On the existence of a limit value in some non expansive optimal control problems, and application to averaging of singularly perturbed systems

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Abstract

We investigate a limit value of an optimal control problem when the horizon converges to infinity. For this aim, we suppose suitable nonexpansive-like assumptions which does not imply that the limit is independent of the initial state as it is usually done in the literature. We apply this new result to study a singularly control perturbed system, and we obtain a description of the limit dynamics in term of a differential inclusion.

1 Introduction

We consider the following optimal control denoted $\Gamma_t(y_0)$:

$$V_t(y_0) := \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds, \tag{1}$$

where $s \mapsto y(s,u,y_0)$ denotes the solution to

$$y'(s) = g(y(s),u(s)), \quad y(0)=y_0. \tag{2}$$

Here $\mathcal{U}$ is the set of measurable controls from $\mathbb{R}_+$ to a given non empty metric space U. Throughout the paper, we will suppose Lipschitz regularity of $g : \mathbb{R}^d \times U \to \mathbb{R}^d$ which implies that for a given control u in $\mathcal{U}$ and a given initial condition y_0, equation (2) has a unique absolutely continuous solution.

The main goal of the paper consists in studying the asymptotic behaviour of $V_t(y_0)$ when t tends to ∞. This problem has been considered in several papers

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(cf for instance in [6, 7, 8]) by approaches ensuring that the limit of $V_t(y_0)$ is independent of y_0. In the present paper we exhibit several examples where the limit exists and depends of y_0. Our aim is to obtain a general result which contains in particular the more easy to state following result, where throughout the paper, $< \cdot, \cdot >$ stands for the canonical scalar product and B is the associated closed unit ball..

Proposition 1.1. *Assume that g is Lipschitz, that there exists a compact set N which is - forward - invariant by the control system (2) and that h is a continuous function which does not depend on u. Assume moreover that :*

$$\forall (y_1, y_2) \in N^2, \ \sup_{u \in U} \inf_{v \in U} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq 0. \tag{3}$$

Then problem (1) has a value when t converges to $+\infty$ i.e. there exists $V(y_0) := \lim_{t \to +\infty} V_t(y_0)$.

Condition (3) means a non expansive property of the control system, while the condition

$$\forall (y_1, y_2) \in N^2, \ \sup_{u \in U} \inf_{v \in U} < y_1 - y_2, g(y_1, u) - g(y_2, v) > \leq -C\|y_1 - y_2\|^2$$

expresses a dissipativity property of the control system. The above dissipativity condition does imply that the limit is independent of y_0 (cf [1]).

The value function (1) can also be characterized through - viscosity - solution of a suitable Hamilton-Jacobi equation. In several articles initiated by the pioneering work [16] the limit of $V_t(y_0)$ is obtained by "passing to the limit" on the Hamilton-Jacobi equation. This required coercivity properties of the Hamiltonian which could be implied by controlability and/or dissipativity of the control system but which are not valid in the nonexpansive case (3). Moreover the PDE approach is out of the scope of the - long enough - present article.

Definition 1.2. *The problem $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ has a limit value if $\lim_{t \to \infty} V_t(y_0)$ exists. Whenever it exists, we denote this limit by $V(y_0)$.*

Our main aim consists in giving one sufficient condition ensuring the existence of the limit value. As a particular case of our main result we obtain proposition (1.1).

It is also of interest to know if approximate optimal controls for the value $V_t(y_0)$ are still approximate optimal controls for the limit value. This leads us to the following definition.

Definition 1.3. *The problem $\Gamma(y_0)$ has a uniform value if it has a limit value $V(y_0)$ and if :*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^{t} h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.$$

Whenever the uniform value exists, the controller can act (approximately) optimally independently of the time horizon. On the contrary, if the limit value exists but the uniform value does not, he really needs to know the time horizon before choosing a control. We will prove that our results do imply the existence of a uniform value. We will be inspired by a recent work in the discrete time case [18].

As an application to our method, we will investigate the averaging phenomenom for the following control singularly perturbed system

$$\left\{ \begin{array}{ll} i) & x'(s) = f\big(x(s), y(s), u(s)\big),\ x(0) = x \\ ii) & \varepsilon y'(s) = g\big(x(s), y(s), u(s)\big)\ y(0) = y, \end{array} \right. \tag{4}$$

where $u \in \mathcal{U}$. This problem has been intensively studied in the literature, we refer the interested reader to [1, 12, 13, 14, 15, 17] and references therein. The main question concerns the behaviour of solutions to (4) when ε goes to 0. Note that with some - rather strong assumptions - the solutions $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ converge to solutions of a system obtained by putting formally $\varepsilon = 0$ in (4). This is the so-called Tichonov type result ([19]). A more interesting case is when one cannot obtain the convergence of $y_\varepsilon(\cdot)$; this is the context of averaging method.

In the present paper we will give assumptions such that for any $T > 0$ the $x_\varepsilon(\cdot)$ part of solutions of (4) are approximated - uniformly on $[0, T]$ by a solution of a dynamical system described by the following differential inclusion

$$x'(t) \in \overline{F}(x(t), y),\ t \in [0, T].$$

Such method for control system was first developed by Gaitsgory [12] and afterwards developed for instance in [1, 14, 17] in the case where the limit dynamical system - the differential inclusion - is independent of y. Once again, the main novelty of our approach lies in the fact that our assumptions do not imply that the limit dynamical system is y independent.

Let us explain now, how the paper is organized. The second section contains some preliminaries and discussions of limit behaviors in examples. In the third section, we state and prove our main result for the existence of the uniform value. The last section concerns applications of our approach to the averaging method for singularly perturbed system.

2 Preliminaries

We now consider the optimal control problems $(\Gamma_t(y_0))_t$ described by (1) and (2).

2.1 Assumptions and Notations

We now describe the assumptions made on g and h.

$$(5) \quad \begin{cases} \text{The function } h : I\!R^d \times U \longrightarrow I\!R \text{ is measurable and bounded} \\ \text{The function } g : I\!R^d \times U \longrightarrow I\!R^d \text{ is measurable} \\ \exists L \geq 0, \forall (y, y') \in I\!R^{2d}, \forall u \in U,\ \|g(y,u) - g(y',u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y,u) \in I\!R^d \times U,\ \|g(y,u)\| \leq a(1 + \|y\|) \end{cases}$$

With these hypotheses, given u in $\mathcal{U}$ equation (2) has a unique absolutely continuous solution $y(\cdot, u, y_0) : I\!R_+ \to I\!R^d$.

Since h is bounded, we will assume without loss of generality from now on that h takes values in $[0, 1]$.

We denote by $G(y_0) := \{y(t, u, y_0), t \geq 0, u \in \mathcal{U}\}$ the reachable set (i.e. the set of states that can be reached starting from y_0).

We denote the average cost induced by u between time 0 and time t by :

$$\gamma_t(y_0, u) = \frac{1}{t} \int_0^t h(y(s, u, y_0), u(s))ds$$

The corresponding Value function satisfies $V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u)$.

2.2 Examples

We present here basic examples. In all these examples, the cost $h(y, u)$ only depends on the state y. We will prove later that the uniform value exists in examples 2, 3 and 4.

• Example 1 : here y lies in $I\!R^2$ seen as the complex plane, there is no control and the dynamic is given by $g(y, u) = i\ y$, where $i^2 = -1$. We clearly have :

$$V_t(y_0) \xrightarrow[t \to \infty]{} \frac{1}{2\pi |y_0|} \int_{|z| = |y_0|} h(z)dz,$$

and since there is no control, the value is uniform.

• Example 2 : in the complex plane again, but now $g(y, u) = i\ y\ u$, where $u \in U$ a given bounded subset of $I\!R$, and h is continuous in y.

• Example 3 : $g(y, u) = -y + u$, where $u \in U$ a given bounded subset of $I\!R^d$, and h is continuous in y.

• Example 4 : in $I\!R^2$. The initial state is $y_0 = (0, 0)$ and the control set is $U = [0, 1]$. For a state $y = (y_1, y_2)$ and a control u, the dynamic is given by $y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

Notice that for any control, $y_1'(s) \geq y_2'(s) \geq 0$, and thus $y_2(t) \leq y_1(t)$ for each $t \geq 0$. One can easily observe that $G(y_0) \subset [0,1]^2$.

If one uses the constant control $u = \varepsilon > 0$, we obtain $y_1(t) = 1 - \exp(-\varepsilon t)$ and $y_2(t) = \varepsilon y_1(t)$. So we have $V_t(y_0) \xrightarrow[t\to\infty]{} 0$.

More generally, if the initial state is $y = (y_1, y_2) \in [0,1]^2$, by choosing a constant control $u = \varepsilon > 0$ small, one can show that the limit value exists and $\lim_{t\to\infty} V_t(y) = y_2$.

Notice that there is no hope here to use an ergodic property, because

$$\{y \in [0,1]^2, \lim_{t\to\infty} V_t(y) = \lim_{t\to\infty} V_t(y_0)\} = [0,1] \times \{0\},$$

and starting from y_0 it is possible to reach no point in $(0,1] \times \{0\}$.

• Example 5 : in $\mathbb{R}^2$, $y_0 = (0,0)$, control set $U = [0,1]$, $g(y,u) = (y_2, u)$, and $h(y_1, y_2) = 0$ if $y_1 \in [1,2]$, $= 1$ otherwise.

We have $u(s) = y_2'(s) = y_1''(s)$, hence we may think of the control u as the acceleration, y_2 as the speed and y_1 as the position of some mobile. If $u = \varepsilon$ constant, then $y_2(t) = \sqrt{2\varepsilon y_1(t)}$ $\forall t \geq 0$.

We have $u \geq 0$, hence the speed cannot decrease. Consequently, the time interval where $y_1(t) \in [1,2]$ cannot be longer than the time interval where $y_1(t) \in [0,1)$, and we have $V_T(y_0) \geq 1/2$ for each T.

One can prove that $V_T(y_0) \xrightarrow[T\to\infty]{} 1/2$ by considering the following controls : choose $\hat{t}$ in $(0,T)$ such that $(2/\hat{t}) + (\hat{t}/2) = T$, make a full acceleration up to $\hat{t}$ and completely stop accelerating after : $u(t) = 1$ for $t < \hat{t}$, and $u(t) = 0$ for $t \geq \hat{t}$.

Consequently the limit value exists and is $1/2$. However, for any control u in $\mathcal{U}$, we either have $y(t,u,y_0) = y_0$ for all t, or $y_1(t,u,y_0) \xrightarrow[t\to\infty]{} +\infty$. So in any case we have $\frac{1}{t}\int_0^t h(y(s,u,y_0),u(s))ds \xrightarrow[t\to\infty]{} 1$. The uniform value does not exist here, although the dynamic is very regular.

3 Existence results for the uniform value

3.1 A technical Lemma

Let us define $V^-(y_0) := \liminf_{t\to+\infty} V_t(y_0)$ and $V^+(y_0) := \limsup_{t\to+\infty} V_t(y_0)$. Adding a parameter $m \geq 0$, we will more generally consider the costs between time m and time $m+t$:

$$\gamma_{m,t}(y_0,u) = \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0),u(s))ds,$$

and the value of the problem where the time interval $[0,m]$ can be devoted to reach a good initial state, is denoted by :

$$V_{m,t}(y_0) = \inf_{u\in\mathcal{U}} \gamma_{m,t}(y_0,u).$$

Of course $\gamma_t(y_0,u) = \gamma_{0,t}(y_0,u)$ and $V_t(y_0) = V_{0,t}(y_0)$.

Lemma 3.1. *For every m_0 in $\mathbb{R}_+$, we have :*

$$\sup_{t>0} \inf_{m\le m_0} V_{m,t}(y_0) \ge V^+(y_0) \ge V^-(y_0) \ge \sup_{t>0} \inf_{m\ge 0} V_{m,t}(y_0).$$

Proof : We first prove $\sup_{t>0} \inf_{m\le m_0} V_{m,t}(y_0) \ge V^+(y_0)$. Suppose by contradiction that it is false. So there exists $\varepsilon > 0$ such that for any $t > 0$ we have $\inf_{m\le m_0} V_{m,t}(y_0) \le V^+(y_0) - \varepsilon$. Hence for any $t > 0$ there exists $m \le m_0$ with $V_{m,t}(y_0) \le V^+(y_0) - (\varepsilon/2)$. Now observe that

$$V_{m,t}(y_0) = \inf_u \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0),u(s))ds = \frac{1}{t}\inf_u\{\int_0^{m_0+t} h(y(s,u,y_0),u(s))ds$$

$$-\int_{m+t}^{m_0+t} h(y(s,u,y_0),u(s))ds - \int_0^m h(y(s,u,y_0),u(s))ds\} \ge \frac{m_0+t}{t}V_{m_0+t}(y_0) - 2\frac{m_0}{t}.$$

Hence

$$\frac{m_0+t}{t}V_{m_0+t}(y_0) - 2\frac{m_0}{t} \le V^+(y_0) - (\varepsilon/2).$$

Passing to the limsup when t goes to $+\infty$ we obtain a contradiction.

We now prove $V^-(y_0) \ge \sup_{t>0} \inf_{m\le 0} V_{m,t}(y_0)$. Assume on the contrary that it is false. Then there exists $\varepsilon > 0$ and $t > 0$ such that $V^-(y_0) + \varepsilon \le \inf_{m\le 0} V_{m,t}(y_0)$. So for any $m \ge 0$, we have $V^-(y_0) + \varepsilon \le V_{m,t}(y_0)$. We will obtain a contradiction by concatenating trajectories. Take $T > 0$, and write $T = lt + r$, with l in $\mathbb{N}$ and r in $[0,t)$. For any control u in $\mathcal{U}$, we have : $T\gamma_T(y_0,u) = t\gamma_{0,t}(y_0,u) + t\gamma_{t,t}(y_0,u)$ $+ ... + t\gamma_{(l-1)t,t}(y_0,u) + r\gamma_{lt,r}(y_0,u) \ge lt(V^-(y_0)+\varepsilon)$. Hence

$$\gamma_T(y_0,u) \ge \frac{T-r}{T}(V^-(y_0)+\varepsilon).$$

So for T large enough we have $V_T(y_0) \ge V^-(y_0) + \varepsilon/2$, hence a contradiction by taking the liminf when $T \to \infty$. □

Remark : it is also easy to show that for each $t_0 \ge 0$, we have $\inf_{m\ge 0} \sup_{t>t_0} V_{m,t}(y_0) \ge V^+(y_0)$.

The following quantity will play a great role in the sequel.

Definition 3.2.

$$V^*(y_0) = \sup_{t>0} \inf_{m\ge 0} V_{m,t}(y_0).$$

3.2 Main results

Let us state the first version of our main result (which clearly implies Proposition 1.1 stated in the introduction)

Proposition 3.3. *Assume that (5) holds true and furthermore :*
(H'1) $h(y,u) = h(y)$ *only depends on the state, and is continuous on* $\mathbb{R}^d$.
(H'2) $G(y_0)$ *is bounded,*
(H'3) $\forall (y_1,y_2) \in G(y_0)^2$, $\sup_{u\in U} \inf_{v\in U} < y_1 - y_2, g(y_1,u) - g(y_2,v) >\leq 0$.
Then the problem $\Gamma(y_0)$ *has a limit value which is* $V^*(y_0)$, *i.e.* $V_t(y_0) \xrightarrow[t\to+\infty]{} V^*(y_0)$. *The convergence of* $(V_t)_t$ *to* V^* *is uniform over* $G(y_0)$, *and we have* $V^*(y_0) = \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(y_0) = \inf_{m\geq 0} \sup_{t\geq 1} V_{m,t}(y_0) = \lim_{m\to\infty,t\to\infty} V_{m,t}(y_0)$. *Moreover the value of* $\Gamma(y_0)$ *is uniform.*

Condition (*H'3*) can be used to show that (cf Proposition 3.5) : $\forall (y_1,y_2) \in G(y_0)^2$, $\forall \varepsilon > 0$, $\forall T \geq 0$, $\forall u \in \mathcal{U}$, $\exists v \in \mathcal{U}$ s.t. : $\forall t \in [0,T]$, $\|y(t,u,y_1) - y(t,v,y_2)\| \leq \|y_1 - y_2\| + \varepsilon$. Proposition 3.3 can be applied to the previous examples 1, 2 and 3, but not to example 4. Notice that in example 5, we have $V^*(y_0) = 0 < 1/2 = \lim_t V_t(y_0)$.

We will prove the following generalization of Proposition 3.3. We put $Z = G(y_0)$, and denote by $\bar{Z}$ its closure in $\mathbb{R}^d$.

Theorem 3.4. *Suppose that (5) holds true and furthermore assume that*
(H1) h *is uniformly continuous in* y *on* $\bar{Z}$ *uniformly in* u. *And for each* y *in* $\bar{Z}$, *either* h *does not depend on* u *or the set* $\{(g(y,u), h(y,u)) \in \mathbb{R}^d \times [0,1],\ u \in U\}$ *is closed.*
(H2) : There exist a continuous function $\Delta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$, *vanishing on the diagonal (*$\Delta(y,y) = 0$ *for each* y*) and symmetric (*$\Delta(y_1,y_2) = \Delta(y_2,y_1)$ *for all* y_1 *and* y_2*), and a function* $\hat{\alpha} :\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *s.t.* $\hat{\alpha}(t) \xrightarrow[t\to 0]{} 0$ *satisfying :*

a) For every sequence $(z_n)_n$ *with values in* Z *and every* $\varepsilon > 0$, *one can find* n *such that* $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$.

b) $\forall (y_1,y_2) \in \bar{Z}^2$, $\forall u \in U$, $\exists v \in U$ *such that*
$D\uparrow \Delta(y_1,y_2)(g(y_1,u), g(y_2,v)) \leq 0$ *and* $h(y_2,v) - h(y_1,u) \leq \hat{\alpha}(\Delta(y_1,y_2))$.

Then we have the same conclusions as in Proposition 3.3. The problem $\Gamma(y_0)$ *has a limit value which is* $V^*(y_0)$. *The convergence of* V_t *to* V^* *is uniform over* Z, *and we have* $V^*(y_0) = \sup_{t\geq 1} \inf_{m\geq 0} V_{m,t}(y_0) = \inf_{m\geq 0} \sup_{t\geq 1} V_{m,t}(y_0) = \lim_{m\to\infty,t\to\infty} V_{m,t}(y_0)$. *Moreover the value of* $\Gamma(y_0)$ *is uniform.*

Remarks :

• Although Δ may not satisfy the triangular inequality nor the separation property, it may be seen as a "distance" adapted to the problem $\Gamma(y_0)$.

• The assumption : "$\{(g(y,u), h(y,u)) \in \mathbb{R}^d \times [0,1],\ u \in U\}$ closed" could be checked for instance if U is compact and if h and g are continuous with respect to (y,u).

• $D\uparrow$ is the contingent epi-derivative (cf [4]) (which reduces to the upper Dini derivative if Δ is Lipschitz), defined by : $D\uparrow\Delta(z)(\alpha) = \liminf_{t\to 0^+, \alpha'\to\alpha} \frac{1}{t}(\Delta(z + t\alpha') - \Delta(z))$. If Δ is differentiable, the condition $D\uparrow \Delta(y_1,y_2)(g(y_1,u), g(y_2,v)) \leq 0$ just reads : $< g(y_1,u), \frac{\partial}{\partial y_1}\Delta(y_1,y_2) > + < g(y_2,v), \frac{\partial}{\partial y_2}\Delta(y_1,y_2) >\leq 0$.

• Proposition 3.3 will be a corollary of Theorem 3.4. It corresponds to the case where : $\Delta(y_1, y_2) = \|y_1 - y_2\|^2$, $G(y_0)$ is bounded, and $h(y, u) = h(y)$ does not depend on u (one can just take $\hat{\alpha}(t) = \sup\{|h(x) - h(y)|, \|x - y\|^2 \leq t\}$).

• $H2a)$ is a precompacity condition. It is satisfied as soon as $G(y_0)$ is bounded. It is also satisfied if Δ satisfies the triangular inequality and the usual precompacity condition : for each $\varepsilon > 0$, there exists a finite subset C of Z s.t. : $\forall z \in Z, \exists c \in C, \Delta(z, c) \leq \varepsilon$. (see lemma 3.13)

• Notice that $H2$ is satisfied with $\Delta = 0$ if we are in the trivial case where $\inf_u h(y, u)$ is constant.

• Theorem 3.4 can be applied to example 4, with $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ (L^1-norm). In this example, we have for each y_1, y_2 and u : $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$ as soon as $t \geq 0$ is small enough.

3.3 Proof of Theorem 3.4

We assume in this section that the hypotheses of Theorem 3.4 are satisfied, and we may assume without loss of generality that $\hat{\alpha}$ is non decreasing and upper semicontinuous (otherwise we replace $\hat{\alpha}(t)$ by $\inf_{\varepsilon>0} \sup_{t' \in [0, t+\varepsilon]} \alpha(t')$).

3.3.1 A non expansion property

We start with a proposition expressing the fact that the problem is non expansive with respect to Δ, the idea being that given two initial conditions y_1 and y_2 and a control to be played at y_1, there exists another control to be played at y_2 such that $t \mapsto \Delta(y(t, u, y_1), y(t, v, y_2))$ will not increase.

Proposition 3.5. *We suppose the hypothesis of Theorem 3.4. Then*

$$(6) \quad \begin{cases} \forall (y_1, y_2) \in \bar{Z}^2,\ \forall T \geq 0,\ \forall \varepsilon > 0, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U}, \\ \forall t \in [0, T],\ \Delta(y(t, u, y_1), y(t, v, y_2)) \leq \Delta(y_1, y_2) + \varepsilon, \\ \quad \textit{and for almost every } t \in [0, T], \\ h(y(t, v, y_2), v(t)) - h(y(t, u, y_1), u(t)) \leq \hat{\alpha}(\Delta(y(t, u, y_1), y(t, v, y_2))). \end{cases}$$

Proof : First fix y_1, y_2 $\varepsilon > 0$, $T > 0$ and u. Let us consider the following set-valued map $\Phi : I\!R_+ \times \bar{Z} \times \bar{Z} \times I\!R \to I\!R^d \times I\!R^d \times I\!R$

$$\Phi(t, x, y, l) := co\, cl\{(g(x, u(t)), g(y, v), 0)) \mid v \in U,\ h(y, v) - h(x, u(t)) \leq \hat{\alpha}(\Delta(x, y))\},$$

where *co* stands for the convex hull and *cl* for the closure. Notice that $\Phi(t, x, y, l)$ does not depend on l. Using (5), *H1)* and *H2)b)*, one can check that Φ is a set valued map which is upper semicontinuous in (x, y, l), measurable in t and with compact convex nonempty values [4, 10]. We also denote $\tilde{\Phi}$ the set valued map defined as Φ but removing the convex hull.

¿From the measurable Viability Theorem [11] (cf also [9] section 6.5), condition (H2) b) implies that the epigraph of Δ (restricted to $\bar{Z}^2 \times I\!R$) is viable for the differential inclusion

$$(7) \qquad (x'(t), y'(t), l'(t)) \in \Phi(t, x(t), y(t), l(t)) \text{ for a. e. } t \geq 0$$

So starting from $(y_1, y_2, \Delta(y_1, y_2))$, there exists a solution $(x(\cdot), y(\cdot), l(\cdot))$ to (7) which stays for any $t \geq 0$ in the epigraph of Δ namely

$$\Delta(x(t), y(t)) \leq l(t) = \Delta(y_1, y_2), \ \forall t \geq 0, \tag{8}$$

by noticing that $l(\cdot)$ is a constant.

¿From the suppositions made on the dynamics g, the trajectory $(x(\cdot), y(\cdot))$ remains in a compact set (included in some large enough ball $B(0, M)$) on the time interval $[0, T]$. Because Δ is uniformly continuous on $B(0, M) \times B(0, M)$, there exists $\eta \in (0, 1)$ with

$$\forall (x, x', y, y') \in B(0, M+1)^4, \ \|x - x'\| + \|y - y'\| < \eta \Longrightarrow |\Delta(x, y) - \Delta(x', y')| < \varepsilon.$$

Thanks to the Wazewski Relaxation Theorem (cf for instance Th. 10.4.4 in [4]) applied to Φ, the trajectory $(x(\cdot), y(\cdot), l(\cdot))$ could be approximated on every compact interval by a trajectory to the differential inclusion defined by $\tilde{\Phi}$. So there exists $(y_1(\cdot), y_2(\cdot), l(\cdot))$ satisfying

$$(y_1'(t), y_2'(t), l'(t)) \in \tilde{\Phi}(t, y_1(t), y_2(t), l(t)) \text{ for a. e. } t \geq 0$$

such that

$$\|x(t) - y_1(t)\| + \|y(t) - y_2(t)\| < \eta, \ \forall t \in [0, T].$$

From the choice of η and the very definition of $\tilde{\Phi}$ we also have for any $t \in [0, T]$

$$\begin{cases} \Delta(y_1(t), y_2(t)) \leq \Delta(x(t), y(t)) + \varepsilon \leq \Delta(y_1, y_2) + \varepsilon \\ h(y_2(t), v(t)) - h(y_1(t), u(t)) \leq \tilde{\alpha}(\Delta(y_1(t), y_2(t))) \end{cases}$$

This completes our proof if, from one hand we observe that $y_1(\cdot) = y(\cdot, u, y_1)$ and from the other hand, we use Filippov's measurable selection Theorem (e.g. Theorem 8.2.10 in [4]) to $\tilde{\Phi}$ for finding a measurable control $v \in \mathcal{U}$ such that $y_2(\cdot) = y(\cdot, v, y_2)$.

QED

3.3.2 The limit value exists

Since $\hat{\alpha}$ is u.s.c. and non decreasing, we obtain the following consequence of Proposition 3.5.

Corollary 3.6. *For every y_1 and y_2 in $G(y_0)$, for all $T > 0$,*

$$|V_T(y_1) - V_T(y_2)| \leq \hat{\alpha}(\Delta(y_1, y_2)).$$

Define now, for each $m \geq 0$, $G^m(y_0)$ as the set of states which can be reached from x_0 before time m :

$$G^m(y_0) = \{y(t, u, y_0), t \leq m, u \in \mathcal{U}\}, \ \text{ so that } \ G(y_0) = \cup_{m \geq 0} G^m(y_0).$$

An immediate consequence of the precompacity hypothesis H2a) is the following

Lemma 3.7. *For every $\varepsilon > 0$, there exists m_0 in $\mathbb{R}_+$ such that :*

$$\forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \ \ \textit{such that} \ \ \Delta(z, z') \leq \varepsilon.$$

Proof : Otherwise for each positive integer m one can find z_m in $G(y_0)$ such that $\Delta(z_m, z) > \varepsilon$ for all z in $G^m(y_0)$. Use *H2a)* to find n such that $\liminf_m \Delta(z_n, z_m) \leq \varepsilon$. Since $z_n \in G(y_0)$, there must exist k such that $z_n \in G^k(y_0)$. But for each $m \geq k$ we have $z_n \in G^m(y_0)$, hence $\Delta(z_m, z_n) > \varepsilon$. We obtain a contradiction.

QED

We can already conclude for the limit value.

Proposition 3.8. $V_t(y_0) \xrightarrow[t\to\infty]{} V^*(y_0)$.

Proof : Because of lemma 3.1, it is sufficient to prove that for every $\varepsilon > 0$, there exists m_0 such that :

$$\sup_{t>0} \inf_{m\leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m\geq 0} V_{m,t}(y_0) + 2\varepsilon$$

Fix ε, and consider $\eta > 0$ such that $\hat{\alpha}(t) \leq \varepsilon$ as soon as $t \leq \eta$. Use lemma 3.7 to find m_0 such that $\forall z \in G(y_0), \exists z' \in G^{m_0}(y_0)$ *s.t.* $\Delta(z, z') \leq \eta$.

Consider any $t > 0$. We have $\inf_{m\geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$, and $\inf_{m\leq m_0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G^{m_0}(y_0)\}$. Let z in $G(y_0)$ be such that $V_t(z) \leq \inf_m V_{m,t}(y_0) + \varepsilon$, and consider $z' \in G^{m_0}(y_0)$ *s.t.* $\Delta(z, z') \leq \eta$. By corollary 3.6, $|V_t(z) - V_t(z')| \leq \hat{\alpha}(\Delta(z, z')) \leq \varepsilon$, so we obtain that

$$\inf_{m\leq m_0} V_{m,t}(y_0) \leq V_t(z') \leq V_t(z) + \varepsilon \leq \inf_m V_{m,t}(y_0) + 2\varepsilon.$$

Passing to the supremum on t, this completes the proof.

QED

Remark 3.9. *Observe that for obtaining the existence of the value, we have used a compactness argument (assumption H2)a)) and condition (6). We did not use explicitly assumption H2)b) which is only used for obtaining (6).*

The rest of the proof is more involved, and is inspired by the proof of Theorem 3.6 in [18].

3.3.3 Auxiliary value functions

The uniform value requires the same control to be good for all time horizons, and we are led to introduce new auxiliary value functions. Given $m \geq 0$ and $n \geq 1$, for any initial state z in $Z = G(y_0)$ and control u in $\mathcal{U}$, we define

$$\nu_{m,n}(z, u) = \sup_{t\in[1,n]} \gamma_{m,t}(z, u), \text{ and } W_{m,n}(z) = \inf_{u\in\mathcal{U}} \nu_{m,n}(z, u).$$

$W_{m,n}$ is the value function of the problem where the controller can use the time interval $[0, m]$ to reach a good state, and then his cost is only the supremum for t in $[1, n]$, of the average cost between time m and $m + t$. Of course, we have $W_{m,n} \geq V_{m,n}$. We write ν_n for $\nu_{0,n}$, and W_n for $W_{0,n}$.

We easily obtain from proposition 3.5, as in corollary 3.6, the following result.

Lemma 3.10. *For every z and z' in Z, for all $m \geq 0$ and $n \geq 1$,*

$$|V_{m,n}(z) - V_{m,n}(z')| \leq \hat{\alpha}(\Delta(z, z')).$$

$$|W_{m,n}(z) - W_{m,n}(z')| \leq \hat{\alpha}(\Delta(z, z')).$$

The following lemma shows that the quantities $W_{m,n}$ are not that high.

Lemma 3.11. $\forall k \geq 1, \forall n \geq 1, \forall m \geq 0, \forall z \in Z$,

$$V_{m,n}(z) \geq \inf_{l \geq m} W_{l,k}(z) - \frac{k}{n}.$$

Proof : Fix k, n, m and z, and put $A = \inf_{l \geq m} W_{l,k}(z)$. Consider any control u in $\mathcal{U}$. For any $i \geq m$, we have

$$\sup_{t \in [1,k]} \gamma_{i,t}(z, u) = \nu_{i,k}(z, u) \geq W_{i,k}(z) \geq A.$$

So we know that for any $i \geq m$, there exists $t(i) \in [1, k]$ such that $\gamma_{i,t(i)}(z, u) \geq A$.

Define now by induction $i_1 = m$, $i_2 = i_1 + t(i_1)$,..., $i_q = i_{q-1} + t(i_{q-1})$, where q is such that $i_q \leq n+m < i_q+t(i_q)$. We have $n\gamma_{m,n}(z, u) \geq \sum_{p=1}^{q-1} t(i_p)A \geq nA-k$, so $\gamma_{m,n}(z, u) \geq A-\frac{k}{n}$. Taking the infimum over all controls, the proof is complete.

QED

We know from Proposition 3.8 that the limit value is given by V^*. We now give other formulas for this limit.

Proposition 3.12. *For every state z in Z,*

$$\inf_{m \geq 0} \sup_{n \geq 1} W_{m,n}(z) = \inf_{m \geq 0} \sup_{n \geq 1} V_{m,n}(z) = V^*(z) = \sup_{n \geq 1} \inf_{m \geq 0} V_{m,n}(z) = \sup_{n \geq 1} \inf_{m \geq 0} W_{m,n}(z).$$

Proof of proposition 3.12 : Fix an initial state z in Z. We already have $V^*(z) = \sup_{t>0} \inf_{m \geq 0} V_{m,t}(z) \geq \sup_{t \geq 1} \inf_{m \geq 0} V_{m,t}(z)$. One can easily check that $\inf_{m \geq 0} V_{m,t}(z) \leq \inf_{m \geq 0} V_{m,2t}(z)$ for each positive t. So

$$V^*(z) \geq \sup_{t \geq 1} \inf_{m \geq 0} V_{m,t}(z) \geq \sup_{t \geq (1/2)} \inf_{m \geq 0} V_{m,t}(z) \geq \ldots \sup_{t>0} \inf_{m \geq 0} V_{m,t}(z) = V^*(z).$$

Consequently $V^*(z) = \sup_{t \geq 1} \inf_{m \geq 0} V_{m,t}(z)$. Moreover because $V_{m,t} \leq W_{m,t}$ we have also $V^*(z) \leq \sup_{t \geq 1} \inf_{m \geq 0} W_{m,t}(z)$.

We now claim that $V^*(z) = \sup_{t\geq 1} \inf_{m\geq 0} W_{m,t}(z)$. It remains to show $V^*(z) \geq \sup_{t\geq 1} \inf_{m\geq 0} W_{m,t}(z)$. From Lemma 3.11, we know that for all $k \geq 1$, $n \geq 1$ and $m \geq 0$, we have $V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z) - \frac{1}{n}$, so $\inf_m V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z) - \frac{1}{n}$. By taking the supremum on n , we obtain

$$V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} V_{m,n}(z) \geq \sup_{n\geq 1} \inf_{m\geq 0} V_{m,nk}(z) \geq \inf_{l\geq 0} W_{l,k}(z).$$

Since k is arbitrary, we have proved our claim.

Since the inequalities

$$\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \geq \inf_{m\geq 0} \sup_{n\geq 1} \; V_{m,n}(z) \geq \sup_{n\geq 1} \; \inf_{m\geq 0} V_{m,n}(z) = V^*(z)$$

are clear, to conclude the proof of the proposition it is enough to show that $\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \leq V^*(z)$.

Fix $\varepsilon > 0$. We have already proved that $V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} W_{m,n}(z)$, so for each $n \geq 1$ there exists $m \geq 0$ such that $W_{m,n}(z) \leq V^*(z) + \varepsilon$. Hence for each n, there exists z'_n in $G(z)$ such that $W_{0,n}(z'_n) \leq V^*(z) + \varepsilon$. We know from Lemma 3.7 that there exists $m_0 \geq 0$ such that : $\forall z' \in G(z), \exists z'' \in G^{m_0}(z)$ *s.t.* $\Delta(z', z'') \leq \varepsilon$. Consequently, for each $n \geq 1$, there exists z''_n in $G^{m_0}(z)$ such that $\Delta(z'_n, z''_n) \leq \varepsilon$, and by lemma 3.10 this implies that

$$W_n(z''_n) \leq W_n(z'_n) + \hat{\alpha}(\varepsilon) \leq V^*(z) + \varepsilon + \hat{\alpha}(\varepsilon).$$

Up to now, we have proved that for every $\varepsilon' > 0$, there exists m_0 such that :

$$\forall n \geq 1, \exists m \leq m_0 \text{ s.t. } W_{m,n}(z) \leq V^*(z) + \varepsilon'.$$

Since all costs lie in $[0, 1]$, it is easy to check that $|W_{m,n}(z) - W_{m',n}(z)| \leq |m - m'|$ for each n, m, m'. Hence there exists a *finite* subset F of $[0, m_0]$ such that : $\forall n \geq 1, \exists m \in F$ s.t. $W_{m,n}(z) \leq V^*(z) + 2\varepsilon'$. Considering $\hat{m}$ in F such that the set $\{n$ positive integer, $W_{\hat{m},n}(z) \leq V^*(z) + 2\varepsilon'\}$ is infinite, and noticing that $W_{m,n}$ is non decreasing in n, we obtain the existence of a unique $\hat{m} \geq 0$ such that $\forall n \geq 1$, $W_{\hat{m},n}(z) \leq V^*(z) + 2\varepsilon'$. Hence ε' being arbitrary, $\inf_{m\geq 0} \sup_{n\geq 1} \; W_{m,n}(z) \leq V^*(z)$, concluding the proof of Proposition 3.12.

QED

We now look for uniform convergence properties. By the precompacity condition $H2a$), it is easy to obtain that :

Lemma 3.13. *For each $\varepsilon > 0$, there exists a finite subset C of Z s.t. : $\forall z \in Z, \exists c \in C, \Delta(z, c) \leq \varepsilon$.*

We know that $(V_n)_n$ simply converges to V^* on Z. Since $|V_n(z) - V_n(z')| \leq \hat{\alpha}(\Delta(z, z'))$ for all n, z and z', we obtain by lemma 3.13 :

Corollary 3.14. *The convergence of* $(V_n)_n$ *to* V^* *is uniform on* Z*.*

We can proceed similarly to obtain other uniform properties. We have

$$V^*(z) = \sup_{n\geq 1} \inf_{m\geq 0} W_{m,n}(z) = \lim_{n\to+\infty} \inf_{m\geq 0} W_{m,n}(z)$$

since $\inf_{m\geq 0} W_{m,n}(z)$ is not decreasing in n. Using lemmas 3.10 and 3.13, we obtain that the convergence is uniform, hence we get :

$$\forall \varepsilon > 0, \exists n_0, \forall z \in Z, \;\; V^*(z) - \varepsilon \leq \inf_{m\geq 0} W_{m,n_0}(z) \leq V^*(z).$$

By Lemma 3.11, we obtain :

$$\forall \varepsilon > 0, \exists n_0, \forall z \in Z, \forall m \geq 0, \forall n \geq 1, V_{m,n}(z) \geq V^*(z) - \varepsilon - \frac{n_0}{n}.$$

Considering n large gives :

$$\forall \varepsilon > 0, \exists K, \forall z \in Z, \forall n \geq K, \;\; \inf_{m\geq 0} V_{m,n}(z) \geq V^*(z) - \varepsilon \tag{9}$$

Write now, for each state z and $m \geq 0$: $h_m(z) = \inf_{m'\leq m} \sup_{n\geq 1} W_{m',n}(z)$. $(h_m)_m$ converges to V^*, and as before, by Lemmas 3.10 and 3.13, we obtain that the convergence is uniform. Consequently,

$$\forall \varepsilon > 0, \exists M \geq 0, \forall z \in Z, \exists m \leq M, \;\; \sup_{n\geq 1} W_{m,n}(z) \leq V^*(z) + \varepsilon. \tag{10}$$

3.3.4 On the existence of a uniform value

In order to prove that $\Gamma(y_0)$ has a uniform value we have to show that for every $\varepsilon > 0$, there exist a control u and a time n_0 such that for every $n \geq n_0$, $\gamma_n(y_0, u) \leq V^*(y_0) + \varepsilon$. In this subsection we adapt the proofs of Lemma 4.1 and Proposition 4.2. in [18]. We start by constructing, for each n, a control which : 1) gives low average costs if one stops the play at *any* large time before n, and 2) after time n, leaves the player with a good "target" cost. This explains the importance of the quantities $\nu_{m,n}$. We start with the following

Lemma 3.15. $\forall \varepsilon > 0, \exists M \geq 0, \exists K \geq 1, \forall z \in Z, \exists m \leq M, \forall n \geq K, \exists u \in \mathcal{U}$ *such that :*

$$\nu_{m,n}(z,u) \leq V^*(z) + \varepsilon/2, \text{ and } V^*(y(m+n,u,z)) \leq V^*(z) + \varepsilon. \tag{11}$$

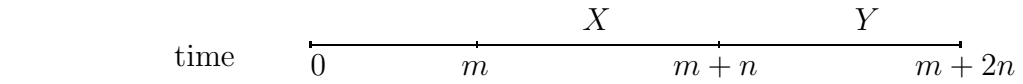
Proof : Fix $\varepsilon > 0$. Take M given by (10), so that $\forall z \in Z, \exists m \leq M, \;\; \sup_{n\geq 1} W_{m,n}(z) \leq V^*(z) + \varepsilon$. Take $K \geq 1$ given by (9) such that : $\forall z \in Z$, $\forall n \geq K$, $\inf_m V_{m,n}(z) \geq V^*(z) - \varepsilon$.

Fix an initial state z in Z. Consider m given by (10), and $n \geq K$. We have to find u in $\mathcal{U}$ satisfying (11).

We have $W_{m,n'}(z) \leq V^*(z) + \varepsilon$ for every $n' \geq 1$, so $W_{m,2n}(z) \leq V^*(z) + \varepsilon$, and we consider a control u which is ε-optimal for $W_{m,2n}(z)$, in the sense that $\nu_{m,2n}(z,u) \leq W_{m,2n}(z) + \varepsilon$. We have :

$$\nu_{m,n}(z,u) \leq \nu_{m,2n}(z,u) \leq W_{m,2n}(z) + \varepsilon \leq V^*(z) + 2\varepsilon.$$

Denoting $X = \gamma_{m,n}(z,u)$ and $Y = \gamma_{m+n,n}(z,u)$.



Since $\nu_{m,2n}(z,u) \leq V^*(z) + 2\varepsilon$, we have $X \leq V^*(z) + 2\varepsilon$, and $(X+Y)/2 = \gamma_{m,2n}(z,u) \leq V^*(z) + 2\varepsilon$. Since $n \geq K$, we also have $X \geq V_{m,n}(z) \geq V^*(z) - \varepsilon$. And $n \geq K$ also gives $V_n(y(m+n,u,z)) \geq V^*(y(m+n,u,z)) - \varepsilon$, so $V^*(y(m+n,u,z)) \leq V_n(y(m+n,u,z)) + \varepsilon \leq Y + \varepsilon$. Writing now $Y/2 = (X+Y)/2 - X/2$ we obtain $Y/2 \leq (V^*(z) + 5\varepsilon)/2$. So $Y \leq V^*(z) + 5\varepsilon$, and finally $V^*(y(m+n,u,z)) \leq V^*(z) + 6\varepsilon$.

QED

We can now conclude the proof of theorem 3.4.

Proposition 3.16. *For every state z in Z and $\varepsilon > 0$, there exists a control u in $\mathcal{U}$ and T_0 such that for every $T \geq T_0$, $\gamma_T(z,u) \leq V^*(z) + \varepsilon$.*

Proof : Fix $\alpha > 0$.

For every positive integer i, put $\varepsilon_i = \frac{\alpha}{2^i}$. Define $M_i = M(\varepsilon_i)$ and $K_i = K(\varepsilon_i)$ given by lemma 3.15 for ε_i. Define also $n_i = \mathrm{Max}\{K_i, \frac{M_{i+1}}{\alpha}\} \geq 1$.

We have : $\forall i \geq 1, \forall z \in Z, \exists\, m(z,i) \leq M_i, \exists u \in \mathcal{U}$, s.t.

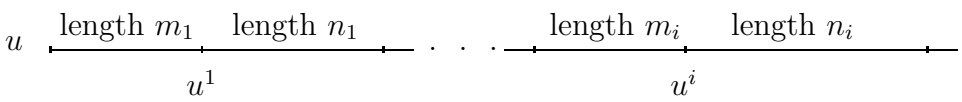
$$\nu_{m(z,i),n_i}(z,u) \leq V^*(z) + \frac{\alpha}{2^{i+1}} \text{ and } V^*(y(m(z,i)+n_i,u,z)) \leq V^*(z) + \frac{\alpha}{2^i}.$$

We now fix the initial state z in Z, and for simplicity write v^* for $V^*(z)$. We define a sequence $(z^i, m_i, u^i)_{i\geq 1}$ by induction :

• first put $z^1 = z$, $m_1 = m(z^1, 1) \leq M_1$, and pick u^1 in $\mathcal{U}$ such that $\nu_{m_1,n_1}(z^1,u^1) \leq V^*(z^1) + \frac{\alpha}{2^2}$, and $V^*(y(m_1+n_1,u^1,z^1)) \leq V^*(z^1) + \frac{\alpha}{2}$.

• for $i \geq 2$, put $z^i = y(m_{i-1}+n_{i-1}, u^{i-1}, z^{i-1})$, $m_i = m(z^i,i) \leq M_i$, and pick u^i in $\mathcal{U}$ such that $\nu_{m_i,n_i}(z^i,u^i) \leq V^*(z^i) + \frac{\alpha}{2^{i+1}}$ and $V^*(y(m_i+n_i,u^i,z^i)) \leq V^*(z^i) + \frac{\alpha}{2^i}$.

Consider finally u in $\mathcal{U}$ defined by concatenation : first u^1 is followed for time t in $[0, m_1+n_1)$, then u^2 is followed for t in $[m_1+n_1, m_2+n_2)$, etc... Since $z^i = y(m_{i-1}+n_{i-1}, u^{i-1}, z^{i-1})$ for each i, we have $y(\sum_{j=1}^{i-1} m_j + n_j, u, z) = z^i$ for each i. For each i we have $n_i \geq M_{i+1}/\alpha \geq m_{i+1}/\alpha$, so an interval with length n_i is much longer than an interval with length m_{i+1}.



For each $i \geq 1$, we have $V^*(z^i) \leq V^*(z^{i-1}) + \frac{\alpha}{2^{i-1}}$. So $V^*(z^i) \leq + \frac{\alpha}{2^{i-1}} + \frac{\alpha}{2^{i-2}} ... + \frac{\alpha}{2} + V^*(z^1) \leq v^* + \alpha - \frac{\alpha}{2^i}$. So $\nu_{m_i,n_i}(z^i, u^i) \leq v^* + \alpha$.

Let now T be large.

- First assume that $T = m_1 + n_1 + ... + m_{i-1} + n_{i-1} + r$, for some positive i and r in $[0, m_i]$. We have :

$$\begin{aligned} \gamma_T(z,u) &= \frac{1}{T}\int_0^T h(y(s,u,z),u(s))ds \\ &\leq \frac{1}{T}\left(\sum_{j=1}^{i-1} n_j\right)(v^*+\alpha) + \frac{m_1}{T} + \frac{1}{T}\left(\sum_{j=2}^{i} m_j\right) \end{aligned}$$

But $m_j \leq \alpha n_{j-1}$ for each j, so

$$\gamma_T(z,u) \leq v^* + 2\alpha + \frac{m_1}{T}.$$

- Assume now that $T = m_1 + n_1 + ... + m_{i-1} + n_{i-1} + m_i + r$, for some positive i and r in $[0, n_i]$. The previous computation shows that :

$$\int_0^{T-r} h(y(s,u,z),u(s))ds \leq m_1 + (T-r)(v^*+2\alpha).$$

Since $\nu_{m_i,n_i}(z^i, u^i) \leq v^* + \alpha$, we obtain :

$$\begin{aligned} T\gamma_T(z,u) &= \int_0^{T-r} h(y(s,u,z),u(s))ds + \int_{T-r}^{T} h(y(s,u,z),u(s))ds, \\ &\leq m_1 + (T-r)(v^*+2\alpha) + r(v^*+\alpha), \\ &\leq m_1 + T(v^*+2\alpha). \end{aligned}$$

Consequently, here also we have :

$$\gamma_T(z,u) \leq v^* + 2\alpha + \frac{m_1}{T}.$$

This concludes the proofs of Proposition 3.16 and consequently, of Theorem 3.4.

QED

4 Averaging methods for control systems with singular perturbations

Now we explain how to build a dynamics satisfied by the cluster points of the family $(x_\varepsilon(\cdot))_\varepsilon$ of solutions to (4).

4.1 Construction of a Limit system

We assume that the functions f and g are bounded continuous on $\mathbb{R}^b \times \mathbb{R}^d \times U$, and Lipschitz in (x, y) uniformly with respect to $u \in U$:

$$(\mathbf{A1}) \quad \begin{array}{l} \|f(x_1, y_1, u) - f(x_2, y_2, u)\| \leq L\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big), \\ \|g(x_1, y_1, u) - g(x_2, y_2, u)\| \leq L\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big). \end{array}$$

Under these assumptions we know that for every fixed control u there exists a unique solution $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ to the problem (4). We call $S_\varepsilon(t, x, y)$ the set of all trajectories of (4) :

$$S_\varepsilon(t, x, y) = \{\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big) \text{ solution of (4)}\}.$$

When we consider the system (4) on $[0, T]$ and we make the change of variable $\tau = \frac{t}{\varepsilon}$ we obtain a new system by setting $X(\tau) = x(\varepsilon\tau)$, $Y(\tau) = y(\varepsilon\tau)$ and $U(\tau) = u(\varepsilon\tau)$ for $\tau \in [0, \frac{T}{\varepsilon}]$

$$(12) \qquad \left\{ \begin{array}{l} X'(\tau) = \varepsilon f\big(X(\tau), Y(\tau), U(\tau)\big),\ X(0) = x, \\ Y'(\tau) = g\big(X(\tau), Y(\tau), U(\tau)\big),\ Y(0) = y. \end{array} \right.$$

When we take formally $\varepsilon = 0$ in (12), we are led to consider the following associated system :

$$(13) \qquad y'(\tau) = g\big(x, y(\tau), u(\tau)\big),\ y(0) = y,$$

where x is fixed in $\mathbb{R}^b$. We denote by $y_x(\cdot, u, y)$ the unique solution of (13) corresponding to the control u and to the initial value y.

We follow an averaging method (cf for instance [13], [15]) : we set, for $(x, y) \in \mathbb{R}^b \times \mathbb{R}^d$, $S > 0$, and any measurable control $u(\cdot)$,

$$A(x, y, S, u) = \frac{1}{S} \int_0^S f\big(x, y_x(\tau, u, y), u(\tau)\big)\, d\tau,$$

$$F(x, y, S) = \{A(x, y, S, u);\ u \in \mathcal{U}\}.$$

We shall make the following hypothesis on the system :

$$(\mathbf{A2}) \left\{ \begin{array}{l} \text{There exist compact sets } M \subset \mathbb{R}^b \text{ and } N \subset \mathbb{R}^d \text{ such that} \\ M \times N \text{ is invariant by (4) for all } \varepsilon. \end{array} \right.$$

$$(\mathbf{A3}) \left\{ \begin{array}{l} \text{There exist } c > 0,\ \Delta \text{ and } \hat{\alpha} \text{ as in Theorem 3.4 such that} \\ \forall (x_1, x_2) \in M^2, \forall (y_1, y_2) \in N^2, \forall S > 0, \forall \varepsilon > 0, \forall u \in \mathcal{U}, \exists v \in \mathcal{U}, \forall t \in [0, S] \\ \Delta(y_{x_1}(t, u, y_1), y_{x_2} x(t, v, y_2)) \leq \Delta(y_1, y_2) + c\|x_1 - x_2\| + \varepsilon, \\ \|f(x_1, y_{x_1}(t, u, y_1), u(t)) - f(x_2, y_{x_2}(t, v, y_2), v(t))\| \leq \hat{\alpha}(\Delta(y_1, y_2)) + c\|x_1 - x_2\|. \end{array} \right.$$

We will denote by $\hat{M}$ an upper bound of $\|f\|$ and $\|g\|$ on the compact set $M \times N$.

Proposition 4.1. *We assume Assumptions* **A1, A2, A3**. *Then there exists a function* $\gamma : I\!R \to I\!R_+$ *with* $\lim_{S\to+\infty} \gamma(S) = 0$ *and a set-valued map* $\bar{F} : M \times N \to \mathbb{R}^b$ *with compact convex nonempty values such that*

$$d(co\ clF(x,y,S), \overline{F}(x,y)) \leq \gamma(S),\ \forall (x,y) \in M \times N,\ \forall S > 0, \tag{14}$$

where d stands for the Hausdorff distance and co cl for the closed convex hull.

Proof : For any $p \in B \subset I\!R^b$ and x, y, u, we define

$$h(x,y,u,p) :=< f(x,y,u), p >$$

$$H(x,y,S,p) := \inf_{a\in F(x,y,S)} < p, a >= \inf_{u\in\mathcal{U}} \frac{1}{S} \int_0^S h(x, y_x(\tau,u,y), u(\tau), p) d\tau.$$

Clearly $H(x,y,S,p) = \inf_{a\in coF(x,y,S)} < p, a >$. For x, y, p fixed, using Theorem 3.4 and Remark 3.9, one can deduce that $H(x,y,S,p)$ converges to some $\overline{H}(x,y,p)$ when $S \to +\infty$.

We claim that the convergence is uniform in $(x,y,p) \in M \times N \times B$. To prove this it is enough to show that the family of functions $(H(\cdot,\cdot,S,\cdot))_{S>0}$ from $M \times N \times B$ to $I\!R$ is equicontinuous and to use Ascoli's Theorem. Fix $\varepsilon > 0$, $S > 0$, $(x_1, x_2) \in M^2$, $(y_1, y_2) \in N^2$ and $(p_1, p_2) \in B^2$. Take u such that $H(x_2, y_2, S, p_2) \geq \frac{1}{S}\int_0^S < f(x_2, y_{x_2}(\tau, u, y_2), u(\tau)), p_2 > d\tau - \varepsilon$. Then there exists some $v \in \mathcal{U}$ satisfying A3. So

$$H(x_1,y_1,S,p_1) - H(x_2,y_2,S,p_2) \leq$$

$$\frac{1}{S}\int_0^S < f(x_1, y_{x_1}(\tau,u,y_1), v(\tau)), p_1 > - < f(x_2, y_{x_2}(\tau,u,y_2), u(\tau)), p_2 > d\tau + \varepsilon$$

$$\leq \frac{1}{S}\int_0^S < f(x_1, y_{x_1}(\tau,u,y_1), v(\tau)), p_1 - p_2 > d\tau +$$

$$\frac{1}{S}\int_0^S < f(x_1, y_{x_1}(\tau,u,y_1), v(\tau)), p_2 > - < f(x_2, y_{x_2}(\tau,u,y_2), u(\tau)), p_2 > d\tau + \varepsilon$$

$$\leq \hat{M}\|p_1 - p_2\| + \hat{\alpha}(\Delta(y_1,y_2)) + c\|x_1 - x_2\| + \varepsilon.$$

Because one can easily obtain the reverse inequality by interchanging x_1, y_1, p_1 and x_2, y_2, p_2 and because ε is arbitrary we have proved our claim.

Observe that because H is bounded positively homogeneous and concave in p so is $\overline{H}$. So we can define the following convex compact subset of $\mathbb{R}^b$ by

$$\overline{F}(x,y) := \{a \in \mathbb{R}^b,\ < a, p > \geq \overline{H}(x,y,p),\ \forall p \in \mathbb{R}^b\}.$$

A straightforward verification shows (14).

QED

4.2 Main result for averaging method

Theorem 4.2. *Suppose that conditions* **A1, A2** *hold true. Assume that there exists a function* $\gamma : I\!R \mapsto I\!R_+$ *with* $\lim_{S\to+\infty}\gamma(S) = 0$ *and a set-valued map* $\bar{F} : M \times N \to \mathbb{R}^b$ *with compact convex nonempty values which is Lipschitz with respect to* x *(with a Lipschitz constant independent of* $y \in N$*) and which satisfies (14).*

Then, for any $x \in M$ *and* $y \in N$ *the solutions of the following differential inclusion*

$$x'(s) \in \overline{F}\big(x(s), y\big),\ x(0) = x. \tag{15}$$

approximate the solutions of the singularly perturbed system (4) in the following sense : For any $\varepsilon > 0$*, and any* $T > 0$ *there exists* $M(T,\varepsilon) > 0$ *with* $\lim_{\varepsilon\to 0} M(T,\varepsilon) = 0$ *such that*

a) For any family of solutions $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ *to (4) there exists a solution* $x(\cdot)$ *to (15) such that*

$$\sup_{t\in[0,T]} \|x_\varepsilon(t) - x(t)\| \le M(T,\varepsilon).$$

b) Conversely fix $x(\cdot)$ *a solution to (15) then for any* ε *small enough there exists a solution* $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ *to (4) such that*

$$\sup_{t\in[0,T]} \|x_\varepsilon(t) - x(t)\| \le M(T,\varepsilon).$$

Proof : The first part of the theorem is a direct consequence of Propositions 4.1 and 4.3. Let us prove the second part which is inspired by a method due to Gaitsgory [12, 13] (cf also [14, 15, 17]).

Fix $T > 0$, $x \in M$ and $y \in N$. We consider $S_\varepsilon > 0$ such that $\lim_{\varepsilon\to 0} S_\varepsilon = +\infty$, $\lim_{\varepsilon\to 0} \varepsilon S_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} \varepsilon S_\varepsilon e^{LS_\varepsilon} = 0$ (this is possible if we take for instance $S_\varepsilon = \ln \frac{1}{L\sqrt{\varepsilon}}$). We divide the interval $[0, \frac{T}{\varepsilon}]$ into subintervals $[\tau_l, \tau_{l+1}]$ having the same length S_ε (except possibly the last one). This corresponds to a division of $[0,T]$ in subintervals $[t_l, t_{l+1}]$ of length $\varepsilon S_\varepsilon$ with $t_l = \varepsilon\tau_l$, $l = 0, 1, \ldots, [T/(\varepsilon S_\varepsilon)]$.

Proof of part a) For any $\varepsilon > 0$, fix $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ a solution to (4) to which is associated a solution $(X_\varepsilon(\cdot), Y_\varepsilon(\cdot))$ to (12). We define a family of points for $l = 0, 1, \ldots, [T/(\varepsilon S_\varepsilon)]$

$$\begin{cases} x_0 := x \\ x_{l+1} := x_l + \varepsilon \int_{\tau_l}^{\tau_{l+1}} f(x_l, Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot), U(s))ds \end{cases} \tag{16}$$

and the corresponding interpolating curves $X_l(\cdot)$

$$\begin{cases} X_l(0) := x \\ X_{l+1}(\tau) := x_l + \varepsilon \int_{\tau_l}^{\tau} f(x_l, Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot)), U(s))ds.\ \tau \in [\tau_l, \tau_{l+1}] \end{cases} \tag{17}$$

We claim that $X_\varepsilon(\cdot)$ is approximated by $X_l(\cdot)$ when $\varepsilon \to 0^+$.

Let us define

$$\begin{cases} \Delta_l(\tau) := \max_{s\in[\tau_l,\tau]} \|X_\varepsilon(\cdot) - X_l(s)\|,\ \tau \in [\tau_l, \tau_{l+1}], \\ D_l(\tau) := \max_{s\in[\tau_l,\tau]} \|Y_\varepsilon(\cdot) - Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot))\|, \\ d_l(\tau) := \max_{s\in[\tau_l,\tau]} \|X_\varepsilon(\cdot) - x_l\| \end{cases}$$

Observe that for $s \in [\tau_l, \tau]$

$$\|Y_\varepsilon(\cdot) - Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot))\| \leq$$

$$\| \int_{\tau_l}^{s} g(X_\varepsilon(\sigma), Y_\varepsilon(\sigma), U(\sigma)) - g(x_l, Y_{x_l}(\sigma-\tau_l, y, U(\tau_l+\cdot)), U(\sigma))d\sigma\|$$

$$\text{(thanks to } \mathbf{A1}) \ \leq L \int_{\tau_l}^{s} \|X_\varepsilon(\sigma) - x_l\| + \|Y_\varepsilon(\sigma) - Y_{x_l}(\sigma-\tau_l, y, U(\tau_l+\cdot))\|d\sigma.$$

Thus

$$D_l(s) \leq L \int_{\tau_l}^{s} d_l(\sigma) + D_l(\sigma)d\sigma s \in [\tau_l, \tau]. \tag{18}$$

which gives using Gronwall's Lemma

$$D_l(s) \leq LS_\varepsilon d_l(\tau) e^{LS_\varepsilon},\ \forall s \in [0,\tau],\ \tau \in [\tau_l, \tau_{l+1}]. \tag{19}$$

Now we will estimate Δ_l. Pick $s \in [\tau_l, \tau]$.

$$\|X_\varepsilon(s) - X_l(s)\| \leq$$

$$\|X_\varepsilon(\tau_l) - x_l\| + \varepsilon \| \int_{\tau_l}^{s} f(X_\varepsilon(\sigma), Y_\varepsilon(\sigma), U(\sigma)) - f(x_l, Y_{x_l}(\sigma-\tau_l, y, U(\tau_l+\cdot)), U(\sigma))d\sigma\|$$

$$\leq \|X_\varepsilon(\tau_l) - x_l\| + \varepsilon L \int_{\tau_l}^{s} \|X_\varepsilon(\sigma) - x_l\| + \|Y_\varepsilon(\sigma) - Y_{x_l}(\sigma-\tau_l, y, U(\tau_l+\cdot))\|d\sigma$$

Thus

$$\Delta_l(s) \leq \Delta_l(\tau_l) + \varepsilon L \int_{\tau_l}^{s} d_l(\sigma) + D_l(\sigma)d\sigma$$

Taking account of (19), we obtain

$$\Delta_l(s) \leq \Delta_l(\tau_l) + \varepsilon L \int_{\tau_l}^{s} (1 + LS_\varepsilon e^{LS_\varepsilon}) d_l(\sigma)d\sigma \tag{20}$$

Since

$$\|X_\varepsilon(s) - x_l\| = \|X_\varepsilon(\tau_l) - x_l + \varepsilon \int_0^s f(X_\varepsilon(\sigma), U(\sigma))d\sigma\| \leq \|X_\varepsilon(\tau_l) - x_l\| + \varepsilon S_\varepsilon M,$$

we get

$$d_l(s) \leq \Delta_l(\tau_l) + \varepsilon S_\varepsilon M \leq \Delta_l(s) + \varepsilon S_\varepsilon M$$

because clearly $\Delta_l(\tau_l) \leq \Delta(s)$. This relation together with (20) enables us to deduce that invoking Gronwall's Lemma

$$\Delta_l(s) \leq (\Delta(\tau_l) + \varepsilon^2 L(1 + LS_\varepsilon e^{LS_\varepsilon})S_\varepsilon^2 M)e^{\varepsilon L(1+LS_\varepsilon e^{LS_\varepsilon})S_\varepsilon},\ \forall s \in [\tau_l, \tau_{l+1}]. \tag{21}$$

Because $\Delta_0(\tau_1) \leq 2\varepsilon M S_\varepsilon$, the relation (21) taken for $s = \tau_{l+1}$ yields by straightforward induction

$$\Delta(\tau_l) \leq 2\varepsilon M T e^{(1+LS_\varepsilon e^{LS_\varepsilon})\varepsilon LT} \tag{22}$$

We have proved our claim saying that $X_\varepsilon(\cdot)$ is approximated by $X_l(\cdot)$ when $\varepsilon \to 0^+$.

Now by Proposition 4.1

$$\frac{1}{S_\varepsilon}\int_{\tau_l}^{\tau_{l+1}} f(x_l, Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot)), U(s))ds \in \overline{F}(x_l, y) + \gamma(S_\varepsilon)B.$$

Choose $v_l \in \overline{F}(x_l, y))$ with

$$\|\frac{1}{S_\varepsilon}\int_{\tau_l}^{\tau_{l+1}} f(x_l, Y_{x_l}(s-\tau_l, y, U(\tau_l+\cdot)), U(s))ds - v_l\| \leq \gamma(S_\varepsilon) \tag{23}$$

Define the points η_l for $l = 0, 1, \ldots, [T/(\varepsilon S_\varepsilon)]$ and the associated interpolating curve $\eta_l(\cdot)$

$$\begin{cases} \eta_0 := x,\ \eta_{l+1} := \eta_l + \varepsilon S_\varepsilon v_l \\ \eta_l(t) := \eta_l + v_l(t - t_l),\ t \in [t_l, t_{l+1}] \end{cases} \tag{24}$$

From the very definition of x_l, η_l and from (23), we get

$$\|x_l - \eta_l\| \leq L\varepsilon S_\varepsilon \gamma(S_\varepsilon) \leq T\gamma(S_\varepsilon) \text{ for } l = 0, 1, \ldots, [T/(\varepsilon S_\varepsilon)]\ . \tag{25}$$

Now

$$d(\eta_l'(t), \overline{F}(\eta_l(t), y)) = d(v_l, \overline{F}(\eta_l(t), y)) \leq d(v_l, \overline{F}(x_l, y)) + \|v_l\|(\|x_l - \eta_l\| + \|\eta_l - \eta_l(t)\|)$$

Thus by (25) we get

$$d(\eta_l'(t), \overline{F}(\eta_l(t), y)) \leq ML(T\gamma(S_\varepsilon) + (S_\varepsilon\varepsilon)M). \tag{26}$$

Using Filippov's Theorem (cf for instance Th. 10.4.1 in [4] or [2, 10]), we can approximate $\eta_l(\cdot)$ by a solution to the differential inclusion (15). There exists $\bar{x}_l(\cdot)$ solution to (15) such that for any $t \in [0, T]$

(27)

$$\|\eta_l(t) - \bar{x}_l(t)\| \leq e^{LT}\int_0^t e^{-Ls} d(\eta_l'(t), \overline{F}(\eta_l(t), y))ds \leq Me^{LT}(T\gamma(S_\varepsilon) + (S_\varepsilon\varepsilon)M).$$

Then

$$\max_{t\in[0,T]} \|x_\varepsilon(t) - \bar{x}_l(t)\| \leq \max_l \max_{t\in[0,T]} (\|X_\varepsilon(t/\varepsilon) - X_l(t/\varepsilon)\| + \|X_l(t/\varepsilon) - \eta_l(t)\| + \|\eta_l(t) - \bar{x}_l(t)\|)$$

$$\leq 2\varepsilon MTe^{(1+LS_\varepsilon e^{LS_\varepsilon})\varepsilon LT} + T\gamma(S_\varepsilon) + 2\varepsilon S_\varepsilon M + Me^{LT}(T\gamma(S_\varepsilon) + (S_\varepsilon\varepsilon)M),$$

in view of (22), (25) and (27) and by the very definition of $\eta_l(\cdot)$ and $X_l(\cdot)$. The proof of part a) is achieved thanks to the choice of S_ε.

Proof of part b) Fix $x(\cdot)$ a solution to (15). For almost any $t \in [t_l, t_{l+1}]$, we have

$$d(x'(t), \overline{F}(x(t_l), y)) \leq L\|x(t_l) - x(t)\| \leq ML\varepsilon S_\varepsilon.$$

Thus, $\overline{F}$ being convex-valued

$$\frac{1}{\varepsilon S_\varepsilon}\int_{t_l}^{t_{l+1}} x'(s)ds \in \overline{F}(x(t_l), y) + \varepsilon S_\varepsilon LMB.$$

Take $v_l \in \overline{F}(x(t_l), y)$ such that

$$\|\int_{t_l}^{t_{l+1}} x'(s)ds - v_l\| \leq \varepsilon S_\varepsilon LM. \tag{28}$$

We define the family of points $(\xi_l)_l$ by

$$\xi_0 := x,\ \xi_{l+1} := \xi_l + \varepsilon S_\varepsilon v_l,\ l = 0, 1 \ldots [T/(\varepsilon S_\varepsilon)]. \tag{29}$$

Then

$$x(t_{l+1}) = x(t_l) + \int_{t_l}^{t_{l+1}} x'(s)ds \in x(t_l) + \varepsilon S_\varepsilon v_l + (\varepsilon S_\varepsilon)^2 LMB.$$

Hence for any $l = 0, 1 \ldots [T/(\varepsilon S_\varepsilon)]$

$$\begin{cases} \|x(t_l) - \xi_l\| \leq \varepsilon S_\varepsilon TML \\ d(v_l, \overline{F}(\xi_l, y)) \leq L^2 MT\varepsilon S_\varepsilon \end{cases} \tag{30}$$

There exist $w_l \in \overline{F}(\xi_l, y))$ with $v_l = w_l + h_l$ and $h_l \in L^2 MT\varepsilon S_\varepsilon B$.

Now we define $((x_l, u_l))_l$ by

$$\begin{cases} x_l := x, \\ x_{l+1} := x_l + \varepsilon \int_{\tau_l}^{\tau_{l+1}} f(x_l, y_{x_l}(s - \tau_l, y, u_l(\tau_l + \cdot)), u_l(\cdot)), u_l(s)ds \end{cases}$$

with $u_l(\cdot)$ is chosen such that

$$\|\frac{1}{S_\varepsilon}\int_{\tau_l}^{\tau_{l+1}} f(x_l, y_{x_l}(s - \tau_l, y, u_l(\tau_l + \cdot), u_l(\cdot)), u_l(s)ds - w_l\| \leq \gamma(S_\varepsilon) + L\|x_l - x_l\|. \tag{31}$$

Such choice of $u_l(\cdot)$ is possible because $w_l \in \overline{F}(\xi_l, y) \subset \overline{F}(x_l, y) + L\|\xi_l - x_l\|B$ and $d(F(x_l, y, S_\varepsilon), \overline{F}(x_l, y)) \leq \gamma(S_\varepsilon)$.

Now we will estimate $d_l := \|\xi_l - x_l\|$. Observe that

$$\xi_{l+1} - x_{l+1} = \xi_l - x_l + \varepsilon S_\varepsilon\{v_l - \frac{1}{S_\varepsilon}\int_{\tau_l}^{\tau_{l+1}} f(x_l, y_{x_l}(s - \tau_l, y, u_l(\tau_l + \cdot)), u_l(\cdot)), u_l(s)ds\}$$

$$\in \xi_l - x_l + \varepsilon S_\varepsilon(v_l - \overline{F}(x_l, y) + \gamma(S_\varepsilon)B) \subset \xi_l - x_l + \varepsilon S_\varepsilon(L\|x_l - \xi_l\| + \gamma(S_\varepsilon))B.$$

Thus

$$d_{l+1} \leq d_l(1 + L\varepsilon S_\varepsilon) + \varepsilon S_\varepsilon \gamma(S_\varepsilon).$$

A straightforward induction gives

$$d_l \leq \gamma(S_\varepsilon)\varepsilon S_\varepsilon \frac{(1 + L\varepsilon S_\varepsilon)^l - 1}{L\varepsilon S_\varepsilon}.$$

Taking into account the elementary inequalities $(1 + \frac{c}{s})^s \leq e^c$ and $\frac{e^c - 1}{c} \leq e^c$, we obtain

$$d_l \leq \gamma(S_\varepsilon)\varepsilon T e^{L\varepsilon T}. \tag{32}$$

Let us define $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ a solution to (4). From the same technique developped in the proof of part a) we obtain that there exists $\kappa(T, \varepsilon)$ with $\lim_{\varepsilon \to 0} \kappa(T, \varepsilon) = 0$ and such that

$$\max_l \|x_\varepsilon(t_l) - x_l\| \leq \kappa(T, \varepsilon). \tag{33}$$

Clearly if $t \in [t_l, t_{l+1}]$ then

$$\|x(t) - x_\varepsilon(t)\| \leq \|x(t_l) - x_\varepsilon(t_l)\| + 2\varepsilon S_\varepsilon M.$$

So

$$\max_{t \in [0,T]} \|x(t) - x_\varepsilon(t)\| \leq 2\varepsilon S_\varepsilon M + \|x(t_l) - \xi_l\| + \|\xi_l - x_l\| + \|x_l - x_\varepsilon(t_l)\|$$

$$\leq 2\varepsilon S_\varepsilon M + \varepsilon S_\varepsilon T M L + \varepsilon T e^{L\varepsilon T}\gamma(S_\varepsilon) + \kappa(T, \varepsilon),$$

thanks to (30),(32) and (33). The proof is complete.

QED

4.3 About the validity of the assumptions for the averaging method

In order to obtain Lipschitz regularity of the set valued map $\overline{F}$ we need to require a more precise assumption on the control system

Proposition 4.3. *Suppose that conditions* **A1, A2** *hold true and assume that there exits* $\overline{F}$ *and* γ *satisfying* 14.

a) If we suppose

$$\begin{array}{l} \exists C > 0,\ \forall (x_1, x_2) \in M^2, \forall y \in N, \forall S > 0, \forall u \in \mathcal{U}, \exists v \in \mathcal{U}, \\ \|\frac{1}{S}\int_0^S f(x_1, y_{x_1}(t, u, y), u(t)) - f(x_2, y_{x_2}(t, v, y), v(t))dt\| \leq C\|x_1 - x_2\|. \end{array} \tag{34}$$

then the set-valued map $x \in M \mapsto \overline{F}(x,y)$ is Lipschitz continuous (with a Lipschitz constant independent of y).

b) If the following assumption

$$(\mathbf{A4}) \left\{ \begin{array}{l} \exists C>0,\ \forall (x_1,x_2)\in M^2, \forall (y_1,y_2)\in N^2, \forall S>0, \forall u\in \mathcal{U}, \exists v\in \mathcal{U}, \forall t\in [0,S] \\ i)\ \|(y_{x_1}(t,u,y_1)-y_{x_2}(t,v,y_2)\| \leq C(\|y_1-y_2\|+\|x_1-x_2\|), \\ ii)\ \|f(x_1,y_{x_1}(t,u,y_1),u(t))-f(x_2,y_{x_2}(t,v,y_2),v(t))\| \leq C(\|y_1-y_2\|+\|x_1-x_2\|). \end{array} \right.$$

holds true then $(x,y)\in M\times N \mapsto \overline{F}(x,y)$ is lipschitz continuous.

Proof : Let us prove a). Fix x_1, x_2, y. For any control $u\in\mathcal{U}$ let us denote v_u the control associated with u by condition (34). Then

$$F(x_1,y,S)=\bigcup_{u\in\mathcal{U}}\{\frac{1}{S}\int_0^S f(x_1,y_{x_1}(t,u,y),u(t))dt\}$$

$$\subset \bigcup_{u\in\mathcal{U}}\{\frac{1}{S}\int_0^S f(x_2,y_{x_2}(t,v_u,y),v_u(t))dt\}+C\|x_1-x_2\|B.$$

Taking the Haussdorff limit when $S\to+\infty$ in the above inclusion we obtain in view of (14) :

$$\overline{F}(x_1,y)\subset \overline{F}(x_2,y)+C\|x_1-x_2\|B.$$

This gives a).

We obtain b) in a similar way by associating to any control u a control v_u given by **A4** (instead of (14)) taking into account that ε is arbitrary.

QED

Now we may summarize our considerations about a way to check the assumptions needed in Theorem 4.2

Proposition 4.4. *Assume that conditions* **A1, A2, A4** *holds true. Then there exists a Lipchitz convex compact valued set-valued map $\overline{F}$ such that trajectories of the associated differential inclusion (15) approximate the solutions of the singularly perturbed control system (4) (in the meaning of Theorem 4.2).*

Proof : Observe first that Assumption **A4** implies **A3** for $\Delta(y_1,y_3)=C|y_1-y_2|$ and $c=C$. So by Proposition 4.1, we are able to define a map $\overline{F}$ satisfying (14). Because **A4** obviously implies (34), we deduce from Proposition 4.3 - a) that $\overline{F}$ is Lipschitz with respect to x.

Now we have checked all the required assumptions of Theorem 4.2 which gives the wished conclusion.

QED

In the literature there exist many works with assumptions implying that the limit system (15) is independent of y. Such assumptions are usually based on controlability or dissipativity properties of the fast dynamics (4) (or equivalently (13)). We refer the reader to [1, 5, 12, 13, 14, 15, 17] and the references therein.

Remarks 4.5. • *The most direct application of our non expansive approach is the case of the system* $x'(t) = f(x(t), y(t))$, $\varepsilon y'(t) = g(y(t), u(t))$ *where* g *satisfies (3).*

• *A straightforward calculus shows that condition (***A4** *) is also satisfied if we assume*

$$\begin{cases} \forall (y_1, y_2) \in N^2,\ \forall u \in U,\ \exists v \in U,\ \forall (x_1, x_2) \in M^2, \\ < y_1 - y_2, g(x_1, y_1, u) - g(x_2, y_2, u) > \leq 0. \end{cases}$$

• *Of course our approach contains also several classical conditions where the limit system is* y *independent, for instance the classical dissipativity condition [1] :*

$$\begin{cases} \exists c > 0,\ \forall (x_1, x_2) \in M^2, \forall (y_1, y_2) \in N^2, \forall u \in U, \\ < y_1 - y_2, g(x_1, y_1, u) - g(x_2, y_2, u) > \leq -c\|y_1 - y_2\|^2. \end{cases}$$

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