Braid groups of non-orientable surfaces and the Fadell-Neuwirth short exact sequence

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Abstract
Let $M$ be a compact, connected non-orientable surface without boundary and of genus $g \geq 3$. We investigate the pure braid groups $P_n(M)$ of $M$, and in particular the possible splitting of the Fadell-Neuwirth short exact sequence

$$1 \rightarrow P_m(M \setminus \{x_1, \ldots, x_n\}) \rightarrow P_{n+m}(M) \xrightarrow{p_*} P_n(M) \rightarrow 1,$$

where $m, n \geq 1$, and $p_*$ is the homomorphism which corresponds geometrically to forgetting the last $m$ strings. This problem is equivalent to that of the existence of a section for the associated fibration $p: F_{n+m}(M) \rightarrow F_n(M)$ of configuration spaces, defined by $p((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})) = (x_1, \ldots, x_n)$. We show that $p$ and $p_*$ admit a section if and only if $n = 1$. Together with previous results, this completes the resolution of the splitting problem for surfaces pure braid groups.

1 Introduction
Braid groups of the plane were defined by Artin in 1925 [A1], and further studied in [A2, A3]. Braid groups of surfaces were studied by Zariski [Z], and were later generalised using the following definition due to Fox [FoN]. Let $M$ be a compact, connected surface, and let $n \in \mathbb{N}$. We denote the set of all ordered $n$-tuples of distinct points of $M$, known as the $n^{th}$ configuration space of $M$, by:

$$F_n(M) = \{(p_1, \ldots, p_n) \mid p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j\}.$$
Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [CG, FH] for example.

The symmetric group $S_n$ on $n$ letters acts freely on $F_n(M)$ by permuting coordinates. The corresponding quotient space will be denoted by $D_n(M)$. Notice that $F_n(M)$ is a regular covering of $D_n(M)$. The $n^{th}$ pure braid group $P_n(M)$ (respectively the $n^{th}$ braid group $B_n(M)$) is defined to be the fundamental group of $F_n(M)$ (respectively of $D_n(M)$). If $m \in \mathbb{N}$, then we may define a homomorphism $p_*: P_{n+m}(M) \to P_n(M)$ induced by the projection $p: F_{n+m}(M) \to F_n(M)$ defined by $p((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})) = (x_1, \ldots, x_n)$. Representing $P_{n+m}(M)$ geometrically as a collection of $n + m$ strings, $p_*$ corresponds to forgetting the last $m$ strings. **We adopt the convention, that unless explicitly stated, all homomorphisms $P_{n+m}(M) \to P_n(M)$ in the text will be this one.** If $M$ is the 2-disc (or the plane $\mathbb{R}^2$), $B_n(M)$ and $P_n(M)$ are respectively the classical Artin braid group $B_n$ and pure braid group $P_n$ [FVB].

If $M$ is without boundary, Fadell and Neuwirth study the map $p_*$ and show ([FaN, Theorem 3]) that it is a locally-trivial fibration. The fibre over a point $(x_1, \ldots, x_n)$ of the base space is $F_n(M\setminus\{x_1, \ldots, x_n\})$ which we interpret as a subspace of the total space via the map $i: F_n(M\setminus\{x_1, \ldots, x_n\}) \to F_n(M)$ defined by $i((y_1, \ldots, y_m)) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$.

Applying the associated long exact sequence in homotopy, we obtain the **pure braid group short exact sequence of Fadell and Neuwirth:**

$$1 \longrightarrow P_m(M\setminus\{x_1, \ldots, x_n\}) \xrightarrow{i_*} P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1,$$

(\text{PBS})

where $n \geq 3$ if $M$ is the sphere $S^2$ [Fa, FVB], $n \geq 2$ if $M$ is the real projective plane $\mathbb{R}P^2$ [VB], and $n \geq 1$ otherwise [FaN], and where $i_*$ and $p_*$ are the homomorphisms induced by the maps $i$ and $p$ respectively. The short exact sequence (PBS) has been widely studied, and may be employed for example to determine presentations of $P_n(M)$ (see Section 2), its centre, and possible torsion. It was also used in recent work on the structure of the mapping class groups [PR] and on Vassiliev invariants for surface braids [GMP].

In the case of $P_n$, and taking $m = 1$, Ker $(p_*)$ is a free group of rank $n$. The short exact sequence (PBS) splits for all $n \geq 1$, and so $P_n$ may be described as a repeated semi-direct product of free groups. This decomposition, known as the ‘combing’ operation, is the principal result of Artin’s classical theory of braid groups [A2], and yields normal forms and a solution to the word problem in $B_n$. More recently, it was used by Falk and Randell to study the lower central series and the residual nilpotence of $P_n$ [FR], and by Rolfsen and Zhu to prove that $P_n$ is bi-orderable [RZ].

The problem of deciding whether such a decomposition exists for braid groups of surfaces is thus fundamental. This was indeed a recurrent and central question during the foundation of the theory and its subsequent development during the 1960’s [Fa, FaN, FVB, VB, Bi]. If the fibre of the fibration is an Eilenberg-MacLane space then the existence of a section for $p_*$ is equivalent to that of a cross-section for $p$ [Ba, Wh] (cf. [GG2]). But with the exception of the construction of sections in certain cases (for $S^2$ [Fa] and the 2-torus $\mathbb{T}^2$ [Bi]), no progress on the possible splitting of (PBS) was recorded for nearly forty years. In the case of orientable surfaces without boundary of genus at least two, the
question of the splitting of (PBS) which was posed explicitly by Birman in 1969 [Bi], was finally resolved by the authors, the answer being positive if and only if \( n = 1 \) [GG1].

As for the non-orientable case, the braid groups of \( \mathbb{R}P^2 \) were first studied by Van Buskirk [VB], and more recently by Wang [Wa] and the authors [GG2, GG3, GG4]. For \( n = 1 \), we have \( P_1(\mathbb{R}P^2) = B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \). Van Buskirk showed that for all \( n \geq 2 \), neither the fibration \( p: F_n(\mathbb{R}P^2) \to F_1(\mathbb{R}P^2) \) nor the homomorphism \( p_\ast: P_n(\mathbb{R}P^2) \to P_1(\mathbb{R}P^2) \) admit a cross-section (for \( p \), this is a manifestation of the fixed point property of \( \mathbb{R}P^2 \), but that the fibration \( p: F_3(\mathbb{R}P^2) \to F_2(\mathbb{R}P^2) \) admits a cross-section, and hence so does the corresponding homomorphism \( p_\ast \). Using coincidence theory, we showed that for \( n = 2, 3 \) and \( m \geq 4 - n \), neither the fibration nor the short exact sequence (PBS) admit a section [GG2]. In [GG3], we gave a complete answer to the splitting problem for \( \mathbb{R}P^2 \): if \( m, n \in \mathbb{N} \), the homomorphism \( p_\ast: P_{n+m}(\mathbb{R}P^2) \to P_n(\mathbb{R}P^2) \) and the fibration \( p: F_{n+m}(\mathbb{R}P^2) \to F_n(\mathbb{R}P^2) \) admit a section if and only if \( n = 2 \) and \( m = 1 \). In other words, Van Buskirk’s values \((n = 2 \text{ and } m = 1)\) are the only ones for which a section exists (both on the geometric and the algebraic level).

In this paper, we study the splitting problem for compact, connected non-orientable surfaces without boundary and of genus \( g \geq 3 \) (every non-orientable compact surface \( M \) without boundary is homeomorphic to the connected sum of \( g \) copies of \( \mathbb{R}P^2 \), \( g \in \mathbb{N} \) being the genus of \( M \)). In the case of the Klein bottle \( \mathbb{K}^2 \) \((g = 2)\), the existence of a non-vanishing vector field implies that there always exists a section, both geometric and algebraic (cf. [FaN]). Our main result is:

**Theorem 1.** Let \( M \) be a compact, connected, non-orientable surface without boundary of genus \( g \geq 3 \), and let \( m, n \in \mathbb{N} \). Then the homomorphism \( p_\ast: P_{n+m}(M) \to P_n(M) \) and the fibration \( p: F_{n+m}(M) \to F_n(M) \) admit a section if and only if \( n = 1 \).

Applying Theorem 1 and the results of [GG1, GG3], we may solve completely the splitting problem for surface pure braid groups:

**Theorem 2.** Let \( m, n \in \mathbb{N} \) and \( r \geq 0 \). Let \( N \) be a compact, connected surface possibly with boundary, let \( \{x_1, \ldots, x_r\} \) be a finite subset in the interior of \( N \), let \( M = N \setminus \{x_1, \ldots, x_r\} \), and let \( p_\ast: P_{n+m}(M) \to P_n(M) \) be the standard projection.

(a) If \( r > 0 \) or if \( M \) has non-empty boundary then \( p_\ast \) admits a section for all \( m \) and \( n \).

(b) Suppose that \( r = 0 \) and that \( M \) is without boundary. Then \( p_\ast \) admits a section if and only if one of the following conditions holds:

(i) \( M \) is \( S^2 \), the 2-torus \( \mathbb{T}^2 \) or the Klein bottle \( \mathbb{K}^2 \) (for all \( m \) and \( n \)).

(ii) \( M = \mathbb{R}P^2 \), \( n = 2 \) and \( m = 1 \).

(iii) \( M \neq \mathbb{R}P^2, S^2, \mathbb{T}^2, \mathbb{K}^2 \) and \( n = 1 \).

The rest of the paper is organised as follows. In Section 2, we determine a presentation of \( P_n(M) \) (Theorem 3). In Section 3, we study the consequences of the existence of a section in the case \( m = 1 \) and \( n \geq 2 \), i.e. \( p_\ast: P_{n+1}(M) \to P_n(M) \). The general strategy of the proof of Theorem 1 is based on the following remark. Suppose that (PBS) splits. If \( H \) is any normal subgroup of \( P_{n+1}(M) \) contained in \( \text{Ker} \ (p_\ast) \), the quotiented short exact sequence \( 1 \to \text{Ker} \ (p_\ast) / H \to P_{n+1}(M) / H \to P_n(M) \to 1 \) must also split. In order to obtain a contradiction, we seek such a subgroup \( H \) for which this short exact sequence does not split. However the choice of \( H \) needed to achieve this may be somewhat delicate:
if $H$ is too ‘small’, the structure of the quotient $P_{n+1}(\mathbb{R}P^2)/H$ remains complicated; on
the other hand, if $H$ is too ‘large’, we lose too much information and cannot reach a conclusion. In Section 4, we first show that we may reduce to the case $m = 1$, and then go on to prove Theorem 1 using the analysis of Section 3. As we shall see in Section 4, it suffices to take $H$ to be Abelianisation of $\ker (p_s)$, in which case the quotient $\ker (p_s)/H$ is a free Abelian group. We will then deduce Theorem 2.

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2 A presentation of $P_n(M)$

Let $M = M_g$ be a compact, connected, non-orientable surface without boundary of genus $g \geq 2$. If $n \in \mathbb{N}$ and $\mathbb{D}^2 \subseteq M$ is a topological disc, the inclusion induces a homomorphism $i: B_n(\mathbb{D}^2) \longrightarrow B_n(M)$. If $\beta \in B_n(\mathbb{D}^2)$ then we shall denote its image $i(\beta)$ simply by $\beta$. For $1 \leq i < j \leq n$, we consider the following elements of $P_n(M)$:

$$B_{i,j} = \sigma_1^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i \cdots \sigma_{j-1}^{-1} \sigma_j \cdots \sigma_n$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of $B_n(\mathbb{D}^2)$. The geometric braid corresponding to $B_{i,j}$ takes the $i^{th}$ string once around the $j^{th}$ string in the positive sense, with all other strings remaining vertical. For each $1 \leq k \leq n$ and $1 \leq l \leq g$, we define a generator $\rho_{k,l}$ which is represented geometrically by a loop based at the $k^{th}$ point and which goes round the $l^{th}$ twisted handle. These elements are illustrated in Figure 1 that represents $M$ minus a disc.

A presentation of the braid groups of non-orientable surfaces was originally given by Scott [S]. Other presentations were later obtained in [Be, GM]. In the following theorem, we derive another presentation of $P_n(M)$.

**Theorem 3.** Let $M$ be a compact, connected, non-orientable surface without boundary of genus $g \geq 2$, and let $n \in \mathbb{N}$. The following constitutes a presentation of the pure braid group $P_n(M)$:

**generators:** $B_{i,j}$, $1 \leq i < j \leq n$, and $\rho_{k,l}$, where $1 \leq k \leq n$ and $1 \leq l \leq g$.

**relations:**

(a) the Artin relations between the $B_{i,j}$ emanating from those of $P_n(\mathbb{D}^2)$:

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j}B_{r,s}B_{i,j} & \text{if } r < i = s < j \\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i = r < s < j \\ B_{s,j}^{-1}B_{r,s}^{-1}B_{s,j}B_{r,s}B_{i,j}B_{r,s}^{-1}B_{s,j} & \text{if } r < i < s < j. \end{cases}$$

(1)
Figure 1: The generators $B_{i,j}$ and $\rho_{k,l}$ of $P_n(M)$, represented geometrically by loops lying in $M$ minus a disc.

(b) for all $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,

$$
\rho_{i,k} B_{i,j} \rho_{i,k}^{-1} = \begin{cases} 
\rho_{j,l} & \text{if } k < l \\
\rho_{j,k}^{-1} B_{i,j} \rho_{j,k} \rho_{i,j} & \text{if } k = l \\
\rho_{j,k}^{-1} B_{i,j} \rho_{j,k} B_{i,j} & \rho_{i,j} B_{i,j} \rho_{j,k}^{-1} B_{i,j} B_{i,j} & \text{if } k > l
\end{cases}
$$

(2)

(c) for all $1 \leq i \leq n$, the ‘surface relations’

$$
\prod_{i=1}^{g} \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i+1} \cdots B_{i,n}.
$$

(d) for all $1 \leq i < j \leq n$, $1 \leq k \leq n$, $k \neq j$, and $1 \leq l \leq g$,

$$
\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} 
B_{i,j} & \text{if } k < i \text{ or } j < k \\
B_{i,j}^{-1} B_{i,j} \rho_{j,l} & \text{if } k = i \\
B_{i,j}^{-1} B_{i,j} B_{i,j}^{-1} B_{i,j} B_{i,j} B_{i,j}^{-1} B_{i,j} \rho_{j,l} & \text{if } i < k < j
\end{cases}
$$

(3)

Proof. We apply induction and standard results concerning the presentation of an extension (see [J, Theorem 1, Chapter 13]). The proof generalises that of [GG3] for $\mathbb{R}P^2$, and is similar in spirit to that of [S].

First note that the given presentation is correct for $n = 1$ ($P_1(M) = \pi_1(M)$ has a presentation $\left\langle \rho_{1,1}, \ldots, \rho_{1,g} \mid \prod_{i=1}^{g} \rho_{1,l}^2 = 1 \right\rangle$). So let $n \geq 1$, and suppose that $P_n(M)$ has
the given presentation. Taking $m = 1$ in (PBS), we have a short exact sequence:

$$1 \longrightarrow \pi_1(M \setminus \{x_1, \ldots, x_n\}, x_{n+1}) \longrightarrow P_{n+1}(M) \xrightarrow{\rho_*} P_n(M) \longrightarrow 1.$$ 

In order to retain the symmetry of the presentation, we take the free group $\text{Ker} (p_*)$ to have the following one-relator presentation:

$$\left\langle \rho_{n+1,1}, \ldots, \rho_{n+1,g}, B_{1,n+1}, \ldots, B_{n,n+1} \middle| \prod_{l=1}^g \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1} \right\rangle.$$ 

Together with these generators of $\text{Ker} (p_*)$, the elements $B_{i,j}$, $1 \leq i < j \leq n$, and $\rho_{k,l}$, $1 \leq k \leq n$ and $1 \leq l \leq g$, of $P_{n+1}(M)$ (which are coset representatives of the generators of $P_n(M)$) form the given generating set of $P_{n+1}(M)$.

There are three classes of relations of $P_{n+1}(M)$ which are obtained as follows. The first consists of the single relation $\prod_{l=1}^g \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1}$ of $\text{Ker} (p_*)$. The second class is obtained by rewriting the relations of the quotient in terms of the coset representatives, and expressing the corresponding element as a word in the generators of $\text{Ker} (p_*)$. In this way, all of the relations of $P_n(M)$ lift directly to relations of $P_{n+1}(M)$, with the exception of the surface relations which become

$$\prod_{l=1}^g \rho_{n+1,l}^2 = B_{1,j} \cdots B_{i-1,j} B_{i,i+1} \cdots B_{i,n} B_{i,n+1} \quad \text{for all } 1 \leq i \leq n.$$ 

Along with the relation of $\text{Ker} (p_*)$, we obtain the complete set of surface relations (relations (c)) for $P_{n+1}(M)$.

The third class of relations is obtained by rewriting the conjugates of the generators of $\text{Ker} (p_*)$ by the coset representatives in terms of the generators of $\text{Ker} (p_*)$:

(i) For all $1 \leq i < j \leq n$ and $1 \leq l \leq n$,

$$B_{i,j} B_{l,n+1} B_{i,j}^{-1} = \begin{cases} B_{l,n+1} & \text{if } l < i \text{ or } j < l \\ B_{i,n+1}^{-1} B_{l,n+1} B_{l,n+1} B_{n,n+1} B_{i,n+1} & \text{if } l = j \\ B_{i,n+1}^{-1} B_{l,n+1} B_{j,n+1} & \text{if } l = i \\ B_{j,n+1}^{-1} B_{i,n+1} B_{n+1} B_{n,n+1} B_{i,n+1} B_{j,n+1}^{-1} B_{i,n+1} B_{j,n+1} & \text{if } i < l < j. \end{cases}$$

(ii) $B_{i,j} \rho_{n+1,1} B_{i,j}^{-1} = \rho_{n+1,1}$ for all $1 \leq i < j \leq n$ and $1 \leq l \leq g$.

(iii) For all $1 \leq i \leq n$ and $1 \leq k, l \leq g$,

$$\rho_{i,k} \rho_{n+1,1} \rho_{i,k}^{-1} = \begin{cases} \rho_{n+1,1} & \text{if } k < l \\ \rho_{n+1,1} B_{i,n+1} \rho_{n+1,1}^{-1} & \text{if } k = l \\ \rho_{n+1,1} B_{i,n+1} \rho_{n+1,1}^{-1} \rho_{n+1,1}^{-1} B_{i,n+1} \rho_{n+1,1}^{-1} & \text{if } k > l. \end{cases}$$

(iv) For all $1 \leq i, k \leq n$ and $1 \leq l \leq g$,

$$\rho_{k,l} \rho_{i,n+1} \rho_{k,l}^{-1} = \begin{cases} B_{i,n+1} & \text{if } k < i \\ \rho_{n+1,1} B_{i,n+1} \rho_{n+1,1}^{-1} & \text{if } k = i \\ \rho_{n+1,1} B_{k,n+1} \rho_{n+1,1}^{-1} B_{k,n+1} \rho_{n+1,1}^{-1} B_{k,n+1} \rho_{n+1,1}^{-1} & \text{if } i < k. \end{cases}$$
Then relations (a) for $P_{n+1}(M)$ are obtained from relations (a) for $P_n(M)$ and relations (i), relations (b) for $P_{n+1}(M)$ are obtained from relations (b) for $P_n(M)$ and relations (iii), and relations (d) for $P_{n+1}(M)$ are obtained from relations (d) for $P_n(M)$, and relations (ii) and (iv).

\[ \square \]

3 Analysis of the case $P_{n+1}(M_g) \longrightarrow P_n(M_g), n \geq 2$

For the whole of this section, we suppose that $g \geq 3$ and $n \geq 2$. By Theorem 3, $P_n(M_g)$ is generated by the union of the $B_{i,j}$, $1 \leq i < j \leq n$, and of the $\rho_{k,l}$, $1 \leq k \leq n$, $1 \leq l \leq g$. Let us consider the homomorphism $p_s : P_{n+1}(M_g) \longrightarrow P_n(M_g)$. In this section, we suppose that $p_s$ admits a section, denoted by $s_a$. Applying (PBS), we thus have a split short exact sequence

\[ 1 \longrightarrow K \longrightarrow P_{n+1}(M_g) \overset{p_s}{\longrightarrow} P_n(M_g) \longrightarrow 1, \]

(4)

where $K = \text{Ker}(p_s) = \pi_1(M_g \setminus \{x_1, \ldots, x_n, x_{n+1}\}$ is a free group of rank $n + g - 1$, generated by $\{B_{1,n+1}, \ldots, B_{n,n+1}, \rho_{n+1,1}, \ldots, \rho_{n+1,g}\}$, and subject to the relation

\[ B_{1,n+1} \cdots B_{n,n+1} = \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2. \]

Let $H = [K, K]$ be the commutator subgroup of $K$. Then $K/H$ is a free Abelian group of rank $n + g - 1$. In what follows, we shall not distinguish notationally between the elements of $K$ and those of $K/H$. The quotient group $K/H$ thus has a basis

\[ B = \{B_{1,n+1}, \ldots, B_{n-1,n+1}, \rho_{n+1,1}, \ldots, \rho_{n+1,g}\}, \]

(5)

and the relation

\[ B_{n,n+1} = B_{1,n+1}^{-1} B_{n-1,n+1}^{-1} \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2 \]

(6)

holds in the Abelian group $K/H$. Since $H$ is normal in $P_{n+1}(M_g)$ and $p_s$ admits a section, it follows from equation (4) that we have a split short exact sequence

\[ 1 \longrightarrow K/H \longrightarrow P_{n+1}(M_g)/H \overset{\overline{p}}{\longrightarrow} P_n(M_g) \longrightarrow 1, \]

where $\overline{p}$ is the homomorphism induced by $p_s$, and $\overline{s}$ is the induced section.

Consider the subset

\[ \Gamma = \{B_{i,j} \rho_{k,l} \mid 1 \leq i < j \leq n, 1 \leq k \leq n, 1 \leq l \leq g\} \]

of $P_{n+1}(M_g)/H$. If $g \in \Gamma$ then $\overline{p}(g) = g \in P_n(M_g)$, and so $g^{-1} \overline{s}(\overline{p}(g)) \in \text{Ker}(\overline{p}) = K/H$. Then the integer coefficients $a_{i,j,r}, \beta_{i,j,l}, \gamma_{j,i,r}, \eta_{j,i,l}$, where $1 \leq r \leq g$ and $1 \leq q \leq n - 1$, are (uniquely) defined by the equations:

\[ \overline{s} \left( B_{i,j} \right) = B_{i,j}^{a_{i,j,1}} \cdots \rho_{n+1,1}^{a_{i,j,n+1}} B_{n+1,n+1}^{B_{i,j,n+1}} \rho_{n+1,1}^{B_{i,j,n+1-1}} \rho_{n+1,1}^{B_{i,j,n+1-1}} \cdots \rho_{n+1,1}^{B_{i,j,n+1-1}}. \]

(7)
There is an equation for each element of $\Gamma$. Most of the elements of $\Gamma$ commute with the elements of the basis $B$ of $K/H$ given in equation (5). We record the list of conjugates of such elements for later use. In what follows, $1 \leq i < j \leq n$, $1 \leq k, m \leq n$ and $1 \leq l, r \leq g$. In $K/H$, we have

$$B_{i,j}B_{m,n+1}B_{i,j}^{-1} = B_{m,n+1}$$

(this follows from equation (1) and the fact that the elements $B_{q,n+1}$, $1 \leq q \leq n$, belong to $K/H$ and so commute pairwise), and

$$B_{i,j}\rho_{n+1,l}B_{i,j}^{-1} = \rho_{n+1,l}$$

by equation (3). Thus $B_{i,j}$ belongs to the centraliser of $K/H$ in $P_{n+1}(M_g)/H$. Also by equation (3), we have

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = \begin{cases} B_{m,n+1} & \text{if } k < m \\ \rho_{n+1,l}B_{m,n+1}\rho_{n+1,l} = B_{m,n+1}^{-1} & \text{if } k = m \\ \rho_{n+1,l}B_{k,n+1}\rho_{n+1,l}B_{k,n+1} = B_{m,n+1}^{-1} & \text{if } k > m, \end{cases}$$

so

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = B_{m,n+1}^{1-2\delta_{k,m}},$$

where $\delta_{k,m}$ is the Kronecker delta. By equation (2), we have

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \begin{cases} \rho_{n+1,r} & \text{if } l < r \\ \rho_{n+1,r}B_{k,n+1}^{-1} & \text{if } l = r \\ \rho_{n+1,r}B_{k,n+1}\rho_{n+1,r}B_{k,n+1} = \rho_{n+1,r} & \text{if } l > r, \end{cases}$$

so

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-\delta_{l,r}}.$$  \hfill (8)

Combining equations (8) and (9), we obtain

$$\rho_{k,r}\rho_{n+1,r}\rho_{k,r}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-1} = \rho_{n+1,r},$$

so

$$\rho_{k,r}\rho_{n+1,r}\rho_{k,r}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-1}.$$  \hfill (9)

Furthermore, by equation (8), $\rho_{k,l}$ commutes with $B_{m,n+1}$, and therefore

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = B_{m,n+1}^{1-\delta_{k,l}}.$$  \hfill (10)

Hence $\rho_{k,l}$ also belongs to the centraliser of $K/H$ in $P_{n+1}(M_g)/H$. From equations (8) and (9), we obtain the following relations:

$$\gamma_{i,k,l}^{\gamma_{i,k,l}} \gamma_{i,k,g}^{\gamma_{i,k,g}} \rho_{j,l} = \rho_{j,l} \gamma_{i,k,l} \gamma_{i,k,g}^{\gamma_{i,k,g}} \text{ for all } 1 \leq j \leq n,$$  \hfill (12)

and

$$B_{1,n+1}^{\gamma_{i,k,l}} \cdots B_{n-1,n+1}^{\gamma_{i,k,g}} \rho_{j,l} = \begin{cases} \rho_{j,l}B_{1,n+1}^{\gamma_{i,k,l}} \cdots B_{n-1,n+1}^{\gamma_{i,k,g}} & \text{if } 1 \leq j \leq n-1 \\ \rho_{j,l}B_{1,n+1}^{\gamma_{i,k,l}} \cdots B_{n-1,n+1}^{\gamma_{i,k,g}} & \text{if } j = n, \end{cases}$$  \hfill (12)
Setting $\eta_{i,k,n} = 0$ for all $1 \leq i \leq n$ and $1 \leq k \leq g$ yields:

$$B_{n-1,n+1}^{\eta_{i,k,n}} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdot \rho_{j,l} = \rho_{j,l} \cdot B_{n+1,n+1}^{\eta_{i,k,n}} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1}$$

for all $1 \leq j \leq n$. \hfill (13)

Equations (12) and (13) will be employed repeatedly in the ensuing calculations.

We now investigate the images under $\mathfrak{s}$ of some of the relations (b)–(d) of Theorem 3 (it turns out that the analysis of the other relations, including (a), will not be necessary for our purposes).

(a) Let $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$. We examine the three possible cases of equation (7) (relation (b) of Theorem 3).

(i) $k < l$: then $\rho_{i,k} \rho_{j,l} = \rho_{j,l} \rho_{i,k}$ in $P_n(M_g)$. The respective images under $\mathfrak{s}$ are:

$$\mathfrak{s}(\rho_{i,k} \rho_{j,l}) = \rho_{i,k} \rho_{n+1,1} \cdots \rho_{i,k} \rho_{n+1,1} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n+1,n+1}^{\eta_{i,k,n-1}} \cdots \rho_{n+1,1} \rho_{n+1,1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1},$$

and

$$\mathfrak{s}(\rho_{j,l} \rho_{i,k}) = \rho_{j,l} \rho_{n+1,1} \cdots \rho_{j,l} \rho_{n+1,1} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n+1,n+1}^{\eta_{i,k,n-1}} \cdots \rho_{n+1,1} \rho_{n+1,1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1}.$$

The relation $\rho_{i,k} \rho_{j,l} = \rho_{j,l} \rho_{i,k}$ in $P_{n+1}(M_g)$ implies that $B_{n+1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n,n+1}^{\eta_{i,k,n-1}}$.

Comparing coefficients of the elements of $B$ in $K/H$ (cf. equation (5)), if $j < n$, we have

$$\begin{cases} 
\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 & \text{and} \\
\gamma_{i,k,l} + 2\eta_{i,k,j} = 0, 
\end{cases}$$

while if $j = n$, applying equation (6) yields

$$B_{n+1,n+1}^{\gamma_{j,l,k} + 2\eta_{j,l,i}} = B_{n+1,n+1}^{\gamma_{i,k,l} + 2\eta_{i,k,j}} = B_{n+1,n+1}^{\gamma_{j,l,k} + 2\eta_{j,l,i}} \cdots B_{n,n+1}^{\gamma_{j,l,k} + 2\eta_{j,l,i}} \cdots B_{n-1,n+1}^{\gamma_{j,l,k} + 2\eta_{j,l,i}},$$

and thus equation (14) also holds for $j = n$. So for all $1 \leq i < j \leq n$ and $1 \leq k < l \leq g$,

$$\begin{cases} 
\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 & \text{and} \\
\gamma_{i,k,l} + 2\eta_{i,k,j} = 0. 
\end{cases}$$

(ii) $k = l$: then $\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \rho_{l,k}^{-1} B_{l,j} \rho_{l,k}$ in $P_n(M_g)$ for all $1 \leq i < j \leq n$ and $1 \leq k \leq g$. The respective images under $\mathfrak{s}$ are:

$$\mathfrak{s}(\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1}) = \rho_{i,k} \rho_{n+1,1} \cdots \rho_{i,k} \rho_{n+1,1} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n+1,n+1}^{\eta_{i,k,n-1}} \cdots \rho_{n+1,1} \rho_{n+1,1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1},$$

and

$$\mathfrak{s}(\rho_{j,l} \rho_{i,k} \rho_{i,k}^{-1}) = \rho_{j,l} \rho_{n+1,1} \cdots \rho_{j,l} \rho_{n+1,1} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdots B_{n+1,n+1}^{\eta_{i,k,n-1}} \cdots \rho_{n+1,1} \rho_{n+1,1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1} \cdots B_{n,n+1}^{\eta_{i,k,n-1}} \cdots B_{n-1,n+1}.$$
and

\[
\mathbf{s} \left( \rho_{j,k}^{-1} B_{ij}^{n+1} \right) = B_{n-1,n+1} \cdots B_{1,n+1}^{n} \rho_{n+1} \cdots \rho_{n+1} \rho_{j,k}. \\
\mathbf{s} \left( \frac{B_{n-1,n+1}}{1} \cdots B_{1,n+1}^{n} \rho_{n+1} \cdots \rho_{n+1} \rho_{j,k}^{-1} \right) = B_{n-1,n+1}^{n} \cdots B_{1,n+1}^{n} \rho_{n+1} \cdots \rho_{n+1} \rho_{j,k}^{-1}.
\]

Since \( \rho_{i,k} \rho_{j,k} \rho_{i,k}^{-1} = \rho_{i,k}^{-1} B_{i,j}^{n+1} (M_{g}) \), we obtain

\[
B_{i,j}^{n+1} = \rho_{i,k} \rho_{j,k} \rho_{i,k}^{-1} = \rho_{i,k}^{-1} B_{i,j}^{n+1} (M_{g}) = B_{i,j}^{n+1}. 
\]

If \( j < n \) then all of the terms in equation (17) are expressed in terms of the basis \( B \) of \( K/H \) of equation (5), and so for all \( 1 \leq i < j \leq n-1, \)

\[
\alpha_{i,j,r} = 0 \quad \text{for all } 1 \leq r \leq g \\
\beta_{i,j,s} = 0 \quad \text{for all } 1 \leq s \leq n-1, \quad s \neq \{i,j\} \\
\beta_{i,j,i} = \gamma_{i,k,k} + 2\eta_{i,k} \\
\beta_{i,j,j} = \gamma_{i,k,k} + 2\eta_{i,k}.
\]

If \( j = n \) then substituting for \( B_{n,n+1} \) in equation (17) using equation (6) and comparing coefficients in \( K/H \) of the elements of \( B \) yields

\[
2(2\eta_{i,k,n} + \gamma_{i,k,k}) = \alpha_{i,n,r} \quad \text{for all } 1 \leq r \leq g \\
(2\eta_{i,k,n} + \gamma_{i,k,k}) = -\beta_{i,n,s} \quad \text{for all } 1 \leq s \leq n-1, \quad s \neq i \\
2(\eta_{i,k,n} - \eta_{i,k,j}) + (\gamma_{i,k,k} - \eta_{i,k,k}) = -\beta_{i,n,i}.
\]

But \( \eta_{i,k,n} = 0, \) so for all \( 1 \leq i \leq n-1 \) and \( 1 \leq k \leq g, \)

\[
\alpha_{i,n,r} = 2\gamma_{i,k,k} \quad \text{for all } 1 \leq r \leq g \\
\beta_{i,n,s} = -\gamma_{i,k,k} \quad \text{for all } 1 \leq s \leq n-1, \quad s \neq i \\
\beta_{i,n,i} = 2\eta_{i,k,n} + (\gamma_{i,k,k} - \eta_{i,k,k}).
\]

(iii) \( k > l: \) then \( \rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \rho_{i,k}^{-1} B_{i,j}^{n+1} \rho_{j,l} B_{i,j}^{n+1} \rho_{j,l}^{-1} B_{i,j} \rho_{j,l} \rho_{i,k} \) in \( P_{n} (M_{g}) \). The respective images under \( \mathbf{s} \) are:

\[
\mathbf{s} \left( \rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} \right) = \rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} B_{i,j}^{n+1} \rho_{j,l} B_{i,j}^{n+1} \rho_{j,l}^{-1} B_{i,j} \rho_{j,l} \rho_{i,k}^{-1}.
\]
and

\[ \pi \left( \rho_{j,k}^{-1} B_{l,j}^{-1} \rho_{j,k} B_{l,j} \rho_{j,k}^{-1} \rho_{l,j} B_{l,j} \rho_{j,k}^{-1} \right) = \]

\[ B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{n-1,n+1}^{-\gamma_{j,k,1}} \rho_{n+1,1}^{-\eta_{j,k,n-1}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} B_{n-1,n+1}^{-\beta_{l,j,n-1}} \cdots B_{n-1,n+1}^{-\beta_{l,j,1}} \rho_{n+1,1}^{-\alpha_{l,j,1}} \cdots \rho_{n+1,1}^{-\alpha_{l,j,1}} B_{n-1,n+1}^{-1}. \]

\[ \rho_{j,k}^{-\eta_{j,k,n-1}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{n-1,n+1}^{-\gamma_{j,k,1}} \rho_{n+1,1}^{-\alpha_{l,j,1}} \cdots \rho_{n+1,1}^{-\alpha_{l,j,1}} B_{n-1,n+1}^{-1}. \]

\[ \rho_{j,k}^{-\eta_{j,k,n-1}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{n-1,n+1}^{-\gamma_{j,k,1}} \rho_{n+1,1}^{-\alpha_{l,j,1}} \cdots \rho_{n+1,1}^{-\alpha_{l,j,1}} B_{n-1,n+1}^{-1}. \]

\[ \rho_{j,k}^{-\eta_{j,k,n-1}} \cdots \rho_{n+1,1}^{-\gamma_{j,k,1}} B_{n-1,n+1}^{-\eta_{j,k,n-1}} \cdots B_{n-1,n+1}^{-\gamma_{j,k,1}} \rho_{n+1,1}^{-\alpha_{l,j,1}} \cdots \rho_{n+1,1}^{-\alpha_{l,j,1}} B_{n-1,n+1}^{-1}. \]

Since \( \rho_{j,k}^{-1} B_{l,j}^{-1} \rho_{j,k} B_{l,j} \rho_{j,k}^{-1} \rho_{l,j} B_{l,j} \rho_{j,k}^{-1} \rho_{l,j} \) in \( P_{n+1}(M_g) \), we see that

\[ B_{n-1,n+1}^{-\gamma_{j,k,l}-2\eta_{j,k,l}} = B_{n-1,n+1}^{2\alpha_{l,j,k}+\gamma_{j,k,l}+2\eta_{j,k,l}}. \]

If \( j < n \), it follows by comparing coefficients of the elements of \( B \) in \( K/H \) that for all \( 1 \leq i < j < n \) and \( 1 \leq l < k \leq g \),

\[ \begin{cases} 
\gamma_{j,k,l} + 2\eta_{j,k,l} = 0 \\
2\alpha_{l,j,k} - 2\alpha_{l,j,k} - \gamma_{j,k,l} + 2\eta_{j,k,l} = 0. 
\end{cases} \tag{25} \]

If \( j = n \) then

\[ B_{n-1,n+1}^{-\gamma_{n,j,k}-2\eta_{n,j,k}} = B_{n-1,n+1}^{-2\alpha_{n,j,k}+\gamma_{n,j,k}+2\eta_{n,j,k}} \\
= B_{n-1,n+1}^{-2\alpha_{n,j,k}+\gamma_{n,j,k}+2\eta_{n,j,k}} \cdots B_{n-1,n+1}^{-2\alpha_{n,j,k}+\gamma_{n,j,k}+2\eta_{n,j,k}}. \]

and comparing coefficients of the elements of \( B \) in \( K/H \), we observe that equations (25) also hold if \( j = n \). So for all \( 1 \leq i < j \leq n \) and \( 1 \leq l < k \leq g \),

\[ \gamma_{j,k,l} + 2\eta_{j,k,l} = 0 \tag{26} \]

\[ 2\alpha_{l,j,k} - 2\alpha_{l,j,k} + \gamma_{j,k,l} + 2\eta_{j,k,l} = 0. \tag{27} \]
(b) Let $1 \leq i \leq n$. Then \[ \prod_{l=1}^{g} \rho_{i,l}^2 = B_{i,1} \cdots B_{i-1,l} B_{i,i+1} \cdots B_{i,n} \] in $P_n(M_g)$ by relation (c) of Theorem 3. For $1 \leq l \leq g$, note that

\[
\mathbb{E} \left( \rho_{i,l}^2 \right) = \rho_{i,l}^2 \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) + \rho_{n+1,1} \rho_{i,l}^2 \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = \rho_{i,l}^2 B_{i,n+1} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right).
\]

As we saw in equations (10) and (11), $\rho_{i,l}^2$ belongs to the centraliser of $K/H$ in $P_{n+1}(M_g)/H$, so

\[
\mathbb{E} \left( \prod_{l=1}^{g} \rho_{i,l}^2 \right) = \left( \prod_{l=1}^{g} \mathbb{E} \left( \rho_{i,l}^2 \right) \right) \left( \prod_{l=1}^{g} B_{i,n+1} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) \right).
\]

Further,

\[
\mathbb{E} \left( B_{1,i} \cdots B_{i-1,l} B_{i,i+1} \cdots B_{i,n} \right) = \mathbb{E} \left( \prod_{l=1}^{g} B_{i,n+1} \right) \prod_{l=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = B_{1,i} \cdots B_{i-1,l} B_{i,i+1} \cdots B_{i,n} \prod_{l=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right).
\]

Now in $P_{n+1}(M_g)/H$, \[ \prod_{l=1}^{g} \rho_{i,l}^2 = B_{i,1} \cdots B_{i-1,l} B_{i,i+1} \cdots B_{i,n} B_{i,n+1}, \] hence

\[
B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right).
\]

Thus for all $1 \leq i \leq n$,

\[
B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right) = B_{i,n+1}^{-2} \sum_{n=1}^{g} \eta_{i,j} - \sum_{n=1}^{g} \gamma_{i,j} \prod_{n=1}^{g} \left( 1 - \rho_{n+1,1} \gamma_{i,j}^{\eta_{i,j}} \right).
\]

\[ \text{(28)} \]

4 Proofs of Theorems 1 and 2

In this section, we use the calculations of Section 3 to prove Theorem 1, from which we shall deduce Theorem 2. 12
Proof of Theorem 1. As we mentioned in the Introduction, the existence of an algebraic section for \( p_n \) is equivalent to that of a cross-section for \( p \).

The case \( n = 1 \) was treated in Theorem 1 of [GG1], using the fact that if \( M = M_g \), where \( g \geq 3 \), then \( M \) is homeomorphic to the connected sum of one or two copies of \( \mathbb{R}P^2 \) with a compact, orientable surface without boundary of genus at least one.

Conversely, suppose that there exist \( m \in \mathbb{N} \) and \( n \geq 2 \) for which the homomorphism \( p_n : P_{n+m}(M) \rightarrow P_n(M) \) admits a section. We shall argue for a contradiction. By [GG1, Proposition 3], it suffices to consider the case \( m = 1 \). We first analyse the general structure of the coefficients \( a_{i,j,r}, \beta_{i,j,q}, \gamma_{k,l,r}, \eta_{k,l,q} \) defined by equation (7).

(a) Taking \( j = n \) in equation (16) implies that \( \gamma_{i,k,l} = 0 \) for all \( 1 \leq i \leq n - 1 \) and \( 1 \leq k < l \leq g \).

(b) By equation (27),
\[
\gamma_{i,k,l} = -2\eta_{i,k,l} - 2(\alpha_{i,j,l} - \alpha_{i,j,k})
\]
for all \( 1 \leq i < j \leq n \) and \( 1 \leq l < k \leq g \). Taking \( j = n \), we obtain
\[
\gamma_{i,k,l} = -2\eta_{i,k,n} - 2(\alpha_{i,n,l} - \alpha_{i,n,k}) = 0
\]
since \( \eta_{i,k,n} = 0 \) by definition and \( \alpha_{i,n,r} = 2\gamma_{i,1,l} \) for all \( 1 \leq i \leq n - 1 \) and \( 1 \leq r \leq g \) by equation (22).

It thus follows from (a) and (b) that
\[
\gamma_{i,k,l} = 0 \quad \text{for all } 1 \leq i \leq n - 1 \text{ and } 1 \leq k, l \leq g, k \neq l. \tag{29}
\]

(c) By equation (22), \( \gamma_{i,k,k} = \frac{1}{2}\alpha_{i,n,1} \) for all \( 1 \leq i \leq n - 1 \) and \( 1 \leq k \leq g \). So
\[
\gamma_{i,k,k} = \gamma_{i,1,1} \quad \text{for all } 1 \leq i \leq n - 1 \text{ and } 1 \leq k \leq g. \tag{30}
\]

(d) By equation (16), for all \( 1 \leq k < l \leq g \) and \( 1 \leq i < j \leq n \), we have
\[
\eta_{i,k,j} = -\frac{1}{2}\gamma_{i,k,l} = 0,
\]
using equation (29). So by taking \( l = g \) we obtain
\[
\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq g - 1.
\]

(e) By equation (27)
\[
\eta_{i,k,j} = \frac{1}{2} \left( 2(\alpha_{i,j,l} - \alpha_{i,j,k}) + \gamma_{i,k,l} \right)
\]
for all \( 1 \leq i < j \leq n \) and \( 1 \leq l < k \leq g \). But \( \gamma_{i,k,l} = 0 \) by equation (29), and \( \alpha_{i,j,l} - \alpha_{i,j,k} = 0 \) by equation (18) if \( j \leq n - 1 \) and by equation (22) if \( j = n \). Setting \( l = 1 \), it follows that
\[
\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 2 \leq k \leq g.
\]
By (d) and (e) we thus have
\[
\eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq g. \tag{31}
\]
Suppose that $1 \leq j < i \leq n - 1$. Then
\[ \eta_{i,k,l} = \frac{1}{2} \gamma_{i,k,l} \quad \text{for all } 1 \leq k < l \leq g, \text{ by equation (26)} \]
\[ = 0 \quad \text{by equation (29)}. \]
So taking $l = g$, we have $\eta_{i,k,j} = 0$ for all $1 \leq k \leq g - 1$. Further, for all $1 \leq l < k \leq g$,
\[ \eta_{i,k,j} = \frac{1}{2} \gamma_{i,k,l} \quad \text{by equation (15)} \]
\[ = 0 \quad \text{by equation (29)}. \]
Hence it follows from equation (31) and (f) that
\[ \eta_{i,k,j} = 0 \quad \text{for all } 1 \leq i, j \leq n - 1, i \neq j, \text{ and } 1 \leq k \leq g. \quad (32) \]

(g) From equation (23), we obtain
\[ \beta_{i,n,s} = -\gamma_{i,1,1} \quad \text{for all } 1 \leq s \leq n - 1, s \neq i. \quad (33) \]
(h) By equations (21) and (32), we see that
\[ \gamma_{i,1,1} = \beta_{i,i+1,i+1} = \cdots = \beta_{i,n-1,n-1} \quad \text{for all } 1 \leq i \leq n - 2. \quad (34) \]
(i) By equations (20) and (32), we obtain
\[ \gamma_{i,1,1} = \beta_{1,i,1} = \cdots = \beta_{i-1,i,i-1} \quad \text{for all } 2 \leq i \leq n - 1. \quad (35) \]
Analysing equation (28), we are now able to complete the proof of Theorem 1 as follows. Let $i \in \{1, \ldots, n - 1\}$. Then the coefficient of $B_{i,n+1}$ yields:
\[ 1 - 2 \sum_{l=1}^{g} \eta_{i,l,i} - \sum_{l=1}^{g} \gamma_{i,l,l} = \sum_{l=1}^{i-1} \beta_{l,i,i} + \sum_{l=i+1}^{n} \beta_{i,l,i} - 2 \sum_{l=1}^{g} \eta_{i,l,i}. \quad (36) \]
Now
\[ \sum_{l=1}^{i-1} \beta_{l,i,i} = \sum_{l=1}^{i-1} \gamma_{l,1,1} \quad \text{by equation (34)}, \]
and
\[ \sum_{l=i+1}^{n} \beta_{i,l,i} = \sum_{l=i+1}^{n} \gamma_{l,1,1} \quad \text{by equation (35)}. \]
So using equation (30), equation (36) becomes
\[ 1 - g \gamma_{i,1,1} = \beta_{i,n,i} + \sum_{l=1}^{n-1} \gamma_{l,1,1} - \gamma_{i,1,1}. \]
Summing over all $i = 1, \ldots, n - 1$, and setting $\Delta = \sum_{i=1}^{n-1} \gamma_{i,1,1}$ and $L = \sum_{i=1}^{n-1} \beta_{i,n,i}$, we obtain
\[ (n + g - 2)\Delta = (n - 1) - L. \quad (37) \]
Now let \( i = n \), and let \( k \in \{1, \ldots, n - 1\} \). Since \( \eta_{n,l,n} = 0 \), the coefficient of \( B_{k,n+1} \) in equation (28) yields:

\[
\sum_{l=1}^{g} \gamma_{n,l,1} - 1 = \sum_{l=1}^{n-1} \beta_{l,n,k} - 2 \sum_{l=1}^{g} \eta_{n,l,k} = \beta_{k,n,k} + \sum_{l \neq k}^{n-1} \beta_{l,n,k} - 2 \sum_{l=1}^{g} \eta_{n,l,k}
\]

\[
= \beta_{k,n,k} - \sum_{l=1}^{n-1} \gamma_{l,1,1} - 2 \sum_{l=1}^{g} \eta_{n,l,k} \quad \text{by equation (33)}
\]

\[
= \beta_{k,n,k} - (\Delta - \gamma_{k,1,1}) + \sum_{l=1}^{g} (-\beta_{k,n,k} + \gamma_{n,l,l} - \gamma_{k,l,l}) \quad \text{by equation (24)}
\]

\[
= (1 - g)\beta_{k,n,k} + \gamma_{k,1,1} - \Delta + \sum_{l=1}^{g} \gamma_{n,l,l} - \sum_{l=1}^{g} \gamma_{k,1,1} \quad \text{by equation (30)}
\]

\[
= (1 - g)\beta_{k,n,k} + (1 - g)\gamma_{k,1,1} - \Delta + \sum_{l=1}^{g} \gamma_{n,l,l}.
\]

Hence \(-1 = (1 - g)\beta_{k,n,k} + (1 - g)\gamma_{k,1,1} - \Delta\). Summing over all \( k = 1, \ldots, n - 1 \), we obtain

\[
(n + g - 2)\Delta = (1 - g)L + (n - 1).
\]

Equating equations (37) and (38), we see that

\[
(n - 1) - L = (1 - g)L + (n - 1).
\]

Since \( g \geq 3 \), it follows that \( L = 0 \), and therefore

\[
\Delta = \frac{n - 1}{(n - 1) + (g - 1)}
\]

by equation (37). This yields a contradiction to the fact that \( \Delta \) is an integer, and thus completes the proof of Theorem 1. \( \square \)

**Remark.** Although some of the relations derived in (a)–(i) do not exist if \( n = 2 \), one may check that the above analysis from equation (36) onwards is also valid in this case (with \( \Delta = \gamma_{1,1,1} \) and \( L = \beta_{1,2,1} \)).

**Proof of Theorem 2.**

(a) If \( r > 0 \) then the result follows applying the methods of the proofs of Proposition 27 and Theorem 6 of [GG1]. If \( r = 0 \) and \( M \) has non-empty boundary, let \( C \) be a boundary component of \( M \). Then \( M' = M \setminus C \) is homeomorphic to a compact surface with a single point deleted (which is the case \( r = 1 \)), so (PBS) splits for \( M' \). The inclusion of \( M' \) in \( M \) not only induces a homotopy equivalence between \( M \) and \( M' \), but also a homotopy equivalence between their \( n \)-th configuration spaces. Therefore their \( n \)-th pure braid groups are isomorphic, and the sequence (PBS) for \( M' \) splits if and only it splits for \( M' \).

(b) Suppose that \( r = 0 \) and that \( M \) is without boundary. If \( M = S^2 \), \( m = 1 \) and \( n \geq 3 \) then the statement follows from [Fa]. The geometric construction of Fadell may be easily generalised to all \( m \in \mathbb{N} \). If \( n \in \{1,2\} \), the result is obvious since \( P_n(S^2) \) is trivial. If \( M = \mathbb{T}^2 \) or \( \mathbb{K}^2 \), the fact that \( p_a \) has a section is a consequence of [FaN] and the fact that \( \mathbb{T}^2 \) and \( \mathbb{K}^2 \)
admit a non-vanishing vector field. If \( M = \mathbb{R}P^2 \) then \( p_* \) admits a section if and only if \( n = 2 \) and \( m = 1 \) by [GG3]. Finally, if \( M \neq \mathbb{R}P^2, S^2, T^2, K^2 \) then \( p_* \) admits a section if and only if \( n = 1 \) by Theorem 1 for the non-orientable case, and by [GG1] for the orientable case.

References

[GG1] D. L. Gonçalves and J. Guaschi, On the structure of surface pure braid groups, J. Pure Appl. Algebra 182 (2003), 33–64 (due to a printer’s error, this article was republished in its entirety with the reference 186 (2004), 187–218).


