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# Braid groups of non-orientable surfaces and the Fadell-Neuwirth short exact sequence

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## Abstract

Let  $M$  be a compact, connected non-orientable surface without boundary and of genus  $g \geq 3$ . We investigate the pure braid groups  $P_n(M)$  of  $M$ , and in particular the possible splitting of the Fadell-Neuwirth short exact sequence

$$1 \longrightarrow P_m(M \setminus \{x_1, \dots, x_n\}) \hookrightarrow P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1,$$

where  $m, n \geq 1$ , and  $p_*$  is the homomorphism which corresponds geometrically to forgetting the last  $m$  strings. This problem is equivalent to that of the existence of a section for the associated fibration  $p: F_{n+m}(M) \longrightarrow F_n(M)$  of configuration spaces, defined by  $p((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})) = (x_1, \dots, x_n)$ . We show that  $p$  and  $p_*$  admit a section if and only if  $n = 1$ . Together with previous results, this completes the resolution of the splitting problem for surfaces pure braid groups.

## 1 Introduction

Braid groups of the plane were defined by Artin in 1925 [A1], and further studied in [A2, A3]. Braid groups of surfaces were studied by Zariski [Z], and were later generalised using the following definition due to Fox [FoN]. Let  $M$  be a compact, connected surface, and let  $n \in \mathbb{N}$ . We denote the set of all ordered  $n$ -tuples of distinct points of  $M$ , known as the  $n^{\text{th}}$  configuration space of  $M$ , by:

$$F_n(M) = \{(p_1, \dots, p_n) \mid p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [CG, FH] for example.

The symmetric group  $S_n$  on  $n$  letters acts freely on  $F_n(M)$  by permuting coordinates. The corresponding quotient space will be denoted by  $D_n(M)$ . Notice that  $F_n(M)$  is a regular covering of  $D_n(M)$ . The  $n^{\text{th}}$  pure braid group  $P_n(M)$  (respectively the  $n^{\text{th}}$  braid group  $B_n(M)$ ) is defined to be the fundamental group of  $F_n(M)$  (respectively of  $D_n(M)$ ). If  $m \in \mathbb{N}$ , then we may define a homomorphism  $p_*: P_{n+m}(M) \rightarrow P_n(M)$  induced by the projection  $p: F_{n+m}(M) \rightarrow F_n(M)$  defined by  $p((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})) = (x_1, \dots, x_n)$ . Representing  $P_{n+m}(M)$  geometrically as a collection of  $n + m$  strings,  $p_*$  corresponds to forgetting the last  $m$  strings. **We adopt the convention, that unless explicitly stated, all homomorphisms  $P_{n+m}(M) \rightarrow P_n(M)$  in the text will be this one.** If  $M$  is the 2-disc (or the plane  $\mathbb{R}^2$ ),  $B_n(M)$  and  $P_n(M)$  are respectively the classical Artin braid group  $B_n$  and pure braid group  $P_n$  [FVB].

If  $M$  is without boundary, Fadell and Neuwirth study the map  $p$ , and show ([FaN, Theorem 3]) that it is a locally-trivial fibration. The fibre over a point  $(x_1, \dots, x_n)$  of the base space is  $F_m(M \setminus \{x_1, \dots, x_n\})$  which we interpret as a subspace of the total space via the map  $i: F_m(M \setminus \{x_1, \dots, x_n\}) \rightarrow F_n(M)$  defined by

$$i((y_1, \dots, y_m)) = (x_1, \dots, x_n, y_1, \dots, y_m).$$

Applying the associated long exact sequence in homotopy, we obtain the *pure braid group short exact sequence of Fadell and Neuwirth*:

$$1 \longrightarrow P_m(M \setminus \{x_1, \dots, x_n\}) \xrightarrow{i_*} P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1, \quad (\mathbf{PBS})$$

where  $n \geq 3$  if  $M$  is the sphere  $\mathbb{S}^2$  [Fa, FVB],  $n \geq 2$  if  $M$  is the real projective plane  $\mathbb{R}P^2$  [VB], and  $n \geq 1$  otherwise [FaN], and where  $i_*$  and  $p_*$  are the homomorphisms induced by the maps  $i$  and  $p$  respectively. The short exact sequence **(PBS)** has been widely studied, and may be employed for example to determine presentations of  $P_n(M)$  (see Section 2), its centre, and possible torsion. It was also used in recent work on the structure of the mapping class groups [PR] and on Vassiliev invariants for surface braids [GMP].

In the case of  $P_n$ , and taking  $m = 1$ ,  $\text{Ker}(p_*)$  is a free group of rank  $n$ . The short exact sequence **(PBS)** splits for all  $n \geq 1$ , and so  $P_n$  may be described as a repeated semi-direct product of free groups. This decomposition, known as the ‘combing’ operation, is the principal result of Artin’s classical theory of braid groups [A2], and yields normal forms and a solution to the word problem in  $B_n$ . More recently, it was used by Falk and Randell to study the lower central series and the residual nilpotence of  $P_n$  [FR], and by Rolfsen and Zhu to prove that  $P_n$  is bi-orderable [RZ].

The problem of deciding whether such a decomposition exists for braid groups of surfaces is thus fundamental. This was indeed a recurrent and central question during the foundation of the theory and its subsequent development during the 1960’s [Fa, FaN, FVB, VB, Bi]. If the fibre of the fibration is an Eilenberg-MacLane space then the existence of a section for  $p_*$  is equivalent to that of a cross-section for  $p$  [Ba, Wh] (cf. [GG2]). But with the exception of the construction of sections in certain cases (for  $\mathbb{S}^2$  [Fa] and the 2-torus  $\mathbb{T}^2$  [Bi]), no progress on the possible splitting of **(PBS)** was recorded for nearly forty years. In the case of orientable surfaces without boundary of genus at least two, the

question of the splitting of **(PBS)** which was posed explicitly by Birman in 1969 [Bi], was finally resolved by the authors, the answer being positive if and only if  $n = 1$  [GG1].

As for the non-orientable case, the braid groups of  $\mathbb{R}P^2$  were first studied by Van Buskirk [VB], and more recently by Wang [Wa] and the authors [GG2, GG3, GG4]. For  $n = 1$ , we have  $P_1(\mathbb{R}P^2) = B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ . Van Buskirk showed that for all  $n \geq 2$ , neither the fibration  $p: F_n(\mathbb{R}P^2) \rightarrow F_1(\mathbb{R}P^2)$  nor the homomorphism  $p_*: P_n(\mathbb{R}P^2) \rightarrow P_1(\mathbb{R}P^2)$  admit a cross-section (for  $p$ , this is a manifestation of the fixed point property of  $\mathbb{R}P^2$ ), but that the fibration  $p: F_3(\mathbb{R}P^2) \rightarrow F_2(\mathbb{R}P^2)$  admits a cross-section, and hence so does the corresponding homomorphism  $p_*$ . Using coincidence theory, we showed that for  $n = 2, 3$  and  $m \geq 4 - n$ , neither the fibration nor the short exact sequence **(PBS)** admit a section [GG2]. In [GG3], we gave a complete answer to the splitting problem for  $\mathbb{R}P^2$ : if  $m, n \in \mathbb{N}$ , the homomorphism  $p_*: P_{n+m}(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$  and the fibration  $p: F_{n+m}(\mathbb{R}P^2) \rightarrow F_n(\mathbb{R}P^2)$  admit a section if and only if  $n = 2$  and  $m = 1$ . In other words, Van Buskirk's values ( $n = 2$  and  $m = 1$ ) are the only ones for which a section exists (both on the geometric and the algebraic level).

In this paper, we study the splitting problem for compact, connected non-orientable surfaces without boundary and of genus  $g \geq 3$  (every non-orientable compact surface  $M$  without boundary is homeomorphic to the connected sum of  $g$  copies of  $\mathbb{R}P^2$ ,  $g \in \mathbb{N}$  being the *genus* of  $M$ ). In the case of the Klein bottle  $\mathbb{K}^2$  ( $g = 2$ ), the existence of a non-vanishing vector field implies that there always exists a section, both geometric and algebraic (cf. [FaN]). Our main result is:

**Theorem 1.** *Let  $M$  be a compact, connected, non-orientable surface without boundary of genus  $g \geq 3$ , and let  $m, n \in \mathbb{N}$ . Then the homomorphism  $p_*: P_{n+m}(M) \rightarrow P_n(M)$  and the fibration  $p: F_{n+m}(M) \rightarrow F_n(M)$  admit a section if and only if  $n = 1$ .*

Applying Theorem 1 and the results of [GG1, GG3], we may solve completely the splitting problem for surface pure braid groups:

**Theorem 2.** *Let  $m, n \in \mathbb{N}$  and  $r \geq 0$ . Let  $N$  be a compact, connected surface possibly with boundary, let  $\{x_1, \dots, x_r\}$  be a finite subset in the interior of  $N$ , let  $M = N \setminus \{x_1, \dots, x_r\}$ , and let  $p_*: P_{n+m}(M) \rightarrow P_n(M)$  be the standard projection.*

- (a) *If  $r > 0$  or if  $M$  has non-empty boundary then  $p_*$  admits a section for all  $m$  and  $n$ .*
- (b) *Suppose that  $r = 0$  and that  $M$  is without boundary. Then  $p_*$  admits a section if and only if one of the following conditions holds:*
  - (i)  *$M$  is  $\mathbb{S}^2$ , the 2-torus  $\mathbb{T}^2$  or the Klein bottle  $\mathbb{K}^2$  (for all  $m$  and  $n$ ).*
  - (ii)  *$M = \mathbb{R}P^2$ ,  $n = 2$  and  $m = 1$ .*
  - (iii)  *$M \neq \mathbb{R}P^2, \mathbb{S}^2, \mathbb{T}^2, \mathbb{K}^2$  and  $n = 1$ .*

The rest of the paper is organised as follows. In Section 2, we determine a presentation of  $P_n(M)$  (Theorem 3). In Section 3, we study the consequences of the existence of a section in the case  $m = 1$  and  $n \geq 2$ , i.e.  $p_*: P_{n+1}(M) \rightarrow P_n(M)$ . The general strategy of the proof of Theorem 1 is based on the following remark. Suppose that **(PBS)** splits. If  $H$  is any normal subgroup of  $P_{n+1}(M)$  contained in  $\text{Ker}(p_*)$ , the quotiented short exact sequence  $1 \rightarrow \text{Ker}(p_*)/H \hookrightarrow P_{n+1}(M)/H \rightarrow P_n(M) \rightarrow 1$  must also split. In order to obtain a contradiction, we seek such a subgroup  $H$  for which this short exact sequence does *not* split. However the choice of  $H$  needed to achieve this may be somewhat delicate:

if  $H$  is too ‘small’, the structure of the quotient  $P_{n+1}(\mathbb{R}P^2)/H$  remains complicated; on the other hand, if  $H$  is too ‘large’, we lose too much information and cannot reach a conclusion. In Section 4, we first show that we may reduce to the case  $m = 1$ , and then go on to prove Theorem 1 using the analysis of Section 3. As we shall see in Section 4, it suffices to take  $H$  to be Abelianisation of  $\text{Ker}(p_*)$ , in which case the quotient  $\text{Ker}(p_*)/H$  is a free Abelian group. We will then deduce Theorem 2.

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## 2 A presentation of $P_n(M)$

Let  $M = M_g$  be a compact, connected, non-orientable surface without boundary of genus  $g \geq 2$ . If  $n \in \mathbb{N}$  and  $\mathbb{D}^2 \subseteq M$  is a topological disc, the inclusion induces a homomorphism  $\iota: B_n(\mathbb{D}^2) \rightarrow B_n(M)$ . If  $\beta \in B_n(\mathbb{D}^2)$  then we shall denote its image  $\iota(\beta)$  simply by  $\beta$ . For  $1 \leq i < j \leq n$ , we consider the following elements of  $P_n(M)$ :

$$B_{i,j} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i,$$

where  $\sigma_1, \dots, \sigma_{n-1}$  are the standard generators of  $B_n(\mathbb{D}^2)$ . The geometric braid corresponding to  $B_{i,j}$  takes the  $i^{\text{th}}$  string once around the  $j^{\text{th}}$  string in the positive sense, with all other strings remaining vertical. For each  $1 \leq k \leq n$  and  $1 \leq l \leq g$ , we define a generator  $\rho_{k,l}$  which is represented geometrically by a loop based at the  $k^{\text{th}}$  point and which goes round the  $l^{\text{th}}$  twisted handle. These elements are illustrated in Figure 1 that represents  $M$  minus a disc.

A presentation of the braid groups of non-orientable surfaces was originally given by Scott [S]. Other presentations were later obtained in [Be, GM]. In the following theorem, we derive another presentation of  $P_n(M)$ .

**Theorem 3.** *Let  $M$  be a compact, connected, non-orientable surface without boundary of genus  $g \geq 2$ , and let  $n \in \mathbb{N}$ . The following constitutes a presentation of the pure braid group  $P_n(M)$ :*

**generators:**  $B_{i,j}$ ,  $1 \leq i < j \leq n$ , and  $\rho_{k,l}$ , where  $1 \leq k \leq n$  and  $1 \leq l \leq g$ .

**relations:**

(a) *the Artin relations between the  $B_{i,j}$  emanating from those of  $P_n(\mathbb{D}^2)$ :*

$$B_{r,s} B_{i,j} B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ B_{i,j}^{-1} B_{r,j}^{-1} B_{i,j} B_{r,j} B_{i,j} & \text{if } r < i = s < j \\ B_{s,j}^{-1} B_{i,j} B_{s,j} & \text{if } i = r < s < j \\ B_{s,j}^{-1} B_{r,j}^{-1} B_{s,j} B_{r,j} B_{i,j} B_{r,j}^{-1} B_{s,j}^{-1} B_{r,j} B_{s,j} & \text{if } r < i < s < j. \end{cases} \quad (1)$$

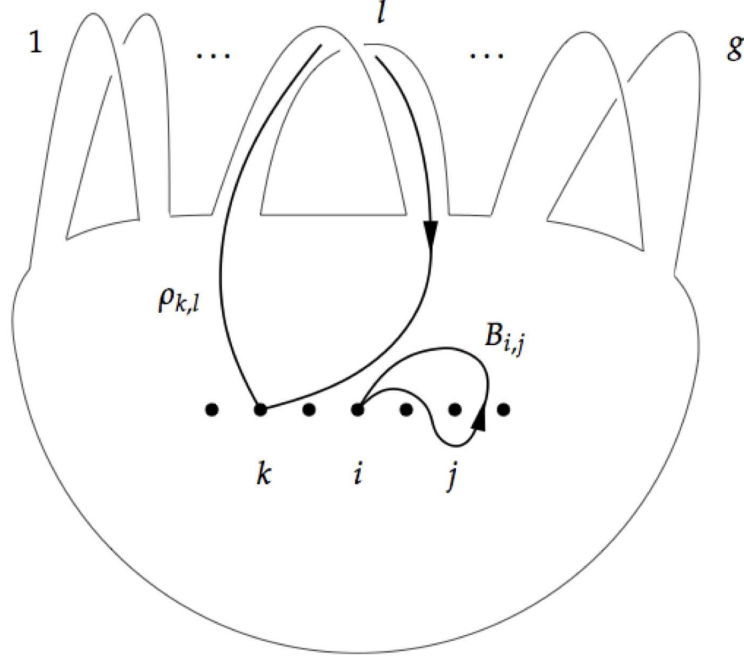


Figure 1: The generators  $B_{i,j}$  and  $\rho_{k,l}$  of  $P_n(M)$ , represented geometrically by loops lying in  $M$  minus a disc.

(b) for all  $1 \leq i < j \leq n$  and  $1 \leq k, l \leq g$ ,

$$\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k < l \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2 & \text{if } k = l \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} & \text{if } k > l \end{cases} \quad (2)$$

(c) for all  $1 \leq i \leq n$ , the 'surface relations'  $\prod_{l=1}^g \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n}$ .

(d) for all  $1 \leq i < j \leq n$ ,  $1 \leq k \leq n$ ,  $k \neq j$ , and  $1 \leq l \leq g$ ,

$$\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k < i \text{ or } j < k \\ \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l} & \text{if } k = i \\ \rho_{j,l}^{-1} B_{k,j}^{-1} \rho_{j,l} B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_{j,l}^{-1} B_{k,j} \rho_{j,l} & \text{if } i < k < j. \end{cases} \quad (3)$$

*Proof.* We apply induction and standard results concerning the presentation of an extension (see [J, Theorem 1, Chapter 13]). The proof generalises that of [GG3] for  $\mathbb{R}P^2$ , and is similar in spirit to that of [S].

First note that the given presentation is correct for  $n = 1$  ( $P_1(M) = \pi_1(M)$  has a presentation  $\langle \rho_{1,1}, \dots, \rho_{1,g} \mid \prod_{l=1}^g \rho_{1,l}^2 = 1 \rangle$ ). So let  $n \geq 1$ , and suppose that  $P_n(M)$  has

the given presentation. Taking  $m = 1$  in **(PBS)**, we have a short exact sequence:

$$1 \longrightarrow \pi_1(M \setminus \{x_1, \dots, x_n\}, x_{n+1}) \longrightarrow P_{n+1}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1.$$

In order to retain the symmetry of the presentation, we take the free group  $\text{Ker}(p_*)$  to have the following one-relator presentation:

$$\left\langle \rho_{n+1,1}, \dots, \rho_{n+1,g}, B_{1,n+1}, \dots, B_{n,n+1} \mid \prod_{l=1}^g \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1} \right\rangle.$$

Together with these generators of  $\text{Ker}(p_*)$ , the elements  $B_{i,j}$ ,  $1 \leq i < j \leq n$ , and  $\rho_{k,l}$ ,  $1 \leq k \leq n$  and  $1 \leq l \leq g$ , of  $P_{n+1}(M)$  (which are coset representatives of the generators of  $P_n(M)$ ) form the given generating set of  $P_{n+1}(M)$ .

There are three classes of relations of  $P_{n+1}(M)$  which are obtained as follows. The first consists of the single relation  $\prod_{l=1}^g \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1}$  of  $\text{Ker}(p_*)$ . The second class is obtained by rewriting the relators of the quotient in terms of the coset representatives, and expressing the corresponding element as a word in the generators of  $\text{Ker}(p_*)$ . In this way, all of the relations of  $P_n(M)$  lift directly to relations of  $P_{n+1}(M)$ , with the exception of the surface relations which become

$$\prod_{l=1}^g \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} B_{i,n+1} \quad \text{for all } 1 \leq i \leq n.$$

Along with the relation of  $\text{Ker}(p_*)$ , we obtain the complete set of surface relations (relations (c)) for  $P_{n+1}(M)$ .

The third class of relations is obtained by rewriting the conjugates of the generators of  $\text{Ker}(p_*)$  by the coset representatives in terms of the generators of  $\text{Ker}(p_*)$ :

(i) For all  $1 \leq i < j \leq n$  and  $1 \leq l \leq n$ ,

$$B_{i,j} B_{l,n+1} B_{i,j}^{-1} = \begin{cases} B_{l,n+1} & \text{if } l < i \text{ or } j < l \\ B_{l,n+1}^{-1} B_{i,n+1}^{-1} B_{l,n+1} B_{i,n+1} B_{l,n+1} & \text{if } l = j \\ B_{j,n+1}^{-1} B_{l,n+1} B_{j,n+1} & \text{if } l = i \\ B_{j,n+1}^{-1} B_{i,n+1}^{-1} B_{j,n+1} B_{i,n+1} B_{l,n+1} B_{i,n+1}^{-1} B_{j,n+1}^{-1} B_{i,n+1} B_{j,n+1} & \text{if } i < l < j. \end{cases}$$

(ii)  $B_{i,j} \rho_{n+1,l} B_{i,j}^{-1} = \rho_{n+1,l}$  for all  $1 \leq i < j \leq n$  and  $1 \leq l \leq g$ .

(iii) for all  $1 \leq i \leq n$  and  $1 \leq k, l \leq g$ ,

$$\rho_{i,k} \rho_{n+1,l} \rho_{i,k}^{-1} = \begin{cases} \rho_{n+1,l} & \text{if } k < l \\ \rho_{n+1,k}^{-1} B_{i,n+1}^{-1} \rho_{n+1,k}^2 & \text{if } k = l \\ \rho_{n+1,k}^{-1} B_{i,n+1}^{-1} \rho_{n+1,k} B_{i,n+1}^{-1} \rho_{n+1,l} B_{i,n+1} \rho_{n+1,k}^{-1} B_{i,n+1} \rho_{n+1,k} & \text{if } k > l. \end{cases}$$

(iv) For all  $1 \leq i, k \leq n$  and  $1 \leq l \leq g$ ,

$$\rho_{k,l} B_{i,n+1} \rho_{k,l}^{-1} = \begin{cases} B_{i,n+1} & \text{if } k < i \\ \rho_{n+1,l}^{-1} B_{i,n+1}^{-1} \rho_{n+1,l} & \text{if } k = i \\ \rho_{n+1,l}^{-1} B_{k,n+1}^{-1} \rho_{n+1,l} B_{k,n+1}^{-1} B_{i,n+1} B_{k,n+1} \rho_{n+1,l}^{-1} B_{k,n+1} \rho_{n+1,l} & \text{if } i < k. \end{cases}$$

Then relations (a) for  $P_{n+1}(M)$  are obtained from relations (a) for  $P_n(M)$  and relations (i), relations (b) for  $P_{n+1}(M)$  are obtained from relations (b) for  $P_n(M)$  and relations (iii), and relations (d) for  $P_{n+1}(M)$  are obtained from relations (d) for  $P_n(M)$ , and relations (ii) and (iv).  $\square$

### 3 Analysis of the case $P_{n+1}(M_g) \longrightarrow P_n(M_g)$ , $n \geq 2$

For the whole of this section, we suppose that  $g \geq 3$  and  $n \geq 2$ . By Theorem 3,  $P_n(M_g)$  is generated by the union of the  $B_{i,j}$ ,  $1 \leq i < j \leq n$ , and of the  $\rho_{k,l}$ ,  $1 \leq k \leq n$ ,  $1 \leq l \leq g$ . Let us consider the homomorphism  $p_*: P_{n+1}(M_g) \longrightarrow P_n(M_g)$ . In this section, we suppose that  $p_*$  admits a section, denoted by  $s_*$ . Applying (PBS), we thus have a split short exact sequence

$$1 \longrightarrow K \longrightarrow P_{n+1}(M_g) \begin{array}{c} \xrightarrow{p_*} \\ \xleftarrow{s_*} \end{array} P_n(M_g) \longrightarrow 1, \quad (4)$$

where  $K = \text{Ker}(p_*) = \pi_1(M_g \setminus \{x_1, \dots, x_n\}, x_{n+1})$  is a free group of rank  $n + g - 1$ , generated by  $\{B_{1,n+1}, \dots, B_{n,n+1}, \rho_{n+1,1}, \dots, \rho_{n+1,g}\}$ , and subject to the relation

$$B_{1,n+1} \cdots B_{n,n+1} = \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2.$$

Let  $H = [K, K]$  be the commutator subgroup of  $K$ . Then  $K/H$  is a free Abelian group of rank  $n + g - 1$ . In what follows, we shall not distinguish notationally between the elements of  $K$  and those of  $K/H$ . The quotient group  $K/H$  thus has a basis

$$\mathcal{B} = \{B_{1,n+1}, \dots, B_{n-1,n+1}, \rho_{n+1,1}, \dots, \rho_{n+1,g}\}, \quad (5)$$

and the relation

$$B_{n,n+1} = B_{1,n+1}^{-1} \cdots B_{n-1,n+1}^{-1} \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2 \quad (6)$$

holds in the Abelian group  $K/H$ . Since  $H$  is normal in  $P_{n+1}(M_g)$  and  $p_*$  admits a section, it follows from equation (4) that we have a split short exact sequence

$$1 \longrightarrow K/H \longrightarrow P_{n+1}(M_g)/H \begin{array}{c} \xrightarrow{\bar{p}} \\ \xleftarrow{\bar{s}} \end{array} P_n(M_g) \longrightarrow 1,$$

where  $\bar{p}$  is the homomorphism induced by  $p_*$ , and  $\bar{s}$  is the induced section.

Consider the subset

$$\Gamma = \{B_{i,j}, \rho_{k,l} \mid 1 \leq i < j \leq n, 1 \leq k \leq n, 1 \leq l \leq g\}$$

of  $P_{n+1}(M_g)/H$ . If  $g \in \Gamma$  then  $\bar{p}(g) = g \in P_n(M_g)$ , and so  $g^{-1} \cdot \bar{s}(\bar{p}(g)) \in \text{Ker}(\bar{p}) = K/H$ . Then the integer coefficients  $\alpha_{i,j,r}, \beta_{i,j,q}, \gamma_{k,l,r}, \eta_{k,l,q}$ , where  $1 \leq r \leq g$  and  $1 \leq q \leq n - 1$ , are (uniquely) defined by the equations:

$$\begin{cases} \bar{s}(B_{i,j}) = B_{i,j} \rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \\ \bar{s}(\rho_{k,l}) = \rho_{k,l} \rho_{n+1,1}^{\gamma_{k,l,1}} \cdots \rho_{n+1,g}^{\gamma_{k,l,g}} B_{1,n+1}^{\eta_{k,l,1}} \cdots B_{n-1,n+1}^{\eta_{k,l,n-1}} \end{cases} \quad (7)$$



There is an equation for each element of  $\Gamma$ . Most of the elements of  $\Gamma$  commute with the elements of the basis  $\mathcal{B}$  of  $K/H$  given in equation (5). We record the list of conjugates of such elements for later use. In what follows,  $1 \leq i < j \leq n$ ,  $1 \leq k, m \leq n$  and  $1 \leq l, r \leq g$ . In  $K/H$ , we have

$$B_{i,j}B_{m,n+1}B_{i,j}^{-1} = B_{m,n+1}$$

(this follows from equation (1) and the fact that the elements  $B_{q,n+1}$ ,  $1 \leq q \leq n$ , belong to  $K/H$  and so commute pairwise), and

$$B_{i,j}\rho_{n+1,l}B_{i,j}^{-1} = \rho_{n+1,l}$$

by equation (3). Thus  $B_{i,j}$  belongs to the centraliser of  $K/H$  in  $P_{n+1}(M_g)/H$ . Also by equation (3), we have

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = \begin{cases} B_{m,n+1} & \text{if } k < m \\ \rho_{n+1,l}^{-1}B_{m,n+1}^{-1}\rho_{n+1,l} = B_{m,n+1}^{-1} & \text{if } k = m \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}B_{k,n+1}^{-1}B_{m,n+1}B_{k,n+1}\rho_{n+1,l}^{-1}B_{k,n+1}\rho_{n+1,l} = B_{m,n+1} & \text{if } k > m, \end{cases}$$

so

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = B_{m,n+1}^{1-2\delta_{k,m}}, \quad (8)$$

where  $\delta_{\cdot,\cdot}$  is the Kronecker delta. By equation (2), we have

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \begin{cases} \rho_{n+1,r} & \text{if } l < r \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}^2 = \rho_{n+1,l}B_{k,n+1}^{-1} & \text{if } l = r \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}B_{k,n+1}^{-1}\rho_{n+1,r}B_{k,n+1}\rho_{n+1,l}^{-1}B_{k,n+1}\rho_{n+1,l} = \rho_{n+1,r} & \text{if } l > r, \end{cases}$$

so

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-\delta_{l,r}}. \quad (9)$$

Combining equations (8) and (9), we obtain

$$\rho_{k,r}^2\rho_{n+1,r}\rho_{k,r}^{-2} = \rho_{k,r}\rho_{n+1,r}B_{k,n+1}^{-1}\rho_{k,r}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-1}B_{k,n+1} = \rho_{n+1,r},$$

so

$$\rho_{k,r}\rho_{n+1,r}\rho_{k,r}^{-1} = \rho_{k,r}^{-1}\rho_{n+1,r}\rho_{k,r}. \quad (10)$$

Furthermore, by equation (8),  $\rho_{k,l}^2$  commutes with  $B_{m,n+1}$ , and therefore

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = \rho_{k,l}^{-1}B_{m,n+1}\rho_{k,l}. \quad (11)$$

Hence  $\rho_{k,l}^2$  also belongs to the centraliser of  $K/H$  in  $P_{n+1}(M_g)/H$ . From equations (8) and (9), we obtain the following relations:

$$\rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} \cdot \rho_{j,l} = \rho_{j,l} \cdot B_{j,n+1}^{-\gamma_{i,k,l}} \rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} \quad \text{for all } 1 \leq j \leq n, \quad (12)$$

and

$$B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \rho_{j,l} = \begin{cases} \rho_{j,l} B_{j,n+1}^{-2\eta_{i,k,j}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} & \text{if } 1 \leq j \leq n-1 \\ \rho_{j,l} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} & \text{if } j = n. \end{cases}$$

Setting  $\eta_{i,k,n} = 0$  for all  $1 \leq i \leq n$  and  $1 \leq k \leq g$  yields:

$$B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdot \rho_{j,l} = \rho_{j,l} \cdot B_{j,n+1}^{-2\eta_{i,k,j}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \quad \text{for all } 1 \leq j \leq n. \quad (13)$$

Equations (12) and (13) will be employed repeatedly in the ensuing calculations.

We now investigate the images under  $\bar{s}$  of some of the relations (b)–(d) of Theorem 3 (it turns out that the analysis of the other relations, including (a), will not be necessary for our purposes).

(a) Let  $1 \leq i < j \leq n$  and  $1 \leq k, l \leq g$ . We examine the three possible cases of equation (7) (relation (b) of Theorem 3).

(i)  $k < l$ : then  $\rho_{i,k}\rho_{j,l} = \rho_{j,l}\rho_{i,k}$  in  $P_n(M_g)$ . The respective images under  $\bar{s}$  are:

$$\begin{aligned} \bar{s}(\rho_{i,k}\rho_{j,l}) &= \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \\ &= \rho_{i,k}\rho_{j,l} B_{j,n+1}^{-\gamma_{i,k,l}-2\eta_{i,k,j}} \rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,l,g}} B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,l,n-1}}, \end{aligned}$$

and

$$\begin{aligned} \bar{s}(\rho_{j,l}\rho_{i,k}) &= \rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \\ &= \rho_{j,l}\rho_{i,k} B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}} \rho_{n+1,1}^{\gamma_{j,l,1}+\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}+\gamma_{i,k,g}} B_{1,n+1}^{\eta_{j,l,1}+\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}+\eta_{i,k,n-1}}. \end{aligned}$$

The relation  $\rho_{i,k}\rho_{j,l} = \rho_{j,l}\rho_{i,k}$  in  $P_{n+1}(M_g)$  implies that  $B_{j,n+1}^{-\gamma_{i,k,l}-2\eta_{i,k,j}} = B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}}$ . Comparing coefficients of the elements of  $\mathcal{B}$  in  $K/H$  (cf. equation (5)), if  $j < n$ , we have

$$\begin{cases} \gamma_{j,l,k} + 2\eta_{j,l,i} = 0 & \text{and} \\ \gamma_{i,k,l} + 2\eta_{i,k,j} = 0, \end{cases} \quad (14)$$

while if  $j = n$ , applying equation (6) yields

$$B_{i,n+1}^{\gamma_{n,l,k}+2\eta_{n,l,i}} = B_{n,n+1}^{\gamma_{i,k,l}+2\eta_{i,k,n}} = B_{1,n+1}^{-(\gamma_{i,k,l}+2\eta_{i,k,n})} \cdots B_{n-1,n+1}^{-(\gamma_{i,k,l}+2\eta_{i,k,n})} \rho_{n+1,1}^{2(\gamma_{i,k,l}+2\eta_{i,k,n})} \cdots \rho_{n+1,g}^{2(\gamma_{i,k,l}+2\eta_{i,k,n})},$$

and thus equation (14) also holds for  $j = n$ . So for all  $1 \leq i < j \leq n$  and  $1 \leq k < l \leq g$ ,

$$\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \quad \text{and} \quad (15)$$

$$\gamma_{i,k,l} + 2\eta_{i,k,j} = 0. \quad (16)$$

(ii)  $k = l$ : then  $\rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1} = \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^2$  in  $P_n(M_g)$  for all  $1 \leq i < j \leq n$  and  $1 \leq k \leq g$ . The respective images under  $\bar{s}$  are:

$$\begin{aligned} \bar{s}(\rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1}) &= \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\ &\quad B_{n-1,n+1}^{-\eta_{i,k,n-1}} \cdots B_{1,n+1}^{-\eta_{i,k,1}} \rho_{n+1,g}^{-\gamma_{i,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{i,k,1}} \rho_{i,k}^{-1} \\ &= \rho_{i,k}\rho_{j,k} B_{j,n+1}^{-\gamma_{i,k,k}} \rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,k,g}} B_{j,n+1}^{-2\eta_{i,k,j}} B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,k,n-1}} \\ &\quad \rho_{i,k}^{-1} B_{i,n+1}^{2\eta_{i,k,i}} B_{n-1,n+1}^{-\eta_{i,k,n-1}} \cdots B_{1,n+1}^{-\eta_{i,k,1}} B_{i,n+1}^{\gamma_{i,k,k}} \rho_{n+1,g}^{-\gamma_{i,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{i,k,1}} \\ &= \rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1} B_{j,n+1}^{-\gamma_{i,k,k}} B_{i,n+1}^{-(\gamma_{i,k,k}+\gamma_{j,k,k})} \rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,k,g}} B_{j,n+1}^{-2\eta_{i,k,j}} B_{i,n+1}^{-2(\eta_{i,k,i}+\eta_{j,k,i})} \\ &\quad B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,k,n-1}} B_{i,n+1}^{2\eta_{i,k,i}} B_{n-1,n+1}^{-\eta_{i,k,n-1}} \cdots B_{1,n+1}^{-\eta_{i,k,1}} B_{i,n+1}^{\gamma_{i,k,k}} \rho_{n+1,g}^{-\gamma_{i,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{i,k,1}} \\ &= \rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} B_{i,n+1}^{-(2\eta_{j,k,i}+\gamma_{j,k,k})} B_{j,n+1}^{-2(\eta_{i,k,i}+\eta_{j,k,i})} \end{aligned}$$

and

$$\begin{aligned}
\bar{s} \left( \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2 \right) &= B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} \rho_{j,k}^{-1} \\
&\quad B_{n-1,n+1}^{-\beta_{i,j,n-1}} \cdots B_{1,n+1}^{-\beta_{i,j,1}} \rho_{n+1,g}^{-\alpha_{i,j,g}} \cdots \rho_{n+1,1}^{-\alpha_{i,j,1}} B_{i,j}^{-1} \\
&\quad \rho_{j,k} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \rho_{j,k} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\
&= \rho_{j,k}^{-1} B_{i,j}^{-1} B_{j,n+1}^{2\eta_{j,k,j}} B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} B_{j,n+1}^{\gamma_{j,k,k}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} B_{n-1,n+1}^{-\beta_{i,j,n-1}} \cdots B_{1,n+1}^{-\beta_{i,j,1}} \\
&\quad \rho_{n+1,g}^{-\alpha_{i,j,g}} \cdots \rho_{n+1,1}^{-\alpha_{i,j,1}} \rho_{j,k}^2 B_{j,n+1}^{-\gamma_{j,k,k}} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{j,n+1}^{-2\eta_{j,k,j}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\
&\quad \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\
&= \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2 \rho_{n+1,1}^{\gamma_{j,k,1} - \alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g} - \alpha_{i,j,g}} B_{1,n+1}^{\eta_{j,k,1} - \beta_{i,j,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1} - \beta_{i,j,n-1}}.
\end{aligned}$$

Since  $\rho_{i,k} \rho_{j,k} \rho_{i,k}^{-1} = \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2$  in  $P_{n+1}(M_g)$ , we obtain

$$B_{i,n+1}^{-(2\eta_{j,k,i} + \gamma_{j,k,k})} B_{j,n+1}^{-(2\eta_{j,k,j} + \gamma_{i,k,k})} = \rho_{n+1,1}^{-\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{-\alpha_{i,j,g}} B_{1,n+1}^{-\beta_{i,j,1}} \cdots B_{n-1,n+1}^{-\beta_{i,j,n-1}}. \quad (17)$$

If  $j < n$  then all of the terms in equation (17) are expressed in terms of the basis  $\mathcal{B}$  of  $K/H$  of equation (5), and so for all  $1 \leq i < j \leq n-1$ ,

$$\alpha_{i,j,r} = 0 \quad \text{for all } 1 \leq r \leq g \quad (18)$$

$$\beta_{i,j,s} = 0 \quad \text{for all } 1 \leq s \leq n-1, s \notin \{i, j\} \quad (19)$$

$$\beta_{i,j,i} = \gamma_{j,k,k} + 2\eta_{j,k,i} \quad (20)$$

$$\beta_{i,j,j} = \gamma_{i,k,k} + 2\eta_{i,k,j}. \quad (21)$$

If  $j = n$  then substituting for  $B_{n,n+1}$  in equation (17) using equation (6) and comparing coefficients in  $K/H$  of the elements of  $\mathcal{B}$  yields

$$\begin{aligned}
2(2\eta_{i,k,n} + \gamma_{i,k,k}) &= \alpha_{i,n,r} \quad \text{for all } 1 \leq r \leq g \\
(2\eta_{i,k,n} + \gamma_{i,k,k}) &= -\beta_{i,n,s} \quad \text{for all } 1 \leq s \leq n-1, s \neq i \\
2(\eta_{i,k,n} - \eta_{n,k,i}) + (\gamma_{i,k,k} - \gamma_{n,k,k}) &= -\beta_{i,n,i}.
\end{aligned}$$

But  $\eta_{i,k,n} = 0$ , so for all  $1 \leq i \leq n-1$  and  $1 \leq k \leq g$ ,

$$\alpha_{i,n,r} = 2\gamma_{i,k,k} \quad \text{for all } 1 \leq r \leq g \quad (22)$$

$$\beta_{i,n,s} = -\gamma_{i,k,k} \quad \text{for all } 1 \leq s \leq n-1, s \neq i \quad (23)$$

$$\beta_{i,n,i} = 2\eta_{n,k,i} + (\gamma_{n,k,k} - \gamma_{i,k,k}). \quad (24)$$

(iii)  $k > l$ : then  $\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} \rho_{j,k} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k}$  in  $P_n(M_g)$ . The respective images under  $\bar{s}$  are:

$$\begin{aligned}
\bar{s} \left( \rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} \right) &= \rho_{i,k} \rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \rho_{j,l} \rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \\
&\quad B_{n-1,n+1}^{-\eta_{i,k,n-1}} \cdots B_{1,n+1}^{-\eta_{i,k,1}} \rho_{n+1,g}^{-\gamma_{i,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{i,k,1}} \rho_{i,k}^{-1} \\
&= \rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} B_{j,n+1}^{-\gamma_{i,k,i} - 2\eta_{i,k,j}} B_{i,n+1}^{-\gamma_{j,l,k} - 2\eta_{j,l,i}} \rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}
\end{aligned}$$

and

$$\begin{aligned}
& \bar{s} \left( \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} \right) = \\
& B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} \rho_{j,k}^{-1} \cdot B_{n-1,n+1}^{-\beta_{i,j,n-1}} \cdots B_{1,n+1}^{-\beta_{i,j,1}} \rho_{n+1,g}^{-\alpha_{i,j,g}} \cdots \rho_{n+1,1}^{-\alpha_{i,j,1}} B_{i,j}^{-1} \cdot \\
& \rho_{j,k} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \cdot B_{n-1,n+1}^{-\beta_{i,j,n-1}} \cdots B_{1,n+1}^{-\beta_{i,j,1}} \rho_{n+1,g}^{-\alpha_{i,j,g}} \cdots \rho_{n+1,1}^{-\alpha_{i,j,1}} B_{i,j}^{-1} \cdot \\
& \rho_{j,l} \rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \cdot B_{i,j} \rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \cdot \\
& B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} \rho_{j,k}^{-1} \cdot B_{i,j} \rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \cdot \\
& \rho_{j,k} \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\
& = B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{j,n+1}^{2\beta_{i,j} + \alpha_{i,j,k}} \cdot \\
& \rho_{n+1,1}^{\gamma_{j,k,1} - 2\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g} - 2\alpha_{i,j,g}} B_{1,n+1}^{\eta_{j,k,1} - 2\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1} - 2\beta_{i,j,n-1}} \cdot \\
& B_{i,j}^{-1} \rho_{j,l} \rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}} B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \cdot B_{i,j} \rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \cdot \\
& B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{1,n+1}^{-\eta_{j,k,1}} \rho_{n+1,g}^{-\gamma_{j,k,g}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} B_{j,n+1}^{-\alpha_{i,j,k} - 2\beta_{i,j,j}} \cdot \\
& \rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \cdot \rho_{n+1,1}^{\gamma_{j,k,1}} \cdots \rho_{n+1,g}^{\gamma_{j,k,g}} B_{1,n+1}^{\eta_{j,k,1}} \cdots B_{n-1,n+1}^{\eta_{j,k,n-1}} \\
& = \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} B_{j,n+1}^{2\alpha_{i,j,l} - 2\alpha_{i,j,k}} \cdot B_{1,n+1}^{\eta_{j,l,1}} \cdots B_{n-1,n+1}^{\eta_{j,l,n-1}} \rho_{n+1,1}^{\gamma_{j,l,1}} \cdots \rho_{n+1,g}^{\gamma_{j,l,g}}.
\end{aligned}$$

Since  $\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k}$  in  $P_{n+1}(M_g)$ , we see that

$$B_{i,n+1}^{-\gamma_{j,l,k} - 2\eta_{j,l,i}} = B_{j,n+1}^{2\alpha_{i,j,l} - 2\alpha_{i,j,k} + \gamma_{i,k,l} + 2\eta_{i,k,j}}.$$

If  $j < n$ , it follows by comparing coefficients of the elements of  $\mathcal{B}$  in  $K/H$  that for all  $1 \leq i < j < n$  and  $1 \leq l < k \leq g$ ,

$$\begin{cases} \gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \\ 2\alpha_{i,j,l} - 2\alpha_{i,j,k} + \gamma_{i,k,l} + 2\eta_{i,k,j} = 0. \end{cases} \quad (25)$$

If  $j = n$  then

$$\begin{aligned}
B_{i,n+1}^{-\gamma_{n,l,k} - 2\eta_{n,l,i}} & = B_{n,n+1}^{2\alpha_{i,n,l} - 2\alpha_{i,n,k} + \gamma_{i,k,l} + 2\eta_{i,k,n}} \\
& = B_{1,n+1}^{-(2\alpha_{i,n,l} - 2\alpha_{i,n,k} + \gamma_{i,k,l} + 2\eta_{i,k,n})} \cdots B_{n-1,n+1}^{-(2\alpha_{i,n,l} - 2\alpha_{i,n,k} + \gamma_{i,k,l} + 2\eta_{i,k,n})} \cdot \\
& \quad \rho_{n+1,1}^{2(2\alpha_{i,n,l} - 2\alpha_{i,n,k} + \gamma_{i,k,l} + 2\eta_{i,k,n})} \cdots \rho_{n+1,g}^{2(2\alpha_{i,n,l} - 2\alpha_{i,n,k} + \gamma_{i,k,l} + 2\eta_{i,k,n})},
\end{aligned}$$

and comparing coefficients of the elements of  $\mathcal{B}$  in  $K/H$ , we observe that equations (25) also hold if  $j = n$ . So for all  $1 \leq i < j \leq n$  and  $1 \leq l < k \leq g$ ,

$$\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \quad (26)$$

$$2\alpha_{i,j,l} - 2\alpha_{i,j,k} + \gamma_{i,k,l} + 2\eta_{i,k,j} = 0. \quad (27)$$

(b) Let  $1 \leq i \leq n$ . Then  $\prod_{l=1}^g \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n}$  in  $P_n(M_g)$  by relation (c) of Theorem 3. For  $1 \leq l \leq g$ , note that

$$\begin{aligned} \bar{s} \left( \rho_{i,l}^2 \right) &= \rho_{i,l} \rho_{n+1,1}^{\gamma_{i,l,1}} \cdots \rho_{n+1,g}^{\gamma_{i,l,g}} B_{1,n+1}^{\eta_{i,l,1}} \cdots B_{n-1,n+1}^{\eta_{i,l,n-1}} \rho_{i,l} \rho_{n+1,1}^{\gamma_{i,l,1}} \cdots \rho_{n+1,g}^{\gamma_{i,l,g}} B_{1,n+1}^{\eta_{i,l,1}} \cdots B_{n-1,n+1}^{\eta_{i,l,n-1}} \\ &= \rho_{i,l}^2 B_{i,n+1}^{-2\eta_{i,l,i} - \gamma_{i,l,l}} \rho_{n+1,1}^{2\gamma_{i,l,1}} \cdots \rho_{n+1,g}^{2\gamma_{i,l,g}} B_{1,n+1}^{2\eta_{i,l,1}} \cdots B_{n-1,n+1}^{2\eta_{i,l,n-1}}. \end{aligned}$$

As we saw in equations (10) and (11),  $\rho_{i,l}^2$  belongs to the centraliser of  $K/H$  in  $P_{n+1}(M_g)/H$ , so

$$\begin{aligned} \bar{s} \left( \prod_{l=1}^g \rho_{i,l}^2 \right) &= \\ \left( \prod_{l=1}^g \rho_{i,l}^2 \right) &\left( \prod_{l=1}^g B_{i,n+1}^{-2\sum_{l=1}^g \eta_{i,l,i} - \sum_{l=1}^g \gamma_{i,l,l}} \rho_{n+1,1}^{2\sum_{l=1}^g \gamma_{i,l,1}} \cdots \rho_{n+1,g}^{2\sum_{l=1}^g \gamma_{i,l,g}} \cdot B_{1,n+1}^{2\sum_{l=1}^g \eta_{i,l,1}} \cdots B_{n-1,n+1}^{2\sum_{l=1}^g \eta_{i,l,n-1}} \right). \end{aligned}$$

Further,

$$\begin{aligned} \bar{s} (B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n}) &= B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} \\ \prod_{l=1}^{i-1} \left( \rho_{n+1,1}^{\alpha_{l,i,1}} \cdots \rho_{n+1,g}^{\alpha_{l,i,g}} B_{1,n+1}^{\beta_{l,i,1}} \cdots B_{n-1,n+1}^{\beta_{l,i,n-1}} \right) &\prod_{l=i+1}^n \left( \rho_{n+1,1}^{\alpha_{l,i,1}} \cdots \rho_{n+1,g}^{\alpha_{l,i,g}} B_{1,n+1}^{\beta_{l,i,1}} \cdots B_{n-1,n+1}^{\beta_{l,i,n-1}} \right) \\ = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} \cdot \rho_{n+1,1}^{\sum_{l=1}^{i-1} \alpha_{l,i,1} + \sum_{l=i+1}^n \alpha_{l,i,1}} \cdots \rho_{n+1,g}^{\sum_{l=1}^{i-1} \alpha_{l,i,g} + \sum_{l=i+1}^n \alpha_{l,i,g}} & \\ B_{1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,1} + \sum_{l=i+1}^n \beta_{l,i,1}} \cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,n-1} + \sum_{l=i+1}^n \beta_{l,i,n-1}} &. \end{aligned}$$

Now in  $P_{n+1}(M_g)/H$ ,  $\prod_{l=1}^g \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} B_{i,n+1}$ , hence

$$\begin{aligned} B_{i,n+1}^{1-2\sum_{l=1}^g \eta_{i,l,i} - \sum_{l=1}^g \gamma_{i,l,l}} \rho_{n+1,1}^{2\sum_{l=1}^g \gamma_{i,l,1}} \cdots \rho_{n+1,g}^{2\sum_{l=1}^g \gamma_{i,l,g}} B_{1,n+1}^{2\sum_{l=1}^g \eta_{i,l,1}} \cdots B_{n-1,n+1}^{2\sum_{l=1}^g \eta_{i,l,n-1}} &= \\ \rho_{n+1,1}^{\sum_{l=1}^{i-1} \alpha_{l,i,1} + \sum_{l=i+1}^n \alpha_{l,i,1}} \cdots \rho_{n+1,g}^{\sum_{l=1}^{i-1} \alpha_{l,i,g} + \sum_{l=i+1}^n \alpha_{l,i,g}} & \\ B_{1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,1} + \sum_{l=i+1}^n \beta_{l,i,1}} \cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,n-1} + \sum_{l=i+1}^n \beta_{l,i,n-1}} &. \end{aligned}$$

Thus for all  $1 \leq i \leq n$ ,

$$\begin{aligned} B_{i,n+1}^{1-2\sum_{l=1}^g \eta_{i,l,i} - \sum_{l=1}^g \gamma_{i,l,l}} &= \rho_{n+1,1}^{\sum_{l=1}^{i-1} \alpha_{l,i,1} + \sum_{l=i+1}^n \alpha_{l,i,1} - 2\sum_{l=1}^g \gamma_{i,l,1}} \cdots \rho_{n+1,g}^{\sum_{l=1}^{i-1} \alpha_{l,i,g} + \sum_{l=i+1}^n \alpha_{l,i,g} - 2\sum_{l=1}^g \gamma_{i,l,g}} \\ B_{1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,1} + \sum_{l=i+1}^n \beta_{l,i,1} - 2\sum_{l=1}^g \eta_{i,l,1}} &\cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1} \beta_{l,i,n-1} + \sum_{l=i+1}^n \beta_{l,i,n-1} - 2\sum_{l=1}^g \eta_{i,l,n-1}}. \quad (28) \end{aligned}$$

## 4 Proofs of Theorems 1 and 2

In this section, we use the calculations of Section 3 to prove Theorem 1, from which we shall deduce Theorem 2.

*Proof of Theorem 1.* As we mentioned in the Introduction, the existence of an algebraic section for  $p_*$  is equivalent to that of a cross-section for  $p$ .

The case  $n = 1$  was treated in Theorem 1 of [GG1], using the fact that if  $M = M_g$ , where  $g \geq 3$ , then  $M$  is homeomorphic to the connected sum of one or two copies of  $\mathbb{R}P^2$  with a compact, orientable surface without boundary of genus at least one.

Conversely, suppose that there exist  $m \in \mathbb{N}$  and  $n \geq 2$  for which the homomorphism  $p_*: P_{n+m}(M) \rightarrow P_n(M)$  admits a section. We shall argue for a contradiction. By [GG1, Proposition 3], it suffices to consider the case  $m = 1$ . We first analyse the general structure of the coefficients  $\alpha_{i,j,r}, \beta_{i,j,q}, \gamma_{k,l,r}, \eta_{k,l,q}$  defined by equation (7).

(a) Taking  $j = n$  in equation (16) implies that  $\gamma_{i,k,l} = 0$  for all  $1 \leq i \leq n-1$  and  $1 \leq k < l \leq g$ .

(b) By equation (27),

$$\gamma_{i,k,l} = -2\eta_{i,k,j} - 2(\alpha_{i,j,l} - \alpha_{i,j,k})$$

for all  $1 \leq i < j \leq n$  and  $1 \leq l < k \leq g$ . Taking  $j = n$ , we obtain

$$\gamma_{i,k,l} = -2\eta_{i,k,n} - 2(\alpha_{i,n,l} - \alpha_{i,n,k}) = 0$$

since  $\eta_{i,k,n} = 0$  by definition and  $\alpha_{i,n,r} = 2\gamma_{i,1,1}$  for all  $1 \leq i \leq n-1$  and  $1 \leq r \leq g$  by equation (22).

It thus follows from (a) and (b) that

$$\gamma_{i,k,l} = 0 \quad \text{for all } 1 \leq i \leq n-1 \text{ and } 1 \leq k, l \leq g, k \neq l. \quad (29)$$

(c) By equation (22),  $\gamma_{i,k,k} = \frac{1}{2}\alpha_{i,n,1}$  for all  $1 \leq i \leq n-1$  and  $1 \leq k \leq g$ . So

$$\gamma_{i,k,k} = \gamma_{i,1,1} \quad \text{for all } 1 \leq i \leq n-1 \text{ and } 1 \leq k \leq g. \quad (30)$$

(d) By equation (16), for all  $1 \leq k < l \leq g$  and  $1 \leq i < j \leq n$ , we have

$$\eta_{i,k,j} = -\frac{1}{2}\gamma_{i,k,l} = 0,$$

using equation (29). So by taking  $l = g$  we obtain

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq g-1.$$

(e) By equation (27)

$$\eta_{i,k,j} = \frac{1}{2} (2(\alpha_{i,j,l} - \alpha_{i,j,k}) + \gamma_{i,k,l})$$

for all  $1 \leq i < j \leq n$  and  $1 \leq l < k \leq g$ . But  $\gamma_{i,k,l} = 0$  by equation (29), and  $\alpha_{i,j,l} - \alpha_{i,j,k} = 0$  by equation (18) if  $j \leq n-1$  and by equation (22) if  $j = n$ . Setting  $l = 1$ , it follows that

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 2 \leq k \leq g.$$

By (d) and (e) we thus have

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq g. \quad (31)$$

(f) Suppose that  $1 \leq j < i \leq n - 1$ . Then

$$\begin{aligned}\eta_{i,k,j} &= -\frac{1}{2}\gamma_{i,k,l} \quad \text{for all } 1 \leq k < l \leq g, \text{ by equation (26)} \\ &= 0 \quad \text{by equation (29)}.\end{aligned}$$

So taking  $l = g$ , we have  $\eta_{i,k,j} = 0$  for all  $1 \leq k \leq g - 1$ . Further, for all  $1 \leq l < k \leq g$ ,

$$\begin{aligned}\eta_{i,k,j} &= -\frac{1}{2}\gamma_{i,k,l} \quad \text{by equation (15)} \\ &= 0 \quad \text{by equation (29)}.\end{aligned}$$

Hence it follows from equation (31) and (f) that

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i, j \leq n - 1, i \neq j, \text{ and } 1 \leq k \leq g. \quad (32)$$

(g) From equation (23), we obtain

$$\beta_{i,n,s} = -\gamma_{i,1,1} \quad \text{for all } 1 \leq s \leq n - 1, s \neq i. \quad (33)$$

(h) By equations (21) and (32), we see that

$$\gamma_{i,1,1} = \beta_{i,i+1,i+1} = \cdots = \beta_{i,n-1,n-1} \quad \text{for all } 1 \leq i \leq n - 2. \quad (34)$$

(i) By equations (20) and (32), we obtain

$$\gamma_{i,1,1} = \beta_{1,i,1} = \cdots = \beta_{i-1,i,i-1} \quad \text{for all } 2 \leq i \leq n - 1. \quad (35)$$

Analysing equation (28), we are now able to complete the proof of Theorem 1 as follows. Let  $i \in \{1, \dots, n - 1\}$ . Then the coefficient of  $B_{i,n+1}$  yields:

$$1 - 2 \sum_{l=1}^g \eta_{i,l,i} - \sum_{l=1}^g \gamma_{i,l,l} = \sum_{l=1}^{i-1} \beta_{l,i,i} + \sum_{l=i+1}^n \beta_{i,l,i} - 2 \sum_{l=1}^g \eta_{i,l,i}. \quad (36)$$

Now

$$\sum_{l=1}^{i-1} \beta_{l,i,i} = \sum_{l=1}^{i-1} \gamma_{l,1,1} \quad \text{by equation (34),}$$

and

$$\sum_{l=i+1}^n \beta_{i,l,i} = \sum_{l=i+1}^n \gamma_{l,1,1} \quad \text{by equation (35).}$$

So using equation (30), equation (36) becomes

$$1 - g\gamma_{i,1,1} = \beta_{i,n,i} + \sum_{l=1}^{n-1} \gamma_{l,1,1} - \gamma_{i,1,1}.$$

Summing over all  $i = 1, \dots, n - 1$ , and setting  $\Delta = \sum_{l=1}^{n-1} \gamma_{l,1,1}$  and  $L = \sum_{i=1}^{n-1} \beta_{i,n,i}$ , we obtain

$$(n + g - 2)\Delta = (n - 1) - L. \quad (37)$$

Now let  $i = n$ , and let  $k \in \{1, \dots, n-1\}$ . Since  $\eta_{n,l,n} = 0$ , the coefficient of  $B_{k,n+1}$  in equation (28) yields:

$$\begin{aligned}
\sum_{l=1}^g \gamma_{n,l,l} - 1 &= \sum_{l=1}^{n-1} \beta_{l,n,k} - 2 \sum_{l=1}^g \eta_{n,l,k} = \beta_{k,n,k} + \sum_{\substack{l=1 \\ l \neq k}}^{n-1} \beta_{l,n,k} - 2 \sum_{l=1}^g \eta_{n,l,k} \\
&= \beta_{k,n,k} - \sum_{\substack{l=1 \\ l \neq k}}^{n-1} \gamma_{l,1,1} - 2 \sum_{l=1}^g \eta_{n,l,k} \quad \text{by equation (33)} \\
&= \beta_{k,n,k} - (\Delta - \gamma_{k,1,1}) + \sum_{l=1}^g (-\beta_{k,n,k} + \gamma_{n,l,l} - \gamma_{k,l,l}) \quad \text{by equation (24)} \\
&= (1-g)\beta_{k,n,k} + \gamma_{k,1,1} - \Delta + \sum_{l=1}^g \gamma_{n,l,l} - \sum_{l=1}^g \gamma_{k,1,1} \quad \text{by equation (30)} \\
&= (1-g)\beta_{k,n,k} + (1-g)\gamma_{k,1,1} - \Delta + \sum_{l=1}^g \gamma_{n,l,l}.
\end{aligned}$$

Hence  $-1 = (1-g)\beta_{k,n,k} + (1-g)\gamma_{k,1,1} - \Delta$ . Summing over all  $k = 1, \dots, n-1$ , we obtain

$$(n+g-2)\Delta = (1-g)L + (n-1). \quad (38)$$

Equating equations (37) and (38), we see that  $(n-1) - L = (1-g)L + (n-1)$ . Since  $g \geq 3$ , it follows that  $L = 0$ , and therefore

$$\Delta = \frac{n-1}{(n-1) + (g-1)}$$

by equation (37). This yields a contradiction to the fact that  $\Delta$  is an integer, and thus completes the proof of Theorem 1.  $\square$

*Remark.* Although some of the relations derived in (a)–(i) do not exist if  $n = 2$ , one may check that the above analysis from equation (36) onwards is also valid in this case (with  $\Delta = \gamma_{1,1,1}$  and  $L = \beta_{1,2,1}$ ).

*Proof of Theorem 2.*

(a) If  $r > 0$  then the result follows applying the methods of the proofs of Proposition 27 and Theorem 6 of [GG1]. If  $r = 0$  and  $M$  has non-empty boundary, let  $C$  be a boundary component of  $M$ . Then  $M' = M \setminus C$  is homeomorphic to a compact surface with a single point deleted (which is the case  $r = 1$ ), so **(PBS)** splits for  $M'$ . The inclusion of  $M'$  in  $M$  not only induces a homotopy equivalence between  $M$  and  $M'$ , but also a homotopy equivalence between their  $n^{\text{th}}$  configuration spaces. Therefore their  $n^{\text{th}}$  pure braid groups are isomorphic, and the sequence **(PBS)** for  $M$  splits if and only if it splits for  $M'$ .

(b) Suppose that  $r = 0$  and that  $M$  is without boundary. If  $M = \mathbb{S}^2$ ,  $m = 1$  and  $n \geq 3$  then the statement follows from [Fa]. The geometric construction of Fadell may be easily generalised to all  $m \in \mathbb{N}$ . If  $n \in \{1, 2\}$ , the result is obvious since  $P_n(\mathbb{S}^2)$  is trivial. If  $M = \mathbb{T}^2$  or  $\mathbb{K}^2$ , the fact that  $p_*$  has a section is a consequence of [FaN] and the fact that  $\mathbb{T}^2$  and  $\mathbb{K}^2$



admit a non-vanishing vector field. If  $M = \mathbb{R}P^2$  then  $p_*$  admits a section if and only if  $n = 2$  and  $m = 1$  by [GG3]. Finally, if  $M \neq \mathbb{R}P^2, \mathbb{S}^2, \mathbb{T}^2, \mathbb{K}^2$  then  $p_*$  admits a section if and only if  $n = 1$  by Theorem 1 for the non-orientable case, and by [GG1] for the orientable case.  $\square$

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