# Three results in Dunkl theory 

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# THREE RESULTS IN DUNKL ANALYSIS 

BÉCHIR AMRI, JEAN-PHILIPPE ANKER \& MOHAMED SIFI<br>In memory of Andrzej Hulanicki (1933-2008), a distinguished polish mathematician, a guide and a friend, who has left many orphans in Wroctaw and around the world. We miss you.


#### Abstract

In this article, we establish first a geometric Paley-Wiener theorem for the Dunkl transform in the crystallographic case. Next we obtain an optimal bound for the $L^{p} \rightarrow L^{p}$ norm of Dunkl translations in dimension 1. Finally we describe more precisely the support of the distribution associated to Dunkl translations in higher dimension.


## 1. Introduction

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^{N}$. It started twenty years ago with Dunkl's seminal work [5] and was further developed by several mathematicians. See for instance the surveys [16, [] and the references cited therein.

In this setting, the Paley-Wiener theorem is known to hold for balls centered at the origin. In [9], a Paley-Wiener theorem was conjectured for convex neighborhoods of the origin, which are invariant under the underlying reflection group, and was partially proved. Our first result in Section 3 is a proof of this conjecture in the crystallographic case, following the third approach in (9).

Generalized translations were introduced in [14] and further studied in [21, 17, 22]. Apart from their abstract definition, we lack precise information, in particular about their integral representation

$$
\left(\tau_{x} f\right)(y)=\int_{\mathbb{R}^{N}} f(z) d \gamma_{x, y}(z),
$$

which was conjectured in [14] and established in few cases, for instance in dimension $N=1$ or when $f$ is radial. Our second result in Section $\square^{6}$ is an optimal bound for the integral

$$
\int_{\mathbb{R}}\left|d \gamma_{x, y}(z)\right|
$$

in dimension $N=1$, improving upon earlier results in [13, 22]. Our bound depends on the multiplicity $k \geq 0$ and tends from below to $\sqrt{2}$, as $k \rightarrow+\infty$. Our third result in

Section 5 deals with the support of the distribution $\gamma_{x, y}$ in higher dimension, that we determine rather precisely in the crystallographic case.

## 2. Background

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the articles [55, 8] or to the surveys [16, (7].

Let $G \subset \mathrm{O}\left(\mathbb{R}^{N}\right)$ be a finite reflection group associated to a reduced root system $R$ and $k: R \rightarrow[0,+\infty)$ a $G$-invariant function (called multiplicity function). Let $R^{+}$be a positive root subsystem, $\Gamma_{+}$the corresponding open positive chamber, $\overline{\Gamma_{+}}$its closure, $\overline{\Gamma^{+}}=\sum_{\alpha \in R^{+}} \mathbb{R}_{+} \alpha$ the dual cone, and let us denote by $x_{+}$the intersection point of any orbit $G . x$ in $\mathbb{R}^{N}$ with $\overline{\Gamma_{+}}$.

The Dunkl operators $T_{\xi}$ on $\mathbb{R}^{N}$ are the following $k$-deformations of directional derivatives $\partial_{\xi}$ by difference operators:

$$
T_{\xi} f(x)=\partial_{\xi} f(x)+\sum_{\alpha \in R^{+}} k(\alpha)\langle\alpha, \xi\rangle \frac{f(x)-f\left(\sigma_{\alpha} \cdot x\right)}{\langle\alpha, x\rangle}
$$

where $\sigma_{\alpha} \cdot x=x-\frac{\langle\alpha, x\rangle}{\left.2|\alpha|\right|^{2}} \alpha$ denotes the reflection with respect to the hyperplane orthogonal to $\alpha$. The Dunkl operators are antisymmetric with respect to the measure $w(x) d x$ with density

$$
w(x)=\prod_{\alpha \in R^{+}}|\langle\alpha, x\rangle|^{2 k(\alpha)} .
$$

The operators $\partial_{\xi}$ and $T_{\xi}$ are intertwined by a Laplace-type operator

$$
\begin{equation*}
V f(x)=\int_{\mathbb{R}^{N}} f(y) d \mu_{x}(y) \tag{1}
\end{equation*}
$$

associated to a family of compactly supported probability measures $\left\{\mu_{x} \mid x \in \mathbb{R}^{N}\right\}$. Specifically, $\mu_{x}$ is supported in the the convex hull

$$
C^{x}=\operatorname{co}(G \cdot x) .
$$

For every $\lambda \in \mathbb{C}^{N}$, the simultaneous eigenfunction problem

$$
T_{\xi} f=\langle\lambda, \xi\rangle f \quad \forall \xi \in \mathbb{R}^{N}
$$

has a unique solution $f(x)=E(\lambda, x)$ such that $E(\lambda, 0)=1$, which is given by

$$
\begin{equation*}
E(\lambda, x)=V\left(e^{\langle\lambda, .\rangle}\right)(x)=\int_{\mathbb{R}^{N}} e^{\langle\lambda, y\rangle} d \mu_{x}(y) \quad \forall x \in \mathbb{R}^{N} \tag{2}
\end{equation*}
$$

Furthermore $\lambda \mapsto E(\lambda, x)$ extends to a holomorphic function on $\mathbb{C}^{N}$ and the following estimate holds:

$$
|E(\lambda, x)| \leq e^{\left\langle(\operatorname{Re} \lambda)_{+}, x+\right\rangle} \quad \forall \lambda \in \mathbb{C}^{N}, \forall x \in \mathbb{R}^{N} .
$$

In dimension $N=1$, these functions can be expressed in terms of Bessel functions. Specifically,

$$
E(\lambda, x)=j_{k-\frac{1}{2}}(\lambda x)+\frac{\lambda x}{2 k+1} j_{k+\frac{1}{2}}(\lambda x)
$$

where

$$
j_{\nu}(z)=\Gamma(\nu+1) \sum_{n=0}^{+\infty} \frac{(-1)^{n}}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2 n}
$$

are normalized Bessel functions.

The Dunkl transform is defined on $L^{1}\left(\mathbb{R}^{N}, w(x) d x\right)$ by

$$
\mathcal{D} f(\xi)=\frac{1}{c} \int_{\mathbb{R}^{N}} f(x) E(-i \xi, x) w(x) d x
$$

where

$$
c=\int_{\mathbb{R}^{N}} e^{-\frac{|x|^{2}}{2}} w(x) d x
$$

We list some known properties of this transform:
(i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}\left(\mathbb{R}^{N}\right)$.
(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of $L^{2}\left(\mathbb{R}^{N}, w(x) d x\right)$.
(iii) (Inversion formula) For every $f \in \mathcal{S}\left(\mathbb{R}^{N}\right)$, and more generally for every $f \in$ $L^{1}\left(\mathbb{R}^{N}, w(x) d x\right)$ such that $\mathcal{D} f \in L^{1}\left(\mathbb{R}^{N}, w(\xi) d \xi\right)$, we have

$$
f(x)=\mathcal{D}^{2} f(-x) \quad \forall x \in \mathbb{R}^{N}
$$

(iv) (Paley-Wiener theorem) The Dunkl transform is a linear isomorphism between the space of smooth functions $f$ on $\mathbb{R}^{N}$ with $\operatorname{supp} f \subset \overline{B(0, R)}$ and the space of entire functions $h$ on $\mathbb{C}^{N}$ such that

$$
\begin{equation*}
\sup _{\xi \in \mathbb{C}^{N}}(1+|\xi|)^{M} e^{-R|\operatorname{Im} \xi|}|h(\xi)|<+\infty \quad \forall M \in \mathbb{N} \tag{3}
\end{equation*}
$$

## 3. A geometric Paley-Wiener theorem

In this section, we prove a geometric version of the Paley-Wiener theorem, which was looked for in [9, 21, 10], under the assumption that $G$ is crystallographic. The proof consists merely in resuming the third approach in [9] and applying it to the convex sets considered in [1], 2, 3, 6] instead of the convex sets considered in [11]. Recall that the second family consists of the convex hulls

$$
C^{\Lambda}=\operatorname{co}(G . \Lambda)
$$

of $G$-orbits $G . \Lambda$ in $\mathbb{R}^{N}$, while the first family consists of the polar sets

$$
C_{\Lambda}=\left\{x \in \mathbb{R}^{N} \mid\langle x, g \cdot \Lambda\rangle \leq 1 \forall g \in G\right\} .
$$



Figure 1. The sets $C^{\Lambda}$ and $C_{\Lambda}$ for the root system $A_{1} \times A_{1}$


Figure 2. The sets $C^{\Lambda}$ and $C_{\Lambda}$ for the root system $B_{2}$
Before stating the geometric Paley-Wiener theorem, let us make some remarks about the sets $C^{\Lambda}$ and $C_{\Lambda}$. Firstly, they are convex, closed, $G$-invariant and the following inclusion holds:

$$
C^{\Lambda} \subset|\Lambda|^{2} C_{\Lambda}
$$

Secondly, we may always assume that $\Lambda=\Lambda_{+}$belongs to the closed positive chamber $\overline{\Gamma_{+}}$ and, in this case, we have

$$
\begin{aligned}
& C^{\Lambda} \cap \overline{\Gamma_{+}}=\overline{\Gamma_{+}} \cap\left(\Lambda-\overline{\Gamma^{+}}\right) \\
& C_{\Lambda} \cap \overline{\Gamma_{+}}=\left\{x \in \overline{\Gamma_{+}} \mid\langle\Lambda, x\rangle \leq 1\right\} .
\end{aligned}
$$

Thirdly, on one hand, every $G$-invariant convex subset in $\mathbb{R}^{N}$ is a union of sets $C^{\Lambda}$ while, on the other hand, every $G$-invariant closed convex subset in $\mathbb{R}^{N}$ is an intersection of sets $C_{\Lambda}$. For instance,

$$
\overline{B(0, R)}=\bigcup_{|\Lambda|=R} C^{\Lambda}=\bigcap_{|\Lambda|=R^{-1}} C_{\Lambda} .
$$

Fourthly, we shall say that $\Lambda \in \overline{\Gamma_{+}}$is admissible if the following equivalent conditions are satisfied:
(i) $\Lambda$ has nonzero projections in each irreducible component of $\left(\mathbb{R}^{N}, R\right)$,
(ii) $C^{\Lambda}$ is a neighborhood of the origin,
(iii) $C_{\Lambda}$ is bounded.

In this case, we may consider the gauge

$$
\chi_{\Lambda}(\xi)=\max _{x \in C_{\Lambda}}\langle x, \xi\rangle=\min \left\{r \in[0,+\infty) \mid \xi \in r C^{\Lambda}\right\}
$$

on $\mathbb{R}^{N}$.
Theorem 3.1. Assume that $\Lambda \in \overline{\Gamma_{+}}$is admissible. Then the Dunkl transform is a linear isomorphism between the space of smooth functions $f$ on $\mathbb{R}^{N}$ with $\operatorname{supp} f \subset C_{\Lambda}$ and the space of entire functions $h$ on $\mathbb{C}^{N}$ such that

$$
\begin{equation*}
\sup _{\xi \in \mathbb{C}^{N}}(1+|\xi|)^{M} e^{-\chi_{\Lambda}(\operatorname{Im} \xi)}|h(\xi)|<+\infty \quad \forall M \in \mathbb{N} \tag{4}
\end{equation*}
$$

Proof. Following [6], this theorem is first proved in the trigonometric case, which explains the restriction to crystallographic groups, and next obtained in the rational case by passing to the limit. The proof of Theorem 3.1 in the trigonometric case is similar to the proof of the Paley-Wiener Theorem in [11, [2], and actually to the initial proof of Helgason for the spherical Fourier transform on symmetric spaces of the noncompact type. This was already observed in [19] and will be developed below for the reader's convenience. The limiting procedure, as far as it is concerned, is described thoroughly in [9] and needs no further explanation.

Thus assume that $h$ is an entire function on $\mathbb{C}^{N}$ satisfying (隹) and, by resuming the proof of [11, Theorem 8.6 (2)], let us show that its inverse Cherednik transform

$$
\begin{equation*}
f(x)=\text { const. } \int_{\mathbb{R}^{N}} h(\xi) \widetilde{E}(i \xi, x) \widetilde{w}(\xi) d \xi \tag{5}
\end{equation*}
$$

vanishes outside $C_{\Lambda}$. Firstly, one may restrict by $G$-equivariance to $x=g_{0} \cdot x_{+}$, where $x_{+} \in \Gamma_{+} \backslash C_{\Lambda}$ and $g_{0}$ denotes the longest element in $G$, which interchanges $\Gamma_{+}$and $-\Gamma_{+}$. Secondly, by expanding

$$
\left\{\prod_{\alpha \in R^{+}}\left(\langle\check{\alpha}, \xi\rangle-k_{\alpha}\right)\right\} \widetilde{E}(\xi, x)=\sum_{g \in G} \sum_{q \in Q^{+}} \mathbf{c}(-g \cdot \xi) \widetilde{E}_{q}(g, g \cdot \xi) e^{\langle g . \xi+\varrho+q, x\rangle}
$$

(55) becomes

$$
f(x)=\text { const. } \sum_{g \in G} \operatorname{det} g \sum_{q \in Q^{+}} f_{g, q}(x) e^{\langle\varrho+q, x\rangle}
$$

where

$$
\begin{equation*}
f_{g, q}(x)=\int_{\mathbb{R}^{N}} h\left(g^{-1} \cdot \xi\right) \widetilde{E}_{q}(g, i \xi) e^{i\langle\xi, x\rangle}\left\{\prod_{\alpha \in R^{+}} \frac{\Gamma\left(i\langle\check{\alpha}, \xi\rangle+k_{\alpha}\right)}{\Gamma(i\langle\tilde{\alpha}, \xi\rangle+1)}\right\} d \xi \tag{6}
\end{equation*}
$$

Thirdly, one shows that all expressions (6) vanish, by shifting the contour of integration from $\mathbb{R}^{N}$ to $\mathbb{R}^{N}+i t g_{0} . \Lambda$ with $t>0$, which produces an exponential factor $e^{-c t}$ with $c=\left\langle\Lambda, x_{+}\right\rangle-1>0$, and by letting $t \rightarrow+\infty$.

Since every $G$-invariant convex compact neighborhood of the origin in $\mathbb{R}^{N}$ is the intersection of admissible sets $C_{\Lambda}$, Theorem 3.1 generalizes as follows.

Corollary 3.2 (Geometric Paley-Wiener Theorem). Let $C$ be a $G$-invariant convex compact neighborhood of the origin in $\mathbb{R}^{N}$ and $\chi(\xi)=\max _{x \in C}\langle x, \xi\rangle$ the dual gauge. Then the Dunkl transform is a linear isomorphism between the space $\mathcal{C}_{C}^{\infty}\left(\mathbb{R}^{N}\right)$ of smooth functions $f$ on $\mathbb{R}^{N}$ with supp $f \subset C$ and the space $\mathcal{H}_{\chi}\left(\mathbb{C}^{N}\right)$ of entire functions $h$ on $\mathbb{C}^{N}$ such that

$$
\sup _{\xi \in \mathbb{C}^{N}}(1+|\xi|)^{M} e^{-\chi(\operatorname{Im} \xi)}|h(\xi)|<+\infty \quad \forall M \in \mathbb{N}
$$

Remark 3.3. Notice that the Dunkl transform $\mathcal{D}$ always maps $\mathcal{C}_{C}^{\infty}\left(\mathbb{R}^{N}\right)$ into $\mathcal{H}_{\chi}\left(\mathbb{C}^{N}\right)$ and that the assumption that $G$ is crystallographic is only used to prove that $\mathcal{D}$ is onto.

## 4. $L^{p}$ BOUNDS FOR GENERALIZED TRANSLATIONS IN DIMENSION 1

Dunkl translations are defined on $\mathcal{S}\left(\mathbb{R}^{N}\right)$ by

$$
\left(\tau_{x} f\right)(y)=\frac{1}{c} \int_{\mathbb{R}^{N}} \mathcal{D} f(\xi) E(i \xi, x) E(i \xi, y) w(\xi) d \xi \quad \forall x, y \in \mathbb{R}^{N}
$$

They have an explicit integral representation [13] in dimension $N=1$ :

$$
\left(\tau_{x} f\right)(y)=\int_{\mathbb{R}} f(z) d \gamma_{x, y}(z)
$$

where

$$
d \gamma_{x, y}(z)= \begin{cases}\gamma(x, y, z)|z|^{2 k} d z & \text { if } x, y \in \mathbb{R}^{*}  \tag{7}\\ d \delta_{y}(z) & \text { if } x=0 \\ d \delta_{x}(z) & \text { if } y=0\end{cases}
$$

is a signed measure such that $\int_{\mathbb{R}} d \gamma_{x, y}(z)=1$. Specifically,

$$
\gamma(x, y, z)=d \sigma(x, y, z) \rho(|x|,|y|,|z|) \mathbb{1}_{I_{|x|,|y|}}(|z|) \quad \forall x, y, z \in \mathbb{R}^{*}
$$

where

$$
\begin{aligned}
d & =\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)}, \\
\sigma(x, y, z) & =1-\frac{x^{2}+y^{2}-z^{2}}{2 x y}+\frac{z^{2}+y^{2}-x^{2}}{2 z y}+\frac{x^{2}+z^{2}-y^{2}}{2 x z} \\
& =\frac{(z+x+y)(z+x-y)(z-x+y)}{2 x y z} \quad \forall x, y, z \in \mathbb{R}^{*}, \\
\rho(a, b, c) & =\frac{\left\{c^{2}-(a-b)^{2}\right\}^{k-1}\left\{(a+b)^{2}-c^{2}\right\}^{k-1}}{(2 a b c)^{2 k-1}} \\
& =\frac{\left(2 b^{2} c^{2}+2 a^{2} c^{2}+2 a^{2} b^{2}-a^{4}-b^{4}-c^{4}\right)^{k-1}}{(2 a b c)^{2 k-1}} \quad \forall a, b, c>0,
\end{aligned}
$$

and $I_{a, b}$ denotes the interval $[|a-b|, a+b]$. Notice the symmetries

$$
\gamma(x, y, z)=\left\{\begin{array}{l}
\gamma(y, x, z)  \tag{8}\\
\gamma(-x,-y,-z) \\
\gamma(-z, y,-x)=\gamma(x,-z,-y)
\end{array}\right.
$$

Proposition 4.1. The following inequality holds, for every $x, y \in \mathbb{R}$ :

$$
\begin{equation*}
\int_{\mathbb{R}}\left|d \gamma_{x, y}(z)\right| \leq A_{k}=\sqrt{2} \frac{\left\{\Gamma\left(k+\frac{1}{2}\right)\right\}^{2}}{\Gamma\left(k+\frac{1}{4}\right) \Gamma\left(k+\frac{3}{4}\right)} . \tag{9}
\end{equation*}
$$

Actually there is equality if $x=y \in \mathbb{R}^{*}$. Moreover $A_{k} \xrightarrow{<} \sqrt{2}$ as $k \rightarrow+\infty$.
Remark 4.2. This result improves earlier bounds obtained in (13] and 22, which were respectively 4 and 3 .

Proof. Let $x, y \in \mathbb{R}^{*}$.
Case 1: Assume that $x y<0$. Then $||x|-|y||=|x+y|$ and $|x|+|y|=|x-y|$, hence $\sigma(x, y, z) \mathbb{1}_{I_{|x|,|y|}}(|z|)=\frac{z+x+y}{z} \frac{(x-y)^{2}-z^{2}}{-2 x y} \mathbb{1}_{||x|,|y|}(|z|)$ and $\gamma_{x, y}$ are positive. Thus

$$
\int_{\mathbb{R}}\left|d \gamma_{x, y}(z)\right|=\int_{\mathbb{R}} d \gamma_{x, y}(z)=1
$$

Case 2: Assume that $x y>0$. By symmetry, we may reduce to $0<x \leq y$. Then

$$
\begin{aligned}
\int_{\mathbb{R}}\left|d \gamma_{x, y}(z)\right| & =\int_{-\infty}^{0}\left|d \gamma_{x, y}(z)\right|+\int_{0}^{+\infty}\left|d \gamma_{x, y}(z)\right| \\
& =2 d \int_{y-x}^{y+x} \frac{x+y}{2 x y z}\left(\frac{z^{2}-x^{2}-y^{2}+2 x y}{2 x y z}\right)^{k}\left(\frac{x^{2}+y^{2}+2 x y-z^{2}}{2 x y z}\right)^{k-1} z^{2 k} d z
\end{aligned}
$$

After performing the change of variables $z=\sqrt{x^{2}+y^{2}-2 x y \cos \theta}$ and setting $y=s x$, we get

$$
\begin{equation*}
\int_{\mathbb{R}}\left|d \gamma_{x, y}(z)\right|=\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)}(1+s) \int_{0}^{\pi} \frac{(1-\cos \theta) \sin ^{2 k-1} \theta}{\sqrt{1+s^{2}-2 s \cos \theta}} d \theta . \tag{10}
\end{equation*}
$$

Denote by $F(s)$ the right hand side of (10). Since

$$
F^{\prime}(s)=\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)}(1-s) \int_{0}^{\pi} \frac{\sin ^{2 k+1} \theta}{\left(1+s^{2}-2 s \cos \theta\right)^{\frac{3}{2}}} d \theta
$$

is nonpositive, $F(s)$ is a decreasing function on $[1,+\infty)$, which reaches its maximum at $s=1$. Let us compute it:

$$
\begin{aligned}
& A_{k}=F(1)=\frac{\sqrt{2} \Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)} \int_{0}^{\pi}(1-\cos \theta)^{k-\frac{1}{2}}(1+\cos \theta)^{k-1} \sin \theta d \theta \\
&=2^{2 k} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)} \int_{0}^{1} t^{k-\frac{1}{2}}(1-t)^{k-1} d t \\
&\left.=2^{2 k} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(k)} B\left(k+\frac{1}{2}, k\right)=2^{2 k} \frac{\left\{\Gamma\left(k+\frac{1}{2}\right)\right\}^{2}}{\sqrt{\pi} \Gamma\left(2 k+\frac{1}{2}\right)}=\sqrt{2} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{4}\right)}\right) \Gamma\left(k+\frac{1}{2}\right) \\
& \Gamma\left(k+\frac{3}{4}\right)
\end{aligned},
$$

after performing the change of variables $t=\frac{1-\cos \theta}{2}$ and using standard properties of the beta and gamma functions.
Finally let us show that $A_{k} \xrightarrow{<} \sqrt{2}$ as $k \rightarrow+\infty$. Write

$$
A_{k}=\sqrt{2} \frac{G\left(k+\frac{1}{4}\right)}{G\left(k+\frac{1}{2}\right)}, \quad \text { where } \quad G(u)=\frac{\Gamma\left(u+\frac{1}{4}\right)}{\Gamma(u)} \quad \forall u>0 .
$$

Since the logarithmic derivative $\frac{\Gamma^{\prime}}{\Gamma}$ of the gamma function is a strictly increasing analytic function on $(0,+\infty)$, the logarithmic derivative

$$
\frac{G^{\prime}(u)}{G(u)}=\frac{\Gamma^{\prime}\left(u+\frac{1}{4}\right)}{\Gamma\left(u+\frac{1}{4}\right)}-\frac{\Gamma^{\prime}(u)}{\Gamma(u)}
$$

is positive. Hence $G$ is an strictly increasing function and $A_{k}<\sqrt{2}$. On the other hand, using Stirling's formula

$$
\Gamma(u) \sim \sqrt{2 \pi} u^{u-\frac{1}{2}} e^{-u} \quad \text { as } \quad u \rightarrow+\infty
$$

we get $G\left(k+\frac{1}{4}\right) \sim G\left(k+\frac{1}{2}\right)$ hence $A_{k} \rightarrow \sqrt{2}$, as $k \rightarrow+\infty$.
As a first consequence, we obtain the $L^{1} \rightarrow L^{1}$ operator norm of Dunkl translations in dimension $N=1$.

Corollary 4.3. Let $x \in \mathbb{R}^{*}$. Then $\tau_{x}$ is a bounded operator on $L^{1}\left(\mathbb{R},|x|^{2 k} d x\right)$, with $\left\|\tau_{x}\right\|_{L^{1} \rightarrow L^{1}}=A_{k}$.
Proof. The inequality $\left\|\tau_{x}\right\|_{L^{1} \rightarrow L^{1}} \leq A_{k}$ follows from (9), together with (8), and it remains for us to prove the converse inequality. By symmetry, we may assume that $x>0$. Since

$$
A_{k}=\lim _{y \rightarrow x} \int_{\mathbb{R}}|\gamma(x, y, z)||z|^{2 k} d z
$$

for every $0<\varepsilon<A_{k}$, there exists $0<\eta<x$ such that, for every $y \in[x-\eta, x+\eta]$,

$$
\begin{equation*}
\int_{\mathbb{R}}|\gamma(x, y, z)||z|^{2 k} d z>A_{k}-\varepsilon \tag{11}
\end{equation*}
$$

Let $f$ be a nonnegative measurable function on $\mathbb{R}$ such that

$$
\operatorname{supp} f \subset[-x-\eta,-x+\eta] \quad \text { and } \quad\|f\|_{L^{1}}=\int_{\mathbb{R}} f(z)|z|^{2 k} d z=1
$$

Since

$$
\left\{\begin{array}{l}
\gamma(x, y, z) \geq 0 \quad \forall y<0, \forall z<0 \\
\gamma(x, y, z) \leq 0 \quad \forall y>0, \forall z<0
\end{array}\right.
$$

we have

$$
\left|\left(\tau_{x} f\right)(y)\right|=\int_{-x-\eta}^{-x+\eta} f(z)|\gamma(x, y, z)||z|^{2 k} d z
$$

Hence, using (8) and (11),

$$
\left\|\tau_{x} f\right\|_{L^{1}}=\int_{\mathbb{R}}\left|\left(\tau_{x} f\right)(y)\right||y|^{2 k} d y=\int_{-x-\eta}^{-x+\eta}\left\{\int_{\mathbb{R}}|\gamma(x,-z,-y)||y|^{2 k} d y\right\} f(z)|z|^{2 k} d z
$$

is bounded from below by $A_{k}-\varepsilon$. Consequently $\left\|\tau_{x}\right\|_{L^{1} \rightarrow L^{1}} \geq A_{k}-\varepsilon$ and we conclude by letting $\varepsilon \rightarrow 0$.

Let us next compute the $L^{2} \rightarrow L^{2}$ operator norm of Dunkl translations.
Lemma 4.4. Let $x \in \mathbb{R}$. Then $\tau_{x}$ is a bounded operator on $L^{2}\left(\mathbb{R},|x|^{2 k} d x\right)$, with $\left\|\tau_{x}\right\|_{L^{2} \rightarrow L^{2}}=1$.
Proof. The proof is straightforward, via the Plancherel formula, and generalizes to higher dimensions. On one hand, the inequality $\left\|\tau_{x}\right\|_{L^{2} \rightarrow L^{2}} \leq 1$ follows from the estimate $|E(i \xi, x)| \leq 1$. On the other hand, let

$$
f_{\varepsilon}(x)=\varepsilon^{k+\frac{1}{2}} f(\varepsilon x)
$$

be a rescaled normalized function in $L^{2}\left(\mathbb{R},|x|^{2 k} d x\right)$. Then

$$
\left\|f_{\varepsilon}\right\|_{L^{2}}=\|f\|_{L^{2}}=1
$$

while

$$
\begin{aligned}
\left\|\tau_{x} f_{\varepsilon}\right\|_{L^{2}}^{2} & =\int_{\mathbb{R}}|E(i \xi, x)|^{2} \varepsilon^{-2 k-1}\left|\mathcal{D} f\left(\varepsilon^{-1} \xi\right)\right|^{2}|\xi|^{2 k} d \xi \\
& =\int_{\mathbb{R}}|E(i \varepsilon \xi, x)|^{2}|\mathcal{D} f(\xi)|^{2}|\xi|^{2 k} d \xi
\end{aligned}
$$

tends to

$$
\int_{\mathbb{R}}|\mathcal{D} f(\xi)|^{2}|\xi|^{2 k} d \xi=\|f\|_{L^{2}}^{2}=1
$$

as $\varepsilon \rightarrow 0$. This concludes the proof of the lemma.
Eventually, Corollary 4.3 and Lemma 4.4 imply the following result, by interpolation and duality.

Corollary 4.5. Let $x \in \mathbb{R}$ and $1 \leq p \leq \infty$. Then $\tau_{x}$ is a bounded operator on $L^{p}\left(\mathbb{R},|x|^{2 k} d x\right)$, with $\left\|\tau_{x}\right\|_{L^{p} \rightarrow L^{p}} \leq A_{k}^{2|1 / \bar{p}-1 / 2|}$.

Remark 4.6. In the product case, where $G=\mathbb{Z}_{2}^{N}$ acts on $\mathbb{R}^{N}$, we have

$$
\left\|\tau_{x}\right\|_{L^{p} \rightarrow L^{p}} \leq A_{k}^{2\left|\frac{1}{p}-\frac{1}{2}\right| N}
$$

for every $x \in \mathbb{R}^{N}$ and $1 \leq p \leq \infty$.

## 5. A support theorem for generalized translations

As mentioned in the introduction, we lack information about Dunkl translations in general. In this section, we locate more precisely the support of the distribution

$$
\left\langle\gamma_{x, y}, f\right\rangle=\left(\tau_{x} f\right)(y)
$$

which is known [21] to be contained in the closed ball of radius $|x|+|y|$.
Theorem 5.1. (i) The distribution $\gamma_{x, y}$ is supported in the spherical shell

$$
\left\{z \in \mathbb{R}^{N}| ||x|-|y||\leq|z| \leq|x|+|y|\} .\right.
$$

(ii) If $G$ is crystallographic, then the support of $\gamma_{x, y}$ is more precisely contained in

$$
\left\{z \in \mathbb{R}^{N} \mid z_{+} \preccurlyeq x_{+}+y_{+}, z_{+} \succcurlyeq y_{+}+g_{0} \cdot x_{+} \text {and } x_{+}+g_{0} \cdot y_{+}\right\} .
$$

Here $g_{0}$ denotes the longest element in $G$, which interchanges the chambers $\Gamma_{+}$and $-\Gamma_{+}$, and $\preccurlyeq$ the partial order on $\mathbb{R}^{N}$ associated to the cone $\overline{\Gamma^{+}}$:

$$
a \preccurlyeq b \quad \Longleftrightarrow \quad b-a \in \overline{\Gamma^{+}} .
$$



Figure 3. Support of $\gamma_{x, y}$ for the root system $A_{1} \times A_{1}$


Figure 4. Support of $\gamma_{x, y}$ for the root system $B_{2}$
Proof. Let $h \in \mathcal{C}_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ be an auxiliary radial function such that

$$
\int_{\mathbb{R}^{N}} h(x) w(x) d x=1
$$

and $\operatorname{supp} h \subset-\operatorname{co}(G . u)$, where $u \in \Gamma_{+}$is a unit vector. For every $\varepsilon>0$ and $x, y, z \in \mathbb{R}^{N}$, set

$$
\gamma_{\varepsilon}(x, y, z)=\frac{1}{c^{2}} \int_{\mathbb{R}^{N}} \mathcal{D} h(\varepsilon \xi) E(i \xi, x) E(i \xi, y) E_{k}(-i \xi, z) w(\xi) d \xi
$$

Firstly, according to (3) and (2),

$$
\xi \longmapsto \mathcal{D h}(\varepsilon \xi) E(i \xi, x) E(i \xi, y)
$$

is an entire function on $\mathbb{C}^{N}$ satisfying

$$
\begin{equation*}
|\mathcal{D} h(\varepsilon \xi) E(i \xi, x) E(i \xi, y)| \leq C_{M}(1+|\xi|)^{-M} e^{-\left\langle g_{0} \cdot\left(x_{+}+y_{+}+\varepsilon u\right),(\operatorname{Im} \xi)_{+}\right\rangle} \tag{12}
\end{equation*}
$$

where $g_{0}$ is the longest element in $G$, which interchanges the chambers $\Gamma_{+}$and $-\Gamma_{+}$. Secondly,

$$
\begin{aligned}
\left\langle\gamma_{x, y}, f\right\rangle & =\frac{1}{c} \int_{\mathbb{R}^{N}} \mathcal{D} f(\xi) E(i \xi, x) E(i \xi, y) w(\xi) d \xi \\
& =\lim _{\varepsilon \rightarrow 0} \frac{1}{c} \int_{\mathbb{R}^{N}} \mathcal{D} h(\varepsilon \xi) \mathcal{D} f(\xi) E(i \xi, x) E(i \xi, y) w(\xi) d \xi \\
& =\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{N}} f(z) \gamma_{\varepsilon}(x, y, z) w(z) d z
\end{aligned}
$$

i.e. the distribution $\gamma_{x, y}$ is the weak limit of the measures $\gamma_{\varepsilon}(x, y, z) w(z) d z$. Thirdly, notice the symmetries

$$
\gamma_{\varepsilon}(x, y, z)=\left\{\begin{array}{l}
\gamma_{\varepsilon}(y, x, z)  \tag{13}\\
\gamma_{\varepsilon}(g \cdot x, g \cdot y, g \cdot z) \quad \forall g \in G \cup\{-\mathrm{Id}\} \\
\gamma_{\varepsilon}(-z, y,-x)=\gamma_{\varepsilon}(x,-z,-y)
\end{array}\right.
$$

If $G$ is crystallogaphic, we use Corollary 3.2 (actually the third version of the PaleyWiener theorem in [D] , and deduce from (12) that the function $z \longmapsto \gamma_{\varepsilon}(x, y, z)$ is supported in

$$
\operatorname{co}\left\{G \cdot\left(x_{+}+y_{+}+\varepsilon u\right)\right\}=\operatorname{co}(G \cdot x)+\operatorname{co}(G . y)+\varepsilon \operatorname{co}(G . u) .
$$

Equivalently,

$$
\gamma_{\varepsilon}(x, y, z) \neq 0 \quad \Longrightarrow \quad z_{+} \prec x_{+}+y_{+}+\varepsilon u
$$

Using the symmetries (13), we see that $\gamma_{\varepsilon}(x, y, z) \neq 0$ implies also

$$
\left\{\begin{array}{lll}
-g_{0} \cdot x_{+} \prec-g_{0} \cdot z_{+}+y_{+}+\varepsilon u & \text { i.e. } & z_{+} \succ x_{+}+g_{0} \cdot y_{+}+\varepsilon g_{0} \cdot u \\
-g_{0} \cdot y_{+} \prec-g_{0} \cdot z_{+}+x_{+}+\varepsilon u & \text { i.e. } & z_{+} \succ g_{0} \cdot x_{+}+y_{+}+\varepsilon g_{0} \cdot u
\end{array}\right.
$$

The conclusion of Theorem 5.1 in the crystallographic case is obtained by letting $\varepsilon \rightarrow 0$.
If $G$ it not crystallographic, we can only use the spherical Paley-Wiener theorem and we obtain this way that $\gamma_{\varepsilon}(x, y, z) \neq 0$ implies

$$
\left\{\begin{array}{l}
|z| \leq|x|+|y|+\varepsilon \\
|x| \leq|z|+|y|+\varepsilon \\
|y| \leq|x|+|z|+\varepsilon
\end{array}\right.
$$

hence

$$
||x|-|y||-\varepsilon \leq|z| \leq|x|+|y|+\varepsilon .
$$

We conclude again by letting $\varepsilon \rightarrow 0$.

## References

[1] J.-Ph. Anker, $L_{p}$ Fourier multipliers on Riemanian symmetric spaces of the noncompact type, Ann. Math. 132 (1990), 597-628
[2] J.-Ph. Anker, The spherical Fourier transform of rapidly decreasing functions (a simple proof of a characterization due to Harish-Chandra, Helgason, Trombi and Varadarajan), J. Funct. Anal. 96 (1991), 331-349
[3] J.-Ph. Anker, Handling the inverse spherical Fourier transform, in Harmonic analysis on reductive groups (Bowdoin College, 1989), W. Barker \& al. (eds.), Progress Math. 101, Birhäuser (1991), 51-56
[4] J.-Ph. Anker, Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces, Duke Math. J. 65 (1992), no. 2, 257-297
[5] C. F. Dunkl, Differential-Difference operators associated to reflextion groups, Trans. Amer. Math. 311 (1989), no. 1, 167-183
[6] C.F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. 43 (1991), 12131227
[7] Harmonic and stochastic analysis of Dunkl processes, P. Graczyk \& al. (eds.), Travaux en cours 71, Hermann (2008)
[8] M.F.E. de Jeu, The Dunkl transform, Invent. Math. 113 (1993), no. 1, 147-162
[9] M.F.E. de Jeu, Paley-Wiener theorems for the Dunkl transform, Trans. Amer. Math. Soc. 358 (2006), no. 10, 4225-4250
[10] M.F.E. de Jeu, Some remarks on a proof of geometrical Paley-Wiener theorems for the Dunkl transform, Integral Transforms Spec. Funct. 18 (2007), no. 5-6, 383-385
[11] E.M. Opdam, Harmonic analysis for cetain representations of graded Hecke algebras, Acta Math. 175 (1995), 75-121
[12] E.M. Opdam, Lecture notes on Dunkl operators for real and complex reflection groups, Math. Soc. Japan Mem. 8 (2000)
[13] M. Rösler, Bessel-type signed hypergroup on $\mathbb{R}$, in Probability measures on groups and related structures XI (Oberwolfach, 1994), H. Heyer \& al. (eds.), World Sci. Publ. (1995), 92-304
[14] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators, Comm. Math. Phys. 192 (1998), no. 3, 519-542
[15] M. Rösler, Positivity of Dunkl's intertwining operator, Duke Math. J. 98 (1999), 445-463
[16] M. Rösler, Dunkl operators : theory and applications, in Orthogonal polynomials and special functions (Leuven, 2002), Lect. Notes Math. 1817, Springer-Verlag (2003), 93-135
[17] M. Rösler, A positive radial product formula for the Dunkl kernel, Trans. Amer. Math. Soc. 355 (2003), no. 6, 2413-2438
[18] M. Rösler \& M.F.E. de Jeu, Asymptotic analysis for the Dunkl kernel, J. Approx. Theory 119 (2002), no. 1, 110-126
[19] Br. Schapira, Contributions to the hypergeometric function theory of Heckman and Opdam (sharp estimates, Schwartz space, heat kernel), Geom. Funct. Anal. 18 (2008), 222-250
[20] K. Trimèche, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual, Integral Transform. Spec. Funct. 12 (2001), no. 4, 349-374
[21] K. Trimèche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, Integral Transform. Spec. Funct. 13 (2002), no. 1, 17-38
[22] S. Thangavelu and Y. Xu, Convolution operator and maximal function for the Dunkl transform, J. Anal. Math. 97 (2005), 25-56

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