consider the process $M^{\psi,q} = (M_a^{\psi,q}, a \geq 0)$ defined by

(8) $$M_a^{\psi,q} = \mathrm{e}^{qx - qZ_a - \psi(q) \int_0^a Z_s \, ds}.$$

THEOREM 2.2. *Let $q \in \mathbb{R}$ such that $q \geq 0$ or $q < 0$ and $\int_{(1,+\infty)} \ell\mathrm{e}^{|q|\ell}\pi(d\ell) < +\infty$.*

(i) *The process $M^{\psi,q}$ is a $\mathcal{F}$-martingale under P_x^ψ.*

(ii) *Let $a, x \geq 0$. On $\mathcal{F}_a$, the probability measure $\mathrm{P}_x^{\psi_q}$ is absolutely continuous with respect to P_x^ψ and*

$$\frac{d\mathrm{P}_x^{\psi_q}{}_{|\mathcal{F}_a}}{d\mathrm{P}_x^\psi{}_{|\mathcal{F}_a}} = M_a^{\psi,q}.$$

Before going into the proof of this theorem, we recall Proposition 2.1 from [2]. For μ a positive measure on $\mathbb{R}$, we set

(9) $$H(\mu) = \sup\{r \in \mathbb{R}; \mu([r, +\infty)) > 0\},$$

the maximal element of its support. For $a < 0$, we set $Z_a = 0$.

PROPOSITION 2.3. *Let μ be a finite positive measure on $\mathbb{R}$ with support bounded from above [i.e., $H(\mu)$ is finite]. Then we have for all $s \in \mathbb{R}$, $x \geq 0$,*

(10)
$$\mathrm{E}_x^{\psi}[\mathrm{e}^{- \int_{\mathbb{R}} Z_{r-s} \mu(dr)}] = \mathrm{e}^{-xw(s)},$$

where the function w is a measurable locally bounded nonnegative solution of the equation

(11)
$$w(s) + \int_s^{+\infty} \psi(w(r))\,dr = \int_{[s,+\infty)} \mu(dr), \qquad s \leq H(\mu) \quad and$$
$$w(s) = 0, \qquad s > H(\mu).$$

If $\psi'(0^+) > -\infty$ or if $\mu(\{H(\mu)\}) > 0$, then (11) has a unique measurable locally bounded nonnegative solution.

PROOF OF THEOREM 2.2.

First case. We consider $q > 0$ such that $\psi(q) \geq 0$.

We have $0 \leq M_a^{\psi,q} \leq \mathrm{e}^{qx}$, thus $M^{\psi,q}$ is bounded. It is clear that $M^{\psi,q}$ is $\mathcal{F}$-adapted.

To check that $M^{\psi,q}$ is a martingale, thanks to the Markov property, it is enough to check that $\mathrm{E}_x^{\psi}[M_a^{\psi,q}] = \mathrm{E}_x^{\psi}[M_0^{\psi,q}] = 1$ for all $a \geq 0$ and all $x \geq 0$. Consider the measure $\nu_q(dr) = q\delta_a(dr) + \psi(q)\mathbf{1}_{[0,a]}(r)\,dr$, where δ_a is the Dirac mass at point a. Notice that $H(\nu_q) = a$ and that $\nu_q(\{H(\nu_q)\}) = q > 0$. Hence, thanks to Proposition 2.3, there exists a unique nonnegative solution w of (11) with $\mu = \nu_q$, and $\mathrm{E}_x^{\psi}[M_a^{\psi,q}] = \mathrm{e}^{-x(w(0)-q)}$. As $q\mathbf{1}_{[0,a]}$ also solves (11) with $\mu = \nu_q$, we deduce that $w = q\mathbf{1}_{[0,a]}$ and that $\mathrm{E}_x[M_a^{\psi,q}] = 1$. Thus, we get that $M^{\psi,q}$ is a bounded martingale.

Let ν be a nonnegative measure on $\mathbb{R}$ with support in $[0,a]$ [i.e., $H(\nu) \leq a$]. Thanks to Proposition 2.3, we have that $\mathrm{E}_x^{\psi}[M_a^{\psi,q}\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)}] = \mathrm{e}^{-x(v(0)-q)}$, where v is the unique nonnegative solution of (11) with $\mu = \nu + \nu_q$. As $M_a^{\psi,q}\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)} \leq M_a^{\psi,q}$, we deduce that $\mathrm{e}^{-x(v(0)-q)} = \mathrm{E}_x^{\psi}[M_a^{\psi,q}\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)}] \leq 1$, that is, $v(0) \geq q$. We set $u = v - q\mathbf{1}_{[0,a]}$, and we deduce that u is nonnegative and solves

(12)
$$u(s) + \int_s^{+\infty} \psi_q(u(r))\,dr = \int_{[s,+\infty)} \nu(dr), \qquad s \leq H(\nu) \quad and$$
$$u(s) = 0, \qquad s > H(\nu).$$

As $\psi(q) \geq 0$, we deduce from the convexity of ψ that $\psi_q'(0) = \psi'(q) \geq 0$. Thanks to Proposition 2.3, we deduce that u is the unique nonnegative solution of (12) and that $\mathrm{e}^{-xu(0)} = \mathrm{E}_x^{\psi_q}[\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)}]$. In particular, we have that for all nonnegative measure ν on $\mathbb{R}$ with support in $[0,a]$,

$$\mathrm{E}_x^{\psi}[M_a^{\psi,q}\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)}] = \mathrm{E}_x^{\psi_q}[\mathrm{e}^{- \int_{\mathbb{R}} Z_r \nu(dr)}].$$

As $\mathrm{e}^{-\int_{\mathbb{R}} Z_r \nu(dr)}$ is $\mathcal{F}_a$-measurable, we deduce from the monotone class theorem that for any nonnegative $\mathcal{F}_a$-measurable random variable W,

$$
(13) \qquad \mathrm{E}_x^\psi[W\mathrm{e}^{qx-qZ_a-\psi(q)\int_0^a Z_r\,dr}] = \mathrm{E}_x^\psi[W M_a^{\psi,q}] = \mathrm{E}_x^{\psi_q}[W].
$$

This proves the second part of the theorem.

Second case. We consider $q \geq 0$ such that $\psi(q) < 0$. Let us remark that this only occurs when ψ is super-critical.

Recall that $q_0 > q^* > 0$ are such that $\psi(q_0) = 0$ and $\psi'(q^*) = 0$. Notice that $\psi_{q^*}'(0) = \psi'(q^*) = 0$, that is, ψ_{q^*} is critical. Let W be any nonnegative random variable $\mathcal{F}_a$-measurable. From the first step, using (13) with $q = q_0$, we get that

$$
\mathrm{E}_x^\psi[W\mathrm{e}^{q_0 x - q_0 Z_a}] = \mathrm{E}_x^{\psi_{q_0}}[W].
$$

Thanks to (13) with ψ_{q^*} instead of ψ and $(q_0 - q^*) \geq 0$ instead of q, and using that $(\psi_{q^*})_{q_0-q^*} = \psi_{q_0}$, we deduce that

$$
\mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{(q_0-q^*)x-(q_0-q^*)Z_a-\psi_{q^*}(q_0-q^*)\int_0^a Z_r\,dr}] = \mathrm{E}_x^{(\psi_{q^*})_{q_0-q^*}}[W] = \mathrm{E}_x^{\psi_{q_0}}[W].
$$

This implies that

$$
\begin{aligned}
\mathrm{E}_x^\psi[W] &= \mathrm{E}_x^{\psi_{q_0}}[W\mathrm{e}^{-q_0 x}\mathrm{e}^{q_0 Z_a}] \\
&= \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-q_0 x}\mathrm{e}^{q_0 Z_a}\mathrm{e}^{(q_0-q^*)x-(q_0-q^*)Z_a-\psi_{q^*}(q_0-q^*)\int_0^a Z_r\,dr}] \\
&= \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-q^* x+q^* Z_a-\psi_{q^*}(q_0-q^*)\int_0^a Z_r\,dr}].
\end{aligned}
$$

As $\psi_{q^*}(q_0 - q^*) = \psi(q_0) - \psi(q^*) = -\psi(q^*) = \psi_{q^*}(-q^*)$, we finally obtain

$$
(14) \qquad \mathrm{E}_x^\psi[W] = \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-q^* x+q^* Z_a-\psi_{q^*}(-q^*)\int_0^a Z_r\,dr}].
$$

If $q < q^*$, as $(\psi_q)_{(q^*-q)} = \psi_{q^*}$ and $\psi_q'(q^* - q) = \psi'(q^*) = 0$, we deduce from (14) with ψ replaced by ψ_q and q^* by $q^* - q$ that

$$
(15) \qquad \mathrm{E}_x^{\psi_q}[W] = \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-(q^*-q)x+(q^*-q)Z_a-\psi_{q^*}(q-q^*)\int_0^a Z_r\,dr}].
$$

If $q > q^*$, formula (13) holds with ψ replaced by ψ_{q^*} and q replaced by $q-q^*$, which also yields equation (15).

Using (14), (15) and that $\psi_{q^*}(-q^*) + \psi(q) = \psi_{q^*}(q - q^*)$, we get that

$$
(16) \qquad
\begin{aligned}
&\mathrm{E}_x^\psi[W\mathrm{e}^{qx-qZ_a-\psi(q)\int_0^a Z_r\,dr}] \\
&\quad = \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-(q^*-q)x+(q^*-q)Z_a-(\psi_{q^*}(-q^*)+\psi(q))\int_0^a Z_r\,dr}] \\
&\quad = \mathrm{E}_x^{\psi_{q^*}}[W\mathrm{e}^{-(q^*-q)x+(q^*-q)Z_a-\psi_{q^*}(q-q^*)\int_0^a Z_r\,dr}] \\
&\quad = \mathrm{E}_x^{\psi_q}[W].
\end{aligned}
$$

Since this holds for any nonnegative $\mathcal{F}_a$-measurable random variable W, this proves (i) and (ii) of the theorem.

Third case. We consider $q < 0$ and assume that $\int_{(1,+\infty)} \ell e^{|q|\ell} \pi(d\ell) < +\infty$. In particular, ψ_q is a conservative branching mechanism, thanks to Remark 2.1.

Let W be any nonnegative $\mathcal{F}_a$-measurable random variable. Using (13) if $\psi_q(-q) \geq 0$ or (16) if $\psi_q(-q) < 0$, with ψ replaced by ψ_q and q by $-q$, we deduce that
$$\mathrm{E}_x^{\psi_q}[W e^{-qx+qZ_a-\psi_q(-q)\int_0^a Z_r\, dr}] = \mathrm{E}_x^{\psi}[W].$$
This implies that
$$\mathrm{E}_x^{\psi_q}[W] = \mathrm{E}_x^{\psi}[W e^{qx-qZ_a+\psi_q(-q)\int_0^a Z_r\, dr}] = \mathrm{E}_x^{\psi}[W e^{qx-qZ_a-\psi(q)\int_0^a Z_r\, dr}].$$
Since this holds for any nonnegative $\mathcal{F}_a$-measurable random variable W, this proves (i) and (ii) of the theorem. $\square$

Finally, we recall some well-known facts on CB. Recall that q_0 is the largest root of $\psi(q) = 0$, $q_0 = 0$ if ψ is (sub)critical and that $q_0 > 0$ if ψ is super-critical. We set

(17) $$\sigma = \int_0^{+\infty} Z_a\, da.$$

For $\lambda \geq 0$, we set

(18) $$\psi^{-1}(\lambda) = \sup\{r \geq 0; \psi(r) = \lambda\},$$

and we call σ the total mass of the CB.

LEMMA 2.4. *Assume that ψ is given by (2) with $\beta \neq 0$ or $\pi \neq 0$ and is conservative.*

(i) *Then P_x^{ψ}-a.s. $Z_\infty = \lim_{a\to+\infty} Z_a$ exists, $Z_\infty \in \{0, +\infty\}$,*

(19) $$\mathrm{P}_x^{\psi}(Z_\infty = 0) = e^{-xq_0},$$

$\{Z_\infty = 0\} = \{\sigma < +\infty\}$, and we have, for $\lambda > 0$,

(20) $$\mathrm{E}_x^{\psi}[e^{-\lambda\sigma}] = e^{-x\psi^{-1}(\lambda)}.$$

(ii) *Let $q > 0$ such that $\psi(q) \geq 0$. Then, the probability measure $\mathrm{P}_x^{\psi_q}$ is absolutely continuous with respect to P_x^{ψ} with*
$$\frac{d\mathrm{P}_x^{\psi_q}}{d\mathrm{P}_x^{\psi}} = M_\infty^{\psi,q},$$
where

(21) $$M_\infty^{\psi,q} = e^{qx-\psi(q)\sigma}\mathbf{1}_{\{\sigma<+\infty\}}.$$

(iii) *If ψ is super-critical then, conditionally on $\{Z_\infty = 0\}$, Z is distributed as $\mathrm{P}^{\psi_{q_0}}$: for any nonnegative random variable measurable w.r.t. $\sigma(Z_a, a \geq 0)$, we have*
$$\mathrm{E}_x^{\psi}[W|Z_\infty = 0] = \mathrm{E}_x^{\psi_{q_0}}[W].$$

PROOF.   For $\lambda > 0$, we set $N_a = \mathrm{e}^{-\lambda Z_a + x u(a,\lambda)}$, where u is the unique non-negative solution of (6). Thanks to (4) and the Markov property, $(N_a, a \geq 0)$ is a bounded martingale under P_x^ψ. Hence, as a goes to infinity, it converges a.s. and in L^1 to a limit, say N_∞. From (6), we get that $\lim_{a \to +\infty} u(a, \lambda) = q_0$. This implies that $Z_\infty = \lim_{a \to +\infty} Z_a$ exists a.s. in $[0, +\infty]$. Since $\mathrm{E}_x^\psi[N_\infty] = 1$, we get $\mathrm{E}_x^\psi[\mathrm{e}^{-\lambda Z_\infty}] = \mathrm{e}^{-q_0 x}$ for all $\lambda > 0$. This implies that P_x^ψ-a.s. $Z_\infty \in \{0, +\infty\}$ and (19).

Clearly, we have $\{Z_\infty = +\infty\} \subset \{\sigma = +\infty\}$. For $q > 0$ such that $\psi(q) \geq 0$, we get that $(M_a^{\psi,q}, a \geq 0)$ is a bounded martingale under P_x^ψ. Hence, as a goes to infinity, it converges a.s. and in L^1 to a limit, say $M_\infty^{\psi,q}$. We deduce that

(22) $$\mathrm{E}_x^\psi[\mathrm{e}^{-\psi(q)\sigma} \mathbf{1}_{\{Z_\infty = 0\}}] = \mathrm{e}^{-qx}.$$

Letting q decrease to q_0, we get that $\mathrm{P}_x^\psi(\sigma < +\infty, Z_\infty = 0) = \mathrm{e}^{-q_0 x} = \mathrm{P}_x^\psi(Z_\infty = 0)$. This implies that P_x^ψ a.s. $\{\sigma = +\infty\} \subset \{Z_\infty = +\infty\}$. We thus deduce that P_x^ψ a.s. $\{Z_\infty = +\infty\} = \{\sigma = +\infty\}$. Notice also that (21) holds.

Notice that (22) readily implies (20). This proves Property (i) of the lemma and (21).

Property (ii) is then a consequence of Theorem 2.2, Property (ii) and the convergence in L^1 of the martingale $(M_a^{\psi,q}, a \geq 0)$ towards $M_\infty^{\psi,q}$.

Property (iii) is a consequence of (ii) with $q = q_0$ and (19). $\square$

3. Lévy continuum random tree.   We recall here the construction of the Lévy continuum random tree (CRT) introduced in [22, 23] and developed later in [16] for critical or sub-critical branching mechanism. We will emphasize on the height process and the exploration process which are the key tools to handle this tree. The results of this section are mainly extracted from [16], except for the next subsection which is extracted from [21].

3.1. *Real trees and their coding by a continuous function.*   Let us first define what a real tree is.

DEFINITION 3.1.   A metric space $(\mathcal{T}, d)$ is a real tree if the following two properties hold for every $v_1, v_2 \in \mathcal{T}$:

(i) (unique geodesic) There is a unique isometric map f_{v_1, v_2} from $[0, d(v_1, v_2)]$ into $\mathcal{T}$ such that

$$f_{v_1, v_2}(0) = v_1 \quad \text{and} \quad f_{v_1, v_2}(d(v_1, v_2)) = v_2.$$

(ii) (no loop) If q is a continuous injective map from $[0, 1]$ into $\mathcal{T}$ such that $q(0) = v_1$ and $q(1) = v_2$, we have
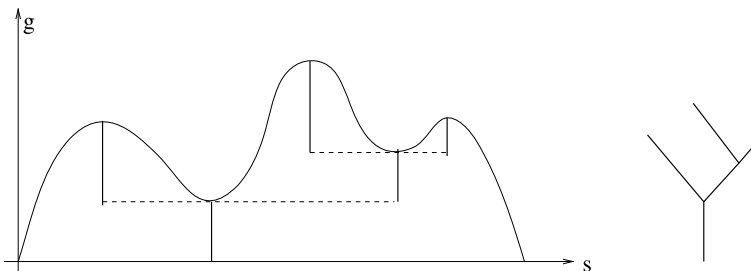
$$q([0, 1]) = f_{v_1, v_2}([0, d(v_1, v_2)]).$$

Fig. 1. *A height process g and its associated real tree.*

A rooted real tree is a real tree $(\mathcal{T}, d)$ with a distinguished vertex $v_\varnothing$ called the root.

Let $(\mathcal{T}, d)$ be a rooted real tree. The range of the mapping f_{v_1,v_2} is denoted by $[\![v_1, v_2,]\!]$ (this is the line between v_1 and v_2 in the tree). In particular, for every vertex $v \in \mathcal{T}$, $[\![v_\varnothing, v]\!]$ is the path going from the root to v which we call the ancestral line of vertex v. More generally, we say that a vertex v is an ancestor of a vertex v' if $v \in [\![v_\varnothing, v']\!]$. If $v, v' \in \mathcal{T}$, there is a unique $a \in \mathcal{T}$ such that $[\![v_\varnothing, v]\!] \cap [\![v_\varnothing, v']\!] = [\![v_\varnothing, a]\!]$. We call a the most recent common ancestor to v and v'. By definition, the degree of a vertex $v \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \setminus \{v\}$. A vertex v is called a leaf if it has degree 1. Finally, we set λ the one-dimensional Hausdorff measure on $\mathcal{T}$.

The coding of a compact real tree by a continuous function is now well known and is a key tool for defining random real trees (see Figure 1). We consider a continuous function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with compact support and such that $g(0) = 0$. We also assume that g is not identically 0. For every $0 \le s \le t$, we set

$$m_g(s,t) = \inf_{u \in [s,t]} g(u)$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$

We then introduce the equivalence relation $s \sim t$ if and only if $d_g(s,t) = 0$. Let $\mathcal{T}_g$ be the quotient space $[0, +\infty)/\sim$. It is easy to check that d_g induces a distance on $\mathcal{T}_g$. Moreover, $(\mathcal{T}_g, d_g)$ is a compact real tree (see [17], Theorem 2.1). We say that g is the height process of the tree $\mathcal{T}_g$.

In order to define a random tree, instead of taking a tree-valued random variable (which implies defining a σ-field on the set of real trees), it suffices to take a continuous stochastic process for g. For instance, when g is a normalized Brownian excursion, the associated real tree is Aldous's CRT (up to a factor 2) [9]. We present now how we can define a height process

that codes a random real trees describing the genealogy of a (sub)critical CB with branching mechanism ψ. This height process is defined via a Lévy process that we first introduce.

3.2. *The underlying Lévy process.* We assume that ψ given by (2) is (sub)critical, that is,

$$\text{(23)} \qquad \alpha := \psi'(0) = \tilde{\alpha} - \int_{(1,+\infty)} \ell\pi(d\ell) \geq 0$$

and that

$$\text{(24)} \qquad \beta > 0 \quad \text{or} \quad \int_{(0,1)} \ell\pi(d\ell) = +\infty.$$

We consider a $\mathbb{R}$-valued Lévy process $X = (X_t, t \geq 0)$ with no negative jumps, starting from 0 and with Laplace exponent ψ under the probability measure $\mathbb{P}^\psi$: for $\lambda \geq 0$ $\mathbb{E}^\psi[\mathrm{e}^{-\lambda X_t}] = \mathrm{e}^{t\psi(\lambda)}$. By assumption (24), X is of infinite variation $\mathbb{P}^\psi$-a.s.

We introduce some processes related to X. Let $\mathcal{J} = \{s \geq 0; X_s \neq X_{s-}\}$ be the set of jump times of X. For $s \in \mathcal{J}$, we denote by

$$\Delta_s = X_s - X_{s-}$$

the size of the jump of X at time s and $\Delta_s = 0$ otherwise. Let $I = (I_t, t \geq 0)$ be the infimum process of X,

$$I_t = \inf_{0 \leq s \leq t} X_s,$$

and let $S = (S_t, t \geq 0)$ be the supremum process,

$$S_t = \sup_{0 \leq s \leq t} X_s.$$

We will also consider for every $0 \leq s \leq t$ the infimum of X over $[s,t]$,

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process $X - I$, and $-I$ is the local time of $X - I$ at 0 (see [12], Chapter VII). Let $\mathbb{N}^\psi$ be the associated excursion measure of the process $X - I$ away from 0. Let $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under $\mathbb{N}^\psi$ [we shall see after Proposition 3.7 that the notation σ is consistent with (17)]. By assumption (24), we have $X_0 = I_0 = 0$ $\mathbb{N}^\psi$-a.e.

Since X is of infinite variation, 0 is also regular for the Markov process $S - X$. The local time, $L = (L_t, t \geq 0)$, of $S - X$ at 0 will be normalized so that

$$\mathbb{E}^\psi[\mathrm{e}^{-\lambda S_{L_t^{-1}}}] = \mathrm{e}^{-t\psi(\lambda)/\lambda},$$

where $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$ (see also [12] Theorem VII.4(ii)).

3.3. *The height process and the Lévy CRT.* For each $t \geq 0$, we consider the reversed process at time t, $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$ by

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \qquad \text{if } 0 \leq s < t,$$

and $\hat{X}_t^{(t)} = X_t$. The two processes $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$ and $(X_s, 0 \leq s \leq t)$ have the same law. Let $\hat{S}^{(t)}$ be the supremum process of $\hat{X}^{(t)}$ and $\hat{L}^{(t)}$ be the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$ with the same normalization as L.

DEFINITION 3.2 ([16], Definition 1.2.1). There exists a lower semi-continuous modification of the process $(\hat{L}^{(t)}, t \geq 0)$. We denote by $(H_t, t \geq 0)$ this modification.

We can also define this process H by approximation: it is a modification of the process

(25) $$H_t^0 = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X_s < I_t^s + \varepsilon\}} \, ds$$

(see [16], Lemma 1.1.3). In general, H takes its values in $[0, +\infty]$, but we have that, a.s. for every $t \geq 0$:

- $H_s < +\infty$ for every $s < t$ such that $X_{s-} \leq I_t^s$;
- $H_t < +\infty$ if $\Delta X_t > 0$

(see [16], Lemma 1.2.1).

We use this process to define a random real-tree that we call the ψ-Lévy CRT via the procedure described above. We will see that this CRT does represent the genealogy of a ψ-CB.

3.4. *The exploration process.* The height process is not Markov in general. But it is a very simple function of a measure-valued Markov process, the so-called exploration process.

If E is a locally compact polish space, let $\mathcal{B}(E)$ [resp., $\mathcal{B}_+(E)$] be the set of real-valued measurable (resp., and nonnegative) functions defined on E endowed with its Borel σ-field, and let $\mathcal{M}(E)$ [resp., $\mathcal{M}_f(E)$] be the set of σ-finite (resp., finite) measures on E, endowed with the topology of vague (resp., weak) convergence. For any measure $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_+(E)$, we write

$$\langle \mu, f \rangle = \int_E f(x)\mu(dx).$$

The exploration process $\rho = (\rho_t, t \geq 0)$ is a $\mathcal{M}_f(\mathbb{R}_+)$-valued process defined as follows: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$, $\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_t^s f(H_s)$ (where $d_s I_t^s$ denotes the Lebesgue–Stieljes integral with respect to the nondecreasing

map $s \mapsto I_t^s$), or equivalently

(26) $$\rho_t(dr) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr) + \beta \mathbf{1}_{[0, H_t]}(r) \, dr.$$

In particular, the total mass of ρ_t is $\langle \rho_t, 1 \rangle = X_t - I_t$.

Recall the definition (9) of $H(\mu)$ for a measure μ with compact support and set by convention $H(0) = 0$.

PROPOSITION 3.3 ([16], Lemma 1.2.2 and formula (1.12)). *Almost surely, for every $t > 0$:*

- $H(\rho_t) = H_t$;
- $\rho_t = 0$ *if and only if* $H_t = 0$;
- *if* $\rho_t \neq 0$, *then* $\mathrm{Supp}\, \rho_t = [0, H_t]$;
- $\rho_t = \rho_{t-} + \Delta_t \delta_{H_t}$, *where* $\Delta_t = 0$ *if* $t \notin \mathcal{J}$.

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ a.s. To state the Markov property of ρ, we must first define the process ρ started at any initial measure $\mu \in \mathcal{M}_f(\mathbb{R}_+)$.

For $a \in [0, \langle \mu, 1 \rangle]$, we define the erased measure $k_a \mu$ by

$$k_a \mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1 \rangle - a) \qquad \text{for } r \geq 0.$$

If $a > \langle \mu, 1 \rangle$, we set $k_a \mu = 0$. In other words, the measure $k_a \mu$ is the measure μ erased by a mass a backward from $H(\mu)$.

For $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$ of the two measures by

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \langle \nu, f(H(\mu) + \cdot) \rangle, \qquad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set for every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and every $t > 0$, $\rho_t^\mu = [k_{-I_t}\mu, \rho_t]$. We say that $(\rho_t^\mu, t \geq 0)$ is the process ρ started at $\rho_0^\mu = \mu$. Unless there is an ambiguity, we shall write ρ_t for ρ_t^μ. Unless it is stated otherwise, we assume that ρ is started at 0.

PROPOSITION 3.4 ([16], Proposition 1.2.3). *The process $(\rho_t, t \geq 0)$ is a càd-làg strong Markov process in $\mathcal{M}_f(\mathbb{R}_+)$.*

REMARK 3.5. From the construction of ρ, we get that a.s. $\rho_t = 0$ if and only if $-I_t \geq \langle \rho_0, 1 \rangle$ and $X_t - I_t = 0$. This implies that 0 is also a regular point for ρ. Notice that $\mathbb{N}^\psi$ is also the excursion measure of the process ρ away from 0, and that σ, the length of the excursion, is $\mathbb{N}^\psi$-a.e. equal to $\inf\{t > 0; \rho_t = 0\}$.

3.5. *Notations.* We consider the set $\mathcal{D}$ of càd-làg processes in $\mathcal{M}_f(\mathbb{R}_+)$, endowed with the Skorohod topology and the Borel σ-field. In what follows, we denote by $\rho = (\rho_t, t \geq 0)$ the canonical process on this set. We still

denote by $\mathbb{P}^\psi$ the probability measure on $\mathcal{D}$ such that the canonical process is distributed as the exploration process associated with the branching mechanism ψ, and by $\mathbb{N}^\psi$ the corresponding excursion measure.

3.6. *Local time of the height process.* The local time of the height process is defined through the next result.

PROPOSITION 3.6 ([16], Lemma 1.3.2 and Proposition 1.3.3). *There exists a jointly measurable process $(L^a_s, a \geq 0, s \geq 0)$ which is continuous and nondecreasing in the variable s such that:*

- *for every $t \geq 0$, $\lim_{\varepsilon \to 0} \sup_{a \geq 0} \mathbb{E}^\psi[\sup_{s \leq t} |\varepsilon^{-1} \int_0^s \mathbf{1}_{\{a < H_r \leq a+\varepsilon\}} \, dr - L^a_s|] = 0$;*
- *for every $t \geq 0$, $\lim_{\varepsilon \to 0} \sup_{a \geq \varepsilon} \mathbb{E}^\psi[\sup_{s \leq t} |\varepsilon^{-1} \int_0^s \mathbf{1}_{\{a-\varepsilon < H_r \leq a\}} \, dr - L^a_s|] = 0$;*
- *$\mathbb{P}^\psi$-a.s., for every $t \geq 0$, $L^0_t = -I_t$;*
- *the occupation time formula holds: for any nonnegative measurable function g on $\mathbb{R}_+$ and any $s \geq 0$, $\int_0^s g(H_r) \, dr = \int_{(0,+\infty)} g(a) L^a_s \, da$.*

Let $T_x = \inf\{t \geq 0; I_t \leq -x\}$. We have the following Ray–Knight theorem which links the ψ-Lévy CRT with the ψ-CB.

PROPOSITION 3.7 ([16], Theorem 1.4.1). *The process $(L^a_{T_x}, a \geq 0)$ is distributed under $\mathbb{P}^\psi$ as Z under P^ψ_x (i.e., is a CB with branching mechanism ψ starting at x).*

Let $\mathbb{P}^\psi_x$ be the distribution of $(\rho_{t \wedge T_x}, t \geq 0)$ under $\mathbb{P}^\psi$. We set $Z_a = L^a_{T_x}$ under $\mathbb{P}^\psi_x$ and $Z_a = L^a_\infty$ under $\mathbb{N}^\psi$ and (under $\mathbb{P}^\psi_x$ or $\mathbb{N}^\psi$)

$$(27) \qquad \sigma(\rho) = \int_0^\infty \mathbf{1}_{\{\rho_t \neq 0\}} \, dt.$$

The occupation time formula implies that

$$(28) \qquad \sigma(\rho) = \int_0^{+\infty} Z_a \, da,$$

which is consistent with notation (17). When there is no confusion, we shall write σ for $\sigma(\rho)$. We call $\sigma(\rho)$ the total mass of the CRT as it represents the total population of the associated CB.

Exponential formula for the Poisson point process of jumps of the inverse subordinator of $-I$ gives (see also the beginning of Section 3.2.2. [16]) that for $\lambda > 0$

$$(29) \qquad \mathbb{N}^\psi[1 - \mathrm{e}^{-\lambda\sigma}] = \psi^{-1}(\lambda).$$

We also recall Lemma 1.6 of [1].

LEMMA 3.8. *Let $\theta > 0$. The excursion measure $\mathbb{N}^{\psi_\theta}$ is absolutely continuous w.r.t. $\mathbb{N}^\psi$ with density $\mathrm{e}^{-\psi(\theta)\sigma}$: for any nonnegative measurable func-*
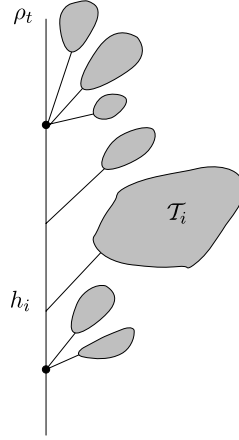
FIG. 2.   *The measure ρ_t and the family $(h_i, \mathcal{I}_i)_{i\in\mathcal{I}}$.*

tion F on the space of excursions, we have

$$\mathbb{N}^{\psi_\theta}[F(\rho)] = \mathbb{N}^\psi[F(\rho)\mathrm{e}^{-\psi(\theta)\sigma}].$$

We recall the Poisson representation of $\mathbb{P}_x^\psi$ based on the excursion measure $\mathbb{N}^\psi$. Let $(\tilde{\alpha}_i, \tilde{\beta}_i)_{i\in\tilde{I}}$ be the excursion intervals of ρ away from 0. For every $i \in \tilde{I}$, $t \geq 0$, we set

$$\tilde{\rho}_t^{(i)} = \rho_{(\tilde{\alpha}_i+t)\wedge\tilde{\beta}_i}.$$

We deduce from Lemma 4.2.4 of [16] the following lemma.

LEMMA 3.9.   *The point measure $\sum_{i\in\tilde{I}}\delta_{\tilde{\rho}^{(i)}}(d\mu)$ is under $\mathbb{P}_x^\psi$ a Poisson measure with intensity $x\mathbb{N}^\psi(d\mu)$.*

To better understand the links between the Lévy CRT and the exploration process, we can combine the Markov property with the other Poisson decomposition of [16], Lemma 4.2.4. Informally speaking, the measure ρ_t is a measure placed on the ancestral line of the individual labelled t which describes how the sub-trees "on the right" of t (i.e., containing individuals $s \geq t$) are grafted along that ancestral line. More precisely, if we denote $(\mathcal{T}_i)_{i\in\mathcal{I}}$ the family of these subtrees and we set h_i the height where the subtree $\mathcal{T}_i$ branches from the ancestral line of t, then the family $(h_i, \mathcal{I}_i)_{i\in\mathcal{I}}$ given ρ_t is distributed as the atoms of a Poisson measure with intensity $\rho_t(dh)\mathbb{N}^\psi[d\mathcal{T}]$ (see Figure 2).

As the measure $\mathbb{N}^\psi$ is an infinite measure, we see that the branching points along the ancestral line of t are of two types (see [17], Theorem 4.6):

- binary nodes (i.e., vertex of degree 3) which are given by the regular part of ρ_t,
- infinite nodes (i.e., vertex of infinite degree) which are given by the atomic part of ρ_t.

By the definition of ρ_t, we see that these infinite nodes are associated with the jumps of the Lévy process X. If such a node corresponds to a jump time s of X, we call ΔX_s the size of the node.

3.7. *The dual process and representation formula.* We shall need the $\mathcal{M}_f(\mathbb{R}_+)$-valued process $\eta = (\eta_t, t \geq 0)$ defined by

$$(30) \qquad \eta_t(dr) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (X_s - I_t^s)\delta_{H_s}(dr) + \beta \mathbf{1}_{[0,H_t]}(r)\,dr.$$

The process η is the dual process of ρ under $\mathbb{N}^\psi$ (see Corollary 3.1.6 in [16]). It represents how the trees "on the left" of t branch along the ancestral line of t.

We recall the Poisson representation of (ρ, η) under $\mathbb{N}^\psi$. Let $\mathcal{N}(dx\,d\ell\,du)$ be a Poisson point measure on $[0, +\infty)^3$ with intensity

$$dx\ell\pi(d\ell)\mathbf{1}_{[0,1]}(u)\,du.$$

For every $a > 0$, let us denote by $\mathbb{M}_a^\psi$ the law of the pair (μ_a, ν_a) of measures on $\mathbb{R}_+$ with finite mass defined by the following: for any $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$(31) \qquad \langle \mu_a, f \rangle = \int \mathcal{N}(dx\,d\ell\,du)\mathbf{1}_{[0,a]}(x)u\ell f(x) + \beta \int_0^a f(x)\,dx,$$

$$(32) \qquad \langle \nu_a, f \rangle = \int \mathcal{N}(dx\,d\ell\,du)\mathbf{1}_{[0,a]}(x)\ell(1-u)f(x) + \beta \int_0^a f(x)\,dx.$$

REMARK 3.10. In particular $\mu_a(dr) + \nu_a(dr)$ is defined as $\mathbf{1}_{[0,a]}(r)d_rW_r$, where W is a subordinator with Laplace exponent $\psi' - \alpha$ where $\alpha = \psi'(0)$ is defined by (23).

We finally set $\mathbb{M}^\psi = \int_0^{+\infty} da\,e^{-\alpha a}\mathbb{M}_a^\psi$.

PROPOSITION 3.11 ([16], Proposition 3.1.3). *For every nonnegative measurable function F on $\mathcal{M}_f(\mathbb{R}_+)^2$,*

$$\mathbb{N}^\psi\left[\int_0^\sigma F(\rho_t, \eta_t)\,dt\right] = \int \mathbb{M}^\psi(d\mu\,d\nu)F(\mu, \nu),$$

where $\sigma = \inf\{s > 0; \rho_s = 0\}$ denotes the length of the excursion.

4. Super-critical Lévy continuum random tree. We shall construct a Lévy CRT with super-critical branching mechanism using a Girsanov formula.

Let $\tilde{\psi}$ be a (sub)critical branching mechanism. The process $Z = (Z_a, a \geq 0)$, where $Z_a = L_{T_x}^a$, is a CB with branching mechanism $\tilde{\psi}$. We have $\mathbb{P}_x^{\tilde{\psi}}$-a.s. $Z_\infty = \lim_{a \to +\infty} Z_a = 0$. We shall call x the initial mass of the $\tilde{\psi}$-CRT under $\mathbb{P}_x^{\tilde{\psi}}$. Formula (28) readily implies the following Girsanov's formula: for any nonnegative measurable function F, and $q \geq 0$,

$$(33) \qquad \mathbb{E}_x^{\tilde{\psi}}[M_\infty^{\tilde{\psi},q} F(\rho)] = \mathbb{E}_x^{\tilde{\psi}_q}[F(\rho)],$$

where $M_\infty^{\tilde{\psi},q}$ is given by (21).

We will use a similar formula (with $q < 0$) to define the exploration process for a super-critical Lévy CRT with branching mechanism ψ. Because super-critical branching process may have an infinite mass, we shall cut it at a given level to construct the corresponding genealogical continuum random tree (see [15] when $\pi = 0$).

For $a \geq 0$, let $\mathcal{M}_f^a = \mathcal{M}_f([0,a])$ be the set of nonnegative measures on $[0,a]$, and let $\mathcal{D}^a$ be the set of càd-làg $\mathcal{M}_f^a$-valued process defined on $[0,+\infty)$ endowed with the Skorohod topology. We now define a projection from $\mathcal{D}$ to $\mathcal{D}^a$. For $\rho = (\rho_t, t \geq 0) \in \mathcal{D}$, we consider the time spent below level a up to time t: $\Gamma_{\rho,a}(t) = \int_0^t \mathbf{1}_{\{H(\rho_s) \leq a\}} \, ds$ and its right continuous inverse

$$(34) \quad C_{\rho,a}(t) = \inf\{r \geq 0; \Gamma_{\rho,a}(r) > t\} = \inf\left\{ r \geq 0; \int_0^r \mathbf{1}_{\{H(\rho_s) \leq a\}} \, ds > t \right\},$$

with the convention that $\inf \varnothing = +\infty$. We define the projector π_a from $\mathcal{D}$ to $\mathcal{D}^a$ by

$$(35) \qquad \pi_a(\rho) = (\rho_{C_{\rho,a}(t)}, t \geq 0),$$

with the convention $\rho_{+\infty} = 0$. By construction we have the following compatibility relation: $\pi_a \circ \pi_b = \pi_a$ for $0 \leq a \leq b$.

Let ψ be a super-critical branching mechanism which we suppose to be conservative, that is, (3) holds. Recall q^* is the unique (positive) root of $\psi'(q) = 0$. In particular the branching mechanism ψ_q is critical if $q = q^*$ and sub-critical if $q > q^*$.

We consider the filtration $\mathcal{H} = (\mathcal{H}_a, a \geq 0)$ where $\mathcal{H}_a$ is the σ-field generated by the càd-làg process $\pi_a(\rho)$ and the class of $\mathbb{P}_x^{\psi_{q^*}}$ negligible sets. Thanks to the second statement of Proposition 3.6, we get that Z is $\mathcal{H}$-adapted. Furthermore the proof of Theorem 1.4.1 in [16] yields that Z is a Markov process w.r.t. the filtration $\mathcal{H}$. In particular the process $M^{\psi_{q^*}, -q^*}$ defined by (8) is thanks to Theorem 2.2 a $\mathcal{H}$-martingale under $\mathbb{P}_x^{\psi_{q^*}}$.

Let $q \geq q^*$. We define the distribution $\mathbb{P}_x^{\psi,a}$ (resp., $\mathbb{N}^{\psi,a}$) of the ψ-CRT cut at level a with initial mass x, as the distribution of $\pi_a(\rho)$ under $M_a^{\psi_q, -q} d\mathbb{P}_x^{\psi_q}$

[resp., $e^{qZ_a + \psi(q) \int_0^a Z_r \, dr} \, d\mathbb{N}^{\psi_q}$]: for any measurable nonnegative function F,

(36) $$\mathbb{E}_x^{\psi,a}[F(\rho)] = \mathbb{E}_x^{\psi_q}[M_a^{\psi_q,-q} F(\pi_a(\rho))],$$

(37) $$\mathbb{N}^{\psi,a}[F(\rho)] = \mathbb{N}^{\psi_q}[e^{qZ_a + \psi(q) \int_0^a Z_r \, dr} F(\pi_a(\rho))].$$

LEMMA 4.1. *The distributions $\mathbb{P}_x^{\psi,a}$ and $\mathbb{N}^{\psi,a}$ do not depend on the choice of $q \geq q^*$.*

PROOF. Let $q > q^*$. For any nonnegative measurable function F, we have

$$\mathbb{E}_x^{\psi_q}[M_a^{\psi_q,-q} F(\pi_a(\rho))] = \mathbb{E}_x^{\psi_q}[e^{-qx + qZ_a + \psi(q) \int_0^a Z_s \, ds} F(\pi_a(\rho))].$$

As $\psi_q = (\psi_{q^*})_{q-q^*}$, we apply Girsanov's formula (33) and the fact that $M^{\psi_{q^*}, q-q^*}$ is a martingale to get

$$\mathbb{E}_x^{\psi_q}[M_a^{\psi_q,-q} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[M_a^{\psi_{q^*}, q-q^*} e^{-qx + qZ_a + \psi(q) \int_0^a Z_s \, ds} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[e^{(q-q^*)x - (q-q^*)Z_a - \psi_{q^*}(q-q^*) \int_0^a Z_s \, ds} e^{-qx + qZ_a + \psi(q) \int_0^a Z_s \, ds} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[e^{-q^* x + q^* Z_a - (\psi(q) - \psi(q^*)) \int_0^a Z_s \, ds} e^{\psi(q) \int_0^a Z_s \, ds} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[M_a^{\psi_{q^*}, -q^*} F(\pi_a(\rho))].$$

Excursion theory then gives the result for the excursion measures. □

Let $\mathcal{W}$ be the set of $\mathcal{D}$-valued processes endowed with the σ-field generated by the coordinate applications.

PROPOSITION 4.2. *Let $(\rho^a, a \geq 0)$ be the canonical process on $\mathcal{W}$. There exists a probability measure $\bar{\mathbb{P}}_x^\psi$ (resp., an excursion measure $\bar{\mathbb{N}}^\psi$) on $\mathcal{W}$, such that, for every $a \geq 0$, the distribution of ρ^a under $\bar{\mathbb{P}}_x^\psi$ (resp., $\bar{\mathbb{N}}^\psi$) is $\mathbb{P}_x^{\psi,a}$ (resp., $\mathbb{N}^{\psi,a}$) and such that, for $0 \leq a \leq b$*

(38) $$\pi_a(\rho^b) = \rho^a \qquad \bar{\mathbb{P}}_x^\psi\text{-a.s. (resp., } \bar{\mathbb{N}}^\psi\text{-a.e.).}$$

PROOF. To prove the existence of such a projective limit, it is enough to check the compatibility relation between $\mathbb{P}_x^{\psi,b}$ and $\mathbb{P}_x^{\psi,a}$ for every $b \geq a \geq 0$.

Let $0 \leq a \leq b$. We get

$$\mathbb{E}_x^{\psi,b}[F(\pi_a(\rho))] = \mathbb{E}_x^{\psi_{q^*}}[M_b^{\psi_{q^*}, -q^*} F(\pi_a \circ \pi_b(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[M_b^{\psi_{q^*}, -q^*} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi_{q^*}}[M_a^{\psi_{q^*}, -q^*} F(\pi_a(\rho))]$$

$$= \mathbb{E}_x^{\psi,a}[F(\rho)],$$

where we used the compatibility relation of the projectors for the second equality and the fact that $M^{\psi_{q^*}, -q^*}$ is a $\mathcal{H}$-martingale for the third equality. We deduce that $\mathbb{P}_x^{\psi, b} \circ \pi_a = \mathbb{P}_x^{\psi, a}$.

This compatibility relation implies the existence of a projective limit $\bar{\mathbb{P}}_x^\psi$. The result is similar for the excursion measure. $\square$

Let us remark that the definitions of $\bar{\mathbb{P}}_x^\psi$ and $\bar{\mathbb{N}}^\psi$ are also valid for a (sub)critical branching mechanism ψ, with the convention $q^* = 0$. In particular, we get the following corollary.

COROLLARY 4.3. *If ψ is (sub)critical, then the law of the process $(\pi_a(\rho), a \geq 0)$ under $\mathbb{P}_x^\psi$ (resp., $\mathbb{N}^\psi$) is $\bar{\mathbb{P}}_x^\psi$ (resp., $\bar{\mathbb{N}}^\psi$).*

By construction the local time at level a of ρ^b for $b \geq a$ does not depend on b, we denote by Z_a its value. Property (ii) of Theorem 2.2 implies that $Z = (Z_a, a \geq 0)$ is under $\bar{\mathbb{P}}_x^\psi$ a CB with branching mechanism ψ. Hence, the probability measure $\bar{\mathbb{P}}_x^\psi$ can be seen as the law of the exploration process that codes the super-critical CRT associated with ψ.

We get the following direct consequence of Properties (i) and (ii) of Lemma 2.4 and of the theory of excursion measures.

COROLLARY 4.4. *Let $q > 0$ such that $\psi(q) \geq 0$. Then, the probability measure $\bar{\mathbb{P}}_x^{\psi_q}$ is absolutely continuous with respect to $\bar{\mathbb{P}}_x^\psi$ with*

$$\frac{d\bar{\mathbb{P}}_x^{\psi_q}}{d\bar{\mathbb{P}}_x^\psi} = M_\infty^{\psi, q} = \mathrm{e}^{qx - \psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

The measure $\bar{\mathbb{N}}^{\psi_q}$ is absolutely continuous with respect to $\bar{\mathbb{N}}^\psi$ with

$$\frac{d\bar{\mathbb{N}}^{\psi_q}}{d\bar{\mathbb{N}}^\psi} = \mathrm{e}^{-\psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

If the total mass of Z, $\sigma = \int_0^{+\infty} Z_a \, da$, is finite, then ρ^a is the projection of a well-defined exploration process.

LEMMA 4.5. *On $\{\sigma < +\infty\}$, there exists $\rho^\infty \in \mathcal{D}$ such that $\rho^a = \pi^a(\rho^\infty)$ for all $a \geq 0$, $\bar{\mathbb{P}}_x^\psi$-a.s. or $\bar{\mathbb{N}}^\psi$-a.e.*

PROOF. It is enough to get the result under $\bar{\mathbb{P}}_x^\psi$.

First we assume that ψ is (sub)critical. Proposition 3.6 implies that $\int_0^t \mathbf{1}_{\{H(\rho_s) \leq a\}} \, ds$ increases to t as a goes to infinity. Using (34), (35) and the right continuity of ρ, we deduce that $\mathbb{P}_x^\psi$-a.s. for all $t \geq 0$, $\lim_{a \to +\infty} \pi^a(\rho)_t = \rho_t$.

Thanks to Corollary 4.3, we deduce that $\bar{\mathbb{P}}_x^\psi$-a.s. for all $t \geq 0$, $\rho_t^\infty = \lim_{a \to +\infty} \pi^a(\rho)_t$ exists and that $\pi_a(\rho^\infty) = \rho^a$.

The case ψ super-critical is then a consequence of Corollary 4.4. $\square$

Without confusion, we shall always write $\mathbb{P}^\psi$ instead of $\bar{\mathbb{P}}^\psi$ and $\mathbb{N}^\psi$ instead of $\bar{\mathbb{N}}^\psi$ and call them the law or the excursion measure of the exploration process of the CRT, whether ψ is super-critical or (sub)critical. And we shall write ρ for the projective limit $(\rho^a, a \geq 0)$ on $\mathcal{W}$, and make the identification $\rho = \rho^\infty \in \mathcal{D}$ when the latter exists, that is, when σ defined by (28) is finite.

Recall ψ^{-1} is given by (18). We now extend formula (29) for general branching mechanism.

LEMMA 4.6. *Let σ be given by (28). We have, for $\lambda \geq 0$,*
$$\mathbb{E}_x^\psi[e^{-\lambda\sigma}] = \exp\left(-x\mathbb{N}^\psi[1 - e^{-\lambda\sigma}]\right) = e^{-x\psi^{-1}(\lambda)}.$$

PROOF. Let $q \geq q^*$. We have
$$\begin{aligned}
\mathbb{E}_x^\psi[e^{-\lambda\int_0^a Z_r\,dr}] &= \mathbb{E}_x^{\psi_q}[M_a^{\psi_q,-q}e^{-\lambda\int_0^a Z_r\,dr}] \\
&= e^{-qx}\mathbb{E}_x^{\psi_q}[e^{qZ_a + (\psi(q)-\lambda)\int_0^a Z_r\,dr}] \\
&= e^{-qx}e^{-x\mathbb{N}^{\psi_q}[1 - e^{qZ_a + (\psi(q)-\lambda)\int_0^a Z_r\,dr}]} \\
&= e^{-qx}e^{-x\mathbb{N}^{\psi_q}[1 - e^{qZ_a + \psi(q)\int_0^a Z_r\,dr}]} \\
&\quad \times e^{-x\mathbb{N}^{\psi_q}[e^{qZ_a + \psi(q)\int_0^a Z_r\,dr}(1 - e^{-\lambda\int_0^a Z_r\,dr})]} \\
&= \mathbb{E}_x^{\psi_q}[M_a^{\psi_q,-q}]e^{-x\mathbb{N}^\psi[1 - e^{-\lambda\int_0^a Z_r\,dr}]} \\
&= e^{-x\mathbb{N}^\psi[1 - e^{-\lambda\int_0^a Z_r\,dr}]},
\end{aligned}$$
where we used (36) for the first equality, (8) for the second, Lemma 3.9 for the third, (37) for the fifth and (1) of Theorem 2.2 for the last. We then let a goes to infinity to get the first equality of the lemma, and use (20) to get the second. $\square$

5. Pruning. We keep notations from Section 3. Recall that $\mathcal{D}$ is the set of càd-làg $\mathcal{M}_f(\mathbb{R}_+)$-valued process, and $\mathcal{W}$ is the set of $\mathcal{D}$-valued processes. Let $R = (\rho^\theta, \theta \geq 0)$ be the canonical process on $\mathcal{W}$.

Let ψ be a (sub)critical branching mechanism. The pruning procedure developed in [6] when $\pi = 0$, [1] when $\beta = 0$ and in [5] or [25] for the general case, yields a probability measure on $\mathcal{W}$, $\tilde{\mathbb{P}}_x^\psi$, such that R is Markov and the law ρ^θ under $\tilde{\mathbb{P}}_x^\psi$ is $\mathbb{P}_x^{\psi_\theta}$ for all $\theta \geq 0$. Furthermore ρ^θ codes for a sub-tree of $\rho^{\theta'}$ if $\theta \geq \theta'$. We recall the construction of $\tilde{\mathbb{P}}_x^\psi$ in Section 5.1.

5.1. *Pruning of (sub)critical CRT.* The main idea of the pruning procedure of a tree coded by an exploration ρ is to put marks on a leaf t (or a branch labeled by t) and more precisely on the measure ρ_t. There are two types of marks: the first ones only lay on the *nodes* of the tree whereas the

other ones lay on the *skeleton* of the tree; each mark appears at a random time. At time θ, we remove all the vertex of the initial tree that contains a mark on their lineage. In terms of exploration processes, we get ρ^θ by a time change of the process ρ that skips all the times t representing individuals that received a mark on their lineage by time θ. We explain more precisely the pruning procedure.

5.1.1. *Marks on the nodes.* Let $(X_t, t \geq 0)$ be the Lévy process with branching mechanism ψ and let ρ be the corresponding exploration process. Recall $(\Delta_s, s \in \mathcal{J})$ denotes the set of the sizes of jumps of X. Conditionally on X, we consider a family

$$(T_s, s \in \mathcal{J})$$

of independent exponential random variables with respective parameter Δ_s. We define the $\mathcal{M}(\mathbb{R}_+^2)$-valued process $M^{(\mathrm{nod})} = (M_t^{(\mathrm{nod})}, t \geq 0)$ by

$$M_t^{(\mathrm{nod})}(dr, dv) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} \delta_{T_s}(dv)\delta_{H_s}(dr).$$

For fixed $\theta \geq 0$, we will consider the $\mathcal{M}(\mathbb{R}_+)$-valued process $M_t^{(\mathrm{nod})}(dr, [0, \theta])$ whose atoms give the marked nodes: each node of infinite degree is marked independently from the others with probability $1 - \mathrm{e}^{-\theta\Delta_s}$, where Δ_s is the mass (i.e., the height of the jump) associated with the node.

REMARK 5.1. Although different from the measure process that defines the marks on the nodes in [1] [formula (12)], this construction gives the same marks (see Introduction of [1]).

REMARK 5.2. The time parameter introduced here allows us to construct a coherent family of marks. Indeed, for $\theta' > \theta$, the atoms of $M_t^{(\mathrm{nod})}(dr, [0, \theta])$ are still atoms of $M_t^{(\mathrm{nod})}(dr, [0, \theta'])$. In other words, there are more and more marked nodes as θ increases, which allows us to construct a "decreasing" tree-valued process in Section 5.1.3.

5.1.2. *Marks on the skeleton.* Let $M^{(\mathrm{ske})} = (M_t^{(\mathrm{ske})}, t \geq 0)$ be a Lévy snake with lifetime H and spatial motion a Poisson point process with intensity

$$2\beta\mathbf{1}_{\{u>0\}}\, du.$$

(See [16] for the definition of a Lévy snake and [5] for the extension to a discontinuous height process H; see also [25].)

In other words, $M^{(\mathrm{ske})}$ is a $\mathcal{M}(\mathbb{R}_+^2)$-valued process such that, conditionally on the exploration process ρ:

- for every $t \geq 0$, $M_t^{(\mathrm{ske})}(dr, du)$ is a Poisson point measure with intensity

$$2\beta\mathbf{1}_{[0, H_t]}(r)\, dr\mathbf{1}_{\{u>0\}}\, du;$$

- for every $0 \leq t \leq t'$, with $H_{t,t'} := \inf_{s \in [t,t']} H_s$, then:
  - the measures $M_t^{(\text{ske})}(dr, du)\mathbf{1}_{r \in [0, H_{t,t'}]}$ and $M_{t'}^{(\text{ske})}(dr, du)\mathbf{1}_{r \in [0, H_{t,t'}]}$ are equal;
  - the random measures $M_t^{(\text{ske})}(dr, du)\mathbf{1}_{r \in [H_{t,t'}, H_t]}$ and $M_{t'}^{(\text{ske})}(dr, du) \times \mathbf{1}_{r \in [H_{t,t'}, H_{t'}]}$ are independent.

5.1.3. *Definition of the pruned processes.* We define the mark process as

(39) $$M^{(\text{mark})} = M^{(\text{nod})} + M^{(\text{ske})}.$$

The process $((\rho_t, M_t^{(\text{mark})}), t \geq 0)$ is called the marked exploration process. It is Markovian (see [25] for its properties). We denote by $\hat{\mathbb{P}}_x^\psi$ its law and by $\hat{\mathbb{N}}^\psi$ the corresponding excursion measure.

For every $\theta > 0$ and $t > 0$, we set

$$m_t^{(\theta)} = M_t^{(\text{mark})}([0, H_t] \times [0, \theta]).$$

The random variable $m_t^{(\theta)}$ is the number of marks at time θ that lay on the lineage of the individual labeled by t. We will only consider the individuals without marks on their lineage. Therefore, we set

(40) $$A_t^{(\theta)} = \int_0^t \mathbf{1}_{\{m_s^{(\theta)}=0\}}\, ds \quad \text{and} \quad C_t^{(\theta)} = \inf\{r \geq 0; A_r^{(\theta)} \geq t\},$$

its right-continuous inverse. Finally, we define $\rho^\theta = (\rho_t^\theta, t \geq 0)$, $M^{(\text{mark}),\theta} = (M_t^{(\text{mark}),\theta}, t \geq 0)$ by

$$\rho_t^\theta = \rho_{C_t^{(\theta)}},$$

$$M_t^{(\text{mark}),\theta}([0, h] \times [0, q]) = M_{C_t^{(\theta)}}^{(\text{mark})}([0, h] \times (\theta, q + \theta]).$$

We shall use in Section 7 the pruning operator Λ_θ defined on the marked exploration process by

(41) $$\Lambda_\theta(\rho, M^{(\text{mark})}) = (\rho^\theta, M^{(\text{mark}),\theta}).$$

Using the lack of memory of the exponential random variables and of properties of Poisson point measure, it is easy to get

LEMMA 5.3. *The process $R = (\rho^\theta, \theta \geq 0)$ is Markov.*

The $\mathcal{W}$-valued process R codes for a decreasing family of CRT, which we shall call a ψ-family of pruned CRT. A direct application of Theorem 1.1 of [5] gives the marginal distribution.

PROPOSITION 5.4. *The marked exploration process $(\rho^\theta, M^{(\text{mark}),\theta})$ under $\mathbb{P}_x^\psi$ (resp., $\mathbb{N}^\psi$) is distributed as $(\rho, M^{(\text{mark})})$ under $\mathbb{P}_x^{\psi_\theta}$ (resp., $\mathbb{N}^{\psi_\theta}$).*

We shall now concentrate on the process R. Let $\tilde{\mathbb{P}}_x^\psi$ be the law of R, and $\tilde{\mathbb{N}}^\psi$ be the corresponding excursion measure.

We deduce the following compatibility relation from the Markov property of R and Proposition 5.4.

COROLLARY 5.5.   *Let $\theta_0 \geq 0$. The law under $\tilde{\mathbb{P}}_x^\psi$ (resp., $\tilde{\mathbb{N}}^\psi$) of the process $(\rho^{\theta_0+\theta}, \theta \geq 0)$ is $\tilde{\mathbb{P}}_x^{\psi_{\theta_0}}$ (resp., $\tilde{\mathbb{N}}^{\psi_\theta}$).*

Let us now recall the special Markov property, Theorem 4.2 of [5], stated for the present context. We fix $\theta > 0$. We want to describe the law of the excursions of ρ "above" the marks, given the process "under" the marks. More precisely, we define O as the interior of the set $\{s \geq 0, m_s^{(\theta)} = 0\}$ and write $O = \bigcup_{i \in I}(\alpha_i, \beta_i)$. For every $i \in I$, we define the exploration process $\rho^{(i)}$ by: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$, $t \geq 0$,

$$\langle \rho_t^{(i)}, f \rangle = \int_{[H_{\alpha_i}, +\infty)} f(x - H_{\alpha_i})\rho_{(\alpha_i+t)\wedge\beta_i}(dx).$$

We have the following theorem.

THEOREM 5.6 (Special Markov property).   *Let $\theta > 0$, and let $(Z_t^\theta, t \geq 0)$ be the CSBP coded by ρ^θ. The point measure*

$$\sum_{i \in I} \delta_{(H_{\alpha_i}, \rho^{(i)})}(dh, d\mu)$$

under $\mathbb{P}_x^\psi$ (or $\mathbb{N}^\psi$) conditionally given $(\rho_t^\theta, t \geq 0)$, is a Poisson point measure of intensity

$$\mathbf{1}_{[0,+\infty)}(h)Z_h^\theta\, dh\left(2\beta\theta\mathbb{N}^\psi(d\mu) + \int_{(0,+\infty)} \pi(dr)(1 - \mathrm{e}^{-\theta r})\mathbb{P}_r^\psi(d\mu)\right).$$

This theorem describes in fact the joint law of $(\rho^{(\theta)}, \rho^{(\theta')})$ for $\theta < \theta'$ and hence the transition probabilities of the process R and of the time-reversed process. In terms of trees, by definition, the tree $\mathcal{T}^{(\theta')}$ is obtained from the tree $\mathcal{T}^{(\theta)}$ by pruning it with the pruning operator $\Lambda_{\theta'-\theta}$. Conversely, to get the tree $\mathcal{T}^{(\theta)}$ from the tree $\mathcal{T}^{(\theta')}$, we pick some individuals of the tree $\mathcal{T}^{(\theta')}$ according to a Poisson point measure and add at these points either a Lévy tree associated with the branching mechanism ψ_θ (first part of the intensity of the Poisson measure), or an infinite node of size r and trees distributed as $\mathbb{P}_r^{\psi_\theta}$ (second part of the intensity of the Poisson measure).

5.2. *Pruning of super-critical CRT.*   We now use the same Girsanov techniques of Section 4 to define a ψ-family of pruned CRT when ψ is supercritical.

Let ψ be a super-critical branching mechanism which we suppose to be conservative, that is, (3) holds. Recall q^* is the unique (positive) root of $\psi'(q) = 0$. In particular the branching mechanism ψ_q is critical if $q = q^*$ and sub-critical if $q > q^*$.

Let $q \geq q^*$. Let $R = (\rho^\theta, \theta \geq 0)$ be the canonical process on $\mathcal{W}$. We set $Z = (L_\infty^a(\rho^0), a \geq 0)$ which is under $\tilde{\mathbb{P}}_x^{\psi_q}(dR)$ a CB with branching mechanism ψ_q. The process Z is also well defined under the excursion measure $\tilde{\mathbb{N}}^{\psi_q}(dR)$. We write $\pi_a(R) = (\pi_a(\rho^\theta), \theta \geq 0)$. Notice that given the marks (i.e., given $M^{(\mathrm{nod})}$ and $M^{(\mathrm{ske})}$), we have $\pi_a(\rho^\theta) = (\pi_a(\rho))^\theta$.

Let $a \geq 0$. We define the distribution $\tilde{\mathbb{P}}_x^{\psi,a}$ (resp., excursion measure $\tilde{\mathbb{N}}^{\psi,a}$) of a ψ-family of pruned CRT cut at level a with initial mass x, as the distribution of $\pi_a(R)$ under $M_a^{\psi_q,-q} d\tilde{\mathbb{P}}_x^{\psi_q}$ [resp., $e^{qZ_a + \psi(q) \int_0^a Z_r \, dr} d\tilde{\mathbb{N}}^{\psi_q}$]: for any measurable nonnegative function F, we have

$$\tilde{\mathbb{P}}_x^{\psi,a}[F(R)] = \tilde{\mathbb{P}}_x^{\psi_q}[M_a^{\psi_q,-q} F(\pi_a(R))]$$

and

$$\tilde{\mathbb{N}}^{\psi,a}[F(\rho)] = \tilde{\mathbb{N}}^{\psi_q}[e^{qZ_a + \psi(q) \int_0^a Z_r \, dr} F(\pi_a(\rho))].$$

Same arguments as for Lemma 4.1 give the following result.

LEMMA 5.7.   *The distributions $\tilde{\mathbb{P}}_x^{\psi,a}$ and $\tilde{\mathbb{N}}^{\psi,a}$ do not depend on the choice of $q \geq q^*$.*

As in Section 4 (see Proposition 4.2) the families of measures $(\tilde{\mathbb{P}}_x^{\psi,a}, x \geq 0)$ and $(\tilde{\mathbb{N}}^{\psi,a}, a \geq 0)$ fulfill a compatibility relation. Hence there exists a projective limit $(R^a, a \geq 0)$ defined on the space of $\mathcal{W}$-valued process such that:

- for every $a \geq 0$, R^a is distributed as $\tilde{\mathbb{P}}_x^{\psi,a}$;
- for every $a < b$, $\pi_a(R^b) = R^a$.

We write $\tilde{\mathbb{P}}_x^\psi$ for the distribution of this projective limit and $\tilde{\mathbb{N}}^\psi$ for the corresponding excursion measure.

By construction the local time at level a of $\pi_b(\rho^\theta)$ for $b \geq a$ does not depend on b, we denote by Z_a^θ its value. Proposition 5.4 and Property (ii) of Theorem 2.2 imply that $Z^\theta = (Z_a^\theta, a \geq 0)$ is under $\tilde{\mathbb{P}}_x^\psi$ a CB with branching mechanism ψ_θ started at x. Following (28), we define $\sigma_\theta = \int_0^\infty Z_a^\theta \, da$. And, when there is no confusion, we write σ for σ_0.

Following Corollaries 4.3, 4.4 and Lemma 4.5, we easily get the following theorem.

THEOREM 5.8.   *Let ψ be a conservative branching mechanism. Let $(R^a, a \geq 0)$ be a $\mathcal{W}$-valued process under $\tilde{\mathbb{P}}_x^\psi$ (resp., $\tilde{\mathbb{N}}^\psi$).*

(1)   *If ψ is (sub)critical, then $(R^a, a \geq 0)$ under $\tilde{\mathbb{P}}_x^\psi$ is distributed as $((\pi_a(\rho^\theta), \theta \geq 0), a \geq 0)$ under $\mathbb{P}_x^\psi$.*

(2) *Let $q > 0$ such that $\psi(q) \geq 0$. Then, the probability measure $\tilde{\mathbb{P}}_x^{\psi_q}$ is absolutely continuous with respect to $\tilde{\mathbb{P}}_x^{\psi}$ with*

$$\frac{d\tilde{\mathbb{P}}_x^{\psi_q}}{d\tilde{\mathbb{P}}_x^{\psi}} = M_\infty^{\psi,q} = \mathrm{e}^{qx - \psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

The measure $\tilde{\mathbb{N}}^{\psi_q}$ is absolutely continuous with respect to $\tilde{\mathbb{N}}^{\psi}$ with

$$\frac{d\tilde{\mathbb{N}}^{\psi_q}}{d\tilde{\mathbb{N}}^{\psi}} = \mathrm{e}^{-\psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

(3) *On $\{\sigma < +\infty\}$, there exists $R^\infty \in \mathcal{W}$ such that $R^a = \pi^a(R^\infty)$ for all $a \geq 0$, $\tilde{\mathbb{P}}_x^{\psi}$-a.s. or $\tilde{\mathbb{N}}^{\psi}$-a.e.*

Without confusion, we shall always write $\mathbb{P}^{\psi}$ instead of $\tilde{\mathbb{P}}^{\psi}$ and $\mathbb{N}^{\psi}$ instead of $\tilde{\mathbb{N}}^{\psi}$ and call them the law or the excursion measure of ψ-pruned family of exploration processes, whether ψ is super-critical or (sub)critical. The ψ-pruned family of exploration processes codes for a ψ-pruned family of continuum random sub-trees.

And we shall write $(\rho^\theta, \theta \geq 0)$ for the projective limit $(R^a, a \geq 0)$, and identify it with $R^\infty \in \mathcal{W}$ when the latter exists, that is, when σ defined by (28) is finite. Notice that if σ_θ is finite, then the exploration process ρ^θ codes for a CRT with finite mass.

5.3. *Properties of the branching mechanism.* Let ψ be a branching mechanism with parameter (α, β, π). Let Θ' be the set of $\theta \in \mathbb{R}$ such that

$$(42) \qquad \int_{(1,+\infty)} \mathrm{e}^{-\theta\ell} \pi(d\ell) < +\infty.$$

We set $\theta_\infty = \inf \Theta'$. Notice that we have either $\Theta' = [\theta_\infty, +\infty)$ or $\Theta' = (\theta_\infty, +\infty)$ and that $\theta_\infty \leq 0$. Notice that ψ_θ exists for every $\theta \in \Theta'$ and is conservative for every $\theta > \theta_\infty$. We set $\Theta = \{\theta \in \Theta'; \psi_\theta \text{ is conservative}\}$. Notice that $\Theta \subset \Theta' \subset \Theta \cup \{\theta_\infty\}$.

For instance, we have the following examples of critical branching mechanisms:

(i) quadratic case: $\psi(u) = \beta u^2$, $\Theta = \Theta' = \mathbb{R}$;
(ii) stable case: $\psi(u) = cu^\alpha$ with $\alpha \in (1, 2)$, $\Theta = \Theta' = [0, +\infty)$;
(iii) $\psi(u) = (u + \mathrm{e}^{-1}) \log(u + \mathrm{e}^{-1}) + \mathrm{e}^{-1}$: $\Theta = \Theta' = [-\mathrm{e}^{-1}, +\infty)$ [Notice that $\psi_{\theta_\infty}(u) = u \log(u)$, $\psi'_{\theta_\infty}(0+) = -\infty$ and ψ_{θ_∞} is conservative.];
(iv) $\psi(u) = u - 1 + \frac{1}{1+u}$ is associated with $(\tilde{\alpha}, \beta, \pi)$ where $\tilde{\alpha} = 2/\mathrm{e}$, $\beta = 0$ and $\pi(d\ell) = \mathrm{e}^{-\ell} \mathbf{1}_{\{\ell > 0\}} d\ell$: $\Theta = \Theta' = (-1, +\infty)$.

For the end of this subsection, we assume that ψ is CRITICAL and that $\beta > 0$ or $\pi \neq 0$. Remark that ψ is a one-to-one function from $[0, +\infty)$ onto $[0, +\infty)$, and we denote by ψ^{-1} its inverse function. For $\theta < 0$ such that

$\theta \in \Theta'$, we define $\bar{\theta} = \psi^{-1}(\psi(\theta))$, or, equivalently, $\bar{\theta}$ is the unique positive real number such that

$$(43) \qquad\qquad \psi(\bar{\theta}) = \psi(\theta).$$

Since ψ is continuous and strictly convex, if $\theta_\infty \in \Theta'$, we have

$$(44) \qquad\qquad \bar{\theta}_\infty = \lim_{\theta \downarrow \theta_\infty} \bar{\theta}.$$

Notice that in this case $\bar{\theta}_\infty$ is finite. If $\theta_\infty \notin \Theta'$, we define $\bar{\theta}_\infty$ using (44).

LEMMA 5.9. *Let ψ be CRITICAL with parameters $(\tilde{\alpha}, \beta, \pi)$ such that $\beta > 0$ or $\pi \neq 0$. If $\theta_\infty \notin \Theta'$ then $\bar{\theta}_\infty = +\infty$.*

PROOF. We assume that $\theta_\infty \notin \Theta'$. It is enough to check that $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$ to get $\bar{\theta}_\infty = +\infty$.

We first consider the case $\theta_\infty = -\infty$. Since $\psi'(0) = 0$ and ψ is strictly convex, we get that $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$.

If $\theta_\infty > -\infty$, then using that (42) does not hold for θ_∞ and monotone convergence theorem, we get that $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$. $\square$

6. A tree-valued process. Let ψ be a branching mechanism. We assume $\theta_\infty < 0$. We write $\mathcal{R}_q = (\rho^{\gamma+q}, \gamma \geq 0)$.

We deduce from Corollary 5.5 that the families of measures $(\mathbb{P}^{\psi_\theta}, \theta \in \Theta)$ and $(\mathbb{N}^{\psi_\theta}, \theta \in \Theta)$ satisfy the following compatibility property: if $\theta' < \theta$, $\theta' \in \Theta$, the process $\mathcal{R}_{\theta-\theta'}$ under $\mathbb{P}^{\psi_{\theta'}}$ (resp., $\mathbb{N}^{\psi_{\theta'}}$) is distributed as $\mathcal{R}_0$ under $\mathbb{P}^{\psi_\theta}$ (resp., $\mathbb{N}^{\psi_\theta}$).

Hence, there exists a projective limit $\mathcal{R} = (\rho^\gamma, \gamma \in \Theta)$ such that, for every $\theta \in \Theta$, the process $(\rho^{\theta+\gamma}, \gamma \geq 0)$ is distributed as $(\rho^\gamma, \gamma \geq 0)$ under $\mathbb{P}^{\psi_\theta}$. We denote by $\mathbf{P}^\psi$ the distribution of the projective limit $\mathcal{R}$, and by $\mathbf{N}^\psi$ the corresponding excursion measure. We still write $\mathcal{R}_\theta$ for $(\rho^{\theta+\gamma}, \gamma \geq 0)$ for all $\theta \in \Theta$.

The process $\mathcal{R} = (\rho^\theta, \theta \in \Theta)$ is Markovian, thanks to Lemma 5.3. It codes for a tree-valued Markov process, which evolves according to a pruning procedure. At time θ, ρ^θ has distribution $\mathbb{P}^{\psi_\theta}$. Recall σ_θ is the mass of the CRT coded by ρ^θ. It is not difficult to check that $\Sigma = (\sigma_\theta, \theta \in \Theta)$ is a nonincreasing Markov process taking values in $[0, +\infty]$ and we shall consider a version of $\mathcal{R}$ such that the process Σ is càd-làg. From the continuity of ψ, we deduce that the Laplace transform of σ_θ given in Lemma 4.6 is continuous, and thus the process Σ is continuous in probability.

See [24] for the distribution of the decreasing rearrangement of the jumps of $(\sigma_\theta, \theta \geq 0)$ in the case of stable trees. We deduce from the pruning procedure that a.s. $\lim_{\theta \to +\infty} \sigma_\theta = 0$. Notice that by considering the time returned process $(\rho^{-\theta}, \theta < \theta_\infty)$, we get a Markovian family of exploration processes coding for a family of increasing CRTs.

REMARK 6.1.   Recall q^* is the unique root of $\psi'(q) = 0$ and that ψ_{q^*} is critical. Using a shift on θ by q^*, that is replacing ψ by ψ_{q^*}, one sees that it is enough, when studying $\mathcal{R}$, to assume that ψ is critical.

LEMMA 6.2.   *Let ψ be a critical branching mechanism with parameter* (α, β, π). *For any $\theta \in \Theta$, and any nonnegative measurable function F defined on the state space of $\mathcal{R}_0$, we have*

$$(45) \qquad \mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{\sigma_\theta < \infty\}}] = \mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\mathbf{1}_{\{\sigma_0 < \infty\}}] = \mathbf{N}^\psi[F(\mathcal{R}_0)\mathrm{e}^{-\psi(\theta)\sigma_0}].$$

PROOF.   The first equality is just the "compatibility property" stated at the beginning of this section.

For $\theta \geq 0$, the second equality is a direct consequence of (ii) from Theorem 5.8.

For $\theta < 0$, let $q = \bar\theta - \theta$. Notice that $\psi_\theta(q) = \psi(\bar\theta) - \psi(q) = 0$ and $(\psi_\theta)_q = \psi_{\bar\theta}$. We deduce from (ii) of Theorem 5.8 that

$$\mathbf{N}^{\psi_{\bar\theta}}[F(\mathcal{R}_0)] = \mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\mathbf{1}_{\{\sigma_0 < \infty\}}].$$

Since $\bar\theta > 0$ and $\psi(\theta) = \psi(\bar\theta)$, we get from (2) of Theorem 5.8 that

$$\mathbf{N}^{\psi_{\bar\theta}}[F(\mathcal{R}_0)] = \mathbf{N}^\psi[F(\mathcal{R}_0)\mathrm{e}^{-\psi(\bar\theta)\sigma_0}] = \mathbf{N}^\psi[F(\mathcal{R}_0)\mathrm{e}^{-\psi(\theta)\sigma_0}].$$

This ends the proof. $\square$

We deduce directly from this lemma the following result on the conditional distribution of the exploration process knowing the total mass of the CRT.

COROLLARY 6.3.   *Let ψ be a branching mechanism with parameter (α, β, π) such that (42) holds. The distribution of $(\rho^{\theta+\gamma}, \gamma \geq 0)$ conditionally on $\{\sigma_\theta = r\}$ does not depend on $\theta \in \Theta$.*

From this point forward, we assume that ψ is CRITICAL and that $\theta_\infty < 0$. The first assumption is not restrictive thanks to Remark 6.1.

Notice that ρ^θ codes for a critical (resp., sub-critical, resp., super-critical) CRT if $\theta = 0$ (resp., $\theta > 0$, resp., $\theta < 0$). In particular, we have $\sigma_\theta < +\infty$ a.s. if $\theta \geq 0$.

We consider the explosion time

$$A = \inf\{\theta \in \Theta, \sigma_\theta < +\infty\},$$

with the convention that $\inf \varnothing = \theta_\infty$. In particular, we have $A \leq 0$ $\mathbf{P}_x^\psi$-a.s. and $\mathbf{N}^\psi$-a.e. Moreover, since the process $(\sigma_\theta, \theta \in \Theta)$ is càd-làg, we have, on $\{A > \theta_\infty\}$, $\sigma_\theta = +\infty$ for every $\theta < A$ and $\sigma_\theta < +\infty$ for every $\theta > A$. For the time reversed process, A is the random time at which the tree gets an infinite mass.

We first give a lemma on the conditional distribution of σ.

LEMMA 6.4. *Let $q \in \Theta$, $q \leq \theta$. We have, for $\lambda \geq 0$,*

$$\mathbf{N}^\psi[\mathrm{e}^{-\lambda\sigma_q}|\rho^\theta] = \mathrm{e}^{-\sigma_\theta\psi_\theta(\psi_q^{-1}(\lambda))}$$

and $\mathbf{N}^\psi[\sigma_q < +\infty|\rho^\theta] = \mathrm{e}^{-\sigma_\theta\psi_\theta(\bar{q}-q)}$, where $\bar{q} = \psi^{-1}(\psi(q))$.

PROOF. Let $\lambda > 0$ and F be a nonnegative measurable function defined on $\mathcal{W}$. We write Z_a^q for the local time at level a of the exploration process ρ^q. Using (17), we have

$$(46) \qquad \mathbf{N}^\psi[\mathrm{e}^{-\lambda\sigma_q}F(\rho^\theta)] = \lim_{a\to\infty} \mathbf{N}^\psi[\mathrm{e}^{-\lambda\int_0^a Z_r^q\,dr}F(\rho^\theta)].$$

We set

$$I_a = \mathbf{N}^\psi[\mathrm{e}^{-\lambda\int_0^a Z_r^q\,dr}F(\rho^\theta)].$$

Let $G(\pi_a(\rho^\theta)) = \mathbf{E}^\psi[F(\rho^\theta)|\pi_a(\rho^\theta)]$. We have, with $\theta' = \theta - q \geq 0$,

$$\begin{aligned}
I_a &= \mathbf{N}^\psi[\mathrm{e}^{-\lambda\int_0^a Z_r^q\,dr}G(\pi_a(\rho^\theta))] \\
&= \mathbf{N}^{\psi_q}[\mathrm{e}^{-\lambda\int_0^a Z_r^0\,dr}G(\pi_a(\rho^{\theta'}))] \\
&= \mathbf{N}^\psi[\mathrm{e}^{-qZ_a^0-(\psi(q)+\lambda)\int_0^a Z_r^0\,dr}G(\pi_a(\rho^{\theta'}))] \\
&= \mathbf{N}^\psi[\mathrm{e}^{-qZ_a^{\theta'}-(\psi(q)+\lambda)\int_0^a Z_r^{\theta'}\,dr-\int_0^a K_h^a Z_h^{\theta'}\,dh}G(\pi_a(\rho^{\theta'}))],
\end{aligned}$$

where for the first equality we conditioned with respect to $\sigma(\pi_a(\rho^q))$, used Girsanov's formula for the third equality and Theorem 5.6 for the last equality with

$$\begin{aligned}
K_h^a &= 2\beta\theta'\mathbf{N}^\psi[1 - \mathrm{e}^{-qZ_{a-h}-(\psi(q)+\lambda)\int_0^{a-h} Z_r\,dr}] \\
&\quad + \int_{(0,+\infty)} \pi(du)(1 - \mathrm{e}^{-\theta'u})\mathbf{E}_u^\psi[1 - \mathrm{e}^{-qZ_{a-h}-(\psi(q)+\lambda)\int_0^{a-h} Z_r\,dr}].
\end{aligned}$$

We set

$$\begin{aligned}
\tilde{K}_h^a &= 2\beta\theta'\mathbf{N}^\psi[\mathrm{e}^{-qZ_{a-h}-\psi(q)\int_0^{a-h} Z_r\,dr}(1 - \mathrm{e}^{-\lambda\int_0^{a-h} Z_r\,dr})] \\
&\quad + \int_{(0,+\infty)} \pi(du)(1 - \mathrm{e}^{-\theta'u})\mathbf{E}_u^\psi[\mathrm{e}^{-qZ_{a-h}-\psi(q)\int_0^{a-h} Z_r\,dr}(1 - \mathrm{e}^{-\lambda\int_0^{a-h} Z_r\,dr})].
\end{aligned}$$

Using again Theorem 5.6 and Girsanov's formula, we get

$$(47) \qquad \begin{aligned}
I_a &= \mathbf{N}^\psi[\mathrm{e}^{-qZ_a^0-\psi(q)\int_0^a Z_r^0\,dr}\mathrm{e}^{-\int_0^a(\tilde{K}_h^a+\lambda)Z_h^{\theta'}\,dh}G(\pi_a(\rho^{\theta'}))] \\
&= \mathbf{N}^{\psi_q}[\mathrm{e}^{-\int_0^a(\tilde{K}_h^a+\lambda)Z_h^{\theta'}\,dh}G(\pi_a(\rho^{\theta'}))] \\
&= \mathbf{N}^\psi[\mathrm{e}^{-\int_0^a(\tilde{K}_h^a+\lambda)Z_h^\theta\,dh}G(\pi_a(\rho^\theta))] \\
&= \mathbf{N}^\psi[\mathrm{e}^{-\int_0^a(\tilde{K}_h^a+\lambda)Z_h^\theta\,dh}F(\rho^\theta)].
\end{aligned}$$

Notice also that, thanks to Girsanov's formula,

$$
\begin{aligned}
\tilde{K}_h^a &= 2\beta\theta' \mathbb{N}^{\psi_q}[1 - \mathrm{e}^{-\lambda \int_0^{a-h} Z_r \, dr}] \\
&\quad + \int_{(0,+\infty)} \pi(du)(\mathrm{e}^{-qu} - \mathrm{e}^{-\theta u}) \mathbb{E}_u^{\psi_q}[1 - \mathrm{e}^{-\lambda \int_0^{a-h} Z_r \, dr}] \\
&= 2\beta\theta' \mathbf{N}^{\psi}[1 - \mathrm{e}^{-\lambda \int_0^{a-h} Z_r^q \, dr}] \\
&\quad + \int_{(0,+\infty)} \pi(du)(\mathrm{e}^{-qu} - \mathrm{e}^{-\theta u}) \mathbf{E}_u^{\psi}[1 - \mathrm{e}^{-\lambda \int_0^{a-h} Z_r^q \, dr}].
\end{aligned}
$$

Using Lemma 4.6, we get

$$
\begin{aligned}
\lim_{a \to \infty} \tilde{K}_h^a &= 2\beta\theta' \mathbf{N}^{\psi}[1 - \mathrm{e}^{-\lambda\sigma_q}] + \int_{(0,+\infty)} \pi(du)(\mathrm{e}^{-qu} - \mathrm{e}^{-\theta u}) \mathbf{E}_u^{\psi}[1 - \mathrm{e}^{-\lambda\sigma_q}] \\
&= \psi_\theta(\psi_q^{-1}(\lambda)) - \psi_q(\psi_q^{-1}(\lambda)) \\
&= \psi_\theta(\psi_q^{-1}(\lambda)) - \lambda.
\end{aligned}
$$

We deduce from (46) and (47) that

$$
\mathbf{N}^{\psi}[\mathrm{e}^{-\lambda\sigma_q} F(\rho^\theta)] = \mathbf{N}^{\psi}[\mathrm{e}^{-\psi_\theta(\psi_q^{-1}(\lambda))\sigma_\theta} F(\rho^\theta)].
$$

Letting then λ go down to 0, we deduce, with $\bar{q} = \psi^{-1}(\psi(q))$, that

$$
\mathbf{N}^{\psi}[\mathbf{1}_{\{\sigma_q < +\infty\}} F(\rho^\theta)] = \mathbf{N}^{\psi}[\mathrm{e}^{-\psi_\theta(\bar{q}-q)\sigma_\theta} F(\rho^\theta)]. \qquad \square
$$

The next theorem gives the distribution of the explosion time A under the measure $\mathbf{N}^{\psi}$. Recall the definition of $\bar{\theta}$ in (43) and (44).

THEOREM 6.5.   *We have, for all $\theta \in [\theta_\infty, +\infty)$,*

(48) $$\mathbf{N}^{\psi}[A > \theta] = \bar{\theta} - \theta$$

and

$$
\mathbf{N}^{\psi}[A = \theta_\infty] = \begin{cases} 0, & \text{if } \theta_\infty \notin \Theta', \\ +\infty, & \text{if } \theta_\infty \in \Theta'. \end{cases}
$$

PROOF.   We have for all $\theta > \theta_\infty$

$$
\begin{aligned}
\mathbf{N}^{\psi}[A > \theta] &= \mathbf{N}^{\psi}[\sigma_\theta = +\infty] \\
&= \mathbb{N}^{\psi_\theta}[\sigma = +\infty] \\
&= \lim_{\lambda \to 0} \mathbb{N}^{\psi_\theta}[1 - \mathrm{e}^{-\lambda\sigma}] \\
&= \lim_{\lambda \to 0} \psi_\theta^{-1}(\lambda) \\
&= \psi_\theta^{-1}(0),
\end{aligned}
$$

where we used (4.6) for the fourth equality. We get, for $t > 0$,

$$\psi_\theta(t) = 0 \iff \psi(t + \theta) = \psi(\theta) \iff t + \theta = \bar{\theta},$$

and thus $\psi_\theta^{-1}(0) = \bar{\theta} - \theta$, which gives the first part of the theorem for $\theta > \theta_\infty$. Making θ decrease to θ_∞ gives the result for θ_∞.

For the second part of the theorem, we apply the second assertion of Lemma 6.4 with $\theta = 0$. We have, for every $q \leq 0$,

$$\mathbf{N}^\psi[\sigma_q < +\infty | \rho] = e^{-\sigma\psi(\bar{q} - q)}.$$

Then we have

$$\mathbf{N}^\psi[A = \theta_\infty | \rho] = \mathbf{N}^\psi[\forall q > \theta_\infty, \sigma_q < +\infty | \rho]$$

$$= \lim_{q \to \theta_\infty} \mathbf{N}^\psi[\sigma_q < +\infty | \rho]$$

$$= \lim_{q \to \theta_\infty} e^{-\sigma\psi(\bar{q} - q)}$$

$$= \begin{cases} 0, & \text{if } \theta_\infty \notin \Theta', \\ e^{-\sigma\psi(\bar{\theta}_\infty - \theta_\infty)}, & \text{if } \theta_\infty \in \Theta', \text{ with } \psi(\bar{\theta}_\infty - \theta_\infty) < +\infty, \end{cases}$$

where the last equality is a consequence of Lemma 5.9. Then integrating with respect to ρ gives the theorem. $\square$

REMARK 6.6. Since ψ^{-1} is smooth, we deduce that the mapping $q \mapsto \bar{q}$ is differentiable with

$$\frac{d\bar{q}}{dq} = \frac{\psi'(q)}{\psi'(\bar{q})}.$$

Thus, when $\theta_\infty \notin \Theta$, we have that the law of A under $\mathbf{N}^\psi$ has a density with respect to the Lebesgue measure on $\mathbb{R}$ given by

$$\mathbf{1}_{\{r \in (\theta_\infty, 0)\}} \left(1 - \frac{\psi'(r)}{\psi'(\bar{r})}\right).$$

THEOREM 6.7. (i) *Let* $\theta \in (\theta_\infty, 0)$. *Under* $\mathbf{N}^\psi$, *conditionally on* $\{A = \theta\}$, *we have for any nonnegative measurable function* F

(49) $$\mathbf{N}^\psi[F(\mathcal{R}_A) | A = \theta] = \psi'(\bar{\theta}) \mathbf{N}^\psi[F(\mathcal{R}_0)\sigma_0 e^{-\psi(\theta)\sigma_0}],$$

and the law of σ_A *is given by the following: for* $\lambda \geq 0$

$$\mathbf{N}^\psi[e^{-\lambda\sigma_A} | A = \theta] = \frac{\psi'(\bar{\theta})}{\psi'(\psi^{-1}(\lambda + \psi(\theta)))}.$$

In particular, we have

$$\mathbf{N}^\psi[\sigma_A < \infty | A = \theta] = 1.$$

(ii) *If* $\theta_\infty \in \Theta$, *we have for any nonnegative measurable function* F

(50) $$\mathbf{N}^\psi[F(\mathcal{R}_A)\mathbf{1}_{\{A = \theta_\infty\}}] = \mathbf{N}^{\psi_{\bar{\theta}_\infty}}[F(\mathcal{R}_0)].$$

In particular, the law of σ_A on the event $\{A = \theta_\infty\}$ is given by

$$\mathbf{N}^\psi[(1 - \mathrm{e}^{-\lambda\sigma_A})\mathbf{1}_{\{A=\theta_\infty\}}] = \psi^{-1}(\lambda + \psi(\theta_\infty)) - \bar{\theta}_\infty.$$

PROOF.   Let F be a nonnegative measurable function defined on the state space of $\mathcal{R}_0$. Using Lemma 6.4, we get for every $\theta_\infty < q \leq \theta < 0$,

$$\mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{A>q\}}] = \mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{\sigma_q=+\infty\}}]$$

$$= \mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{N}^\psi[\sigma_q = +\infty|\rho^\theta]]$$

$$= \mathbf{N}^\psi[F(\mathcal{R}_\theta)(1 - \mathrm{e}^{-\sigma_\theta\psi_\theta(\bar{q}-q)})]$$

$$= \mathbf{N}^\psi[F(\mathcal{R}_\theta)(1 - \mathrm{e}^{-\sigma_\theta(\psi(\theta+\bar{q}-q)-\psi(\theta))})].$$

Thus, we get that the mapping

$$q \mapsto \mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{A>q\}}]$$

is differentiable if it is finite. As $d\bar{q}/dq = \psi'(q)/\psi'(\bar{q})$, we get

$$\frac{d}{dq}\mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{A>q\}}]$$

$$= \psi'(\bar{q} - q + \theta)\left(\frac{d\bar{q}}{dq} - 1\right)\mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\sigma_0\mathrm{e}^{-\sigma_0(\psi(\bar{q}-q+\theta)-\psi(\theta))}]$$

$$= \psi'(\bar{q} - q + \theta)\frac{\psi'(q) - \psi'(\bar{q})}{\psi'(\bar{q})}\mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\sigma_0\mathrm{e}^{-\sigma_0(\psi(\bar{q}-q+\theta)-\psi(\theta))}].$$

Finally, using that σ is right continuous, we have

$$\frac{\mathbf{N}^\psi[F(\mathcal{R}_A), A \in d\theta]}{d\theta} = -\frac{d}{dq}(\mathbf{N}^\psi[F(\mathcal{R}_\theta)\mathbf{1}_{\{A>q\}}])_{|q=\theta}$$

$$= (\psi'(\bar{\theta}) - \psi'(\theta))\mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\sigma_0\mathbf{1}_{\{\sigma_0<+\infty\}}].$$

We deduce from Lemma 6.2 that

$$\mathbf{N}^\psi[F(\mathcal{R}_A)|A = \theta] = \frac{\mathbf{N}^{\psi_\theta}[F(\mathcal{R}_0)\sigma_0\mathbf{1}_{\{\sigma_0<+\infty\}}]}{\mathbf{N}^{\psi_\theta}[\sigma_0\mathbf{1}_{\{\sigma_0<+\infty\}}]} = \frac{\mathbf{N}^\psi[F(\mathcal{R}_0)\sigma_0\mathrm{e}^{-\psi(\theta)\sigma_0}]}{\mathbf{N}^\psi[\sigma_0\mathrm{e}^{-\psi(\theta)\sigma_0}]}.$$

This proves (49) but for the normalizing constant. It also implies that

$$\mathbf{N}^\psi[\mathrm{e}^{-\lambda\sigma_A}|A = \theta] = \frac{\mathbb{N}^{\psi_\theta}[\sigma\mathrm{e}^{-\lambda\sigma}]}{\mathbb{N}^{\psi_\theta}[\sigma\mathbf{1}_{\{\sigma<+\infty\}}]}.$$

Notice that $\psi_\theta^{-1}(r) = \psi^{-1}(r + \psi(\theta)) - \theta$ for $r \geq 0$. We get from Lemma 4.6 that, for $r \geq 0$,

$$\mathbb{N}^{\psi_\theta}[\sigma\mathrm{e}^{-r\sigma}] = \frac{d}{dr}\mathbb{N}^{\psi_\theta}[1 - \mathrm{e}^{-r\sigma}] = (\psi_\theta^{-1})'(r) = \frac{1}{\psi'(\psi^{-1}(r + \psi(\theta)))}.$$

In particular, we deduce the value of the normalizing constant,

$$\mathbf{N}^\psi[\sigma_0\mathrm{e}^{-\psi(\theta)\sigma_0}] = \mathbb{N}^{\psi_\theta}[\sigma\mathbf{1}_{\{\sigma<+\infty\}}] = 1/\psi'(\bar{\theta}).$$

We also get

$$\mathbf{N}^\psi[\mathrm{e}^{-\lambda\sigma_A}|A=\theta] = \frac{\psi'(\bar\theta)}{\psi'(\psi^{-1}(\lambda+\psi(\theta)))}.$$

This ends the proof of the first part.

For the second part of the theorem, we consider the case $\theta_\infty \in \Theta$. Let us first remark that, since the process $(\sigma_\theta, \theta \in \Theta)$ is continuous in probability, we have

$$\{A=\theta_\infty\} = \{\sigma_{\theta_\infty} < +\infty\}.$$

We then apply Girsanov's formula (45) twice to get

$$\begin{aligned}
\mathbf{N}^\psi[F(\mathcal{R}_A)\mathbf{1}_{\{A=\theta_\infty\}}] &= \mathbf{N}^\psi[F(\mathcal{R}_{\theta_\infty})\mathbf{1}_{\{\sigma_{\theta_\infty}<+\infty\}}]\\
&= \mathbf{N}^\psi[F(\mathcal{R}_0)\mathrm{e}^{-\psi(\theta_\infty)\sigma_0}]\\
&= \mathbf{N}^\psi[F(\mathcal{R}_0)\mathrm{e}^{-\psi(\bar\theta_\infty)\sigma_0}]\\
&= \mathbf{N}^\psi[F(\mathcal{R}_{\bar\theta_\infty})\mathbf{1}_{\{\sigma_{\bar\theta_\infty}<+\infty\}}]\\
&= \mathbf{N}^{\psi_{\bar\theta_\infty}}[F(\mathcal{R}_0)],
\end{aligned}$$

where we used for the last equality that $\sigma_{\bar\theta_\infty} < +\infty$ $\mathbf{N}^\psi$-a.e. and (45).

For $F(\mathcal{R}) = 1 - \mathrm{e}^{-\lambda\sigma}$, we obtain

$$\begin{aligned}
\mathbf{N}^\psi[(1-\mathrm{e}^{-\lambda\sigma_A})\mathbf{1}_{\{A=\theta_\infty\}}] &= \mathbf{N}^{\psi_{\bar\theta_\infty}}[1-\mathrm{e}^{-\lambda\sigma_0}]\\
&= \psi_{\bar\theta_\infty}^{-1}(\lambda)\\
&= \psi^{-1}(\lambda+\psi(\bar\theta_\infty)) - \bar\theta_\infty. \qquad \square
\end{aligned}$$

We deduce the next corollary from (49).

COROLLARY 6.8. *Let $\theta_\infty < \theta < 0$. The distribution of $\mathcal{R}_A = (\rho^{A+\gamma}, \gamma \geq 0)$ conditionally on $\{\sigma_A = r, A = \theta\}$ does not depend on θ.*

7. Pruning of an infinite tree. We want here to define an infinite tree via a spinal description of this tree. What we call a spinal description of a tree is a representation of the tree where a particular branch is considered (the spine) and the subtrees that are grafted along that branch are then described. The usual, well-known spinal descriptions of a CRT are Bismut decomposition (see [17]) where the spine is picked "at random" among all the possible branches, and Williams decomposition (see [3]) where the spine is chosen to be the highest branch of the tree. We describe next the Bismut decomposition and show how such a decomposition can uniquely define a tree. Then we define the infinite tree by such a decomposition.

7.1. *Bismut decomposition of a Lévy tree.* Let ψ be a (sub)critical branching mechanism. Recall the definition of the mark process $M^{(\mathrm{mark})}$ of Sec-

tion 5.1.3. For a marked exploration process $(\rho, M^{\mathrm{mark}})$ recall that η is defined by (30) and notice that $(\eta_{(\sigma-t)-}, M^{(\mathrm{mark})}_{\sigma-t}, t \in [0, \sigma])$ is distributed as $(\rho, M^{(\mathrm{mark})})$ under the excursion measure thanks to Corollary 3.1.6 in [16] and definition of $M^{(\mathrm{mark})}$.

We recall that the family of pruned exploration processes $\mathcal{R} = (\rho^\theta, \theta \geq 0)$ is constructed from the exploration process ρ (which is equal to ρ^0) and the measure-valued process $M^{(\mathrm{mark})}$.

Let $T \geq 0$. We define under $\mathbb{N}^\psi$ the processes $(\rho^{T\to}, M^{(\mathrm{mark}),T\to})$ and $(\rho^{\leftarrow T}, M^{(\mathrm{mark}),\leftarrow T})$ by the following: for every $t \geq 0$,

$$(\rho^{T\to}_t, M^{(\mathrm{mark}),T\to}_t) = (\rho_{(T+t)\wedge\sigma}, M^{(\mathrm{mark})}_{(T+t)\wedge\sigma}),$$

$$(\rho^{\leftarrow T}_t, M^{(\mathrm{mark}),\leftarrow T}_t) = (\eta_{(T-t)\vee 0}, M^{(\mathrm{mark})}_{(T-t)\vee 0}),$$

where ρ is the canonical exploration process and η its dual process.

Bismut decomposition describes in terms of Poisson point processes the former processes when T is "uniformly distributed" on $[0, \sigma]$.

First we must extend the definition of the measure $\mathbb{M}^\psi(d\mu, d\nu)$ of (31) and (32) to get the marks into account. Let

$$\mathcal{N}(dx, d\ell, du) = \sum_{i \in I} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

be a Poisson point measure with intensity

$$dx\, \ell\pi(d\ell)\mathbf{1}_{[0,1]}(u)\, du.$$

Conditionally on $\mathcal{N}$, let $(T_i, i \in I)$ be a family of independent exponential random variables of respective parameter ℓ_i. Finally, let $\tilde{\mathcal{N}}(dk, db) = \sum_{j \in J} \delta_{(k_j, b_j)}(dk, db)$ be an independent Poisson point measure on $[0, +\infty)^2$ with intensity $2\beta dk\, db$. We then define the spine (μ_a, ν_a, m_a) which are three measures given by

$$\mu_a(dx) = \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i)u_i\ell_i\delta_{x_i}(dx) + \mathbf{1}_{[0,a]}(x)\beta dx,$$

$$\nu_a(dx) = \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i)(1 - u_i)\ell_i\delta_{x_i}(dx) + \mathbf{1}_{[0,a]}(x)\beta dx,$$

$$m_a(dx, dq) = \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i)\delta_{x_i}(dx)\delta_{T_i}(dq) + \sum_{j \in J} \mathbf{1}_{[0,a]}(k_j)\delta_{k_j}(dx)\delta_{b_j}(dq).$$

We denote by $\tilde{\mathbb{M}}^\psi_a$ the law of the triple (μ_a, ν_a, m_a), and we set $\tilde{\mathbb{M}}^\psi = \int_0^{+\infty} da\, e^{-\psi'(0)a}\tilde{\mathbb{M}}^\psi_a$.

Let us denote by $\mathbb{P}^{\psi,*}_{\mu,m}$ the law of the pair $(\rho, M^{(\mathrm{mark})})$ starting from (μ, m) where ρ is an exploration process associated with ψ and stopped when it first reaches 0. It is easy to adapt Lemma 3.4 of [17] to get the following theorem.

THEOREM 7.1 (Bismut decomposition). *For every nonnegative measurable functionals F and G,*

(51)
$$\mathbb{N}^{\psi}\left[\int_0^{\sigma} ds F(\rho^{s\rightarrow}, M^{(\mathrm{mark}),s\rightarrow})G(\rho^{\leftarrow s}, M^{(\mathrm{mark}),\leftarrow s})\right]$$
$$= \int \tilde{\mathbb{M}}^{\psi}(d\mu, d\nu, dm)\mathbb{E}^{\psi,*}_{\mu,m}[F]\mathbb{E}^{\psi,*}_{\nu,m}[G].$$

Informally speaking, the latter theorem describes a spinal decomposition of the tree. We first pick an individual s "uniformly." The height of that individual is "distributed" as $dae^{-\psi'(0)a}$. Then, conditionally on that height, the measures ρ_s, η_s and m_s have law $\tilde{\mathbb{M}}^{\psi}_a$. Eventually, conditionally on those measures, the marked exploration processes on the right and on the left (reversed in time for that one) of the individual s are independent and distributed as marked exploration processes started respectively from (ρ_s, m_s) and (η_s, m_s), stopped when they first reach 0.

Let us now state the Poisson representation of the probability measure $\mathbb{P}^{\psi,*}_{\mu,m}$. Let $(\alpha_i, \beta_i)_{i\in I}$ be the excursion intervals of the total mass process $(\langle\rho_t, 1\rangle, t \geq 0)$ above its minimum under $\mathbb{P}^{\psi,*}_{\mu,m}$. Let $(U_i, i \in I)$ be a family of independent random variables, independent of ρ and uniformly distributed on $[0, 1]$. For every $i \in I$, we set $x_i = H_{\alpha_i}$. Then we define u_i by

$$u_i = \begin{cases} \rho_{\alpha_i}(\{x_i\})/\mu(\{x_i\}), & \text{if } \mu(\{x_i\}) > 0, \\ U_i, & \text{if } \mu(\{x_i\}) = 0. \end{cases}$$

Finally, we define the measure-valued process ρ^i by the following: for every $t \geq 0$ and every $f \in \mathcal{B}_+(\mathbb{R}_+)$,

$$\langle\rho_t^i, f\rangle = \int_{(x_i, +\infty)} f(x - x_i)\rho_{(\alpha_i+t)\wedge\beta_i}(dx),$$

and we define the measure valued-process $M^{(\mathrm{mark}),i}$ by the following: for every $t \geq$ and every $f \in \mathcal{B}_+(\mathbb{R}^2_+)$,

$$\langle M_t^{(\mathrm{mark}),i}, f\rangle = \int_{(x_i, +\infty)\times\mathbb{R}_+} f(x - x_i, \theta)M^{(\mathrm{mark})}_{(\alpha_i+t)\wedge\beta_i}(dx, d\theta).$$

It is easy to adapt Lemma 4.2.4 from [16] to get the following proposition.

PROPOSITION 7.2. *The point measure $\sum_{i\in I}\delta_{(x_i,u_i,\rho^i,M^{(\mathrm{mark}),i})}$ is under $\mathbb{P}^{\psi,*}_{\mu,m}$ a Poisson point measure with intensity*

$$\mu(dx)\,du\mathbf{1}_{[0,1]}(u)\mathbb{N}^{\psi}(d\rho, dM^{(\mathrm{mark})}).$$

7.2. *Reconstruction of the exploration process from a spinal decomposition.* Conversely, given the spinal decomposition of Bismut theorem, we reconstruct the initial exploration process, but we must add the time indices

of the excursions at the node (which in the previous Section are called u_i).
We shall also add the mark process [see its definition (39)].

Let μ and ν be two finite measures such that $\operatorname{Supp}\mu = \operatorname{Supp}\nu = [0, H]$
and m a point measure on $[0, H] \times \mathbb{R}_+$. Let $\{(\rho^i, M^{(\mathrm{mark}),i}), i \in J_g\}$ and
$\{(\rho^i, M^{(\mathrm{mark}),i}), i \in J_d\}$ be two families of marked exploration processes (see
Section 5.1.3). Let $\{(x_i, u_i), i \in J_g \cup J_d\}$ be a family of nonnegative real num-
bers. The measures μ and ν must be seen as the measures $\rho_0^{s\rightarrow}$ and $\rho_0^{\leftarrow s}$ of
Theorem 7.1, the x_i's are the heights of the branching points along the
chosen branch, the ρ^i's are the exploration processes that arise from the de-
composition of the processes $\rho^{s\rightarrow}$ and $\rho^{\leftarrow s}$ above their minimum and the u_i's
are additional features that order the excursions that are attached at the
same level. The measure m and the processes $M^{(\mathrm{mark}),i}$ will allow us to
reconstruct the mark process.

For every $i \in J_g \cup J_d$, we set σ^i the length of the process ρ^i. We define

$$(52) \qquad \mathcal{L}_g = \sum_{i \in J_g} \sigma^i, \qquad \mathcal{L}_d = \sum_{i \in J_d} \sigma^i \quad \text{and} \quad \mathcal{L} = \mathcal{L}_g + \mathcal{L}_d.$$

The variable $\mathcal{L}$ represents the total length of the excursion whereas $\mathcal{L}_g$ plays
the same role as s in the left-hand side of Theorem 7.1. For every $i \in J_g$, we
set

$$t^i = \sum_{j \in J_g, x_j < x_i} \sigma^j + \sum_{j \in J_g, x_j = x_i \text{ and } u_j > u_i} \sigma^j,$$

and, for every $i \in J_d$, we set

$$t^i = \mathcal{L}_g + \sum_{j \in J_d, x_j > x_i} \sigma^j + \sum_{j \in J_d, x_j = x_i \text{ and } u_j > u_i} \sigma^j,$$

which is the time of the beginning of the excursion ρ^i.

For every $t > 0$, we define the measure ρ_t by

$$\rho_t(dx) = \begin{cases} \rho_{t-t^i}^i(x_i + dx) + \mu(dx)\mathbf{1}_{[0,x_i)}(x) + (u_i\nu(\{x_i\}) + \mu(\{x_i\}))\delta_{x_i}(dx), \\ \qquad \text{if } t < \mathcal{L}_g, t^i \leq t < t^i + \sigma^i, \\ \mu, \qquad \text{if } t = \mathcal{L}_g, \\ \rho_{t-t^i}^i(x_i + dx) + \mu(dx)\mathbf{1}_{[0,x_i)}(x) + u_i\mu(\{x_i\})\delta_{x_i}(dx), \\ \qquad \text{if } \mathcal{L}_g < t < \mathcal{L}, t^i \leq t < t^i + \sigma^i, \\ 0, \qquad \text{if } t \geq \mathcal{L}. \end{cases}$$

We also define the mark process $M^{(\mathrm{mark})}(dx, dv)$ by

$$\begin{cases} M_{t-t^i}^{(\mathrm{mark}),i}(x_i + dx, dv) + m(dx, dv)\mathbf{1}_{[0,x_i]}(x), \\ \qquad \text{if } t < \mathcal{L}_g \text{ or } \mathcal{L}_g < t < \mathcal{L}, t^i \leq t < t^i + \sigma^i, \\ m, \qquad \text{if } t = \mathcal{L}_g, \\ 0, \qquad \text{if } t \geq \mathcal{L}. \end{cases}$$

We say that the process $(\rho, M^{(\mathrm{mark})}) = ((\rho_t, M_t^{(\mathrm{mark})}), t \geq 0)$ is the marked exploration process associated with the family

(53)
$$\mathcal{G} = (\mu, \nu, m, (x_i, u_i, (\rho^i, M^{(\mathrm{mark}),i}), i \in J_g),$$
$$(x_i, u_i, (\rho^i, M^{(\mathrm{mark}),i}), i \in J_d)).$$

From Bismut decomposition, Theorem 7.1, Proposition 7.2 and the construction of the mark process, Section 5.1.3, we get the following reconstruction corollary.

COROLLARY 7.3. *Let ψ be a (sub)critical branching mechanism. Let (μ, ν, m) be distributed according to $\tilde{\mathbb{M}}^\psi$. Let $\sum_{i \in J_g} \delta_{(x_i, u_i, \rho^i, M^{(\mathrm{mark}),i})}$ and $\sum_{i \in J_d} \delta_{(x_i, u_i, \rho^i, M^{(\mathrm{mark}),i})}$ be conditionally on (μ, ν, m) independent Poisson point measures with respective intensity*

$$\mu(dx)\mathbf{1}_{[0,1]}(u)\,du\mathbb{N}^\psi(d\rho, dM^{(\mathrm{mark})})$$

and

$$\nu(dx)\mathbf{1}_{[0,1]}(u)\,du\mathbb{N}^\psi(d\rho, dM^{(\mathrm{mark})}).$$

Then the marked exploration process associated with the family $\mathcal{G}$ given by (53) is distributed as $(\rho, M^{(\mathrm{mark})})$ under $\mathbb{N}^\psi[\sigma d(\rho, M)]$.

REMARK 7.4. If we start with an exploration process ρ, pick s at random (conditionally on ρ) on $[0, \sigma]$, then the decomposition of $\rho^{s\to}$ and $\rho^{\leftarrow s}$ as excursions above their minimum gives a family $\mathcal{G}$. The exploration process $\tilde{\rho}$ associated with $\mathcal{G}$ given by the previous construction is not ρ. Indeed, each excursion of $\tilde{\rho}$ "on the left" of s is time-reversed with respect to those of ρ. However, the trees coded by ρ and $\tilde{\rho}$ are the same.

We can also reconstruct the pruned exploration process by pruning $\mathcal{G}$. Let $\theta > 0$. We define the lowest mark lying on the spine as

(54) $\xi_\theta = \sup\{x; m([0, x] \times [0, \theta]) = 0\}.$

We set $\mu^\theta = \mu\mathbf{1}_{[0,\xi_\theta)}$, $\nu^\theta = \nu\mathbf{1}_{[0,\xi_\theta)}$, $m^\theta(dx, dq) = m(dx, \theta + dq)\mathbf{1}_{[0,\xi_\theta)}(x)$, for $\delta \in \{g, d\}$ $J_\delta^\theta = \{i \in J_\delta; x_i < \xi_\theta\}$ and

(55)
$$\mathcal{G}_\theta = (\mu^\theta, \nu^\theta, m^\theta, (x_i, u_i, \Lambda_\theta(\rho^i, M^{(\mathrm{mark}),i}), i \in J_g^\theta),$$
$$(x_i, u_i, \Lambda_\theta(\rho^i, M^{(\mathrm{mark}),i}), i \in J_d^\theta)),$$

where the pruning operator Λ_θ is defined in (41).

PROPOSITION 7.5. *Under the hypothesis of Corollary 7.3, let $(\rho^\theta, M^{(\mathrm{mark}),\theta})$ be the marked exploration process associated with the family $\mathcal{G}_\theta$ given by (55). The process $(\rho^\theta, \theta \geq 0)$ is distributed as $\mathcal{R}_0$ under $\mathbf{N}^\psi[\sigma_0 d\mathcal{R}].$*

PROOF. Let us remark that, by construction, $(\rho^\theta, M^{(\mathrm{mark}),\theta}) = \Lambda_\theta(\rho, M^{(\mathrm{mark})})$. The proposition now follows from Corollary 7.3. $\square$

7.3. *The infinite tree and its pruning.* Let ψ be a critical branching mechanism.

We build a marked continuum random tree associated with the branching mechanism ψ using a spine decomposition with an infinite spine. Intuitively, if the CRT dies in finite time (which corresponds to the case H continuous) this infinite CRT can be seen as the CRT conditioned to nonextinction.

Let

$$\mathcal{N}(dx, d\ell, du) = \sum_{i \in I} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

be a Poisson point measure with intensity

$$dx \, \ell \pi(d\ell) \mathbf{1}_{[0,1]}(u) \, du.$$

Conditionally on $\mathcal{N}$, let $(T_i, i \in I)$ be a family of independent exponential random variables of respective parameter ℓ_i. Finally, let $\tilde{\mathcal{N}}(dk, db) = \sum_{j \in J} \delta_{(k_j, b_j)}(dk, db)$ be an independent Poisson point measure on $[0, +\infty)^2$ with intensity $2\beta dk \, db$. We define the following random measures:

$$\mu^*(dx) = \sum_{i \in I} u_i \ell_i \delta_{x_i}(dx) + \beta dx,$$

$$\nu^*(dx) = \sum_{i \in I} (1 - u_i) \ell_i \delta_{x_i}(dx) + \beta dx,$$

$$m^*(dx, dq) = \sum_{i \in I} \delta_{x_i}(dx) \delta_{T_i}(dq) + \sum_{j \in J} \delta_{k_j}(dx) \delta_{b_j}(dq).$$

The measure (μ^*, ν^*, m^*) corresponds to the the measure (μ_a, ν_a, m_a) of Section 7.1 but for an infinite spine. Let

$$\sum_{i \in J_g} \delta_{(x_i, u_i, \rho^i, M^{(\mathrm{mark}), i})} \quad \text{and} \quad \sum_{i \in J_d} \delta_{(x_i, u_i, \rho^i, M^{(\mathrm{mark}), i})}$$

be conditionally on (μ^*, ν^*, m^*) independent Poisson point measures with intensity

$$\nu^*(dx) \mathbf{1}_{[0,1]}(u) \, du \mathbb{N}^\psi(d\rho, dM^{(\mathrm{mark})})$$

and

$$\mu^*(dx) \mathbf{1}_{[0,1]}(u) \, du \mathbb{N}^\psi(d\rho, dM^{(\mathrm{mark})}).$$

We set

$$\mathcal{G}^* = (\mu^*, \nu^*, m^*, (x_i, u_i, (\rho^i, M^{(\mathrm{mark}), i}), i \in J_g), (x_i, u_i, (\rho^i, M^{(\mathrm{mark}), i}), i \in J_d)),$$

which describes the decomposition of an infinite marked tree as marked subtrees that are attached along its infinite spine. Let $\theta > 0$. Following the end of Section 7.2, we now extend the pruning procedure to this infinite tree by

letting $\mathcal{G}_\theta^*$ be constructed from $\mathcal{G}^*$ as $\mathcal{G}_\theta$ given by (55) from $\mathcal{G}$ given by (53)

$$\xi_\theta^* = \sup\{x; m^*([0,x] \times [0,\theta]) = 0\}, \qquad J_\delta^\theta = \{i \in J_\delta; x_i < \xi_\theta^*\}$$
$$\text{for } \delta \in \{g, d\},$$

$$\mu^{*,\theta} = \mu^* \mathbf{1}_{[0,\xi_\theta^*)}, \qquad \nu^{*,\theta} = \nu^* \mathbf{1}_{[0,\xi_\theta^*)},$$

$$m^{*,\theta}(dx, dq) = m^*(dx, \theta + dq)\mathbf{1}_{[0,\xi_\theta^*)}(x),$$

$$\mathcal{G}_\theta^* = (\mu^{*,\theta}, \nu^{*,\theta}, m^{*,\theta}, (x_i, u_i, \Lambda_\theta(\rho^i, M^{(\mathrm{mark}),i}), i \in J_g^\theta),$$
$$(x_i, u_i, \Lambda_\theta(\rho^i, M^{(\mathrm{mark}),i}), i \in J_d^\theta)).$$

We have the following lemma.

LEMMA 7.6. *Let $\theta > 0$. The probability distribution of the spine $(\mu^{*,\theta}, \nu^{*,\theta}, m^{*,\theta})$ is $\psi'(\theta)\tilde{\mathbb{M}}^{\psi_\theta}$.*

PROOF. As ψ is critical, we deduce from (23) that

$$\psi'(\theta) = 2\beta\theta + \int_{(0,+\infty)} (1 - \mathrm{e}^{-\theta\ell})\ell\pi(d\ell).$$

We deduce from the theory of marked Poisson point measures that

$$\mathcal{N}^\theta(dx, d\ell, du) = \sum_{i \in I} \mathbf{1}_{\{T_i > \theta\}} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

is a Poisson point measure with intensity $dx\,\ell\mathrm{e}^{-\theta\ell}\pi(d\ell)\mathbf{1}_{[0,1]}(u)\,du$. Since ξ_θ^* is independent of $\mathcal{N}^\theta$, we deduce that, conditionally on ξ_θ^*, $(\mu^{*,\theta}, \nu^{*,\theta}, m^{*,\theta})$ is distributed according to $\tilde{\mathbb{M}}_{\xi_\theta^*}^{\psi_\theta}$. Notice then that ξ_θ^* is the minimum of $T_1 = \inf\{x_i; T_i \leq \theta, i \in I\}$ and $T_2 = \inf\{k_j; b_j \leq \theta, j \in J\}$, which are two independent exponential random variables, which are also independent of $\mathcal{N}^\theta$. The exponential distribution of T_1 has parameter $\int_{(0,+\infty)} (1 - \mathrm{e}^{-\theta\ell})\ell\pi(d\ell)$, and the exponential distribution of T_2 has parameter $2\beta\theta$. Thus ξ_θ^* has an exponential distribution with parameter $\psi'(\theta)$, which gives the result. $\square$

Let $(\rho^{\theta,*}, M^{(\mathrm{mark}),\theta,*})$ be the marked exploration process associated with $\mathcal{G}_\theta^*$. We set $\mathcal{R}_\theta^* = (\rho^{\theta+q,*}, q \geq 0)$ and denote by $\mathbf{E}^\psi$ its law. The next proposition tells us that $\mathcal{R}_\theta^*$ under $\mathbf{E}^\psi$ is, up to a normalizing constant, the size biased "distribution" of $\mathcal{R}_\theta$ under $\mathbf{N}^\psi$.

PROPOSITION 7.7. *Let ψ be a critical branching mechanism. For every positive measurable functional F and every $\theta > 0$, we have*

$$\psi'(\theta)\mathbf{N}^\psi[\sigma_\theta F(\mathcal{R}_\theta)] = \mathbf{E}^\psi[F(\mathcal{R}_\theta^*)].$$

PROOF.   Let F be a positive measurable functional. As $\mathcal{R}$ is constructed from $(\rho, M^{(\mathrm{mark})})$, there exists a positive measurable functional G such that

$$F(\mathcal{R}) = G(\rho, M^{(\mathrm{mark})}).$$

Moreover, there exists another positive functional $\tilde{G}$ such that, for every $s \geq 0$,

$$G(\rho, M^{(\mathrm{mark})}) = \tilde{G}((\rho^{s\rightarrow}, M^{(\mathrm{mark}),s\rightarrow}), (\rho^{\leftarrow s}, M^{(\mathrm{mark}),\leftarrow s})).$$

Then by Bismut decomposition, we have

$$
\begin{aligned}
\psi'(\theta)&\mathbf{N}^\psi[\sigma_\theta F(\mathcal{R}_\theta)]\\
&= \psi'(\theta)\mathbf{N}^{\psi_\theta}[\sigma F(\mathcal{R})]\\
&= \psi'(\theta)\mathbf{N}^{\psi_\theta}\left[\int_0^\sigma ds\tilde{G}((\rho^{s\rightarrow}, M^{(\mathrm{mark}),s\rightarrow}), (\rho^{\leftarrow s}, M^{(\mathrm{mark}),\leftarrow s}))\right]\\
&= \int \psi'(\theta)\tilde{\mathbb{M}}^{\psi_\theta}(d\mu, d\nu, dm)\mathbb{E}_{\mu,m}^{\psi_\theta,*} \otimes \mathbb{E}_{\nu,m}^{\psi_\theta,*}[\tilde{G}].
\end{aligned}
$$

Then we conclude using Lemma 7.6 and the fact that $\mathbb{N}^{\psi_\theta}(d\rho, M^{(\mathrm{mark})})$ is the distribution of $\Lambda_\theta(\rho, M^{(\mathrm{mark})})$ under $\mathbb{N}^\psi(d\rho, dM^{(\mathrm{mark})})$. $\square$

8. Distribution identity.   Let ψ be a critical branching mechanism with parameter (α, β, π). We assume that $\theta_\infty < 0$. Recall $\mathcal{R} = (\mathcal{R}_\theta, \theta \in \Theta)$ is defined in Section 6 and $\mathcal{R}_\theta^*$ in Section 7.3.

THEOREM 8.1.   *Let $\theta \in (\theta_\infty, 0)$. Conditionally on $\{A = \theta\}$, $\mathcal{R}_A$ is distributed as $\mathcal{R}_{\bar{\theta}}^*$.*

PROOF.   Let F be a nonnegative measurable function defined on $\mathcal{W}$. We have, for $\theta < 0$,

$$
\begin{aligned}
\mathbf{N}^\psi[F(\mathcal{R}_A)|A = \theta] &= \psi'(\bar{\theta})\mathbf{N}^\psi[F(\mathcal{R}_0)\sigma_0 \mathrm{e}^{-\psi(\theta)\sigma_0}]\\
&= \psi'(\bar{\theta})\mathbf{N}^{\psi_{\bar{\theta}}}[\sigma_0 F(\mathcal{R}_0)]\\
&= \psi'(\bar{\theta})\mathbf{N}^\psi[\sigma_{\bar{\theta}} F(\mathcal{R}_{\bar{\theta}})]\\
&= \mathbf{E}^\psi[F(\mathcal{R}_{\bar{\theta}}^*)],
\end{aligned}
$$

where we used (49) for the first equality, Girsanov's formula (45) (with θ replaced by $\bar{\theta}$) for the second, the invariance of the distribution of $\mathcal{R}$ by the shift for the third and Proposition 7.7 for the last. $\square$

If $u \in (0, \bar{\theta}_\infty)$, let $\check{u}$ be the unique negative real number such that

$$\bar{\check{u}} = u.$$

We deduce from Theorem 6.5 and Remark 6.6 the following corollary.

COROLLARY 8.2.   *Let us suppose that $\theta_\infty \notin \Theta$.*

Let U be a positive "random" variable with (nonnegative) "density" w.r.t., the Lebesgue measure given by

$$\left(1 - \frac{\psi'(r)}{\psi'(\check{r})}\right)\mathbf{1}_{\{r \in (0, \bar{\theta}_\infty)\}}.$$

Assume that U is independent of $\mathcal{G}^*$. Then $\mathcal{R}_A$ is distributed under $\mathbf{N}^\psi$ as $\mathcal{R}^*_U$.

This corollary can be viewed as a continuous analog of Proposition 26 of [11].

9. The quadratic case. We consider $\psi(\lambda) = \beta\lambda^2$ for some $\beta > 0$. We have $\Theta = \Theta' = \mathbb{R}$ (see the definition in Section 5.3) and $\psi_\theta(\lambda) = \beta(\lambda^2 + 2\theta\lambda)$. Recall $\bar{\theta}$ is defined by (1). So we have $\bar{\theta} = |\theta|$. From Theorem 6.5, we get $\mathbf{N}^\psi[A \geq \theta] = \bar{\theta} - \theta = 2|\theta|$ for $\theta < 0$ and $\mathbf{N}^\psi[A \geq \theta] = 0$ for $\theta \geq 0$. Thus under $\mathbf{N}^\psi$, the explosion time A is distributed as 2 times the Lebesgue measure on $(-\infty, 0)$. We deduce from Theorem 6.7 the Laplace transform of the total mass of the CRT before explosion: for $\lambda \geq 0$,

$$\mathbf{N}^\psi[e^{-\lambda\sigma_A}|A = \theta] = \frac{\sqrt{\beta\theta^2}}{\sqrt{\lambda + \beta\theta^2}}.$$

In particular the distribution of σ_A conditionally on $\{A = \theta\}$ is the gamma distribution with parameter $(\beta\theta^2, 1/2)$.

Very similar computations as those in the proof of Theorem 6.7 yield that for all $s, t \geq 0$, $\theta < 0$, $\lambda, \kappa \geq 0$

(56)
$$\mathbf{N}^\psi[e^{-\lambda\sigma_{A+s} - \kappa\sigma_{A+s+t}}|A = \theta]$$

$$= \frac{\sqrt{\beta(|\theta| + s)^2}}{\sqrt{\lambda + \beta(|\theta| + s)^2}} \frac{\sqrt{\beta t^2} + \sqrt{\lambda + \beta(|\theta| + s)^2}}{\sqrt{\kappa + (\sqrt{\beta t^2} + \sqrt{\lambda + \beta(|\theta| + s)^2})^2}}.$$

We denote by σ^*_θ the total mass or length (see definition (52)) of $\mathcal{L}$) of the pruned infinite tree $\mathcal{G}^*_\theta$. Notice that, thanks to Proposition 7.7, σ^*_θ has the size biased distribution of σ_θ (the total mass of the CRT with branching mechanism ψ_θ) under $\mathbf{N}^\psi$. More precisely, we have for any nonnegative measurable function, for $\theta > 0$,

(57)
$$2\beta\theta\mathbf{N}^\psi[\sigma_\theta F(\sigma_{\theta+q}, q \geq 0)] = \mathbf{E}^\psi[F(\sigma^*_{\theta+q}, q \geq 0)].$$

As the process $\Sigma = (\sigma_\theta, \theta \in \mathbb{R})$ is Markov, we get that $\Sigma^* = (\sigma^*_\theta, \theta \geq 0)$ is Markov. Notice that a.s. $\sigma^*_0 = +\infty$. Direct computations or using (56) and Theorem 8.1 yield that for all $\theta, q, \lambda, \kappa \geq 0$

$$\mathbf{E}^\psi[e^{-\lambda\sigma^*_\theta - \kappa\sigma^*_{\theta+q}}] = \frac{\sqrt{\beta\theta^2}}{\sqrt{\lambda + \beta\theta^2}} \frac{\sqrt{\beta q^2} + \sqrt{\lambda + \beta\theta^2}}{\sqrt{\kappa + (\sqrt{\beta q^2} + \sqrt{\lambda + \beta\theta^2})^2}}.$$

Let $\tau = (\tau_\theta, \theta \geq 0)$ be the first passage process of a standard Brownian motion $(B_u, u \geq 0)$: $\tau_\theta = \inf\{u \geq 0, B_u \geq \theta\}$. It is a stable subordinator with index $1/2$, and more precisely with no drift, no killing and Lévy measure $(2\pi x^3)^{-1/2}\, dx$ on $(0, \infty)$: for $\lambda \geq 0$, $\mathbb{E}[e^{-\lambda \tau_\theta}] = e^{-\theta\sqrt{2\lambda}}$. The distribution of τ_θ has density

$$\frac{\theta}{\sqrt{2\pi x^3}} e^{-\theta^2/2x} \mathbf{1}_{\{x>0\}}.$$

We get the following result.

PROPOSITION 9.1.   *We have:*

- *under* $\mathbf{E}^\psi$, $(2\beta\sigma_\theta^*, \theta \geq 0)$ *is distributed as* $(1/\tau_\theta, \theta \geq 0)$;
- *under* $\mathbf{N}^\psi$, $(2\beta\sigma_{A+\theta}, \theta \geq 0)$ *is distributed as* $(1/(V + \tau_\theta), \theta \geq 0)$ *where* V *is independent of* τ *and its "distribution" has density w.r.t. the Lebesgue measure given by* $\sqrt{2/(\pi v)}\mathbf{1}_{\{v>0\}}$.

The proof of this result is postponed to the end of this section.
Notice that (45) implies that for $\theta \geq 0$,

$$\mathbf{N}^\psi[F(\sigma_q, q \geq 0)e^{-\psi(\theta)\sigma_0}] = \mathbf{N}^\psi[F(\sigma_{q+\theta}, q \geq 0)].$$

In particular, we deduce from this, (57) and the fact that τ is a process with independent and stationary increments the following result (notice that the size bias effect vanish, as we condition by $\sigma_0 = 1$).

COROLLARY 9.2.   *Let* $\beta = 1/2$. *Conditionally on* $\sigma_0 = 1$, *we have that* $(\sigma_\theta, \theta \geq 0)$ *is under the excursion measure* $\mathbf{N}^\psi$ *distributed as* $(1/(1 + \tau_\theta), \theta \geq 0)$.

We thus recover a well-known result from Aldous and Pitman [10] on the size process of a tagged fragment for a self-similar fragmentation (see [14]) with index $1/2$, no erosion and binary dislocation measure ν defined on pairs (s_1, s_2) such that $s_1 \geq s_2 \geq 0$ and $s_1 + s_2 = 1$ by

$$\nu(s_1 \in dx) = (2\pi x^3(1 - x)^3)^{-1/2}\mathbf{1}_{\{x>1/2\}}\, dx,$$

which correspond to the fragmentation of the CRT (see also the end of [6, 13] or [24]).

PROOF OF PROPOSITION 9.1.   Let $\lambda, \kappa, \theta, q$ be positive. As we did not find any reference for the computation of

$$I = \mathbb{E}[e^{-\lambda/\tau_\theta - \kappa/\tau_{\theta+q}}],$$

we shall give it here. Using that τ is a subordinator, we have

$$I = \mathbb{E}[e^{-\lambda/\tau_\theta - \kappa/(\tau_\theta + \tau_q')}],$$

where τ' is an independent copy of τ. We set $p = \sqrt{2\lambda + \theta^2}$ and $J = 2\pi\frac{p}{\theta}I$. We get

$$J = 2\pi \frac{p}{\theta} \frac{\theta q}{2\pi} \int_{\mathbb{R}^2_+} \mathrm{e}^{-\lambda/x - \kappa/(x+y) - \theta^2/2x - q^2/2y} \frac{dx\,dy}{(xy)^{3/2}}$$

$$= pq \int_{\mathbb{R}^2_+} \mathrm{e}^{-\kappa/(x+y) - p^2/2x - q^2/2y} \frac{dx\,dy}{(xy)^{3/2}}$$

$$= pq \int_{\mathbb{R}^2_+} \mathrm{e}^{-\kappa z u/(1+u) - zup^2/2 - zq^2/2} \frac{dz\,du}{\sqrt{u}}$$

$$= pq \int_{\mathbb{R}_+} \frac{u+1}{u^2 p^2/2 + u(p^2/2 + q^2/2 + \kappa) + q^2/2} \frac{du}{\sqrt{u}}$$

$$= 2\gamma \int_{\mathbb{R}_+} \frac{u+1}{u^2 + u(1 + \gamma^2 + \kappa') + \gamma^2} \frac{du}{\sqrt{u}},$$

where we used the change of variable $zu = 1/x$ and $z = 1/y$ for the third equality, $\kappa' = 2\kappa/p^2$ and $\gamma = q/p$ for the last. Let a, b such that $a + b = 1 + \gamma^2 + \kappa'$ and $ab = \gamma^2$. Notice that

$$\frac{u+1}{u^2 + u(1 + \gamma^2 + \kappa') + \gamma^2} = \frac{a-1}{a-b}\frac{1}{u+a} + \frac{1-b}{a-b}\frac{1}{u+b}.$$

Then we get

$$J = 2\gamma\frac{a-1}{a-b}\int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+a)} + 2\gamma\frac{1-b}{a-b}\int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+b)}$$

$$= 2\gamma\frac{1}{a-b}\left(\frac{a-1}{\sqrt{a}} + \frac{1-b}{\sqrt{b}}\right)\int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+1)}$$

$$= 2\gamma\frac{\sqrt{ab}+1}{\sqrt{ab}}\frac{1}{\sqrt{a}+\sqrt{b}}\pi$$

$$= 2\pi\frac{\gamma+1}{\sqrt{(1+\gamma)^2 + \kappa'}}.$$

Therefore, we obtain

$$I = \frac{\theta}{p}\frac{\gamma+1}{\sqrt{(1+\gamma)^2 + \kappa'}} = \frac{\theta}{\sqrt{\theta^2 + 2\lambda}}\frac{q + \sqrt{\theta^2 + 2\lambda}}{\sqrt{2\kappa + (q + \sqrt{\theta^2 + 2\lambda})^2}}.$$

We deduce that the two processes, $(2\beta\sigma_\theta^*, \theta \geq 0)$ and $(1/\tau_\theta, \theta \geq 0)$, have the same two-dimensional marginals. Since they are Markov processes, they have the same distribution. This proves the first part of the theorem.

Let U be a positive "random" variable whose "distribution" given by 2 times the Lebesgue measure on $(0, +\infty)$ which is independent of τ. The

"distribution" of $V = \tau_U$ has density w.r.t. the Lebesgue measure given by $\sqrt{2/(\pi v)}\mathbf{1}_{\{v>0\}}$. The second part is then a direct consequence of Corollary 8.2. $\square$

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