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# A CONTINUUM-TREE-VALUED MARKOV PROCESS

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

ABSTRACT. We present a construction of a Lévy continuum random tree (CRT) associated with a super-critical continuous state branching process using the so-called exploration process and a Girsanov's theorem. We also extend the pruning procedure to this super-critical case. Let  $\psi$  be a critical branching mechanism. We set  $\psi_\theta(\cdot) = \psi(\cdot + \theta) - \psi(\theta)$ . Let  $\Theta = (\theta_\infty, +\infty)$  or  $\Theta = [\theta_\infty, +\infty)$  be the set of values of  $\theta$  for which  $\psi_\theta$  is a branching mechanism. The pruning procedure allows to construct a decreasing Lévy-CRT-valued Markov process  $(\mathcal{T}_\theta, \theta \in \Theta)$ , such that  $\mathcal{T}_\theta$  has branching mechanism  $\psi_\theta$ . It is sub-critical if  $\theta > 0$  and super-critical if  $\theta < 0$ . We then consider the explosion time  $A$  of the CRT: the smaller (negative) time  $\theta$  for which  $\mathcal{T}_\theta$  has finite mass. We describe the law of  $A$  as well as the distribution of the CRT just after this explosion time. The CRT just after explosion can be seen as a CRT conditioned not to be extinct which is pruned with an independent intensity related to  $A$ . We also study the evolution of the CRT-valued process after the explosion time. This extends results from Aldous and Pitman on Galton-Watson trees. For the particular case of the quadratic branching mechanism, we show that after explosion the total mass of the CRT behaves like the inverse of a stable subordinator with index  $1/2$ . This result is related to the size of the tagged fragment for the fragmentation of Aldous' CRT.

## 1. INTRODUCTION

Continuous state branching processes (CB in short) are non-negative real valued Markov processes first introduced by Jirina [19] that satisfy a branching property: the process  $(Z_t, t \geq 0)$  is a CB if its law when starting from  $x+x'$  is equal to the law of the sum of two independent copies of  $Z$  starting respectively from  $x$  and  $x'$ . The law of such a process is characterized by the so-called branching mechanism  $\psi$  via its Laplace functionals. The branching mechanism  $\psi$  of a CB is given by

$$\psi(\lambda) = \tilde{\alpha}\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left[ e^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell \leq 1\}} \right],$$

where  $\tilde{\alpha} \in \mathbb{R}$ ,  $\beta \geq 0$  and  $\pi$  is a Radon measure on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} (1 \wedge \ell^2) \pi(d\ell) < +\infty$ . The CB is said to be respectively sub-critical, critical, super-critical when  $\psi'(0) > 0$ ,  $\psi'(0) = 0$  or  $\psi'(0) < 0$ . We will write (sub)critical for critical or sub-critical. Notice that  $\psi$  is smooth and strictly convex if  $\beta > 0$  or  $\pi \neq 0$ .

It is shown in [20] that all these CBs can be obtained as the limit of renormalized sequences of Galton-Watson processes. A genealogical tree is naturally associated with a Galton-Watson process and the question of existence of such a genealogical structure for CB arises naturally. This question has given birth to the theory of continuum random trees (CRT), first introduced in the pioneer work of Aldous [7, 6, 8]. A continuum random tree (called Lévy CRT) that

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codes the genealogy of a general (sub)critical branching process has been constructed in [21, 22] and studied further in [16]. The main tool of this approach is the so-called exploration process  $(\rho_s, s \in \mathbb{R}^+)$ , where  $\rho_s$  is a measure on  $\mathbb{R}^+$ , which codes for the CRT. For (sub)critical quadratic branching mechanism ( $\pi = 0$ ), the measure  $\rho_s$  is just the Lebesgue measure over an interval  $[0, H_s]$ , and the so-called height process  $(H_s, s \in \mathbb{R}^+)$  is a Brownian motion with drift reflected at 0. In [15], a CRT is built for super-critical quadratic branching mechanism using Girsanov theorem for Brownian motion.

We propose here a construction for general super-critical Lévy tree, using the exploration process, based on ideas from [15]. We first build the super-critical tree up to a given level  $a$ . This tree can be coded by an exploration process and its law is absolutely continuous with respect to the law of a (sub)critical Lévy tree, whose leaves above level  $a$  are removed. Moreover, this family of processes (indexed by parameter  $a$ ) satisfies a compatibility property and hence there exists a projective limit which can be seen as the law of the CRT associated with the super-critical CB. This construction enables us to use most of the results known for (sub)critical CRT. Notice that another construction of a Lévy CRT that does not make use of the exploration process has been proposed in [18] as the limit, for the Gromov-Hausdorff metric, of a sequence of discrete trees. This construction also holds in the super-critical case but is not easy to use to derive properties for super-critical CRT.

In a second time, we want to construct a “decreasing” tree-valued Markov process. To begin with, if  $\psi$  is (sub)critical, for  $\theta > 0$  we can construct, via the pruning procedure of [4], from a Lévy CRT  $\mathcal{T}$  associated with  $\psi$  a sub-tree  $\mathcal{T}_\theta$  associated with the branching mechanism  $\psi_\theta$  defined by

$$\forall \lambda \geq 0, \quad \psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta).$$

By [1, 23], we can even construct a “decreasing” family of Lévy CRTs  $(\mathcal{T}_\theta, \theta \geq 0)$  such that  $\mathcal{T}_\theta$  is associated with  $\psi_\theta$  for every  $\theta \geq 0$ .

In this paper, we consider a critical branching mechanism  $\psi$  and denote by  $\Theta$  the set of real numbers  $\theta$  (including negative ones) for which  $\psi_\theta$  is a well-defined branching mechanism (see Section 5.3 for some examples). Notice that  $\Theta = [\theta_\infty, +\infty)$  or  $(\theta_\infty, +\infty)$  for some  $\theta_\infty \in [-\infty, 0]$ . We then extend the pruning procedure of [4] to super-critical branching mechanisms in order to define a Lévy CRT-valued process  $(\mathcal{T}_\theta, \theta \in \Theta)$  such that

- For every  $\theta \in \Theta$ , the Lévy CRT  $\mathcal{T}_\theta$  is associated with the branching mechanism  $\psi_\theta$ .
- All the trees  $\mathcal{T}_\theta, \theta \in \Theta$  have a common root.
- The tree-valued process  $(\mathcal{T}_\theta, \theta \in \Theta)$  is decreasing in the sense that for  $\theta < \theta'$ ,  $\mathcal{T}_{\theta'}$  is a sub-tree of  $\mathcal{T}_\theta$ .

Let  $\rho^\theta$  be the exploration process that codes for  $\mathcal{T}_\theta$ . We denote by  $\mathbf{N}^\psi$  the excursion measure of the process  $(\rho^\theta, \theta \in \Theta)$ , that is under  $\mathbf{N}^\psi$ , each  $\rho^\theta$  is the excursion of an exploration process associated with  $\psi_\theta$ . Let  $\sigma_\theta$  denote the length of this excursion. The quantity  $\sigma_\theta$  corresponds also to the total mass of the tree  $\mathcal{T}_\theta$ . We say that the tree  $\mathcal{T}_\theta$  is finite (under  $\mathbf{N}^\psi$ ) if  $\sigma_\theta$  is finite (or equivalently if the total mass of the associated CB is finite). By construction, we have that the trees  $\mathcal{T}_\theta$  for  $\theta \geq 0$  are associated with (sub)critical branching mechanisms and hence are a.e. finite. On the other hand, the trees  $\mathcal{T}_\theta$  for negative  $\theta$  are associated with super-critical branching mechanisms. We define the explosion time

$$A = \inf\{\theta \in \Theta, \sigma_\theta < +\infty\}.$$

For  $\theta \in \Theta$ , we define  $\bar{\theta}$  as the unique non-negative real number such that  $\psi(\bar{\theta}) = \psi(\theta)$  (notice that  $\bar{\theta} = \theta$  if  $\theta \geq 0$ ). We give the distribution of  $A$  under  $\mathbf{N}^\psi$  (Theorem 6.5). In particular

we have, for all  $\theta \in [\theta_\infty, +\infty)$ ,

$$\mathbf{N}^\psi[A > \theta] = \bar{\theta} - \theta.$$

We also give the distribution of the trees after the explosion time  $(\mathcal{T}_\theta, \theta \geq A)$  (Theorem 6.7 and Corollary 8.2). Of particular interest is the distribution of the tree at its explosion time,  $\mathcal{T}_A$ .

In [10], Aldous and Pitman constructed a tree-valued process by pruning a discrete Galton-Watson tree. They also constructed another tree-valued process by pruning an infinite tree (a (sub)critical Galton-Watson tree conditioned on non-extinction) and proved some identities in law between these two processes in the case of Poisson offspring distributions. In the same spirit, we also construct another tree-valued Markov process  $(\mathcal{T}_\theta^*, \theta \geq 0)$  associated with a critical branching mechanism  $\psi$ . In the case of a.s. extinction (that is when  $\int^{+\infty} \frac{dv}{\psi(v)} < +\infty$ ),  $\mathcal{T}_0^*$  is distributed as  $\mathcal{T}_0$  conditioned to survival. The tree  $\mathcal{T}_0^*$  is constructed via a spinal decomposition along an infinite spine. Then, we define the continuum-tree-valued Markov process  $(\mathcal{T}_\theta^*, \theta \geq 0)$  again by a pruning procedure. Let  $\theta \in (\theta_\infty, 0)$ . We prove that under the excursion measure  $\mathbf{N}^\psi$ , given  $A = \theta$ , the process  $(\mathcal{T}_{\theta+u}, u \geq 0)$  is distributed as the process  $(\mathcal{T}_{\theta+u}^*, u \geq 0)$  (Theorem 8.1).

When the branching mechanism is quadratic,  $\psi(\lambda) = \lambda^2/2$ , some explicit computations can be carried out. Let  $\sigma_\theta^*$  be the total mass of  $\mathcal{T}_\theta^*$  and  $\tau = (\tau_\theta, \theta \geq 0)$  be the first passage process of a standard Brownian motion, that is a stable subordinator with index 1/2. We get (Proposition 9.1) that  $(\sigma_\theta^*, \theta \geq 0)$  is distributed as  $(1/\tau_\theta, \theta \geq 0)$  and that  $(\sigma_{A+\theta}, \theta \geq 0)$  is distributed as  $(1/(V + \tau_\theta), \theta \geq 0)$  for some random variable  $V$  independent of  $\tau$ . Let us recall that the pruning procedure of the tree can be used to construct some fragmentation processes (see [5, 1, 23]) and the process  $(\sigma_\theta, \theta \geq 0)$ , conditionally on  $\sigma_0 = 1$ , represents then the evolution of a tagged fragment. We hence recover a well known result of Aldous-Pitman [9]: conditionally on  $\sigma_0 = 1$ ,  $(\sigma_\theta, \theta \geq 0)$  is distributed as  $(1/(1 + \tau_\theta), \theta \geq 0)$  (see Corollary 9.2).

The paper is organized as follows. In Section 2, we introduce an exponential martingale of a CB and give a Girsanov formula for CBs. We recall in Section 3 the construction of a (sub)critical Lévy CRT via the exploration process and some useful properties of this exploration process. Then, we construct in Section 4 the super-critical Lévy CRT via a Girsanov theorem involving the same martingale as in Section 2. We recall in Section 5 the pruning procedure for critical or sub-critical CRTs and extend this procedure to super-critical CRTs. We construct in Section 6 the tree-valued process  $(\mathcal{T}_\theta, \theta \in \Theta)$ , or more precisely the family of exploration processes  $(\rho^\theta, \theta \geq 0)$  which codes for it. We also give the law of the explosion time  $A$  and the law of the tree at this time. In Section 7, we construct an infinite tree and the corresponding pruned sub-trees  $(\mathcal{T}_\theta^*, \theta \geq 0)$ , which are given by a spinal representation using exploration processes. We prove in Section 8 that the process  $(\mathcal{T}_{A+u}, u \geq 0)$  is distributed as the process  $(\mathcal{T}_{U+u}^*, u \geq 0)$  where  $U$  is a positive random time independent of  $(\mathcal{T}_\theta^*, \theta \geq 0)$ . We finally make the explicit computations for the quadratic case in Section 9.

Notice that all the results in the following Sections are stated using exploration processes which code for the CRT, instead of the CRT directly.

## 2. GIRSANOV FORMULA FOR CONTINUOUS BRANCHING PROCESS

**2.1. Continuous branching process.** Let  $\psi$  be a branching mechanism of a CB: for  $\lambda \geq 0$ ,

$$(1) \quad \psi(\lambda) = \tilde{\alpha}\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left[ e^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell \leq 1\}} \right],$$

where  $\tilde{\alpha} \in \mathbb{R}$ ,  $\beta \geq 0$  and  $\pi$  is a Radon measure on  $(0, +\infty)$  such that  $\int_{(0,+\infty)} (1 \wedge \ell^2) \pi(d\ell) < +\infty$ . We shall say that  $\psi$  has parameter  $(\tilde{\alpha}, \beta, \pi)$ .

We shall assume that  $\beta \neq 0$  or  $\pi \neq 0$ . We have  $\psi(0) = 0$  and  $\psi'(0^+) = \tilde{\alpha} - \int_{(1,+\infty)} \ell \pi(d\ell) \in [-\infty, +\infty)$ . In particular, we have  $\psi'(0^+) = -\infty$  if and only if  $\int_{(1,+\infty)} \ell \pi(d\ell) = +\infty$ . We say that  $\psi$  is conservative if for all  $\varepsilon > 0$

$$(2) \quad \int_0^\varepsilon \frac{1}{|\psi(u)|} du = +\infty.$$

Notice that (2) is fulfilled if  $\psi'(0^+) > -\infty$  that is if  $\int_{(1,+\infty)} \ell \pi(d\ell) < +\infty$ . If  $\psi$  is conservative, the CB associated with  $\psi$  does not explode in finite time a.s.

Let  $\mathbb{P}_x^\psi$  be the law of a CB  $Z = (Z_a, a \geq 0)$  started at  $x \geq 0$  and with branching mechanism  $\psi$ , and let  $\mathbb{E}_x^\psi$  be the corresponding expectation. The process  $Z$  is a Feller process and thus càd-làg. Let  $\mathcal{F} = (\mathcal{F}_a, a \geq 0)$  be the filtration generated by  $Z$  completed the usual way. For every  $\lambda > 0$ , for every  $a \geq 0$ , we have

$$(3) \quad \mathbb{E}_x^\psi \left[ e^{-\lambda Z_a} \right] = e^{-xu(a,\lambda)},$$

where function  $u$  is the unique non-negative solution of

$$(4) \quad u(a, \lambda) + \int_0^a \psi(u(s, \lambda)) ds = \lambda, \quad \lambda \geq 0, \quad a \geq 0.$$

This equation is equivalent to

$$(5) \quad \int_{u(a,\lambda)}^\lambda \frac{dr}{\psi(r)} = a \quad \lambda \geq 0, \quad a \geq 0.$$

If (2) holds, then the process is conservative: a.s. for all  $a \geq 0$ ,  $Z_a < +\infty$ .

It is easy to check that  $\mathbb{E}_x^\psi[Z_a] = x e^{-a\psi'(0^+)}$ . The CB  $Z$  or the branching mechanism  $\psi$  is called critical (resp. super-critical, resp. sub-critical) if  $\psi'(0^+) = 0$  (resp.  $\psi'(0^+) < 0$ , resp.  $\psi'(0^+) > 0$ ). We shall write  $\psi$  is (sub)critical if it is critical or sub-critical.

Let  $q_0$  be the largest root of  $\psi(q) = 0$ . Since  $\psi(0) = 0$ , we have  $q_0 \geq 0$ . If  $\psi$  is (sub)critical, since  $\psi$  is strictly convex, we get that  $q_0 = 0$ . If  $\psi$  is super-critical, if we denote by  $q^* > 0$  the only real number such that  $\psi'(q^*) = 0$ , we have  $q_0 > q^* > 0$ . See Lemma 2.4 for the interpretation of  $q_0$ .

If  $f$  is a function defined on  $[\gamma, +\infty)$ , then for  $\theta \geq \gamma$ , we set for  $\lambda \geq \gamma - \theta$ :

$$f_\theta(\lambda) = f(\theta + \lambda) - f(\theta).$$

If  $\nu$  is a measure on  $(0, +\infty)$ , then for  $q \in \mathbb{R}$ , we set

$$(6) \quad \nu^{(q)}(d\ell) = e^{-q\ell} \nu(d\ell).$$

*Remark 2.1.* If  $\pi^{(q)}((1, +\infty)) < +\infty$  for some  $q < 0$ , then  $\psi$  given by (1) is well defined on  $[q, +\infty)$  and, for  $\theta \in [q, +\infty)$ ,  $\psi_\theta$  is a branching mechanism with parameter  $(\tilde{\alpha} + 2\beta\theta +$

$\int_{(0,1]} \pi(d\ell) \ell(1 - e^{-\theta\ell}), \beta, \pi^{(\theta)}$ . Notice that for all  $\theta > q$ ,  $\psi_\theta$  is conservative. And, if the additional assumption

$$\int_{(1,+\infty)} \ell \pi^{(q)}(d\ell) = \int_{(1,+\infty)} \ell e^{q\ell} \pi(d\ell) < +\infty$$

holds, then  $|(\psi_q)'(0+)| < +\infty$  and  $\psi_q$  is conservative.

**2.2. Girsanov formula.** Let  $Z = (Z_a, a \geq 0)$  be a conservative CB with branching mechanism  $\psi$  given by (1) with  $\beta \neq 0$  or  $\pi \neq 0$ , and let  $(\mathcal{F}_a, a \geq 0)$  be its natural filtration. Let  $q \in \mathbb{R}$  such that  $q \geq 0$  or  $q < 0$  and  $\int_{(1,+\infty)} \ell e^{q\ell} \pi(d\ell) < +\infty$ . Then, thanks to Remark 2.1,  $\psi(q)$  and  $\psi_q$  are well defined and  $\psi_q$  is conservative. Then we consider the process  $M^{\psi,q} = (M_a^{\psi,q}, a \geq 0)$  defined by

$$(7) \quad M_a^{\psi,q} = e^{qx - qZ_a - \psi(q) \int_0^a Z_s ds}.$$

**Theorem 2.2.** *Let  $q \in \mathbb{R}$  such that  $q \geq 0$  or  $q < 0$  and  $\int_{(1,+\infty)} \ell e^{q\ell} \pi(d\ell) < +\infty$ .*

- (i) *The process  $M^{\psi,q}$  is a  $\mathcal{F}$ -martingale under  $P_x^\psi$ .*
- (ii) *Let  $a, x \geq 0$ . On  $\mathcal{F}_a$ , the probability measure  $P_x^{\psi,q}$  is absolutely continuous with respect to  $P_x^\psi$  and*

$$\frac{dP_x^{\psi,q} |_{\mathcal{F}_a}}{dP_x^\psi |_{\mathcal{F}_a}} = M_a^{\psi,q}.$$

Before going into the proof of this theorem, we recall Proposition 2.1 from [2]. For  $\mu$  a positive measure on  $\mathbb{R}$ , we set

$$(8) \quad H(\mu) = \sup\{r \in \mathbb{R}; \mu([r, +\infty)) > 0\},$$

the maximal element of its support. For  $a < 0$ , we set  $Z_a = 0$ .

**Proposition 2.3.** *Let  $\mu$  be a finite positive measure on  $\mathbb{R}$  with support bounded from above (i.e.  $H(\mu)$  is finite). Then we have for all  $s \in \mathbb{R}$ ,  $x \geq 0$ ,*

$$(9) \quad \mathbb{E}_x^\psi \left[ e^{-\int Z_{r-s} \mu(dr)} \right] = e^{-xw(s)},$$

where the function  $w$  is a measurable locally bounded non-negative solution of the equation

$$(10) \quad w(s) + \int_s^{+\infty} \psi(w(r)) dr = \int_{[s,+\infty)} \mu(dr), \quad s \leq H(\mu) \quad \text{and} \quad w(s) = 0, \quad s > H(\mu).$$

If  $\psi'(0^+) > -\infty$  or if  $\mu(\{H(\mu)\}) > 0$ , then (10) has a unique measurable locally bounded non-negative solution.

*Proof of Theorem 2.2. First case.* We consider  $q > 0$  such that  $\psi(q) \geq 0$ .

We have  $0 \leq M_a^{\psi,q} \leq e^{qx}$ , thus  $M^{\psi,q}$  is bounded. It is clear that  $M^{\psi,q}$  is  $\mathcal{F}$ -adapted.

To check that  $M^{\psi,q}$  is a martingale, thanks to the Markov property, it is enough to check that  $\mathbb{E}_x^\psi[M_a^{\psi,q}] = \mathbb{E}_x^\psi[M_0^{\psi,q}] = 1$  for all  $a \geq 0$ . Consider the measure  $\nu_q(dr) = q\delta_a(dr) + \psi(q)\mathbf{1}_{[0,a]}(r)dr$ , where  $\delta_a$  is the Dirac mass at point  $a$ . Notice that  $H(\nu_q) = a$  and that  $\nu_q(\{H(\nu_q)\}) = q > 0$  hence, thanks to Proposition 2.3, there exists a unique non-negative solution  $w$  of (10) with  $\mu = \nu_q$ , and  $\mathbb{E}_x^\psi[M_a^{\psi,q}] = e^{-x(w(0)-q)}$ . As  $q\mathbf{1}_{[0,a]}$  also solves (10) with  $\mu = \nu_q$ , we deduce that  $w = q\mathbf{1}_{[0,a]}$  and that  $\mathbb{E}_x[M_a^{\psi,q}] = 1$ . Thus, we get that  $M^{\psi,q}$  is a bounded martingale.

Let  $\nu$  be a non-negative measure on  $\mathbb{R}$  with support in  $[0, a]$  (i.e.  $H(\nu) \leq a$ ). Thanks to Proposition 2.3, we have that  $\mathbb{E}_x^\psi [M_a^{\psi, q} e^{-\int Z_r \nu(dr)}] = e^{-x(v(0)-q)}$ , where  $v$  is the unique non-negative solution of (10) with  $\mu = \nu + \nu_q$ . As  $M_a^{\psi, q} e^{-\int Z_r \nu(dr)} \leq M_a^{\psi, q}$ , we deduce that  $e^{-x(v(0)-q)} = \mathbb{E}_x^\psi [M_a^{\psi, q} e^{-\int Z_r \nu(dr)}] \leq 1$ , that is  $v(0) \geq q$ . We set  $u = v - q$ , and we deduce that  $u$  is non-negative and solves

$$(11) \quad u(s) + \int_s^{+\infty} \psi_q(u(r)) dr = \int_{[s, +\infty)} \nu(dr), \quad s \leq H(\nu) \quad \text{and} \quad u(s) = 0, \quad s > H(\nu).$$

As  $\psi(q) \geq 0$ , we deduce from the convexity of  $\psi$  that  $\psi'_q(0) = \psi'(q) \geq 0$ . Thanks to Proposition 2.3, we deduce that  $u$  is the unique non-negative solution of (11) and that  $e^{-xu(0)} = \mathbb{E}_x^{\psi_q} [e^{-\int Z_r \nu(dr)}]$ . In particular, we have that for all non-negative measure  $\nu$  on  $\mathbb{R}$  with support in  $[0, a]$ ,

$$\mathbb{E}_x^\psi [M_a^{\psi, q} e^{-\int Z_r \nu(dr)}] = \mathbb{E}_x^{\psi_q} [e^{-\int Z_r \nu(dr)}].$$

As  $e^{-\int Z_r \nu(dr)}$  is  $\mathcal{F}_a$ -measurable, we deduce from the monotone class theorem that for any non-negative  $\mathcal{F}_a$ -measurable random variable  $W$ ,

$$(12) \quad \mathbb{E}_x^\psi [W e^{qx - qZ_a - \psi(q) \int_0^a Z_r dr}] = \mathbb{E}_x^\psi [W M_a^{\psi, q}] = \mathbb{E}_x^{\psi_q} [W].$$

This proves the second part of the theorem.

**Second case.** We consider  $q \geq 0$  such that  $\psi(q) < 0$ . Let us remark that this only occurs when  $\psi$  is super-critical.

Recall that  $q_0 > q^* > 0$  are such that  $\psi(q_0) = 0$  and  $\psi'(q^*) = 0$ . Notice that  $\psi'_{q^*}(0) = \psi'(q^*) = 0$ , that is  $\psi_{q^*}$  is critical. Let  $W$  be any non-negative random variable  $\mathcal{F}_a$ -measurable. From the first step, using (12) with  $q = q_0$ , we get that

$$\mathbb{E}_x^\psi [W e^{q_0 x - q_0 Z_a}] = \mathbb{E}_x^{\psi_{q_0}} [W].$$

Thanks to (12) with  $\psi_{q^*}$  instead of  $\psi$  and  $(q_0 - q^*) \geq 0$  instead of  $q$ , and using that  $(\psi_{q^*})_{q_0 - q^*} = \psi_{q_0}$ , we deduce that

$$\mathbb{E}_x^{\psi_{q^*}} [W e^{(q_0 - q^*)x - (q_0 - q^*)Z_a - \psi_{q^*}(q_0 - q^*) \int_0^a Z_r dr}] = \mathbb{E}_x^{(\psi_{q^*})_{q_0 - q^*}} [W] = \mathbb{E}_x^{\psi_{q_0}} [W].$$

This implies that

$$\begin{aligned} \mathbb{E}_x^\psi [W] &= \mathbb{E}_x^{\psi_{q_0}} [W e^{-q_0 x} e^{q_0 Z_a}] \\ &= \mathbb{E}_x^{\psi_{q^*}} [W e^{-q_0 x} e^{q_0 Z_a} e^{(q_0 - q^*)x - (q_0 - q^*)Z_a - \psi_{q^*}(q_0 - q^*) \int_0^a Z_r dr}] \\ &= \mathbb{E}_x^{\psi_{q^*}} [W e^{-q^* x + q^* Z_a - \psi_{q^*}(q_0 - q^*) \int_0^a Z_r dr}]. \end{aligned}$$

As  $\psi_{q^*}(q_0 - q^*) = \psi(q_0) - \psi(q^*) = -\psi(q^*) = \psi_{q^*}(-q^*)$ , we finally obtain

$$(13) \quad \mathbb{E}_x^\psi [W] = \mathbb{E}_x^{\psi_{q^*}} [W e^{-q^* x + q^* Z_a - \psi_{q^*}(-q^*) \int_0^a Z_r dr}].$$

If  $q < q^*$ , as  $(\psi_q)_{(q^* - q)} = \psi_{q^*}$  and  $\psi'_q(q^* - q) = \psi'(q^*) = 0$ , we deduce from (13) with  $\psi$  replaced by  $\psi_q$  and  $q^*$  by  $q^* - q$  that

$$(14) \quad \mathbb{E}_x^{\psi_q} [W] = \mathbb{E}_x^{\psi_{q^*}} [W e^{-(q^* - q)x + (q^* - q)Z_a - \psi_{q^*}(q - q^*) \int_0^a Z_r dr}].$$

If  $q > q^*$ , formula (12) holds with  $\psi$  replaced by  $\psi_{q^*}$  and  $q$  replaced by  $q - q^*$ , which also yields equation (14).

Using (13), (14) and that  $\psi_{q^*}(-q^*) + \psi(q) = \psi_{q^*}(q - q^*)$ , we get that

$$(15) \quad \begin{aligned} \mathbb{E}_x^\psi \left[ W e^{qx - qZ_a - \psi(q) \int_0^a Z_r dr} \right] &= \mathbb{E}_x^{\psi_{q^*}} \left[ W e^{-(q^* - q)x + (q^* - q)Z_a - (\psi_{q^*}(-q^*) + \psi(q)) \int_0^a Z_r dr} \right] \\ &= \mathbb{E}_x^{\psi_{q^*}} \left[ W e^{-(q^* - q)x + (q^* - q)Z_a - \psi_{q^*}(q - q^*) \int_0^a Z_r dr} \right] \\ &= \mathbb{E}_x^{\psi_q} [W]. \end{aligned}$$

Since this holds for any non-negative  $\mathcal{F}_a$ -measurable random variable  $W$ , this proves (i) and (ii) of the theorem.

**Third case.** We consider  $q < 0$  and assume that  $\int_{(1, +\infty)} \ell e^{|\ell|} \pi(d\ell) < +\infty$ . In particular,  $\psi_q$  is a conservative branching mechanism, thanks to Remark 2.1.

Let  $W$  be any non-negative  $\mathcal{F}_a$ -measurable random variable. Using (12) if  $\psi_q(-q) \geq 0$  or (15) if  $\psi_q(-q) < 0$ , with  $\psi$  replaced by  $\psi_q$  and  $q$  by  $-q$ , we deduce that

$$\mathbb{E}_x^{\psi_q} [W e^{-qx + qZ_a - \psi_q(-q) \int_0^a Z_r dr}] = \mathbb{E}_x^\psi [W].$$

This implies that

$$\mathbb{E}_x^{\psi_q} [W] = \mathbb{E}_x^\psi [W e^{qx - qZ_a + \psi_q(-q) \int_0^a Z_r dr}] = \mathbb{E}_x^\psi [W e^{qx - qZ_a - \psi(q) \int_0^a Z_r dr}].$$

Since this holds for any non-negative  $\mathcal{F}_a$ -measurable random variable  $W$ , this proves (i) and (ii) of the theorem.  $\square$

Finally, we recall some well known facts on CB. Recall that  $q_0$  is the largest root of  $\psi(q) = 0$ ,  $q_0 = 0$  if  $\psi$  is (sub)critical and that  $q_0 > 0$  if  $\psi$  is super-critical. We set

$$(16) \quad \sigma = \int_0^{+\infty} Z_a da.$$

For  $\lambda \geq 0$ , we set

$$(17) \quad \psi^{-1}(\lambda) = \sup\{r \geq 0; \psi(r) = \lambda\}.$$

**Lemma 2.4.** *Assume that  $\psi$  is given by (1) with  $\beta \neq 0$  or  $\pi \neq 0$  and is conservative.*

(i) *Then  $\mathbb{P}_x^\psi$ -a.s.  $Z_\infty = \lim_{a \rightarrow +\infty} Z_a$  exists,  $Z_\infty \in \{0, +\infty\}$ ,*

$$(18) \quad \mathbb{P}_x^\psi(Z_\infty = 0) = e^{-xq_0},$$

*$\{Z_\infty = 0\} = \{\sigma < +\infty\}$  and we have: for  $\lambda > 0$ ,*

$$(19) \quad \mathbb{E}_x^\psi \left[ e^{-\lambda\sigma} \right] = e^{-x\psi^{-1}(\lambda)}.$$

(ii) *Let  $q \geq 0$  such that  $\psi(q) \geq 0$ . Then, the probability measure  $\mathbb{P}_x^{\psi_q}$  is absolutely continuous with respect to  $\mathbb{P}_x^\psi$  with*

$$\frac{d\mathbb{P}_x^{\psi_q}}{d\mathbb{P}_x^\psi} = M_\infty^{\psi, q},$$

where

$$(20) \quad M_\infty^{\psi, q} = e^{qx - \psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

(iii) *If  $\psi$  is super-critical then, conditionally on  $\{Z_\infty = 0\}$ ,  $Z$  is distributed as  $\mathbb{P}^{\psi_{q_0}}$ : for any non-negative random variable measurable w.r.t.  $\sigma(Z_a, a \geq 0)$ , we have*

$$\mathbb{E}_x^\psi [W | Z_\infty = 0] = \mathbb{E}_x^{\psi_{q_0}} [W].$$

*Proof.* For  $\lambda > 0$ , we set  $N_a = e^{-\lambda Z_a + xu(\lambda, a)}$ , where  $u$  is the unique non-negative solution of (5). Thanks to (3) and the Markov property,  $(N_a, a \geq 0)$  is a bounded martingale under  $\mathbb{P}_x^\psi$ . Hence, as  $a$  goes to infinity, it converges a.s. and in  $L^1$  to a limit, say  $N_\infty$ . From (5), we get that  $\lim_{a \rightarrow +\infty} u(\lambda, a) = q_0$ . This implies that  $Z_\infty = \lim_{a \rightarrow +\infty} Z_a$  exists a.s. in  $[0, +\infty]$ . Since  $\mathbb{E}_x^\psi[N_\infty] = 1$ , we get  $\mathbb{E}_x^\psi[e^{-\lambda Z_\infty}] = e^{-q_0 x}$  for all  $\lambda > 0$ . This implies that  $\mathbb{P}_x^\psi$ -a.s.  $Z_\infty \in \{0, +\infty\}$  and (18).

Clearly, we have  $\{Z_\infty = +\infty\} \subset \{\sigma = +\infty\}$ . For  $q \geq 0$  such that  $\psi(q) \geq 0$ , we get that  $(M_a^{\psi, q}, a \geq 0)$  is a bounded martingale under  $\mathbb{P}_x^\psi$ . Hence, as  $a$  goes to infinity, it converges a.s. and in  $L^1$  to a limit, say  $M_\infty^{\psi, q}$ . We deduce that

$$(21) \quad \mathbb{E}_x^\psi \left[ e^{-\psi(q)\sigma} \mathbf{1}_{\{Z_\infty=0\}} \right] = e^{-qx}.$$

Letting  $q$  decrease to  $q_0$ , we get that  $\mathbb{P}_x^\psi(\sigma < +\infty, Z_\infty = 0) = e^{-q_0 x} = \mathbb{P}(Z_\infty = 0)$ . This implies that  $\mathbb{P}_x^\psi$  a.s.  $\{\sigma = +\infty\} \subset \{Z_\infty = +\infty\}$ . We thus deduce that  $\mathbb{P}_x^\psi$  a.s.  $\{Z_\infty = +\infty\} = \{\sigma = +\infty\}$ . Notice also that (20) holds.

Notice that (21) readily implies (19). This proves Property (i) of the theorem and (20).

Property (ii) is then a consequence of Theorem 2.2, Property (ii) and the convergence in  $L^1$  of the martingale  $(M_a^{\psi, q}, a \geq 0)$  towards  $M_\infty^{\psi, q}$ .

Property (iii) is a consequence of (ii) with  $q = q_0$  and (18). □

### 3. LÉVY CONTINUUM RANDOM TREE

We recall here the construction of the Lévy continuum random tree (CRT) introduced in [22, 21] and developed later in [16] for critical or sub-critical branching mechanism. We will emphasize on the height process and the exploration process which are the key tools to handle this tree. The results of this section are mainly extracted from [16].

**3.1. The underlying Lévy process.** We assume that  $\psi$  given by (1) is (sub)critical, i.e.

$$(22) \quad \alpha := \psi'(0) = \tilde{\alpha} - \int_{(1, +\infty)} \ell \pi(d\ell) \geq 0$$

and that

$$(23) \quad \beta > 0 \quad \text{and} \quad \int_{(0, 1)} \ell \pi(d\ell) = +\infty.$$

We consider a  $\mathbb{R}$ -valued Lévy process  $X = (X_t, t \geq 0)$  with no negative jumps, starting from 0 and with Laplace exponent  $\psi$  under the probability measure  $\mathbb{P}^\psi$ : for  $\lambda \geq 0$   $\mathbb{E}^\psi \left[ e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}$ . By assumption (23),  $X$  is of infinite variation  $\mathbb{P}^\psi$ -a.s.

We introduce some processes related to  $X$ . Let  $\mathcal{J} = \{s \geq 0; X_s \neq X_{s-}\}$  be the set of jumping times of  $X$ . For  $s \in \mathcal{J}$ , we denote by

$$\Delta_s = X_s - X_{s-}$$

the jump of  $X$  at time  $s$  and  $\Delta_s = 0$  otherwise. Let  $I = (I_t, t \geq 0)$  be the infimum process of  $X$ ,  $I_t = \inf_{0 \leq s \leq t} X_s$ , and let  $S = (S_t, t \geq 0)$  be the supremum process,  $S_t = \sup_{0 \leq s \leq t} X_s$ . We will also consider for every  $0 \leq s \leq t$  the infimum of  $X$  over  $[s, t]$ :

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process  $X - I$ , and  $-I$  is the local time of  $X - I$  at 0 (see [11], chap. VII). Let  $\mathbb{N}^\psi$  be the associated excursion measure of the process  $X - I$  away from 0. Let  $\sigma = \inf\{t > 0; X_t - I_t = 0\}$  be the length of the excursion of  $X - I$  under  $\mathbb{N}^\psi$  (we shall see after Proposition 3.6 that the notation  $\sigma$  is consistent with (16)). By assumption (23), we have  $X_0 = I_0 = 0$   $\mathbb{N}^\psi$ -a.e.

Since  $X$  is of infinite variation, 0 is also regular for the Markov process  $S - X$ . The local time,  $L = (L_t, t \geq 0)$ , of  $S - X$  at 0 will be normalized so that

$$\mathbb{E}^\psi[e^{-\lambda S_{L_t^{-1}}}] = e^{-t\psi(\lambda)/\lambda},$$

where  $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$  (see also [11] Theorem VII.4 (ii)).

**3.2. The height process and the Lévy CRT.** For each  $t \geq 0$ , we consider the reversed process at time  $t$ ,  $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$  by:

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if } 0 \leq s < t,$$

and  $\hat{X}_t^{(t)} = X_t$ . The two processes  $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$  and  $(X_s, 0 \leq s \leq t)$  have the same law. Let  $\hat{S}^{(t)}$  be the supremum process of  $\hat{X}^{(t)}$  and  $\hat{L}^{(t)}$  be the local time at 0 of  $\hat{S}^{(t)} - \hat{X}^{(t)}$  with the same normalization as  $L$ .

**Definition 3.1** ([16], Definition 1.2.1, Lemma 1.2.1 and Lemma 1.2.4). *There exists a  $[0, +\infty]$ -valued lower semi-continuous process  $H = (H_t, t \geq 0)$ , called the height process, such that  $H_0 = 0$  and for all  $t \geq 0$ , a.s.  $H_t = \hat{L}_t^{(t)}$ . And a.s. for all  $s < t$  such that  $X_{s-} \leq I_t^s$  and for  $s = t$  if  $\Delta_t > 0$  then  $H_t < +\infty$  and for all  $t' > t \geq 0$ , the process  $H$  takes all the values between  $H_t$  and  $H_{t'}$  on the time interval  $[t, t']$ .*

The height process  $(H_t, t \in [0, \sigma])$  under  $\mathbb{N}^\psi$  codes a continuous genealogical structure, the Lévy CRT, via the following procedure.

- (i) To each  $t \in [0, \sigma]$  corresponds a vertex at generation  $H_t$ .
- (ii) Vertex  $t$  is an ancestor of vertex  $t'$  if  $H_t = H_{t,t'}$ , where

$$(24) \quad H_{t,t'} = \inf\{H_u, u \in [t \wedge t', t \vee t']\}.$$

In general  $H_{t,t'}$  is the generation of the last common ancestor of  $t$  and  $t'$ .

- (iii) We put  $d(t, t') = H_t + H_{t'} - 2H_{t,t'}$  and identify  $t$  and  $t'$  ( $t \sim t'$ ) if  $d(t, t') = 0$ .

The Lévy CRT coded by  $H$  is then the quotient set  $[0, \sigma]/\sim$ , equipped with the distance  $d$  and the genealogical relation specified in (ii).

**3.3. The exploration process.** The height process is not Markov in general. But it is a very simple function of a measure-valued Markov process, the so-called exploration process.

If  $E$  is a polish space, let  $\mathcal{B}(E)$  (resp.  $\mathcal{B}_+(E)$ ) be the set of real-valued measurable (resp. and non-negative) functions defined on  $E$  endowed with its Borel  $\sigma$ -field, and let  $\mathcal{M}(E)$  (resp.  $\mathcal{M}_f(E)$ ) be the set of  $\sigma$ -finite (resp. finite) measures on  $E$ , endowed with the topology of vague (resp. weak) convergence. For any measure  $\mu \in \mathcal{M}(E)$  and  $f \in \mathcal{B}_+(E)$ , we write

$$\langle \mu, f \rangle = \int f(x) \mu(dx).$$

The exploration process  $\rho = (\rho_t, t \geq 0)$  is a  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process defined as follows: for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $\langle \rho_t, f \rangle = \int_{[0,t]} d_s I_t^s f(H_s)$ , or equivalently

$$(25) \quad \rho_t(dr) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr) + \beta \mathbf{1}_{[0, H_t]}(r) dr.$$

In particular, the total mass of  $\rho_t$  is  $\langle \rho_t, 1 \rangle = X_t - I_t$ .

Recall (8) and set by convention  $H(0) = 0$ .

**Proposition 3.2** ([16], Lemma 1.2.2 and formula (1.12)). *Almost surely, for every  $t > 0$ ,*

- $H(\rho_t) = H_t$ ,
- $\rho_t = 0$  if and only if  $H_t = 0$ ,
- if  $\rho_t \neq 0$ , then  $\text{Supp } \rho_t = [0, H_t]$ .
- $\rho_t = \rho_{t-} + \Delta_t \delta_{H_t}$ , where  $\Delta_t = 0$  if  $t \notin \mathcal{J}$ .

In the definition of the exploration process, as  $X$  starts from 0, we have  $\rho_0 = 0$  a.s. To state the Markov property of  $\rho$ , we must first define the process  $\rho$  started at any initial measure  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ .

For  $a \in [0, \langle \mu, 1 \rangle]$ , we define the erased measure  $k_a \mu$  by

$$k_a \mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1 \rangle - a), \quad \text{for } r \geq 0.$$

If  $a > \langle \mu, 1 \rangle$ , we set  $k_a \mu = 0$ . In other words, the measure  $k_a \mu$  is the measure  $\mu$  erased by a mass  $a$  backward from  $H(\mu)$ .

For  $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$ , and  $\mu$  with compact support, we define the concatenation  $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$  of the two measures by:

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \langle \nu, f(H(\mu) + \cdot) \rangle, \quad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set for every  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  and every  $t > 0$ ,  $\rho_t^\mu = [k_{-I_t} \mu, \rho_t]$ . We say that  $(\rho_t^\mu, t \geq 0)$  is the process  $\rho$  started at  $\rho_0^\mu = \mu$ . Unless there is an ambiguity, we shall write  $\rho_t$  for  $\rho_t^\mu$ . Unless it is stated otherwise, we assume that  $\rho$  is started at 0.

**Proposition 3.3** ([16], Proposition 1.2.3). *The process  $(\rho_t, t \geq 0)$  is a càd-làg strong Markov process in  $\mathcal{M}_f(\mathbb{R}_+)$ .*

*Remark 3.4.* From the construction of  $\rho$ , we get that a.s.  $\rho_t = 0$  if and only if  $-I_t \geq \langle \rho_0, 1 \rangle$  and  $X_t - I_t = 0$ . This implies that 0 is also a regular point for  $\rho$ . Notice that  $\mathbb{N}^\psi$  is also the excursion measure of the process  $\rho$  away from 0, and that  $\sigma$ , the length of the excursion, is  $\mathbb{N}^\psi$ -a.e. equal to  $\inf\{t > 0; \rho_t = 0\}$ .

**3.4. Notations.** We consider the set  $\mathcal{D}$  of càd-làg processes in  $\mathcal{M}_f(\mathbb{R}_+)$ , endowed with the Skorohod topology and the Borel  $\sigma$ -field. In what follows, we denote by  $\rho = (\rho_t, t \geq 0)$  the canonical process on this set. We still denote by  $\mathbb{P}^\psi$  the probability measure on  $\mathcal{D}$  such that the canonical process is distributed as the exploration process associated with the branching mechanism  $\psi$ , and by  $\mathbb{N}^\psi$  the corresponding excursion measure.

**3.5. Local time of the height process.** The local time of the height process is defined through the next result.

**Proposition 3.5** ([16], Lemma 1.3.2 and Proposition 1.3.3). *There exists a jointly measurable process  $(L_s^a, a \geq 0, s \geq 0)$  which is continuous and non-decreasing in the variable  $s$  such that:*

- For every  $t \geq 0$ ,  $\limsup_{\varepsilon \rightarrow 0} \sup_{a \geq 0} \mathbb{E}^\psi \left[ \sup_{s \leq t} \left| \varepsilon^{-1} \int_0^s \mathbf{1}_{\{a < H_r \leq a + \varepsilon\}} dr - L_s^a \right| \right] = 0$ .

- For every  $t \geq 0$ ,  $\limsup_{\varepsilon \rightarrow 0} \mathbb{E}^\psi \left[ \sup_{a \geq \varepsilon} \left| \varepsilon^{-1} \int_0^s \mathbf{1}_{\{a-\varepsilon < H_r \leq a\}} dr - L_s^a \right| \right] = 0$ .
- $\mathbb{P}^\psi$ -a.s., for every  $t \geq 0$ ,  $L_t^0 = -I_t$ .
- The occupation time formula holds: for any non-negative measurable function  $g$  on  $\mathbb{R}_+$  and any  $s \geq 0$ ,  $\int_0^s g(H_r) dr = \int_{(0,+\infty)} g(a) L_s^a da$ .

Let  $T_x = \inf\{t \geq 0; I_t \leq -x\}$ . We have the following Ray-Knight theorem.

**Proposition 3.6** ([16], Theorem 1.4.1). *The process  $(L_{T_x}^a, a \geq 0)$  is distributed under  $\mathbb{P}^\psi$  as  $Z$  under  $\mathbb{P}_x^\psi$  (i.e. is a CB with branching mechanism  $\psi$  starting at  $x$ ).*

Let  $\mathbb{P}_x^\psi$  be the distribution of  $(\rho_{t \wedge T_x}, t \geq 0)$  under  $\mathbb{P}^\psi$ . We set  $Z_a = L_{T_x}^a$  under  $\mathbb{P}_x^\psi$  and  $Z_a = L_\infty^a$  under  $\mathbb{N}^\psi$  and (under  $\mathbb{P}_x^\psi$  or  $\mathbb{N}^\psi$ )

$$(26) \quad \sigma(\rho) = \int_0^\infty \mathbf{1}_{\{\rho_t \neq 0\}} dt.$$

The occupation time formula implies that  $\sigma(\rho) = \int_0^{+\infty} Z_a da$ , which is consistent with notation (16). When there is no confusion, we shall write  $\sigma$  for  $\sigma(\rho)$ .

Exponential formula for the Poisson point process of jumps of the inverse subordinator of  $-I$  gives (see also the beginning of Section 3.2.2. [16]) that for  $\lambda > 0$

$$(27) \quad \mathbb{N}^\psi \left[ 1 - e^{-\lambda \sigma} \right] = \psi^{-1}(\lambda).$$

We also recall Lemma 1.6 of [1].

**Lemma 3.7.** *Let  $\theta > 0$ . The excursion measure  $\mathbb{N}^{\psi_\theta}$  is absolutely continuous w.r.t.  $\mathbb{N}^\psi$  with density  $e^{-\psi(\theta)\sigma}$ : for any non-negative measurable function  $F$  on the space of excursions, we have*

$$\mathbb{N}^{\psi_\theta} [F(\rho)] = \mathbb{N}^\psi \left[ F(\rho) e^{-\psi(\theta)\sigma} \right].$$

We recall the Poisson representation of  $\mathbb{P}_x^\psi$  based on the excursion measure  $\mathbb{N}^\psi$ . Let  $(\tilde{\alpha}_i, \tilde{\beta}_i)_{i \in \tilde{I}}$  be the excursion intervals of  $\rho$  away from 0. For every  $i \in \tilde{I}$ ,  $t \geq 0$ , we set

$$\tilde{\rho}_t^{(i)} = \rho_{(\tilde{\alpha}_i + t) \wedge \tilde{\beta}_i}.$$

We deduce from Lemma 4.2.4 of [16] the following lemma.

**Lemma 3.8.** *The point measure  $\sum_{i \in \tilde{I}} \delta_{\tilde{\rho}^{(i)}}(d\mu)$  is under  $\mathbb{P}_x^\psi$  a Poisson measure with intensity  $x \mathbb{N}^\psi(d\mu)$ .*

**3.6. The dual process and representation formula.** We shall need the  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process  $\eta = (\eta_t, t \geq 0)$  defined by

$$(28) \quad \eta_t(dr) = \sum_{\substack{0 < s \leq t \\ X_s < I_t^s}} (X_s - I_t^s) \delta_{H_s}(dr) + \beta \mathbf{1}_{[0, H_t]}(r) dr.$$

The process  $\eta$  is the dual process of  $\rho$  under  $\mathbb{N}^\psi$  (see Corollary 3.1.6 in [16]).

We recall the Poisson representation of  $(\rho, \eta)$  under  $\mathbb{N}^\psi$ . Let  $\mathcal{N}(dx dl du)$  be a Poisson point measure on  $[0, +\infty)^3$  with intensity

$$dx \ell \pi(dl) \mathbf{1}_{[0,1]}(u) du.$$

For every  $a > 0$ , let us denote by  $\mathbb{M}_a^\psi$  the law of the pair  $(\mu_a, \nu_a)$  of measures on  $\mathbb{R}_+$  with finite mass defined by: for any  $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$(29) \quad \langle \mu_a, f \rangle = \int \mathcal{N}(dx d\ell du) \mathbf{1}_{[0,a]}(x) u \ell f(x) + \beta \int_0^a f(x) dx,$$

$$(30) \quad \langle \nu_a, f \rangle = \int \mathcal{N}(dx d\ell du) \mathbf{1}_{[0,a]}(x) \ell (1-u) f(x) + \beta \int_0^a f(x) dx.$$

*Remark 3.9.* In particular  $\mu_a(dr) + \nu_a(dr)$  is defined as  $\mathbf{1}_{[0,a]}(r) d_r W_r$ , where  $W$  is a subordinator with Laplace exponent  $\psi' - \alpha$  where  $\alpha = \psi'(0)$  is defined by (22).

We finally set  $\mathbb{M}^\psi = \int_0^{+\infty} da e^{-\alpha a} \mathbb{M}_a^\psi$ .

**Proposition 3.10** ([16], Proposition 3.1.3). *For every non-negative measurable function  $F$  on  $\mathcal{M}_f(\mathbb{R}_+)^2$ ,*

$$\mathbb{N}^\psi \left[ \int_0^\sigma F(\rho_t, \eta_t) dt \right] = \int \mathbb{M}^\psi(d\mu d\nu) F(\mu, \nu),$$

where  $\sigma = \inf\{s > 0; \rho_s = 0\}$  denotes the length of the excursion.

#### 4. SUPER-CRITICAL LÉVY CONTINUUM RANDOM TREE

We shall construct a Lévy CRT with super-critical branching mechanism using a Girsanov formula.

Let  $\tilde{\psi}$  be a (sub)critical branching mechanism. The process  $Z = (Z_a, a \geq 0)$ , where  $Z_a = L_{T_x}^a$ , is a CB with branching mechanism  $\tilde{\psi}$ . We have  $\mathbb{P}_x^{\tilde{\psi}}$ -a.s.  $Z_\infty = \lim_{a \rightarrow +\infty} Z_a = 0$ . We shall call  $x$  the initial mass of the  $\tilde{\psi}$ -CRT under  $\mathbb{P}_x^{\tilde{\psi}}$ . The occupation time formula (see Proposition 3.5) and (26) imply that  $\mathbb{P}_x^{\tilde{\psi}}$ -a.s. and  $\mathbb{N}^{\tilde{\psi}}$ -a.e., we have

$$(31) \quad \sigma = \int_0^{+\infty} Z_a da.$$

This readily implies the following Girsanov formula: for any non-negative measurable function  $F$ , and  $q \geq 0$ ,

$$(32) \quad \mathbb{E}_x^{\tilde{\psi}} \left[ M_\infty^{\tilde{\psi}, q} F(\rho) \right] = \mathbb{E}_x^{\tilde{\psi}^q} [F(\rho)],$$

where  $M_\infty^{\tilde{\psi}, q}$  is given by (20).

We will use a similar formula (with  $q < 0$ ) to define the exploration process for a super-critical Lévy CRT with branching mechanism  $\psi$ . Because super-critical branching process may have an infinite mass, we shall cut it at a given level to construct the corresponding genealogical continuum random tree, see [15] when  $\pi = 0$ .

For  $a \geq 0$ , let  $\mathcal{M}_f^a = \mathcal{M}_f([0, a])$  be the set of non-negative measures on  $[0, a]$  and let  $\mathcal{D}^a$  be the set of càd-làg  $\mathcal{M}_f^a$ -valued process defined on  $[0, +\infty)$  endowed with the Skorohod topology. We now define a projection from  $\mathcal{D}$  to  $\mathcal{D}^a$ . For  $\rho = (\rho_t, t \geq 0) \in \mathcal{D}$ , we consider the time spent below level  $a$  up to time  $t$ :  $\Gamma_{\rho, a}(t) = \int_0^t \mathbf{1}_{\{H(\rho_s) \leq a\}} ds$  and its right continuous inverse

$$(33) \quad C_{\rho, a}(t) = \inf\{r \geq 0; \Gamma_{\rho, a}(r) > t\} = \inf\{r \geq 0; \int_0^r \mathbf{1}_{\{H(\rho_s) \leq a\}} ds > t\},$$

with the convention that  $\inf \emptyset = +\infty$ . We define the projector  $\pi_a$  from  $\mathcal{D}$  to  $\mathcal{D}^a$  by

$$(34) \quad \pi_a(\rho) = (\rho_{C_{\rho, a}(t)}, t \geq 0),$$

with the convention  $\rho_{+\infty} = 0$ . By construction we have the following compatibility relation:  $\pi_a \circ \pi_b = \pi_a$  for  $0 \leq a \leq b$ .

Let  $\psi$  be a super-critical branching mechanism which we suppose to be conservative, i.e. (2) holds. Recall  $q^*$  is the unique (positive) root of  $\psi'(q) = 0$ . In particular the branching mechanism  $\psi_q$  is critical if  $q = q^*$  and sub-critical if  $q > q^*$ .

We consider the filtration  $\mathcal{H} = (\mathcal{H}_a, a \geq 0)$  where  $\mathcal{H}_a$  is the  $\sigma$ -field generated by the càd-làg process  $\pi_a(\rho)$  and the class of  $\mathbb{P}_x^{\psi_{q^*}}$  negligible sets. Thanks to the second statement of Proposition 3.5, we get that  $Z$  is  $\mathcal{H}$ -adapted. Furthermore the proof of Theorem 1.4.1 in [16] yields that  $Z$  is a Markov process w.r.t. the filtration  $\mathcal{H}$ . In particular the process  $M^{\psi_{q^*}, -q^*}$  defined by (7) is thanks to Theorem 2.2 a  $\mathcal{H}$ -martingale under  $\mathbb{P}_x^{\psi_{q^*}}$ .

Let  $q \geq q^*$ . We define the distribution  $\mathbb{P}_x^{\psi, a}$  (resp.  $\mathbb{N}^{\psi, a}$ ) of the  $\psi$ -CRT cut at level  $a$  with initial mass  $x$ , as the distribution of  $\pi_a(\rho)$  under  $M_a^{\psi_q, -q} d\mathbb{P}_x^{\psi_q}$  (resp.  $e^{qZ_a + \psi(q)} \int_0^a Z_r dr d\mathbb{N}^{\psi_q}$ ): for any measurable non-negative function  $F$ ,

$$(35) \quad \mathbb{E}_x^{\psi, a} [F(\rho)] = \mathbb{E}_x^{\psi_q} \left[ M_a^{\psi_q, -q} F(\pi_a(\rho)) \right],$$

$$(36) \quad \mathbb{N}^{\psi, a} [F(\rho)] = \mathbb{N}^{\psi_q} \left[ e^{qZ_a + \psi(q)} \int_0^a Z_r dr F(\pi_a(\rho)) \right].$$

**Lemma 4.1.** *The distributions  $\mathbb{P}_x^{\psi, a}$  and  $\mathbb{N}^{\psi, a}$  do not depend on the choice of  $q \geq q^*$ .*

*Proof.* Let  $q > q^*$ . For any non-negative measurable function  $F$ , we have

$$\mathbb{E}_x^{\psi_q} \left[ M_a^{\psi_q, -q} F(\pi_a(\rho)) \right] = \mathbb{E}_x^{\psi_q} \left[ e^{-qx + qZ_a + \psi(q)} \int_0^a Z_s ds F(\pi_a(\rho)) \right].$$

As  $\psi_q = (\psi_{q^*})_{q-q^*}$ , we apply Girsanov formula (32) and the fact that  $M^{\psi_{q^*}, q-q^*}$  is a martingale to get

$$\begin{aligned} & \mathbb{E}_x^{\psi_q} \left[ M_a^{\psi_q, -q} F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi_{q^*}} \left[ M_a^{\psi_{q^*}, q-q^*} e^{-qx + qZ_a + \psi(q)} \int_0^a Z_s ds F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi_{q^*}} \left[ e^{(q-q^*)x - (q-q^*)Z_a - \psi_{q^*}(q-q^*)} \int_0^a Z_s ds e^{-qx + qZ_a + \psi(q)} \int_0^a Z_s ds F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi_{q^*}} \left[ e^{-q^*x + q^*Z_a - (\psi(q) - \psi(q^*))} \int_0^a Z_s ds e^{\psi(q)} \int_0^a Z_s ds F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi_{q^*}} \left[ M_a^{\psi_{q^*}, -q^*} F(\pi_a(\rho)) \right]. \end{aligned}$$

Excursion theory then gives the result for the excursion measures.  $\square$

Let  $\mathcal{W}$  be the set of  $\mathcal{D}$ -valued processes endowed with the  $\sigma$ -field generated by the coordinate applications.

**Proposition 4.2.** *Let  $(\rho^a, a \geq 0)$  be the canonical process on  $\mathcal{W}$ . There exists a probability measure  $\bar{\mathbb{P}}_x^\psi$  (resp. an excursion measure  $\bar{\mathbb{N}}^\psi$ ) on  $\mathcal{W}$ , such that, for every  $a \geq 0$ , the distribution of  $\rho^a$  under  $\bar{\mathbb{P}}_x^\psi$  (resp.  $\bar{\mathbb{N}}^\psi$ ) is  $\mathbb{P}_x^{\psi, a}$  (resp.  $\mathbb{N}^{\psi, a}$ ) and such that, for  $0 \leq a \leq b$*

$$(37) \quad \pi_a(\rho^b) = \rho^a \quad \bar{\mathbb{P}}_x^\psi\text{-a.s. (resp. } \bar{\mathbb{N}}^\psi\text{-a.e.)}$$

*Proof.* To prove the existence of such a projective limit, it is enough to check the compatibility relation between  $\mathbb{P}_x^{\psi, b}$  and  $\mathbb{P}_x^{\psi, a}$  for every  $b \geq a \geq 0$ .

Let  $0 \leq a \leq b$ . We get

$$\begin{aligned} \mathbb{E}_x^{\psi,b} [F(\pi_a(\rho))] &= \mathbb{E}_x^{\psi,q^*} \left[ M_b^{\psi,q^*,-q^*} F(\pi_a \circ \pi_b(\rho)) \right] \\ &= \mathbb{E}_x^{\psi,q^*} \left[ M_b^{\psi,q^*,-q^*} F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi,q^*} \left[ M_a^{\psi,q^*,-q^*} F(\pi_a(\rho)) \right] \\ &= \mathbb{E}_x^{\psi,a} [F(\rho)], \end{aligned}$$

where we used the compatibility relation of the projectors for the second equality and the fact that  $M^{\psi,q^*,-q^*}$  is a  $\mathcal{H}$ -martingale for the third equality. We deduce that  $\mathbb{P}_x^{\psi,b} \circ \pi_a = \mathbb{P}_x^{\psi,a}$ .

This compatibility relation implies the existence of a projective limit  $\bar{\mathbb{P}}_x^\psi$ . The result is similar for the excursion measure.  $\square$

Let us remark that the definitions of  $\bar{\mathbb{P}}_x^\psi$  and  $\bar{\mathbb{N}}^\psi$  are also valid for a (sub)critical branching mechanism  $\psi$ , with the convention  $q^* = 0$ . In particular, we get the following corollary.

**Corollary 4.3.** *If  $\psi$  is (sub)critical, then the law of the process  $(\pi_a(\rho), a \geq 0)$  under  $\mathbb{P}_x^\psi$  (resp.  $\mathbb{N}^\psi$ ) is  $\bar{\mathbb{P}}_x^\psi$  (resp.  $\bar{\mathbb{N}}^\psi$ ).*

By construction the local time at level  $a$  of  $\rho^b$  for  $b \geq a$  does not depend on  $b$ , we denote by  $Z_a$  its value. Property (ii) of Theorem 2.2 implies that  $Z = (Z_a, a \geq 0)$  is under  $\bar{\mathbb{P}}_x^\psi$  a CB with branching mechanism  $\psi$ . Hence, the probability measure  $\bar{\mathbb{P}}_x^\psi$  can be seen as the law of the exploration process that codes the super-critical CRT associated with  $\psi$ .

We get the following direct consequence of Properties (i) and (ii) of Lemma 2.4 and of the theory of excursion measures.

**Corollary 4.4.** *Let  $q \geq 0$  such that  $\psi(q) \geq 0$ . Then, the probability measure  $\bar{\mathbb{P}}_x^{\psi,q}$  is absolutely continuous with respect to  $\bar{\mathbb{P}}_x^\psi$  with*

$$\frac{d\bar{\mathbb{P}}_x^{\psi,q}}{d\bar{\mathbb{P}}_x^\psi} = M_\infty^{\psi,q} = e^{qx - \psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

*The measure  $\bar{\mathbb{N}}^{\psi,q}$  is absolutely continuous with respect to  $\bar{\mathbb{N}}^\psi$  with*

$$\frac{d\bar{\mathbb{N}}^{\psi,q}}{d\bar{\mathbb{N}}^\psi} = e^{-\psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

If the total mass of  $Z$ ,  $\sigma = \int_0^{+\infty} Z_a da$ , is finite, then  $\rho^\sigma$  is the projection of a well defined exploration process.

**Lemma 4.5.** *On  $\{\sigma < +\infty\}$ , there exists  $\rho^\infty \in \mathcal{D}$  such that  $\rho^a = \pi^a(\rho^\infty)$  for all  $a \geq 0$ ,  $\bar{\mathbb{P}}_x^\psi$ -a.s. or  $\bar{\mathbb{N}}^\psi$ -a.e.*

*Proof.* It is enough to get the result under  $\bar{\mathbb{P}}_x^\psi$ .

First we assume that  $\psi$  is (sub)critical. Proposition 3.5 implies that  $\int_0^t \mathbf{1}_{\{H(\rho_s) \leq a\}} ds$  increases to  $t$  as  $a$  goes to infinity. Using (33), (34) and the right continuity of  $\rho$ , we deduce that  $\mathbb{P}_x^\psi$ -a.s. for all  $t \geq 0$ ,  $\lim_{a \rightarrow +\infty} \pi^a(\rho)_t = \rho_t$ .

Thanks to Corollary 4.3, we deduce that  $\bar{\mathbb{P}}_x^\psi$ -a.s. for all  $t \geq 0$ ,  $\rho_t^\infty = \lim_{a \rightarrow +\infty} \pi^a(\rho)_t$  exists and that  $\pi_a(\rho^\infty) = \rho^a$ .

The case  $\psi$  super-critical is then a consequence of Corollary 4.4.  $\square$

Without confusion, we shall always write  $\mathbb{P}^\psi$  instead of  $\bar{\mathbb{P}}^\psi$  and  $\mathbb{N}^\psi$  instead of  $\bar{\mathbb{N}}^\psi$  and call them the law or the excursion measure of the exploration process of the CRT, whether  $\psi$  is super-critical or (sub)critical. And we shall write  $\rho$  for the projective limit  $(\rho^a, a \geq 0)$  on  $\mathcal{W}$ , and make the identification  $\rho = \rho^\infty \in \mathcal{D}$  when the latter exists that is when  $\sigma$  defined by (31) is finite.

Recall  $\psi^{-1}$  is given by (17). We now extend formula (27) for general branching mechanism.

**Lemma 4.6.** *Let  $\sigma$  be given by (31). We have for  $\lambda \geq 0$ :*

$$\mathbb{E}_x^\psi \left[ e^{-\lambda \sigma} \right] = \exp \left( -x \mathbb{N}^\psi [1 - e^{-\lambda \sigma}] \right) = e^{-x \psi^{-1}(\lambda)}.$$

*Proof.* Let  $q \geq q^*$ . We have

$$\begin{aligned} \mathbb{E}_x^\psi \left[ e^{-\lambda \int_0^a Z_r dr} \right] &= \mathbb{E}_x^{\psi_q} \left[ M_a^{\psi_q, -q} e^{-\lambda \int_0^a Z_r dr} \right] \\ &= e^{-qx} \mathbb{E}_x^{\psi_q} \left[ e^{qZ_a + (\psi(q) - \lambda) \int_0^a Z_r dr} \right] \\ &= e^{-qx} e^{-x \mathbb{N}^{\psi_q} [1 - e^{qZ_a + (\psi(q) - \lambda) \int_0^a Z_r dr}]} \\ &= e^{-qx} e^{-x \mathbb{N}^{\psi_q} [1 - e^{qZ_a + \psi(q) \int_0^a Z_r dr}]} e^{-x \mathbb{N}^{\psi_q} [e^{qZ_a + \psi(q) \int_0^a Z_r dr} (1 - e^{-\lambda \int_0^a Z_r dr})]} \\ &= \mathbb{E}_x^{\psi_q} \left[ M_a^{\psi_q, -q} e^{-x \mathbb{N}^{\psi_q} [1 - e^{-\lambda \int_0^a Z_r dr}]} \right] \\ &= e^{-x \mathbb{N}^\psi [1 - e^{-\lambda \int_0^a Z_r dr}]}, \end{aligned}$$

where we used (35) for the first equality, (7) for the second, Lemma 3.8 for the third, (36) for the fifth, and (1) of Theorem 2.2 for the last. We then let  $a$  goes to infinity to get the first equality of the lemma, and use (19) to get the second.  $\square$

## 5. PRUNING

We keep notations from Section 3. Recall  $\mathcal{D}$  is the set of càd-làg  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process, and  $\mathcal{W}$  is the set of  $\mathcal{D}$ -valued processes. Let  $R = (\rho^\theta, \theta \geq 0)$  be the canonical process on  $\mathcal{W}$ .

Let  $\psi$  be a (sub)critical branching mechanism. The pruning procedure developed in [5] when  $\pi = 0$ , [1] when  $\beta = 0$  and in [4] or [23] for the general case, yields a probability measure on  $\mathcal{W}$ ,  $\tilde{\mathbb{P}}_x^\psi$ , such that  $R$  is Markov and the law  $\rho^\theta$  under  $\tilde{\mathbb{P}}_x^\psi$  is  $\mathbb{P}_x^{\psi_\theta}$  for all  $\theta \geq 0$ . Furthermore  $\rho^\theta$  codes for a sub-tree of  $\rho^{\theta'}$  if  $\theta \geq \theta'$ . We recall the construction of  $\tilde{\mathbb{P}}_x^\psi$  in Section 5.1.

**5.1. Pruning of (sub)critical CRT.** The main idea of the pruning procedure of a tree coded by an exploration  $\rho$  is to put marks on a leaf  $t$  (or a branch labeled by  $t$ ) and more precisely on the measure  $\rho_t$ . There are two types of marks: the first ones only lay on the **nodes** of the tree whereas the other ones lay on the **skeleton** of the tree; and each mark appears at a random time. At time  $\theta$ , we get  $\rho^\theta$  by removing all the leaves from  $\rho$  which received a mark by time  $\theta$ . We explain more precisely the pruning procedure.

**5.1.1. Marks on the nodes.** Let  $(X_t, t \geq 0)$  be the Lévy process with branching mechanism  $\psi$  and let  $\rho$  be the corresponding exploration process. We denote by  $(\Delta_s, s \in \mathcal{I})$  the set of jumps of  $X$ . Conditionally on  $X$ , we consider a family

$$(T_s, s \in \mathcal{I})$$

of independent exponential random variables with respective parameter  $\Delta_s$ . We define the  $\mathcal{M}(\mathbb{R}_+^2)$ -valued process  $M^{(nod)} = (M_t^{(nod)}, t \geq 0)$  by

$$M_t^{(nod)}(dr, dv) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} \delta_{T_s}(dv) \delta_{H_s}(dr).$$

For fixed  $\theta \geq 0$ , we will consider the  $\mathcal{M}(\mathbb{R}_+)$ -valued process  $M_t^{(nod)}(dr, [0, \theta])$  whose atoms give the marked nodes : each node is marked independently from the others with probability  $1 - e^{-\theta \Delta_s}$  where  $\Delta_s$  is the mass (i.e. the height of the jump) associated with the node.

*Remark 5.1.* Although different from the measure process that defines the marks on the nodes in [1] (formula (12)), this construction gives the same marks (see Introduction of [1]).

5.1.2. *Marks on the skeleton.* Let  $M^{(ske)} = (M_t^{(ske)}, t \geq 0)$  be a Lévy snake with lifetime  $H$  and spatial motion a Poisson point process with intensity

$$2\beta \mathbf{1}_{\{u > 0\}} du.$$

(See [16] for the definition of a Lévy snake and [4] for the extension to a discontinuous height process  $H$ , see also [23]).

In other words,  $M^{(ske)}$  is a  $\mathcal{M}(\mathbb{R}_+^2)$ -valued process such that, conditionally on the exploration process  $\rho$ ,

- For every  $t \geq 0$ ,  $M_t^{(ske)}(dr, du)$  is a Poisson point measure with intensity

$$2\beta \mathbf{1}_{[0, H_t]}(r) dr \mathbf{1}_{\{u > 0\}} du,$$

- For every  $0 \leq t \leq t'$ , with  $H_{t,t'} := \inf_{s \in [t, t']} H_s$ , then:

- The measures  $M_t^{(ske)}(dr, du) \mathbf{1}_{r \in [0, H_{t,t'}]}$  and  $M_{t'}^{(ske)}(dr, du) \mathbf{1}_{r \in [0, H_{t,t'}]}$  are equal,
- The random measures  $M_t^{(ske)}(dr, du) \mathbf{1}_{r \in [H_{t,t'}, H_t]}$  and  $M_{t'}^{(ske)}(dr, du) \mathbf{1}_{r \in [H_{t,t'}, H_{t'}]}$  are independent.

5.1.3. *Definition of the pruned processes.* We define the mark process as

$$(38) \quad M^{(mark)} = M^{(nod)} + M^{(ske)}.$$

The process  $((\rho_t, M_t^{(mark)}), t \geq 0)$  is called the marked exploration process. It is Markovian, see [23] for its properties. We denote by  $\hat{\mathbb{P}}_x^\psi$  its law and by  $\hat{\mathbb{N}}^\psi$  the corresponding excursion measure.

For every  $\theta > 0$  and  $t > 0$ , we set

$$m_t^{(\theta)} = M_t^{(mark)}([0, H_t] \times [0, \theta]).$$

The random variable  $m_t^{(\theta)}$  is the number of marks at time  $\theta$  that lay on the branch labeled by  $t$ . We will only consider the branches without marks. Therefore, we set

$$(39) \quad A_t^{(\theta)} = \int_0^t \mathbf{1}_{\{m_s^{(\theta)} = 0\}} ds \quad \text{and} \quad C_t^{(\theta)} = \inf\{r \geq 0; A_r^{(\theta)} \geq t\}$$

its right-continuous inverse. Finally, we define  $\rho^\theta = (\rho_t^\theta, t \geq 0)$ ,  $M^{(mark),\theta} = (M_t^{(mark),\theta}, t \geq 0)$  by

$$\begin{aligned} \rho_t^\theta &= \rho_{C_t^{(\theta)}}, \\ M_t^{(mark),\theta}([0, h] \times [0, q]) &= M_{C_t^{(\theta)}}^{(mark)}([0, h] \times (\theta, q + \theta)). \end{aligned}$$

We shall use in Section 7 the pruning operator  $\Lambda_\theta$  defined on the marked exploration process by

$$(40) \quad \Lambda_\theta(\rho, M^{(mark)}) = (\rho^\theta, M^{(mark),\theta}).$$

Using the lack of memory of the exponential random variables and of properties of Poisson point measure, it is easy to get that

**Lemma 5.2.** *The process  $R = (\rho^\theta, \theta \geq 0)$  is Markov.*

The  $\mathcal{W}$ -valued process  $R$  codes for a decreasing family of CRT, which we shall call a  $\psi$ -family of pruned CRT. A direct application of Theorem 1.1 of [4] gives the marginal distribution.

**Proposition 5.3.** *The marked exploration process  $(\rho^\theta, M^{(mark),\theta})$  under  $\mathbb{P}_x^\psi$  (resp.  $\mathbb{N}^\psi$ ) is distributed as  $(\rho, M^{(mark)})$  under  $\mathbb{P}_x^{\psi_\theta}$  (resp.  $\mathbb{N}^{\psi_\theta}$ ).*

We shall now concentrate on the process  $R$ . Let  $\tilde{\mathbb{P}}_x^\psi$  be the law of  $R$  and  $\tilde{\mathbb{N}}^\psi$  be the corresponding excursion measure.

We deduce the following compatibility relation from the Markov property of  $R$  and Proposition 5.3.

**Corollary 5.4.** *Let  $\theta_0 \geq 0$ . The law under  $\tilde{\mathbb{P}}_x^{\psi_{\theta_0}}$  (resp.  $\tilde{\mathbb{N}}^{\psi_{\theta_0}}$ ) of the process  $(\rho^{\theta_0+\theta}, \theta \geq 0)$  is  $\tilde{\mathbb{P}}_x^{\psi_{\theta_0}}$  (resp.  $\tilde{\mathbb{N}}^{\psi_{\theta_0}}$ ).*

Let us now recall the special Markov property, Theorem 4.2 of [4], stated for the present context. We fix  $\theta > 0$ . We want to describe the law of the excursions of  $\rho$  ‘‘above’’ the marks, given the process ‘‘under’’ the marks. More precisely, we define  $O$  as the interior of the set  $\{s \geq 0, m_s^{(\theta)} = 0\}$  and write  $O = \bigcup_{i \in I} (\alpha_i, \beta_i)$ . For every  $i \in I$ , we define the exploration process  $\rho^{(i)}$  by: for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $t \geq 0$ ,

$$\langle \rho_t^{(i)}, f \rangle = \int_{[H_{\alpha_i}, +\infty)} f(x - H_{\alpha_i}) \rho_{(\alpha_i+t) \wedge \beta_i}(dx).$$

We have the following theorem.

**Theorem 5.5** (Special Markov Property). *Let  $\theta > 0$  and let  $(Z_t^\theta, t \geq 0)$  be the CSBP coded by  $\rho^\theta$ . The point measure*

$$\sum_{i \in I} \delta_{(H_{\alpha_i}, \rho^{(i)})}(dh, d\mu)$$

*under  $\mathbb{P}_x^\psi$  (or  $\mathbb{N}^\psi$ ) conditionally given  $(\rho_t^\theta, t \geq 0)$ , is a Poisson point measure of intensity*

$$\mathbf{1}_{[0, +\infty)}(h) Z_h^\theta dh \left( 2\beta\theta \mathbb{N}^\psi(d\mu) + \int_{(0, +\infty)} \pi(dr) (1 - e^{-\theta r}) \mathbb{P}_r^\psi(d\mu) \right).$$

**5.2. Pruning of super-critical CRT.** We now use the same Girsanov techniques of Section 4 to define a  $\psi$ -family of pruned CRT when  $\psi$  is super-critical.

Let  $\psi$  be a super-critical branching mechanism which we suppose to be conservative, i.e. (2) holds. Recall  $q^*$  is the unique (positive) root of  $\psi'(q) = 0$ . In particular the branching mechanism  $\psi_q$  is critical if  $q = q^*$  and sub-critical if  $q > q^*$ .

Let  $q \geq q^*$ . Let  $R = (\rho^\theta, \theta \geq 0)$  be the canonical process on  $\mathcal{W}$ . We set  $Z = (L_\infty^a(\rho^0), a \geq 0)$  which is under  $\tilde{\mathbb{P}}_x^{\psi_q}(dR)$  a CB with branching mechanism  $\psi_q$ . The process  $Z$  is also well defined under the excursion measure  $\tilde{\mathbb{N}}^{\psi_q}(dR)$ . We write  $\pi_a(R) = (\pi_a(\rho^\theta), \theta \geq 0)$ . Notice that given the marks (i.e. given  $M^{(nod)}$  and  $M^{(ske)}$ ), we have  $\pi_a(\rho^\theta) = (\pi_a(\rho))^\theta$ .

Let  $a \geq 0$ . We define the distribution  $\tilde{\mathbb{P}}_x^{\psi,a}$  (resp. excursion measure  $\tilde{\mathbb{N}}^{\psi,a}$ ) of a  $\psi$ -family of pruned CRT cut at level  $a$  with initial mass  $x$ , as the distribution of  $\pi_a(R)$  under  $M_a^{\psi_q, -q} d\tilde{\mathbb{P}}_x^{\psi_q}$  (resp.  $e^{qZ_a + \psi(q)} \int_0^a Z_r dr d\tilde{\mathbb{N}}^{\psi_q}$ ): for any measurable non-negative function  $F$ , we have:

$$\tilde{\mathbb{P}}_x^{\psi,a} [F(R)] = \tilde{\mathbb{P}}_x^{\psi_q} \left[ M_a^{\psi_q, -q} F(\pi_a(R)) \right]$$

and

$$\tilde{\mathbb{N}}^{\psi,a} [F(\rho)] = \tilde{\mathbb{N}}^{\psi_q} \left[ e^{qZ_a + \psi(q)} \int_0^a Z_r dr F(\pi_a(\rho)) \right].$$

Same arguments as for Lemma 4.1 give the following result.

**Lemma 5.6.** *The distributions  $\tilde{\mathbb{P}}_x^{\psi,a}$  and  $\tilde{\mathbb{N}}^{\psi,a}$  do not depend on the choice of  $q \geq q^*$ .*

As in Section 4, see Proposition 4.2, the families of measures  $(\tilde{\mathbb{P}}_x^{\psi,a}, x \geq 0)$  and  $(\tilde{\mathbb{N}}^{\psi,a}, a \geq 0)$  fulfill a compatibility relation. Hence there exists a projective limit  $(R^a, a \geq 0)$  defined on the space of  $\mathcal{W}$ -valued process such that

- For every  $a \geq 0$ ,  $R^a$  is distributed as  $\tilde{\mathbb{P}}_x^{\psi,a}$ ,
- For every  $a < b$ ,  $\pi_a(R^b) = R^a$ .

We write  $\tilde{\mathbb{P}}_x^\psi$  for the distribution of this projective limit and  $\tilde{\mathbb{N}}^\psi$  for the corresponding excursion measure.

By construction the local time at level  $a$  of  $\pi_b(\rho^\theta)$  for  $b \geq a$  does not depend on  $b$ , we denote by  $Z_a^\theta$  its value. Proposition 5.3 and Property (ii) of Theorem 2.2 imply that  $Z^\theta = (Z_a^\theta, a \geq 0)$  is under  $\tilde{\mathbb{P}}_x^\psi$  a CB with branching mechanism  $\psi_\theta$  started at  $x$ . Following (31), we define  $\sigma_\theta = \int_0^\infty Z_a^\theta da$ . And, when there is no confusion, we write  $\sigma$  for  $\sigma_0$ .

Following Corollaries 4.3, 4.4 and Lemma 4.5, we easily get the following theorem.

**Theorem 5.7.** *Let  $\psi$  be a conservative branching mechanism. Let  $(R^a, a \geq 0)$  be a  $\mathcal{W}$ -valued process under  $\tilde{\mathbb{P}}_x^\psi$  (resp.  $\tilde{\mathbb{N}}^\psi$ ).*

- (1) *If  $\psi$  is (sub)critical, then  $(R^a, a \geq 0)$  under  $\tilde{\mathbb{P}}_x^\psi$  is distributed as  $((\pi_a(\rho^\theta), \theta \geq 0), a \geq 0)$  under  $\mathbb{P}_x^\psi$ .*
- (2) *Let  $q \geq 0$  such that  $\psi(q) \geq 0$ . Then, the probability measure  $\tilde{\mathbb{P}}_x^{\psi,q}$  is absolutely continuous with respect to  $\tilde{\mathbb{P}}_x^\psi$  with*

$$\frac{d\tilde{\mathbb{P}}_x^{\psi,q}}{d\tilde{\mathbb{P}}_x^\psi} = M_\infty^{\psi,q} = e^{qx - \psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

*The measure  $\tilde{\mathbb{N}}^{\psi,q}$  is absolutely continuous with respect to  $\tilde{\mathbb{N}}^\psi$  with*

$$\frac{d\tilde{\mathbb{N}}^{\psi,q}}{d\tilde{\mathbb{N}}^\psi} = e^{-\psi(q)\sigma} \mathbf{1}_{\{\sigma < +\infty\}}.$$

- (3) On  $\{\sigma < +\infty\}$ , there exists  $R^\infty \in \mathcal{W}$  such that  $R^a = \pi^a(R^\infty)$  for all  $a \geq 0$ ,  $\tilde{\mathbb{P}}_x^\psi$ -a.s. or  $\tilde{\mathbb{N}}^\psi$ -a.e.

Without confusion, we shall always write  $\mathbb{P}^\psi$  instead of  $\tilde{\mathbb{P}}^\psi$  and  $\mathbb{N}^\psi$  instead of  $\tilde{\mathbb{N}}^\psi$  and call them the law or the excursion measure of  $\psi$ -pruned family of exploration processes, whether  $\psi$  is super-critical or (sub)critical. The  $\psi$ -pruned family of exploration processes codes for a  $\psi$ -pruned family of continuum random sub-trees.

And we shall write  $(\rho^\theta, \theta \geq 0)$  for the projective limit  $(R^a, a \geq 0)$ , and identify it with  $R^\infty \in \mathcal{W}$  when the latter exists, that is when  $\sigma$  defined by (31) is finite. Notice that if  $\sigma_\theta$  is finite then the exploration process  $\rho^\theta$  codes for a CRT with finite mass.

**5.3. Properties of the branching mechanism.** Let  $\psi$  be a branching mechanism with parameter  $(\alpha, \beta, \pi)$ . Let  $\Theta'$  be the set of  $\theta \in \mathbb{R}$  such that

$$(41) \quad \int_{(1, +\infty)} e^{-\theta \ell} \pi(d\ell) < +\infty.$$

We set  $\theta_\infty = \inf \Theta'$ . Notice that we have either  $\Theta' = [\theta_\infty, +\infty)$  or  $\Theta' = (\theta_\infty, +\infty)$  and that  $\theta_\infty \leq 0$ . Notice that  $\psi_\theta$  exists for every  $\theta \in \Theta'$  and is conservative for every  $\theta > \theta_\infty$ . We set  $\Theta = \{\theta \in \Theta'; \psi_\theta \text{ is conservative}\}$ . Notice that  $\Theta \subset \Theta' \subset \Theta \cup \{\theta_\infty\}$ .

For instance, we have the following examples of critical branching mechanisms:

- i) Quadratic case:  $\psi(u) = \beta u^2$ ,  $\Theta = \Theta' = \mathbb{R}$ .
- ii) Stable case:  $\psi(u) = cu^\alpha$  with  $\alpha \in (1, 2)$ ,  $\Theta = \Theta' = [0, +\infty)$ .
- iii)  $\psi(u) = (u + e^{-1}) \log(u + e^{-1}) + e^{-1}$ :  $\Theta = \Theta' = [-e^{-1}, +\infty)$ . (Notice that  $\psi_{\theta_\infty}(u) = u \log(u)$ ,  $\psi'_{\theta_\infty}(0+) = -\infty$  and  $\psi_{\theta_\infty}$  is conservative.)
- iv)  $\psi(u) = u - 1 + \frac{1}{1+u}$  is associated with  $(\tilde{\alpha}, \beta, \pi)$  where  $\tilde{\alpha} = 2/e$ ,  $\beta = 0$  and  $\pi(d\ell) = e^{-\ell} \mathbf{1}_{\{\ell > 0\}} d\ell$ :  $\Theta = \Theta' = (-1, +\infty)$ .

For the end of this subsection, we assume that  $\psi$  is CRITICAL and that  $\beta > 0$  or  $\pi \neq 0$ . Remark that  $\psi$  is a one-to-one function from  $[0, +\infty)$  onto  $[0, +\infty)$  and we denote by  $\psi^{-1}$  its inverse function. For  $\theta < 0$  such that  $\theta \in \Theta'$ , we define  $\bar{\theta} = \psi^{-1}(\psi(\theta))$  or equivalently  $\bar{\theta}$  is the unique positive real number such that

$$(42) \quad \psi(\bar{\theta}) = \psi(\theta).$$

Since  $\psi$  is continuous and strictly convex, if  $\theta_\infty \in \Theta'$ , we have

$$(43) \quad \bar{\theta}_\infty = \lim_{\theta \downarrow \theta_\infty} \bar{\theta}.$$

Notice that in this case  $\bar{\theta}_\infty$  is finite. If  $\theta_\infty \notin \Theta'$ , we define  $\bar{\theta}_\infty$  using (43).

**Lemma 5.8.** *Let  $\psi$  be CRITICAL with parameters  $(\tilde{\alpha}, \beta, \pi)$  such that  $\beta > 0$  or  $\pi \neq 0$ . If  $\theta_\infty \notin \Theta'$  then  $\bar{\theta}_\infty = +\infty$ .*

*Proof.* We assume that  $\theta_\infty \notin \Theta'$ . It is enough to check that  $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$  to get  $\bar{\theta}_\infty = +\infty$ .

We first consider the case  $\theta_\infty = -\infty$ . Since  $\psi'(0) = 0$  and  $\psi$  is strictly convex, we get that  $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$ .

If  $\theta_\infty > -\infty$ , then using that (41) does not hold for  $\theta_\infty$  and monotone convergence theorem, we get that  $\lim_{\theta \downarrow \theta_\infty} \psi(\theta) = +\infty$ .  $\square$

## 6. A TREE-VALUED PROCESS

Let  $\psi$  be a branching mechanism. We assume  $\theta_\infty < 0$ . We write  $\mathcal{R}_q = (\rho^{\gamma+q}, \gamma \geq 0)$ .

We deduce from Corollary 5.4 that the families of measures  $(\mathbb{P}^{\psi_\theta}, \theta \in \Theta)$  and  $(\mathbb{N}^{\psi_\theta}, \theta \in \Theta)$  satisfy the following compatibility property: if  $\theta' < \theta$ ,  $\theta' \in \Theta$ , the process  $\mathcal{R}_{\theta-\theta'}$  under  $\mathbb{P}^{\psi_{\theta'}}$  (resp.  $\mathbb{N}^{\psi_{\theta'}}$ ) is distributed as  $\mathcal{R}_0$  under  $\mathbb{P}^{\psi_\theta}$  (resp.  $\mathbb{N}^{\psi_\theta}$ ).

Hence, there exists a projective limit  $\mathcal{R} = (\rho^\gamma, \gamma \in \Theta)$  such that, for every  $\theta \in \Theta$ , the process  $(\rho^{\theta+\gamma}, \gamma \geq 0)$  is distributed as  $(\rho^\gamma, \gamma \geq 0)$  under  $\mathbb{P}^{\psi_\theta}$ . We denote by  $\mathbb{P}^\psi$  the distribution of the projective limit  $\mathcal{R}$ , and by  $\mathbf{N}^\psi$  the corresponding excursion measure. We still write  $\mathcal{R}_\theta$  for  $(\rho^{\theta+\gamma}, \gamma \geq 0)$  for all  $\theta \in \Theta$ .

The process  $\mathcal{R} = (\rho^\theta, \theta \in \Theta)$  is Markovian, thanks to Lemma 5.2. It codes for a tree-valued Markov process, which evolves according to a pruning procedure. At time  $\theta$ ,  $\rho^\theta$  has distribution  $\mathbb{P}^{\psi_\theta}$ . Recall  $\sigma_\theta$  is the mass of the CRT coded by  $\rho^\theta$ . It is not difficult to check that  $\Sigma = (\sigma_\theta, \theta \in \Theta)$  is a non-increasing Markov process taking values in  $[0, +\infty]$  and we shall consider a version of  $\mathcal{R}$  such that the process  $\Sigma$  is càd-làg. From the continuity of  $\psi$ , we deduce that the Laplace transform of  $\sigma_\theta$  given in Lemma 4.6 is continuous, and thus the process  $\Sigma$  is continuous in probability.

See [14] for the distribution of the decreasing rearrangement of the jumps of  $(\sigma_\theta, \theta \geq 0)$ . We deduce from the pruning procedure that a.s.  $\lim_{\theta \rightarrow +\infty} \sigma_\theta = 0$ . Notice that by considering the time returned process  $(\rho^{-\theta}, \theta < \theta_\infty)$ , we get a Markovian family of exploration processes coding for a family of increasing CRTs.

*Remark 6.1.* Recall  $q^*$  is the unique root of  $\psi'(q) = 0$  and that  $\psi_{q^*}$  is critical. Using a shift on  $\theta$  by  $q^*$ , that is replacing  $\psi$  by  $\psi_{q^*}$ , one sees that it is enough, when studying  $\mathcal{R}$ , to assume that  $\psi$  is critical.

**Lemma 6.2.** *Let  $\psi$  be a critical branching mechanism with parameter  $(\alpha, \beta, \pi)$ . For any  $\theta \in \Theta$ , and any non-negative measurable function  $F$  defined on the state space of  $\mathcal{R}_0$ , we have*

$$(44) \quad \mathbf{N}^\psi [F(\mathcal{R}_\theta) \mathbf{1}_{\{\sigma_\theta < \infty\}}] = \mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \mathbf{1}_{\{\sigma_0 < \infty\}}] = \mathbf{N}^\psi [F(\mathcal{R}_0) e^{-\psi(\theta)\sigma_0}].$$

*Proof.* The first equality is just the 'compatibility property' stated at the beginning of section.

For  $\theta \geq 0$ , the second equality is a direct consequence of (ii) from Theorem 5.7.

For  $\theta < 0$ , let  $q = \bar{\theta} - \theta$ . Notice that  $\psi_\theta(q) = \psi(\bar{\theta}) - \psi(q) = 0$  and  $(\psi_\theta)_q = \psi_{\bar{\theta}}$ . We deduce from (ii) of Theorem 5.7 that

$$\mathbf{N}^{\psi_{\bar{\theta}}} [F(\mathcal{R}_0)] = \mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \mathbf{1}_{\{\sigma_0 < \infty\}}].$$

Since  $\bar{\theta} > 0$  and  $\psi(\theta) = \psi(\bar{\theta})$ , we get from (2) of Theorem 5.7 that

$$\mathbf{N}^{\psi_{\bar{\theta}}} [F(\mathcal{R}_0)] = \mathbf{N}^\psi [F(\mathcal{R}_0) e^{-\psi(\bar{\theta})\sigma_0}] = \mathbf{N}^\psi [F(\mathcal{R}_0) e^{-\psi(\theta)\sigma_0}].$$

This ends the proof.  $\square$

We deduce directly from this lemma the following result on the conditional distribution of the exploration process knowing the total mass of the CRT.

**Corollary 6.3.** *Let  $\psi$  be a branching mechanism with parameter  $(\alpha, \beta, \pi)$  such that (41) holds. The distribution of  $(\rho^{\theta+\gamma}, \gamma \geq 0)$  conditionally on  $\{\sigma_\theta = r\}$  does not depend on  $\theta \in \Theta$ .*

We assume from now-on that  $\psi$  is CRITICAL and that  $\theta_\infty < 0$ . The first assumption is not restrictive thanks to Remark 6.1.

Notice that  $\rho^\theta$  codes for a critical (resp. sub-critical, resp. super-critical) CRT if  $\theta = 0$  (resp.  $\theta > 0$ , resp.  $\theta < 0$ ). In particular, we have  $\sigma_\theta < +\infty$  a.s. if  $\theta \geq 0$ .

We consider the explosion time

$$A = \inf\{\theta \in \Theta, \sigma_\theta < +\infty\},$$

with the convention that  $\inf \emptyset = \theta_\infty$ . In particular, we have  $A \leq 0$   $\mathbf{P}_x^\psi$ -a.s. and  $\mathbf{N}^\psi$ -a.e. Moreover, since the process  $(\sigma_\theta, \theta \in \Theta)$  is càd-làg, we have, on  $\{A > \theta_\infty\}$ ,  $\sigma_\theta = +\infty$  for every  $\theta < A$  and  $\sigma_\theta < +\infty$  for every  $\theta \geq A$ . For the time returned process,  $A$  is the random time at which the tree gets an infinite mass.

We first give a lemma on the conditional distribution of  $\sigma$ .

**Lemma 6.4.** *Let  $q \in \Theta$ ,  $q \leq \theta$ . We have, for  $\lambda \geq 0$ ,*

$$\mathbf{N}^\psi[e^{-\lambda\sigma_q} | \rho^\theta] = e^{-\sigma_\theta \psi_\theta(\psi_q^{-1}(\lambda))}$$

and  $\mathbf{N}^\psi[\sigma_q < +\infty | \rho^\theta] = e^{-\sigma_\theta \psi_\theta(\bar{q}-q)}$ , where  $\bar{q} = \psi^{-1}(\psi(q))$ .

*Proof.* Let  $\lambda > 0$  and  $F$  be a non-negative measurable function defined on  $\mathcal{W}$ . We write  $Z_a^q$  for the local time at level  $a$  of the exploration process  $\rho^q$ . Using (16), we have

$$(45) \quad \mathbf{N}^\psi[e^{-\lambda\sigma_q} F(\rho^\theta)] = \lim_{a \rightarrow \infty} \mathbf{N}^\psi[e^{-\lambda \int_0^a Z_r^q dr} F(\rho^\theta)].$$

We set

$$I_a = \mathbf{N}^\psi[e^{-\lambda \int_0^a Z_r^q dr} F(\rho^\theta)].$$

Let  $G(\pi_a(\rho^\theta)) = \mathbf{E}^\psi[F(\rho^\theta) | \pi_a(\rho^\theta)]$ . We have, with  $\theta' = \theta - q \geq 0$ ,

$$\begin{aligned} I_a &= \mathbf{N}^\psi[e^{-\lambda \int_0^a Z_r^q dr} G(\pi_a(\rho^\theta))] \\ &= \mathbf{N}^{\psi_q}[e^{-\lambda \int_0^a Z_r^0 dr} G(\pi_a(\rho^{\theta'}))] \\ &= \mathbf{N}^\psi[e^{-qZ_a^0 - (\psi(q)+\lambda) \int_0^a Z_r^0 dr} G(\pi_a(\rho^{\theta'}))] \\ &= \mathbf{N}^\psi[e^{-qZ_a^{\theta'} - (\psi(q)+\lambda) \int_0^a Z_r^{\theta'} dr - \int_0^a K_h^a Z_h^{\theta'} dh} G(\pi_a(\rho^{\theta'}))], \end{aligned}$$

where for the first equality we conditioned with respect to  $\sigma(\pi_a(\rho^q))$ , used Girsanov formula for the third equality and Theorem 5.5 for the last equality with

$$\begin{aligned} K_h^a &= 2\beta\theta' \mathbf{N}^\psi \left[ 1 - e^{-qZ_{a-h} - (\psi(q)+\lambda) \int_0^{a-h} Z_r dr} \right] \\ &\quad + \int_{(0,+\infty)} \pi(du) (1 - e^{-\theta'u}) \mathbf{E}_u^\psi \left[ 1 - e^{-qZ_{a-h} - (\psi(q)+\lambda) \int_0^{a-h} Z_r dr} \right]. \end{aligned}$$

We set

$$\begin{aligned} \tilde{K}_h^a &= 2\beta\theta' \mathbf{N}^\psi \left[ e^{-qZ_{a-h} - \psi(q) \int_0^{a-h} Z_r dr} (1 - e^{-\lambda \int_0^{a-h} Z_r dr}) \right] \\ &\quad + \int_{(0,+\infty)} \pi(du) (1 - e^{-\theta'u}) \mathbf{E}_u^\psi \left[ e^{-qZ_{a-h} - \psi(q) \int_0^{a-h} Z_r dr} (1 - e^{-\lambda \int_0^{a-h} Z_r dr}) \right]. \end{aligned}$$

Using again Theorem 5.5 and Girsanov formula, we get

$$\begin{aligned}
I_a &= \mathbf{N}^\psi \left[ e^{-qZ_a^0 - \psi(q) \int_0^a Z_r^0 dr} e^{-\int_0^a (\tilde{K}_h^a + \lambda) Z_h^{\theta'} dh} G(\pi_a(\rho^{\theta'})) \right] \\
&= \mathbf{N}^{\psi_q} \left[ e^{-\int_0^a (\tilde{K}_h^a + \lambda) Z_h^{\theta'} dh} G(\pi_a(\rho^{\theta'})) \right] \\
&= \mathbf{N}^\psi \left[ e^{-\int_0^a (\tilde{K}_h^a + \lambda) Z_h^\theta dh} G(\pi_a(\rho^\theta)) \right] \\
(46) \quad &= \mathbf{N}^\psi \left[ e^{-\int_0^a (\tilde{K}_h^a + \lambda) Z_h^\theta dh} F(\rho^\theta) \right].
\end{aligned}$$

Notice also that, thanks to Girsanov formula,

$$\begin{aligned}
\tilde{K}_h^a &= 2\beta\theta' \mathbf{N}^{\psi_q} \left[ 1 - e^{-\lambda \int_0^{a-h} Z_r dr} \right] + \int_{(0, +\infty)} \pi(du) (e^{-qu} - e^{-\theta u}) \mathbf{E}_u^{\psi_q} \left[ 1 - e^{-\lambda \int_0^{a-h} Z_r dr} \right] \\
&= 2\beta\theta' \mathbf{N}^\psi \left[ 1 - e^{-\lambda \int_0^{a-h} Z_r^q dr} \right] + \int_{(0, +\infty)} \pi(du) (e^{-qu} - e^{-\theta u}) \mathbf{E}_u^\psi \left[ 1 - e^{-\lambda \int_0^{a-h} Z_r^q dr} \right].
\end{aligned}$$

Using Lemma 4.6, we get

$$\begin{aligned}
\lim_{a \rightarrow \infty} \tilde{K}_h^a &= 2\beta\theta' \mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma_q} \right] + \int_{(0, +\infty)} \pi(du) (e^{-qu} - e^{-\theta u}) \mathbf{E}_u^\psi \left[ 1 - e^{-\lambda \sigma_q} \right] \\
&= \psi_\theta(\psi_q^{-1}(\lambda)) - \psi_q(\psi_q^{-1}(\lambda)) \\
&= \psi_\theta(\psi_q^{-1}(\lambda)) - \lambda.
\end{aligned}$$

We deduce from (45) and (46), that

$$\mathbf{N}^\psi [e^{-\lambda \sigma_q} F(\rho^\theta)] = \mathbf{N}^\psi [e^{-\psi_\theta(\psi_q^{-1}(\lambda)) \sigma_\theta} F(\rho^\theta)].$$

Letting then  $\lambda$  goes down to 0, we deduce, with  $\bar{q} = \psi^{-1}(\psi(q))$ , that

$$\mathbf{N}^\psi [\mathbf{1}_{\{\sigma_q < +\infty\}} F(\rho^\theta)] = \mathbf{N}^\psi [e^{-\psi_\theta(\bar{q}-q) \sigma_\theta} F(\rho^\theta)].$$

□

The next theorem gives the distribution of the explosion time  $A$  under the measure  $\mathbf{N}^\psi$ .

**Theorem 6.5.** *We have, for all  $\theta \in [\theta_\infty, +\infty)$ ,*

$$(47) \quad \mathbf{N}^\psi [A > \theta] = \bar{\theta} - \theta$$

and

$$\mathbf{N}^\psi [A = \theta_\infty] = \begin{cases} 0 & \text{if } \theta_\infty \notin \Theta', \\ +\infty & \text{if } \theta_\infty \in \Theta'. \end{cases}$$

*Proof.* We have for all  $\theta > \theta_\infty$

$$\begin{aligned}
\mathbf{N}^\psi [A > \theta] &= \mathbf{N}^\psi [\sigma_\theta = +\infty] \\
&= \mathbf{N}^{\psi_\theta} [\sigma = +\infty] \\
&= \lim_{\lambda \rightarrow 0} \mathbf{N}^{\psi_\theta} [1 - e^{-\lambda \sigma}] \\
&= \lim_{\lambda \rightarrow 0} \psi_\theta^{-1}(\lambda) \\
&= \psi_\theta^{-1}(0),
\end{aligned}$$

where we used (4.6) for the fourth equality. We get, for  $t > 0$ ,

$$\psi_\theta(t) = 0 \iff \psi(t + \theta) = \psi(\theta) \iff t + \theta = \bar{\theta},$$

and thus  $\psi_{\theta}^{-1}(0) = \bar{\theta} - \theta$ , which gives the first part of the theorem for  $\theta > \theta_{\infty}$ . Making  $\theta$  decrease to  $\theta_{\infty}$  gives the result for  $\theta_{\infty}$ .

For the second part of the theorem, we apply the second assertion of Lemma 6.4 with  $\theta = 0$ . We have, for every  $q \leq 0$ ,

$$\mathbf{N}^{\psi}[\sigma_q < +\infty | \rho] = e^{-\sigma\psi(\bar{q}-q)}.$$

Then, we have

$$\begin{aligned} \mathbf{N}^{\psi}[A = \theta_{\infty} | \rho] &= \mathbf{N}^{\psi}[\forall q > \theta_{\infty}, \sigma_q < +\infty | \rho] \\ &= \lim_{q \rightarrow \theta_{\infty}} \mathbf{N}^{\psi}[\sigma_q < +\infty | \rho] \\ &= \lim_{q \rightarrow \theta_{\infty}} e^{-\sigma\psi(\bar{q}-q)} \\ &= \begin{cases} 0 & \text{if } \theta_{\infty} \notin \Theta' \\ e^{-\sigma\psi(\bar{\theta}_{\infty}-\theta_{\infty})} & \text{if } \theta_{\infty} \in \Theta', \text{ with } \psi(\bar{\theta}_{\infty} - \theta_{\infty}) < +\infty, \end{cases} \end{aligned}$$

where the last equality is a consequence of Lemma 5.8. Then, integrating with respect to  $\rho$  gives the theorem.  $\square$

*Remark 6.6.* Since  $\psi^{-1}$  is smooth, we deduce that the mapping  $q \mapsto \bar{q}$  is differentiable with

$$\frac{d\bar{q}}{dq} = \frac{\psi'(q)}{\psi'(\bar{q})}.$$

Thus, when  $\theta_{\infty} \notin \Theta$ , we have that the law of  $A$  under  $\mathbf{N}^{\psi}$  has a density with respect to the Lebesgue measure on  $\mathbb{R}$  given by

$$\mathbf{1}_{\{r \in (\theta_{\infty}, 0)\}} \left( 1 - \frac{\psi'(r)}{\psi'(\bar{r})} \right).$$

**Theorem 6.7.** (i) *Let  $\theta \in (\theta_{\infty}, 0)$ . Under  $\mathbf{N}^{\psi}$ , conditionally on  $\{A = \theta\}$ , we have for any non-negative measurable function  $F$*

$$(48) \quad \mathbf{N}^{\psi}[F(\mathcal{R}_A) | A = \theta] = \psi'(\bar{\theta}) \mathbf{N}^{\psi}[F(\mathcal{R}_0) \sigma_0 e^{-\psi(\theta)\sigma_0}],$$

and the law of  $\sigma_A$  is given by: for  $\lambda \geq 0$

$$\mathbf{N}^{\psi}[e^{-\lambda\sigma_A} | A = \theta] = \frac{\psi'(\bar{\theta})}{\psi'(\psi^{-1}(\lambda + \psi(\theta)))}.$$

(ii) *If  $\theta_{\infty} \in \Theta$ , we have for any non-negative measurable function  $F$*

$$(49) \quad \mathbf{N}^{\psi}[F(\mathcal{R}_A) \mathbf{1}_{\{A=\theta_{\infty}\}}] = \mathbf{N}^{\psi_{\bar{\theta}_{\infty}}}[F(\mathcal{R}_0)].$$

In particular, the law of  $\sigma_A$  on the event  $\{A = \theta_{\infty}\}$  is given by

$$\mathbf{N}^{\psi}\left[\left(1 - e^{-\lambda\sigma_A}\right) \mathbf{1}_{\{A=\theta_{\infty}\}}\right] = \psi^{-1}(\lambda + \psi(\theta_{\infty})) - \bar{\theta}_{\infty}.$$

*Proof.* Let  $F$  be a non-negative measurable function defined on the state space of  $\mathcal{R}_0$ . Using Lemma 6.4, we get for every  $\theta_{\infty} < q \leq \theta < 0$ ,

$$\begin{aligned} \mathbf{N}^{\psi}[F(\mathcal{R}_\theta) \mathbf{1}_{\{A>q\}}] &= \mathbf{N}^{\psi}[F(\mathcal{R}_\theta) \mathbf{1}_{\{\sigma_q = +\infty\}}] \\ &= \mathbf{N}^{\psi}[F(\mathcal{R}_\theta) \mathbf{N}^{\psi}[\sigma_q = +\infty | \rho^\theta]] \\ &= \mathbf{N}^{\psi}[F(\mathcal{R}_\theta) (1 - e^{-\sigma_\theta \psi^\theta(\bar{q}-q)})] \\ &= \mathbf{N}^{\psi}[F(\mathcal{R}_\theta) (1 - e^{-\sigma_\theta (\psi(\theta + \bar{q} - q) - \psi(\theta))})]. \end{aligned}$$

Thus, we get that the mapping

$$q \mapsto \mathbf{N}^\psi [F(\mathcal{R}_\theta) \mathbf{1}_{\{A > q\}}]$$

is differentiable if it is finite. As  $d\bar{q}/dq = \psi'(q)/\psi'(\bar{q})$ , we get

$$\begin{aligned} \frac{d}{dq} \mathbf{N}^\psi [F(\mathcal{R}_\theta) \mathbf{1}_{\{A > q\}}] &= \psi'(\bar{q} - q + \theta) \left( \frac{d\bar{q}}{dq} - 1 \right) \mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \sigma_0 e^{-\sigma_0(\psi(\bar{q}-q+\theta)-\psi(\theta))}] \\ &= \psi'(\bar{q} - q + \theta) \frac{\psi'(q) - \psi'(\bar{q})}{\psi'(\bar{q})} \mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \sigma_0 e^{-\sigma_0(\psi(\bar{q}-q+\theta)-\psi(\theta))}]. \end{aligned}$$

Finally, using that  $\sigma$  is right continuous, we have

$$\begin{aligned} \frac{\mathbf{N}^\psi [F(\mathcal{R}_A), A \in d\theta]}{d\theta} &= -\frac{d}{dq} \left( \mathbf{N}^\psi [F(\mathcal{R}_\theta) \mathbf{1}_{\{A > q\}}] \right) \Big|_{q=\theta} \\ &= (\psi'(\bar{\theta}) - \psi'(\theta)) \mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \sigma_0 \mathbf{1}_{\{\sigma_0 < +\infty\}}]. \end{aligned}$$

We deduce from Lemma 6.2 that

$$\mathbf{N}^\psi [F(\mathcal{R}_A) | A = \theta] = \frac{\mathbf{N}^{\psi_\theta} [F(\mathcal{R}_0) \sigma_0 \mathbf{1}_{\{\sigma_0 < +\infty\}}]}{\mathbf{N}^{\psi_\theta} [\sigma_0 \mathbf{1}_{\{\sigma_0 < +\infty\}}]} = \frac{\mathbf{N}^\psi [F(\mathcal{R}_0) \sigma_0 e^{-\psi(\theta)\sigma_0}]}{\mathbf{N}^\psi [\sigma_0 e^{-\psi(\theta)\sigma_0}]}.$$

This proves (48) but for the normalizing constant. It also implies that

$$\mathbf{N}^\psi [e^{-\lambda\sigma_A} | A = \theta] = \frac{\mathbf{N}^{\psi_\theta} [\sigma e^{-\lambda\sigma}]}{\mathbf{N}^{\psi_\theta} [\sigma \mathbf{1}_{\{\sigma < +\infty\}}]}.$$

Notice that  $\psi_\theta^{-1}(r) = \psi^{-1}(r + \psi(\theta)) - \theta$  for  $r \geq 0$ . We get from Lemma 4.6 that, for  $r \geq 0$ ,

$$\mathbf{N}^{\psi_\theta} [\sigma e^{-r\sigma}] = \frac{d}{dr} \mathbf{N}^{\psi_\theta} [1 - e^{-r\sigma}] = (\psi_\theta^{-1})'(r) = \frac{1}{\psi'(\psi^{-1}(r + \psi(\theta)))}.$$

In particular, we deduce the value of the normalizing constant:

$$\mathbf{N}^\psi [\sigma_0 e^{-\psi(\theta)\sigma_0}] = \mathbf{N}^{\psi_\theta} [\sigma \mathbf{1}_{\{\sigma < +\infty\}}] = 1/\psi'(\bar{\theta}).$$

We also get

$$\mathbf{N}^\psi [e^{-\lambda\sigma_A} | A = \theta] = \frac{\psi'(\bar{\theta})}{\psi'(\psi^{-1}(\lambda + \psi(\theta)))}.$$

This ends the proof of the first part.

For the second part of the theorem, we consider the case  $\theta_\infty \in \Theta$ . Let us first remark that, since the process  $(\sigma_\theta, \theta \in \Theta)$  is continuous in probability, we have

$$\{A = \theta_\infty\} = \{\sigma_{\theta_\infty} < +\infty\}.$$

We then apply Girsanov formula (44) twice to get

$$\begin{aligned} \mathbf{N}^\psi [F(\mathcal{R}_A) \mathbf{1}_{\{A = \theta_\infty\}}] &= \mathbf{N}^\psi [F(\mathcal{R}_{\theta_\infty}) \mathbf{1}_{\{\sigma_{\theta_\infty} < +\infty\}}] \\ &= \mathbf{N}^\psi [F(\mathcal{R}_0) e^{-\psi(\theta_\infty)\sigma_0}] \\ &= \mathbf{N}^\psi [F(\mathcal{R}_0) e^{-\psi(\bar{\theta}_\infty)\sigma_0}] \\ &= \mathbf{N}^\psi [F(\mathcal{R}_{\bar{\theta}_\infty}) \mathbf{1}_{\{\sigma_{\bar{\theta}_\infty} < +\infty\}}] \\ &= \mathbf{N}^{\psi_{\bar{\theta}_\infty}} [F(\mathcal{R}_0)], \end{aligned}$$

where we used for the last equality that  $\sigma_{\bar{\theta}_\infty} < +\infty$   $\mathbf{N}^\psi$ -a.e. and (44).

For  $F(\mathcal{R}) = 1 - e^{-\lambda\sigma}$ , we obtain

$$\begin{aligned} \mathbf{N}^\psi \left[ \left( 1 - e^{-\lambda\sigma_A} \right) \mathbf{1}_{\{A=\theta_\infty\}} \right] &= \mathbf{N}^{\psi_{\bar{\theta}_\infty}} \left[ 1 - e^{-\lambda\sigma_0} \right] \\ &= \psi_{\bar{\theta}_\infty}^{-1}(\lambda) \\ &= \psi^{-1}(\lambda + \psi(\bar{\theta}_\infty)) - \bar{\theta}_\infty. \end{aligned}$$

□

We deduce the next corollary from (48).

**Corollary 6.8.** *Let  $\theta_\infty < \theta < 0$ . The distribution of  $\mathcal{R}_A = (\rho^{A+\gamma}, \gamma \geq 0)$  conditionally on  $\{\sigma_A = r, A = \theta\}$  does not depend on  $\theta$ .*

## 7. PRUNING OF AN INFINITE TREE

**7.1. Bismut decomposition of a Lévy tree.** Let  $\psi$  be a (sub)critical branching mechanism. Recall definition of mark process  $M^{(mark)}$  of Section 5.1.3. For a marked exploration process  $(\rho, M^{(mark)})$  recall that  $\eta$  is defined by (28) and notice that  $(\eta_{(\sigma-t)-}, M_{\sigma-t}^{(mark)}, t \in [0, \sigma])$  is distributed as  $(\rho, M^{(mark)})$  under the excursion measure thanks to Corollary 3.1.6 in [16] and definition of  $M^{(mark)}$ .

We recall that the family of pruned exploration processes  $\mathcal{R} = (\rho^\theta, \theta \geq 0)$  is constructed from the exploration process  $\rho$  (which is equal to  $\rho^0$ ) and the measure-valued process  $M^{(mark)}$ .

Let  $T \geq 0$ . We define under  $\mathbf{N}^\psi$  the processes  $(\rho^{T \rightarrow}, M^{(mark), T \rightarrow})$  and  $(\rho^{\leftarrow T}, M^{(mark), \leftarrow T})$  by: for every  $t \geq 0$ ,

$$\begin{aligned} (\rho_t^{T \rightarrow}, M_t^{(mark), T \rightarrow}) &= (\rho_{(T+t) \wedge \sigma}, M_{(T+t) \wedge \sigma}^{(mark)}), \\ (\rho_t^{\leftarrow T}, M_t^{(mark), \leftarrow T}) &= (\eta_{(T-t) \vee 0}, M_{(T-t) \vee 0}^{(mark)}) \end{aligned}$$

where  $\rho$  is the canonical exploration process and  $\eta$  its dual process.

Bismut decomposition describes in terms of Poisson point process the former processes when  $T$  is “uniformly distributed” on  $[0, \sigma]$ .

First we must extend the definition of the measure  $\mathbb{M}^\psi(d\mu, d\nu)$  of (29) and (30) to get into account the marks. Let

$$\mathcal{N}(dx, d\ell, du) = \sum_{i \in I} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

be a Poisson point measure with intensity

$$dx \ell \pi(d\ell) \mathbf{1}_{[0,1]}(u) du.$$

Conditionally on  $\mathcal{N}$ , let  $(T_i, i \in I)$  be a family of independent exponential random variables of respective parameter  $\ell_i$ . Finally, let  $\tilde{\mathcal{N}}(dk, db) = \sum_{j \in J} \delta_{(k_j, b_j)}(dk, db)$  be an independent Poisson point measure on  $[0, +\infty)^2$  with intensity  $2\beta dk db$ . We then define the spine  $(\mu_a, \nu_a, m_a)$  which are three measures given by

$$\begin{aligned} \mu_a(dx) &= \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i) u_i \ell_i \delta_{x_i}(dx) + \mathbf{1}_{[0,a]}(x) \beta dx, \\ \nu_a(dx) &= \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i) (1 - u_i) \ell_i \delta_{x_i}(dx) + \mathbf{1}_{[0,a]}(x) \beta dx, \\ m_a(dx, dq) &= \sum_{i \in I} \mathbf{1}_{[0,a]}(x_i) \delta_{x_i}(dx) \delta_{T_i}(dq) + \sum_{j \in J} \mathbf{1}_{[0,a]}(k_j) \delta_{k_j}(dx) \delta_{b_j}(dq). \end{aligned}$$

We denote by  $\tilde{\mathbb{M}}_a^\psi$  the law of the triple  $(\mu_a, \nu_a, m_a)$  and we set  $\tilde{\mathbb{M}}^\psi = \int_0^{+\infty} da e^{-\psi'(0)a} \mathbb{M}_a^\psi$ .

Let us denote by  $\mathbb{P}_{\mu, m}^{\psi, *}$  the law of the pair  $(\rho, M^{(mark)})$  starting from  $(\mu, m)$  where  $\rho$  is an exploration process associated with  $\psi$  and stopped when it first reaches 0. It is easy to adapt Lemma 3.4 of [17] to get the following theorem.

**Theorem 7.1** (Bismut decomposition). *For every non-negative measurable functionals  $F$  and  $G$ ,*

$$(50) \quad \mathbb{N}^\psi \left[ \int_0^\sigma ds F(\rho^{s\rightarrow}, M^{(mark), s\rightarrow}) G(\rho^{\leftarrow s}, M^{(mark), \leftarrow s}) \right] = \int \tilde{\mathbb{M}}^\psi(d\mu, d\nu, dm) \mathbb{E}_{\mu, m}^{\psi, *} [F] \mathbb{E}_{\nu, m}^{\psi, *} [G].$$

Let us now state the Poisson representation of the probability measure  $\mathbb{P}_{\mu, m}^{\psi, *}$ . Let  $(\alpha_i, \beta_i)_{i \in I}$  be the excursion intervals of the total mass process  $(\langle \rho_t, 1 \rangle, t \geq 0)$  above its minimum under  $\mathbb{P}_{\mu, m}^{\psi, *}$ . Let  $(U_i, i \in I)$  be a family of independent random variables, independent of  $\rho$  and uniformly distributed on  $[0, 1]$ . For every  $i \in I$ , we set  $x_i = H_{\alpha_i}$ . Then we define  $u_i$  by

$$u_i = \begin{cases} \rho_{\alpha_i}(\{x_i\}) / \mu(\{x_i\}) & \text{if } \mu(\{x_i\}) > 0 \\ U_i & \text{if } \mu(\{x_i\}) = 0. \end{cases}$$

Finally, we define the measure-valued process  $\rho^i$  by: for every  $t \geq 0$  and every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,

$$\langle \rho_t^i, f \rangle = \int_{(x_i, +\infty)} f(x - x_i) \rho_{(\alpha_i + t) \wedge \beta_i}(dx)$$

and the measure valued-process  $M^{(mark), i}$  by: for every  $t \geq 0$  and every  $f \in \mathcal{B}_+(\mathbb{R}_+^2)$ ,

$$\langle M_t^{(mark), i}, f \rangle = \int_{(x_i, +\infty) \times \mathbb{R}_+} f(x - x_i, \theta) M_{(\alpha_i + t) \wedge \beta_i}^{(mark)}(dx, d\theta).$$

It is easy to adapt Lemma 4.2.4 from [16] to get the following proposition.

**Proposition 7.2.** *The point measure  $\sum_{i \in I} \delta_{(x_i, u_i, \rho^i, M^{(mark), i})}$  is under  $\mathbb{P}_{\mu, m}^{\psi, *}$  a Poisson point measure with intensity*

$$\mu(dx) du \mathbf{1}_{[0, 1]}(u) \mathbb{N}^\psi(d\rho, dM^{(mark)}).$$

**7.2. Reconstruction of the exploration process from a spinal decomposition.** Conversely, given the spinal decomposition of Bismut theorem, we reconstruct the initial exploration process, but we must add the time indices of the excursions at the node (which in the previous Section are called  $u_i$ ). We shall also add the mark process (see its definition (38)).

Let  $\mu$  and  $\nu$  be two finite measures such that  $\text{Supp } \mu = \text{Supp } \nu = [0, H]$  and  $m$  a point measure on  $[0, H] \times \mathbb{R}_+$ . Let  $\{(\rho^i, M^{(mark), i}), i \in J_g\}$  and  $\{(\rho^i, M^{(mark), i}), i \in J_d\}$  be two families of marked exploration processes (see Section 5.1.3). Let  $\{(x_i, u_i), i \in J_g \cup J_d\}$  be a family of non-negative real numbers. The measures  $\mu$  and  $\nu$  must be seen as the measures  $\rho_0^{s\rightarrow}$  and  $\rho_0^{\leftarrow s}$  of Theorem 7.1, the  $x_i$ 's are the heights of the branching points along the chosen branch, the  $\rho^i$ 's are the exploration processes that arise from the decomposition of the processes  $\rho^{s\rightarrow}$  and  $\rho^{\leftarrow s}$  above their minimum, and the  $u_i$ 's are additional features that order the excursions that are attached at the same level. The measure  $m$  and the processes  $M^{(mark), i}$  will allow us to reconstruct the mark process.

For every  $i \in J_g \cup J_d$ , we set  $\sigma^i$  the length of the process  $\rho^i$ . We define

$$(51) \quad \mathcal{L}_g = \sum_{i \in J_g} \sigma^i, \quad \mathcal{L}_d = \sum_{i \in J_d} \sigma^i \quad \text{and} \quad \mathcal{L} = \mathcal{L}_g + \mathcal{L}_d.$$

The variable  $\mathcal{L}$  represents the total length of the excursion whereas  $\mathcal{L}_g$  plays the same role as  $s$  in the left hand-side of (7.1). For every  $i \in J_g$ , we set

$$t^i = \sum_{j \in J_g, x_j < x_i} \sigma^j + \sum_{j \in J_g, x_j = x_i \text{ and } u_j > u_i} \sigma^j,$$

and, for every  $i \in J_d$ , we set

$$t^i = \mathcal{L}_g + \sum_{j \in J_d, x_j > x_i} \sigma^j + \sum_{j \in J_d, x_j = x_i \text{ and } u_j > u_i} \sigma^j,$$

which is the time of the beginning of the excursion  $\rho^i$ .

For every  $t > 0$ , we define the measure  $\rho_t$  by

$$\rho_t(dx) = \begin{cases} \rho_{t-t^i}^i(x_i + dx) + \mu(dx) \mathbf{1}_{[0, x_i]}(x) \\ \quad + (u_i \nu(\{x_i\}) + \mu(\{x_i\})) \delta_{x_i}(dx) & \text{if } t < \mathcal{L}_g, t^i \leq t < t^i + \sigma^i, \\ \mu & \text{if } t = \mathcal{L}_g, \\ \rho_{t-t^i}^i(x_i + dx) \\ \quad + \mu(dx) \mathbf{1}_{[0, x_i]}(x) + u_i \mu(\{x_i\}) \delta_{x_i}(dx) & \text{if } \mathcal{L}_g < t < \mathcal{L}, t^i \leq t < t^i + \sigma^i, \\ 0 & \text{if } t \geq \mathcal{L}. \end{cases}$$

We also define the mark process  $M^{(mark)}(dx, dv)$  by

$$\begin{cases} M_{t-t^i}^{(mark), i}(x_i + dx, dv) + m(dx, dv) \mathbf{1}_{[0, x_i]}(x) & \text{if } t < \mathcal{L}_g \text{ or } \mathcal{L}_g < t < \mathcal{L}, t^i \leq t < t^i + \sigma^i, \\ m & \text{if } t = \mathcal{L}_g, \\ 0 & \text{if } t \geq \mathcal{L}. \end{cases}$$

We say that the process  $(\rho, M^{(mark)}) = ((\rho_t, M_t^{(mark)}), t \geq 0)$  is the marked exploration process associated with the family

$$(52) \quad \mathcal{G} = (\mu, \nu, m, (x_i, u_i, (\rho^i, M^{(mark), i}), i \in J_g), (x_i, u_i, (\rho^i, M^{(mark), i}), i \in J_d)).$$

From Bismut decomposition, Theorem 7.1, Proposition 7.2 and the construction of the mark process, Section 5.1.3, we get the following reconstruction corollary.

**Corollary 7.3.** *Let  $\psi$  be a (sub)critical branching mechanism. Let  $(\mu, \nu, m)$  be distributed according to  $\tilde{\mathbb{M}}^\psi$ . Let  $\sum_{i \in J_g} \delta_{(x_i, u_i, \rho^i, M^{(mark), i})}$  and  $\sum_{i \in J_d} \delta_{(x_i, u_i, \rho^i, M^{(mark), i})}$  be conditionally on  $(\mu, \nu, m)$  independent Poisson point measures with respective intensity*

$$\mu(dx) \mathbf{1}_{[0, 1]}(u) du \mathbb{N}^\psi(d\rho, dM^{(mark)}) \quad \text{and} \quad \nu(dx) \mathbf{1}_{[0, 1]}(u) du \mathbb{N}^\psi(d\rho, dM^{(mark)}).$$

*Then the marked exploration process associated with the family  $\mathcal{G}$  given by (52) is distributed as  $(\rho, M^{(mark)})$  under  $\mathbb{N}^\psi[\sigma d(\rho, M)]$ .*

*Remark 7.4.* If we start with an exploration process  $\rho$ , pick  $s$  at random (conditionally on  $\rho$ ) on  $[0, \sigma]$ , then the decomposition of  $\rho^{s \rightarrow}$  and  $\rho^{\leftarrow s}$  as excursions above their minimum gives a family  $\mathcal{G}$ . The exploration process  $\tilde{\rho}$  associated with  $\mathcal{G}$  given by the previous construction is not  $\rho$ . Indeed, each excursion of  $\tilde{\rho}$  ‘‘on the left’’ of  $s$  is time-reversed with respect to those of  $\rho$ . However, the trees coded by  $\rho$  and  $\tilde{\rho}$  are the same.

We can also reconstruct the pruned exploration process by pruning  $\mathcal{G}$ . Let  $\theta > 0$ . We define the lowest mark lying on the spine as

$$(53) \quad \xi_\theta = \sup\{x; m([0, x] \times [0, \theta]) = 0\}.$$

We set  $\mu^\theta = \mu \mathbf{1}_{[0, \xi_\theta]}$ ,  $\nu^\theta = \nu \mathbf{1}_{[0, \xi_\theta]}$ ,  $m^\theta(dx, dq) = m(dx, \theta + dq) \mathbf{1}_{[0, \xi_\theta]}(x)$ , for  $\delta \in \{g, d\}$   $J_\delta^\theta = \{i \in J_\delta; x_i < \xi_\theta\}$  and

$$(54) \quad \mathcal{G}_\theta = (\mu^\theta, \nu^\theta, m^\theta, (x_i, u_i, \Lambda_\theta(\rho^i, M^{(mark), i}), i \in J_g^\theta), (x_i, u_i, \Lambda_\theta(\rho^i, M^{(mark), i}), i \in J_d^\theta)),$$

where the pruning operator  $\Lambda_\theta$  is defined in (40).

**Proposition 7.5.** *Under the hypothesis of Corollary 7.3, let  $(\rho^\theta, M^{(mark), \theta})$  be the marked exploration process associated with the family  $\mathcal{G}_\theta$  given by (54). The process  $(\rho^\theta, \theta \geq 0)$  is distributed as  $\mathcal{R}_0$  under  $\mathbf{N}^\psi[\sigma_0 d\mathcal{R}]$ .*

*Proof.* Let us remark that, by construction,  $(\rho^\theta, M^{(mark), \theta}) = \Lambda_\theta(\rho, M^{(mark)})$ . The proposition now follows from Corollary 7.3.  $\square$

**7.3. The infinite tree and its pruning.** Let  $\psi$  be a critical branching mechanism.

In [3], another spinal decomposition, called Williams decomposition, of the exploration process is given where the spine is not chosen randomly as for Bismut decomposition but corresponds to the highest branch of the tree. Inspired by this decomposition we build a marked continuum random tree with an infinite spine. Intuitively, if the CRT dies in finite time (which corresponds to the case  $H$  continuous) this infinite CRT can be seen as the CRT conditioned to non-extinction.

Let

$$\mathcal{N}(dx, d\ell, du) = \sum_{i \in I} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

be a Poisson point measure with intensity

$$dx \ell \pi(d\ell) \mathbf{1}_{[0, 1]}(u) du.$$

Conditionally on  $\mathcal{N}$ , let  $(T_i, i \in I)$  be a family of independent exponential random variables of respective parameter  $\ell_i$ . Finally, let  $\tilde{\mathcal{N}}(dk, db) = \sum_{j \in J} \delta_{(k_j, b_j)}(dk, db)$  be an independent Poisson point measure on  $[0, +\infty)^2$  with intensity  $2\beta dk db$ . We define the following random measures:

$$\begin{aligned} \mu^*(dx) &= \sum_{i \in I} u_i \ell_i \delta_{x_i}(dx) + \beta dx, \\ \nu^*(dx) &= \sum_{i \in I} (1 - u_i) \ell_i \delta_{x_i}(dx) + \beta dx, \\ m^*(dx, dq) &= \sum_{i \in I} \delta_{x_i}(dx) \delta_{T_i}(dq) + \sum_{j \in J} \delta_{k_j}(dx) \delta_{b_j}(dq). \end{aligned}$$

The measure  $(\mu^*, \nu^*, m^*)$  corresponds to the the measure  $(\mu_a, \nu_a, m_a)$  of Section 7.1 but for an infinite spine. Let  $\sum_{i \in J_g} \delta_{(x_i, u_i, \rho^i, M^{(mark), i})}$  and  $\sum_{i \in J_d} \delta_{(x_i, u_i, \rho^i, M^{(mark), i})}$  be conditionally on  $(\mu^*, \nu^*, m^*)$  independent Poisson point measures with intensity

$$\nu^*(dx) \mathbf{1}_{[0, 1]}(u) du \mathbf{N}^\psi(d\rho, dM^{(mark)}) \quad \text{and} \quad \mu^*(dx) \mathbf{1}_{[0, 1]}(u) du \mathbf{N}^\psi(d\rho, dM^{(mark)}).$$

We set

$$\mathcal{G}^* = (\mu^*, \nu^*, m^*, (x_i, u_i, (\rho^i, M^{(mark), i}), i \in J_g), (x_i, u_i, (\rho^i, M^{(mark), i}), i \in J_d)),$$

which describes the decomposition of an infinite marked tree as marked sub-trees that are attached along its infinite spine. Let  $\theta > 0$ . Following the end of Section 7.2, we now extend the pruning procedure to this infinite tree by letting  $\mathcal{G}_\theta^*$  be constructed from  $\mathcal{G}^*$  as  $\mathcal{G}_\theta$  given by (54) from  $\mathcal{G}$  given by (52):

$$\begin{aligned}\xi_\theta^* &= \sup\{x; m^*([0, x] \times [0, \theta]) = 0\}, & J_\delta^\theta &= \{i \in J_\delta; x_i < \xi_\theta^*\} \quad \text{for } \delta \in \{g, d\}, \\ \mu^{*,\theta} &= \mu^* \mathbf{1}_{[0, \xi_\theta^*]}, & \nu^{*,\theta} &= \nu^* \mathbf{1}_{[0, \xi_\theta^*]}, & m^{*,\theta}(dx, dq) &= m^*(dx, \theta + dq) \mathbf{1}_{[0, \xi_\theta^*]}(x), \\ \mathcal{G}_\theta^* &= (\mu^{*,\theta}, \nu^{*,\theta}, m^{*,\theta}, (x_i, u_i, \Lambda_\theta(\rho^i, M^{(\text{mark}),i}), i \in J_g^\theta), (x_i, u_i, \Lambda_\theta(\rho^i, M^{(\text{mark}),i}), i \in J_d^\theta)).\end{aligned}$$

We have the following lemma.

**Lemma 7.6.** *Let  $\theta > 0$ . The probability distribution of the spine  $(\mu^{*,\theta}, \nu^{*,\theta}, m^{*,\theta})$  is  $\psi'(\theta) \tilde{\mathbb{M}}^{\psi_\theta}$ .*

*Proof.* As  $\psi$  is critical, we deduce from (22) that

$$\psi'(\theta) = 2\beta\theta + \int_{(0, +\infty)} (1 - e^{-\theta\ell}) \ell \pi(d\ell).$$

We deduce from the theory of marked Poisson point measure that

$$\mathcal{N}^\theta(dx, d\ell, du) = \sum_{i \in I} \mathbf{1}_{\{T_i > \theta\}} \delta_{(x_i, \ell_i, u_i)}(dx, d\ell, du)$$

is a Poisson point measure with intensity  $dx \ell e^{-\theta\ell} \pi(d\ell) \mathbf{1}_{[0,1]}(u) du$ . Notice then that  $\xi_\theta^*$  is the minimum of  $T_1 = \inf\{x_i; T_i \leq \theta, i \in I\}$  and  $T_2 = \inf\{k_j; b_j \leq \theta, j \in J\}$ , which are two independent exponential random variable, which are also independent of  $\mathcal{N}^\theta$ . The exponential distribution of  $T_1$  has parameter  $\int_{(0, +\infty)} (1 - e^{-\theta\ell}) \ell \pi(d\ell)$  and the exponential distribution of  $T_2$  has parameter  $2\beta\theta$ . Thus  $\xi_\theta^*$  has an exponential distribution with parameter  $\psi'(\theta)$ . Since  $\xi_\theta^*$  is independent of  $\mathcal{N}^\theta$  we get the result.  $\square$

Let  $(\rho^{\theta,*}, M^{(\text{mark}),\theta,*})$  be the marked exploration process associated with  $\mathcal{G}_\theta^*$ . We set  $\mathcal{R}_\theta^* = (\rho^{\theta+q,*}, q \geq 0)$  and denote by  $\mathbf{E}^\psi$  its law. The next proposition tells us that  $\mathcal{R}_\theta^*$  under  $\mathbf{E}^\psi$  is, up to a normalizing constant, the size biased “distribution” of  $\mathcal{R}_\theta$  under  $\mathbf{N}^\psi$ .

**Proposition 7.7.** *Let  $\psi$  be a critical branching mechanism. For every positive measurable functional  $F$  and every  $\theta > 0$ , we have*

$$\psi'(\theta) \mathbf{N}^\psi[\sigma_\theta F(\mathcal{R}_\theta)] = \mathbf{E}^\psi[F(\mathcal{R}_\theta^*)].$$

*Proof.* Let  $F$  be a positive measurable functional. As  $\mathcal{R}$  is constructed from  $(\rho, M^{(\text{mark})})$ , there exists a positive measurable functional  $G$  such that

$$F(\mathcal{R}) = G(\rho, M^{(\text{mark})}).$$

Moreover, there exists an other positive functional  $\tilde{G}$  such that, for every  $s \geq 0$ ,

$$G(\rho, M^{(\text{mark})}) = \tilde{G}((\rho^{s \rightarrow}, M^{(\text{mark}),s \rightarrow}), (\rho^{\leftarrow s}, M^{(\text{mark}),\leftarrow s})).$$

Then by Bismut decomposition, we have

$$\begin{aligned}\psi'(\theta) \mathbf{N}^\psi[\sigma_\theta F(\mathcal{R}_\theta)] &= \psi'(\theta) \mathbf{N}^{\psi_\theta}[\sigma F(\mathcal{R})] \\ &= \psi'(\theta) \mathbf{N}^{\psi_\theta} \left[ \int_0^\sigma ds \tilde{G}((\rho^{s \rightarrow}, M^{(\text{mark}),s \rightarrow}), (\rho^{\leftarrow s}, M^{(\text{mark}),\leftarrow s})) \right] \\ &= \int \psi'(\theta) \tilde{\mathbb{M}}^{\psi_\theta}(d\mu, d\nu, dm) \mathbb{E}_{\mu,m}^{\psi_\theta,*} \otimes \mathbb{E}_{\nu,m}^{\psi_\theta,*}[\tilde{G}].\end{aligned}$$

Then we conclude using Lemma 7.6 and the fact that  $\mathbb{N}^{\psi_\theta}(d\rho, M^{(mark)})$  is the distribution of  $\Lambda_\theta(\rho, M^{(mark)})$  under  $\mathbb{N}^\psi(d\rho, dM^{(mark)})$ .  $\square$

## 8. DISTRIBUTION IDENTITY

Let  $\psi$  be a critical branching mechanism with parameter  $(\alpha, \beta, \pi)$ . We assume that  $\theta_\infty < 0$ . Recall  $\mathcal{R} = (\mathcal{R}_\theta, \theta \in \Theta)$  is defined in Section 6 and  $\mathcal{R}_\theta^*$  in Section 7.3.

**Theorem 8.1.** *Let  $\theta \in (\theta_\infty, 0)$ . Conditionally on  $\{A = \theta\}$ ,  $\mathcal{R}_A$  is distributed as  $\mathcal{R}_\theta^*$ .*

*Proof.* Let  $F$  be a non-negative measurable function defined on  $\mathcal{W}$ . We have, for  $\theta < 0$ ,

$$\begin{aligned} \mathbf{N}^\psi[F(\mathcal{R}_A)|A = \theta] &= \psi'(\bar{\theta})\mathbf{N}^\psi[F(\mathcal{R}_0)\sigma_0 e^{-\psi(\theta)\sigma_0}] \\ &= \psi'(\bar{\theta})\mathbf{N}^{\psi_{\bar{\theta}}}[\sigma_0 F(\mathcal{R}_0)] \\ &= \psi'(\bar{\theta})\mathbf{N}^\psi[\sigma_{\bar{\theta}} F(\mathcal{R}_{\bar{\theta}})] \\ &= \mathbf{E}^\psi[F(\mathcal{R}_{\bar{\theta}}^*)], \end{aligned}$$

where we used (48) for the first equality, Girsanov formula (44) (with  $\theta$  replaced by  $\bar{\theta}$ ) for the second, the invariance of the distribution of  $\mathcal{R}$  by the shift for the third and Proposition 7.7 for the last one.  $\square$

If  $u \in (0, \bar{\theta}_\infty)$ , let  $\bar{u}$  be the unique negative real number such that

$$\bar{\bar{u}} = u.$$

We deduce from Theorem 6.5 and Remark 6.6 the following corollary.

**Corollary 8.2.** *Let us suppose that  $\theta_\infty \notin \Theta$ .*

*Let  $U$  be a positive “random” variable with (non-negative) “density” w.r.t. the Lebesgue measure given by*

$$\left(1 - \frac{\psi'(r)}{\psi'(\bar{r})}\right) \mathbf{1}_{\{r \in (0, \bar{\theta}_\infty)\}}.$$

*Assume that  $U$  is independent of  $\mathcal{G}^*$ . Then  $\mathcal{R}_A$  is distributed under  $\mathbf{N}^\psi$  as  $\mathcal{R}_U^*$ .*

This corollary can be viewed as a continuous analogue of Proposition 26 of [10].

## 9. THE QUADRATIC CASE

We consider  $\psi(\lambda) = \beta\lambda^2$  for some  $\beta > 0$ . We have  $\Theta = \Theta' = \mathbb{R}$ , see definition in Section 5.3, and  $\psi_\theta(\lambda) = \beta\lambda^2 + 2\theta\lambda$ . Recall  $\bar{\theta}$  is defined by (43). So we have  $\bar{\theta} = |\theta|$ . From Theorem 6.5, we get  $\mathbf{N}^\psi[A \geq \theta] = \bar{\theta} - \theta = 2|\theta|$ . Thus under  $\mathbf{N}^\psi$ , the explosion time  $A$  is distributed as 2 times the Lebesgue measure on  $(-\infty, 0)$ . We deduce from Theorem 6.7 the Laplace transform of the total mass of the CRT before explosion: for  $\lambda \geq 0$ ,

$$\mathbf{N}^\psi[e^{-\lambda\sigma_A} | A = \theta] = \frac{\sqrt{\beta\theta^2}}{\sqrt{\lambda + \beta\theta^2}}.$$

In particular the distribution of  $\sigma_A$  conditionally on  $\{A = \theta\}$  is the gamma distribution with parameter  $(\beta\theta^2, 1/2)$ .

Very similar computations as those in the proof of Theorem 6.7 yield that for all  $s, t \geq 0$ ,  $\theta < 0$ ,  $\lambda, \kappa \geq 0$

$$(55) \quad \mathbf{N}^\psi[e^{-\lambda\sigma_{A+s} - \kappa\sigma_{A+s+t}} | A = \theta] = \frac{\sqrt{\beta(|\theta| + s)^2}}{\sqrt{\lambda + \beta(|\theta| + s)^2}} \frac{\sqrt{\beta t^2} + \sqrt{\lambda + \beta(|\theta| + s)^2}}{\sqrt{\kappa + (\sqrt{\beta t^2} + \sqrt{\lambda + \beta(|\theta| + s)^2})^2}}.$$

We denote by  $\sigma_\theta^*$  the total mass or length (see definition (51) of  $\mathcal{L}$ ) of the pruned infinite tree  $\mathcal{G}_\theta^*$ . Notice that, thanks to Proposition 7.7,  $\sigma_\theta^*$  has the size biased distribution of  $\sigma_\theta$  (the total mass of the CRT with branching mechanism  $\psi_\theta$ ) under  $\mathbf{N}^\psi$ . More precisely, we have for any non-negative measurable function: for  $\theta > 0$ ,

$$(56) \quad 2\beta\theta\mathbf{N}^\psi[\sigma_\theta F(\sigma_{\theta+q}, q \geq 0)] = \mathbf{E}^\psi[F(\sigma_{\theta+q}^*, q \geq 0)].$$

As the process  $\Sigma = (\sigma_\theta, \theta \in \mathbb{R})$  is Markov, we get that  $\Sigma^* = (\sigma_\theta^*, \theta \geq 0)$  is Markov. Notice that a.s.  $\sigma_0^* = +\infty$ . Direct computations or using (55) and Theorem 8.1 yield that for all  $\theta, q, \lambda, \kappa \geq 0$

$$\mathbf{E}^\psi[e^{-\lambda\sigma_\theta^* - \kappa\sigma_{\theta+q}^*}] = \frac{\sqrt{\beta\theta^2}}{\sqrt{\lambda + \beta\theta^2}} \frac{\sqrt{\beta q^2} + \sqrt{\lambda + \beta\theta^2}}{\sqrt{\kappa + (\sqrt{\beta q^2} + \sqrt{\lambda + \beta\theta^2})^2}}.$$

Let  $\tau = (\tau_\theta, \theta \geq 0)$  be the first passage process of a standard Brownian motion  $(B_u, u \geq 0)$ :  $\tau_\theta = \inf\{u \geq 0, B_u \geq \theta\}$ . It is a stable subordinator with index  $1/2$ , and more precisely with no drift, no killing, and Lévy measure  $(2\pi x^3)^{-1/2} dx$  on  $(0, \infty)$ : for  $\lambda \geq 0$ ,  $\mathbb{E}[e^{-\lambda\tau_\theta}] = e^{-\theta\sqrt{2\lambda}}$ . The distribution of  $\tau_\theta$  has density

$$\frac{\theta}{\sqrt{2\pi x^3}} e^{-\theta^2/2x} \mathbf{1}_{\{x>0\}}.$$

We get the following result.

**Proposition 9.1.** *We have:*

- Under  $\mathbf{E}^\psi$ ,  $(2\beta\sigma_\theta^*, \theta \geq 0)$  is distributed as  $(1/\tau_\theta, \theta \geq 0)$ .
- Under  $\mathbf{N}^\psi$ ,  $(2\beta\sigma_{A+\theta}, \theta \geq 0)$  is distributed as  $(1/(V + \tau_\theta), \theta \geq 0)$  where  $V$  is independent of  $\tau$  and its “distribution” has density w.r.t. the Lebesgue measure given by  $\sqrt{2/(\pi v)} \mathbf{1}_{\{v>0\}}$ .

The proof of this result is postponed to the end of this Section.

Notice that (44) implies that for  $\theta \geq 0$ ,

$$\mathbf{N}^\psi \left[ F(\sigma_q, q \geq 0) e^{-\psi(\theta)\sigma_0} \right] = \mathbf{N}^\psi [F(\sigma_{q+\theta}, q \geq 0)].$$

In particular, we deduce from this, (56) and the fact that  $\tau$  is a process with independent and stationary increments the following result (notice that the size bias effect vanish, as we condition by  $\sigma_0 = 1$ ).

**Corollary 9.2.** *Let  $\beta = 1/2$ . Conditionally on  $\sigma_0 = 1$ , we have that  $(\sigma_\theta, \theta \geq 0)$  is under the excursion measure  $\mathbf{N}^\psi$  distributed as  $(1/(1 + \tau_\theta), \theta \geq 0)$ .*

We thus recover a well-known result from Aldous and Pitman [9] on the size process of a tagged fragment for a self-similar fragmentation (see [13]) with index  $1/2$ , no erosion and binary dislocation measure  $\nu(s_1 \in dx) = (2\pi x^3(1-x)^3)^{-1/2} \mathbf{1}_{\{x \in (0, 1/2)\}} dx$ , which correspond to the fragmentation of the CRT, see also the end of [12], [5] or [14].

*Proof of Proposition 9.1.* Let  $\lambda, \kappa, \theta, q$  be positive. As we didn't find any reference for the computation of

$$I = \mathbb{E}[e^{-\lambda/\tau_\theta - \kappa/\tau_{\theta+q}}],$$

we shall give it here. Using that  $\tau$  is a subordinator, we have

$$I = \mathbb{E}[e^{-\lambda/\tau_\theta - \kappa/(\tau_\theta + \tau'_q)}].$$

where  $\tau'$  is an independent copy of  $\tau$ . We set  $p = \sqrt{2\lambda + \theta^2}$  and  $J = 2\pi \frac{p}{\theta} I$ . We get

$$\begin{aligned}
J &= 2\pi \frac{p}{\theta} \frac{\theta q}{2\pi} \int_{\mathbb{R}_+^2} e^{-\lambda/x - \kappa/(x+y) - \theta^2/2x - q^2/2y} \frac{dxdy}{(xy)^{3/2}} \\
&= pq \int_{\mathbb{R}_+^2} e^{-\kappa/(x+y) - p^2/2x - q^2/2y} \frac{dxdy}{(xy)^{3/2}} \\
&= pq \int_{\mathbb{R}_+^2} e^{-\kappa zu/(1+u) - zu p^2/2 - zq^2/2} \frac{dzdu}{\sqrt{u}} \\
&= pq \int_{\mathbb{R}_+} \frac{u+1}{u^2 p^2/2 + u(p^2/2 + q^2/2 + \kappa) + q^2/2} \frac{du}{\sqrt{u}} \\
&= 2\gamma \int_{\mathbb{R}_+} \frac{u+1}{u^2 + u(1 + \gamma^2 + \kappa') + \gamma^2} \frac{du}{\sqrt{u}},
\end{aligned}$$

where we used the change of variable  $zu = 1/x$  and  $z = 1/y$  for the third equality,  $\kappa' = 2\kappa/p^2$  and  $\gamma = q/p$  for the last. Let  $a, b$  such that  $a + b = 1 + \gamma^2 + \kappa'$  and  $ab = \gamma^2$ . Notice that

$$\frac{u+1}{u^2 + u(1 + \gamma^2 + \kappa') + \gamma^2} = \frac{a-1}{a-b} \frac{1}{u+a} + \frac{1-b}{a-b} \frac{1}{u+b}.$$

Then we get

$$\begin{aligned}
J &= 2\gamma \frac{a-1}{a-b} \int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+a)} + 2\gamma \frac{1-b}{a-b} \int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+b)} \\
&= 2\gamma \frac{1}{a-b} \left( \frac{a-1}{\sqrt{a}} + \frac{1-b}{\sqrt{b}} \right) \int_{\mathbb{R}_+} \frac{du}{\sqrt{u}(u+1)} \\
&= 2\gamma \frac{\sqrt{ab} + 1}{\sqrt{ab}} \frac{1}{\sqrt{a} + \sqrt{b}} \pi \\
&= 2\pi \frac{\gamma + 1}{\sqrt{(1 + \gamma)^2 + \kappa'}}.
\end{aligned}$$

Therefore, we obtain

$$I = \frac{\theta}{p} \frac{\gamma + 1}{\sqrt{(1 + \gamma)^2 + \kappa'}} = \frac{\theta}{\sqrt{\theta^2 + 2\lambda}} \frac{q + \sqrt{\theta^2 + 2\lambda}}{\sqrt{2\kappa + (q + \sqrt{\theta^2 + 2\lambda})^2}}.$$

We deduce that the two processes  $(2\beta\sigma_\theta^*, \theta \geq 0)$  and  $(1/\tau_\theta, \theta \geq 0)$  have the same two-dimensional marginals. Since they are Markov process, they have the same distribution. This proves the first part of the theorem.

Let  $U$  be a positive “random” variable “distribution” given by 2 times the Lebesgue measure on  $(0, +\infty)$  which is independent of  $\tau$ . The “distribution” of  $V = \tau_U$  has density w.r.t. the Lebesgue measure given by  $\sqrt{2}/(\pi v) \mathbf{1}_{\{v>0\}}$ . The second part is then a direct consequence of Corollary 8.2.  $\square$

## REFERENCES

- [1] R. ABRAHAM and J.-F. DELMAS. Fragmentation associated with Lévy processes using snake. *Probab. Th. Rel. Fields*, 141:113–154, 2008.

- [2] R. ABRAHAM and J.-F. DELMAS. Changing the branching mechanism of a continuous state branching process using immigration. *Ann. Inst. Henri Poincaré*, 45(1):226–238, 2009.
- [3] R. ABRAHAM and J.-F. DELMAS. Williams’ decomposition of the Lévy continuous random tree and simultaneous extinction probability for populations with neutral mutations. *Stoch. Process. and Appl.*, 119:1124–1143, 2009.
- [4] R. ABRAHAM, J.-F. DELMAS, and G. VOISIN. Pruning a Lévy continuum random tree. *Preprint arXiv:0804.1027*, 2008.
- [5] R. ABRAHAM and L. SERLET. Poisson snake and fragmentation. *Elect. J. of Probab.*, 7, 2002.
- [6] D. ALDOUS. The continuum random tree II: an overview. In *Proc. Durham Symp. Stochastic Analysis*, pages 23–70. Cambridge univ. press edition, 1990.
- [7] D. ALDOUS. The continuum random tree I. *Ann. Probab.*, 19(1):1–28, 1991.
- [8] D. ALDOUS. The continuum random tree III. *Ann. Probab.*, 21(1):248–289, 1993.
- [9] D. ALDOUS and J. PITMAN. The standard additive coalescent. *Ann. Probab.*, 26(4):1703–1726, 1998.
- [10] D. ALDOUS and J. PITMAN. Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(5):637–686, 1998.
- [11] J. BERTOIN. *Lévy processes*. Cambridge University Press, Cambridge, 1996.
- [12] J. BERTOIN. Self-similar fragmentations. *Ann. Inst. Henri Poincaré*, 38(3):319–340, 2000.
- [13] J. BERTOIN. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [14] G. U. BRAVO. The falling apart of the tagged fragment and the asymptotic disintegration of the Brownian height fragmentation. *Preprint arXiv:0811.4754*, 2008.
- [15] J.-F. DELMAS. Height process for super-critical continuous state branching process. *Markov Proc. and Rel. Fields.*, 14(2):309–326, 2008.
- [16] T. DUQUESNE and J.-F. LE GALL. *Random trees, Lévy processes and spatial branching processes*, volume 281. Astérisque, 2002.
- [17] T. DUQUESNE and J.-F. LE GALL. Probabilistic and fractal aspects of Lévy trees. *Probab. Th. Rel. Fields*, 131(4):553–603, 2005.
- [18] T. DUQUESNE and M. WINKEL. Growth of Lévy trees. *Probab. Th. Rel. Fields*, 139(3-4):313–371, 2007.
- [19] M. JIRINA. Stochastic branching processes with continuous state space. *Czech. Math. J.*, 83(8):292–312, 1958.
- [20] J. LAMPERTI. The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:271–288, 1967.
- [21] J.-F. LE GALL and Y. LE JAN. Branching processes in Lévy processes: Laplace functionals of snake and superprocesses. *Ann. Probab.*, 26:1407–1432, 1998.
- [22] J.-F. LE GALL and Y. LE JAN. Branching processes in Lévy processes: The exploration process. *Ann. Probab.*, 26:213–252, 1998.
- [23] G. VOISIN. Dislocation measure of the fragmentation of a general Lévy tree. *Preprint arXiv:0903.4024*.

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