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THREE RESULTS IN DUNKL ANALYSIS

BÉCHIR AMRI, JEAN-PHILIPPE ANKER & MOHAMED SIFI

*In memory of Andrzej Hulanicki (1933–2008),
a distinguished polish mathematician, a guide and a friend,
who has left many orphans in Wrocław and around the world.
We miss you.*

ABSTRACT. In this article, we establish first a geometric Paley–Wiener theorem for the Dunkl transform in the crystallographic case. Next we obtain an optimal bound for the $L^p \rightarrow L^p$ norm of Dunkl translations in dimension 1. Finally we describe more precisely the support of the distribution associated to Dunkl translations in higher dimension.

1. INTRODUCTION

Dunkl theory generalizes classical Fourier analysis on \mathbb{R}^N . It started twenty years ago with Dunkl’s seminal work [5] and was further developed by several mathematicians. See for instance the surveys [16, 7] and the references cited therein.

In this setting, the Paley–Wiener theorem is known to hold for balls. In [9], a Paley–Wiener theorem was conjectured for convex sets and partially proved. Our first result in Section 3 is a proof of this conjecture in the crystallographic case, following the third approach in [9].

Generalized translations were introduced in [14] and further studied in [21, 17, 22]. Apart from their abstract definition, we lack precise information, in particular about their integral representation

$$(\tau_x f)(y) = \int_{\mathbb{R}^N} f(z) d\gamma_{x,y}(z),$$

which was conjectured in [14] and established in few cases, for instance in dimension $N = 1$ or when f is radial. Our second result in Section 4 is an optimal bound for the integral

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)|$$

in dimension $N = 1$, improving upon earlier results in [13, 22]. Our bound depends on the multiplicity $k \geq 0$ and tends from below to $\sqrt{2}$, as $k \rightarrow +\infty$. Our third result in Section 5 deals with the support of the distribution $\gamma_{x,y}$ in higher dimension, that we determine rather precisely in the crystallographic case.

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2. BACKGROUND

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the articles [5, 8] or to the surveys [16, 7].

Let $G \subset O(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system R and $k : R \rightarrow [0, +\infty)$ a G -invariant function (called multiplicity function). Let R^+ be a positive root subsystem, Γ_+ the corresponding open positive chamber, $\overline{\Gamma_+}$ its closure, $\overline{\Gamma^+} = \sum_{\alpha \in R^+} \mathbb{R}_+ \alpha$ the dual cone, and let us denote by x_+ the intersection point of any orbit $G.x$ in \mathbb{R}^N with $\overline{\Gamma_+}$.

The Dunkl operators T_ξ on \mathbb{R}^N are the following k -deformations of directional derivatives ∂_ξ by difference operators :

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R^+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha \cdot x)}{\langle \alpha, x \rangle},$$

where $\sigma_\alpha \cdot x = x - \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha$ denotes the reflection with respect to the hyperplane orthogonal to α . The Dunkl operators are antisymmetric with respect to the measure $w(x) dx$ with density

$$w(x) = \prod_{\alpha \in R^+} |\langle \alpha, x \rangle|^{2k(\alpha)}.$$

The operators ∂_ξ and T_ξ are intertwined by a Laplace-type operator

$$(1) \quad Vf(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y)$$

associated to a family of compactly supported probability measures $\{\mu_x | x \in \mathbb{R}^N\}$. Specifically, μ_x is supported in the the convex hull

$$C^x = \text{co}(G.x).$$

For every $\lambda \in \mathbb{C}^N$, the simultaneous eigenfunction problem

$$T_\xi f = \langle \lambda, \xi \rangle f \quad \forall \xi \in \mathbb{R}^N$$

has a unique solution $f(x) = E(\lambda, x)$ such that $E(\lambda, 0) = 1$, which is given by

$$(2) \quad E(\lambda, x) = V(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} d\mu_x(y) \quad \forall x \in \mathbb{R}^N.$$

Furthermore $\lambda \mapsto E(\lambda, x)$ extends to a holomorphic function on \mathbb{C}^N and the following estimate holds :

$$|E(\lambda, x)| \leq e^{\langle (\text{Re } \lambda)_+, x_+ \rangle} \quad \forall \lambda \in \mathbb{C}^N, \forall x \in \mathbb{R}^N.$$

In dimension $N = 1$, these functions can be expressed in terms of Bessel functions. Specifically,

$$E(\lambda, x) = j_{k-\frac{1}{2}}(\lambda x) + \frac{\lambda x}{2k+1} j_{k+\frac{1}{2}}(\lambda x),$$

where

$$j_\nu(z) = \Gamma(\nu+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n}$$

are normalized Bessel functions.

The Dunkl transform is defined on $L^1(\mathbb{R}^N, w(x)dx)$ by

$$\mathcal{D}f(\xi) = \frac{1}{c} \int_{\mathbb{R}^N} f(x) E(-i\xi, x) w(x) dx,$$

where

$$c = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} w(x) dx.$$

We list some known properties of this transform :

- (i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.
- (ii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^N, w(x)dx)$.
- (iii) (*Inversion formula*) For every $f \in \mathcal{S}(\mathbb{R}^N)$, and more generally for every $f \in L^1(\mathbb{R}^N, w(x)dx)$ such that $\mathcal{D}f \in L^1(\mathbb{R}^N, w(\xi)d\xi)$, we have

$$f(x) = \mathcal{D}^2 f(-x) \quad \forall x \in \mathbb{R}^N.$$

- (iv) (*Paley–Wiener theorem*) The Dunkl transform is a linear isomorphism between the space of smooth functions f on \mathbb{R}^N with $\text{supp } f \subset \overline{B(0, R)}$ and the space of entire functions h on \mathbb{C}^N such that

$$\sup_{\xi \in \mathbb{C}^N} (1 + |\xi|)^M e^{-R|\text{Im } \xi|} |h(\xi)| < +\infty \quad \forall M \in \mathbb{N}.$$

3. A GEOMETRIC PALEY–WIENER THEOREM

In this section, we prove a geometric version of the Paley–Wiener theorem, which was looked for in [9, 21, 10], under the assumption that G is crystallographic. The proof consists merely in resuming the third approach in [9] and applying it to the convex sets considered in [1, 2, 3, 4] instead of the convex sets considered in [11]. Recall that the second family consists of the convex hulls

$$C^\Lambda = \text{co}(G.\Lambda)$$

of G -orbits $G.\Lambda$ in \mathbb{R}^N , while the first family consists of the polar sets

$$C_\Lambda = \{ x \in \mathbb{R}^N \mid \langle x, g.\Lambda \rangle \leq 1 \ \forall g \in G \}.$$

Notice that we may always assume that $\Lambda = \Lambda_+$ belongs to the closed positive chamber $\overline{\Gamma}_+$ and, in this case, that

$$\begin{aligned} C^\Lambda \cap \overline{\Gamma}_+ &= \overline{\Gamma}_+ \cap (\Lambda - \overline{\Gamma}^+), \\ C_\Lambda \cap \overline{\Gamma}_+ &= \{ x \in \overline{\Gamma}_+ \mid \langle \Lambda, x \rangle \leq 1 \}. \end{aligned}$$

Let us finally introduce the gauge

$$\chi_\Lambda(\xi) = \max_{x \in C_\Lambda} \langle x, \xi \rangle = \min \{ r \in [0, +\infty) \mid \xi \in r C^\Lambda \}$$

on \mathbb{R}^N .

Theorem 3.1. *The Dunkl transform is a linear isomorphism between the space of smooth functions f on \mathbb{R}^N with $\text{supp } f \subset C_\Lambda$ and the space of entire functions h on \mathbb{C}^N such that*

$$\sup_{\xi \in \mathbb{C}^N} (1 + |\xi|)^M e^{-\chi_\Lambda(\text{Im } \xi)} |h(\xi)| < +\infty \quad \forall M \in \mathbb{N}.$$

Proof. Following [9] this theorem is first proved in the trigonometric case, which explains the restriction to crystallographic groups, and next obtained in the rational case by passing to the limit. As observed in [19], the proof of Theorem 3.1 in the trigonometric case is similar to the proof of the Paley–Wiener Theorem in [11, 12], and actually to the initial proof of Helgason for the spherical Fourier transform on symmetric spaces of the noncompact type. The limiting procedure, as far as it is concerned, is thoroughly described in [9]. \square

Since every G -invariant closed convex set in \mathbb{R}^N is the intersection of a family of subsets C_Λ , Theorem 3.1 generalizes as follows.

Corollary 3.2. *Let C be a G -invariant convex compact neighborhood of the origin in \mathbb{R}^N and $\chi(\xi) = \max_{x \in C} \langle x, \xi \rangle$ the dual gauge. Then the Dunkl transform is a linear isomorphism between the space $\mathcal{C}_C^\infty(\mathbb{R}^N)$ of smooth functions f on \mathbb{R}^N with $\text{supp } f \subset C$ and the space $\mathcal{H}_\chi(\mathbb{C}^N)$ of entire functions h on \mathbb{C}^N such that*

$$\sup_{\xi \in \mathbb{C}^N} (1 + |\xi|)^M e^{-\chi(\text{Im } \xi)} |h(\xi)| < +\infty \quad \forall M \in \mathbb{N}.$$

Remark 3.3. *Notice that the Dunkl transform \mathcal{D} always maps $\mathcal{C}_C^\infty(\mathbb{R}^N)$ into $\mathcal{H}_\chi(\mathbb{C}^N)$ and that the assumption that G is crystallographic is only used to prove that \mathcal{D} is onto.*

4. L^p BOUNDS FOR GENERALIZED TRANSLATIONS IN DIMENSION 1

Dunkl translations are defined on $\mathcal{S}(\mathbb{R}^N)$ by

$$(\tau_x f)(y) = \frac{1}{c} \int_{\mathbb{R}^N} \mathcal{D}f(\xi) E(i\xi, x) E(i\xi, y) w(\xi) d\xi \quad \forall x, y \in \mathbb{R}^N.$$

They have an explicit integral representation [13] in dimension $N = 1$:

$$(\tau_x f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z),$$

where

$$(3) \quad d\gamma_{x,y}(z) = \begin{cases} \gamma(x, y, z) |z|^{2k} dz & \text{if } x, y \in \mathbb{R}^* \\ d\delta_y(z) & \text{if } x = 0 \\ d\delta_x(z) & \text{if } y = 0 \end{cases}$$

is a signed measure such that $\int_{\mathbb{R}} d\gamma_{x,y}(z) = 1$. Specifically,

$$\gamma(x, y, z) = d \sigma(x, y, z) \rho(|x|, |y|, |z|) \mathbb{1}_{I_{|x|, |y|}}(|z|) \quad \forall x, y, z \in \mathbb{R}^*,$$

where

$$\begin{aligned}
d &= \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)}, \\
\sigma(x, y, z) &= 1 - \frac{x^2+y^2-z^2}{2xy} + \frac{z^2+y^2-x^2}{2zy} + \frac{x^2+z^2-y^2}{2xz} \\
&= \frac{(z+x+y)(z+x-y)(z-x+y)}{2xyz} \quad \forall x, y, z \in \mathbb{R}^*, \\
\rho(a, b, c) &= \frac{\{c^2 - (a-b)^2\}^{k-1} \{(a+b)^2 - c^2\}^{k-1}}{(2abc)^{2k-1}} \\
&= \frac{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)^{k-1}}{(2abc)^{2k-1}} \quad \forall a, b, c > 0,
\end{aligned}$$

and $I_{a,b}$ denotes the interval $[|a-b|, a+b]$. Notice the symmetries

$$(4) \quad \gamma(x, y, z) = \begin{cases} \gamma(y, x, z), \\ \gamma(-x, -y, -z), \\ \gamma(-z, y, -x) = \gamma(x, -z, -y). \end{cases}$$

Proposition 4.1. *The following inequality holds, for every $x, y \in \mathbb{R}$:*

$$(5) \quad \int_{\mathbb{R}} |d\gamma_{x,y}(z)| \leq A_k = \sqrt{2} \frac{\{\Gamma(k+\frac{1}{2})\}^2}{\Gamma(k+\frac{1}{4})\Gamma(k+\frac{3}{4})}.$$

Actually there is equality if $x=y \in \mathbb{R}^*$. Moreover $A_k \xrightarrow{k \rightarrow +\infty} \sqrt{2}$.

Remark 4.2. *This result improves earlier bounds obtained in [13] and [22], which were respectively 4 and 3.*

Proof. Let $x, y \in \mathbb{R}^*$.

Case 1: Assume that $xy < 0$. Then $||x| - |y|| = |x+y|$ and $|x| + |y| = |x-y|$, hence $\sigma(x, y, z) \mathbb{1}_{I_{|x|,|y|}}(|z|) = \frac{z+x+y}{z} \frac{(x-y)^2 - z^2}{-2xy} \mathbb{1}_{I_{|x|,|y|}}(|z|)$ and $\gamma_{x,y}$ are positive. Thus

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)| = \int_{\mathbb{R}} d\gamma_{x,y}(z) = 1.$$

Case 2: Assume that $xy > 0$. By symmetry, we may reduce to $0 < x \leq y$. Then

$$\begin{aligned}
\int_{\mathbb{R}} |d\gamma_{x,y}(z)| &= \int_{-\infty}^0 |d\gamma_{x,y}(z)| + \int_0^{+\infty} |d\gamma_{x,y}(z)| \\
&= 2d \int_{y-x}^{y+x} \frac{x+y}{2xyz} \left(\frac{z^2 - x^2 - y^2 + 2xy}{2xyz} \right)^k \left(\frac{x^2 + y^2 + 2xy - z^2}{2xyz} \right)^{k-1} z^{2k} dz.
\end{aligned}$$

After performing the change of variables $z = \sqrt{x^2 + y^2 - 2xy \cos \theta}$ and setting $y = sx$, we get

$$(6) \quad \int_{\mathbb{R}} |d\gamma_{x,y}(z)| = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} (1+s) \int_0^\pi \frac{(1-\cos \theta) \sin^{2k-1} \theta}{\sqrt{1+s^2-2s \cos \theta}} d\theta.$$

Denote by $F(s)$ the right hand side of (6). Since

$$F'(s) = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} (1-s) \int_0^\pi \frac{\sin^{2k+1} \theta}{(1+s^2-2s \cos \theta)^{\frac{3}{2}}} d\theta$$

is nonpositive, $F(s)$ is a decreasing function on $[1, +\infty)$, which reaches its maximum at $s=1$. Let us compute it :

$$\begin{aligned} A_k = F(1) &= \frac{\sqrt{2}\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} \int_0^\pi (1-\cos\theta)^{k-\frac{1}{2}} (1+\cos\theta)^{k-1} \sin\theta \, d\theta \\ &= 2^{2k} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} \int_0^1 t^{k-\frac{1}{2}} (1-t)^{k-1} \, dt \\ &= 2^{2k} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} B(k+\frac{1}{2}, k) = 2^{2k} \frac{\{\Gamma(k+\frac{1}{2})\}^2}{\sqrt{\pi}\Gamma(2k+\frac{1}{2})} = \sqrt{2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{4})} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{3}{4})}, \end{aligned}$$

after performing the change of variables $t = \frac{1-\cos\theta}{2}$ and using standard properties of the beta and gamma functions.

Finally let us show that $A_k \xrightarrow{<} \sqrt{2}$ as $k \rightarrow +\infty$. Write

$$A_k = \sqrt{2} \frac{G(k+\frac{1}{4})}{G(k+\frac{1}{2})}, \quad \text{where } G(u) = \frac{\Gamma(u+\frac{1}{4})}{\Gamma(u)} \quad \forall u > 0.$$

Since the logarithmic derivative $\frac{\Gamma'}{\Gamma}$ of the gamma function is a strictly increasing analytic function on $(0, +\infty)$, the logarithmic derivative

$$\frac{G'(u)}{G(u)} = \frac{\Gamma'(u+\frac{1}{4})}{\Gamma(u+\frac{1}{4})} - \frac{\Gamma'(u)}{\Gamma(u)}$$

is positive. Hence G is a strictly increasing function and $A_k < \sqrt{2}$. On the other hand, using Stirling's formula

$$\Gamma(u) \sim \sqrt{2\pi} u^{u-\frac{1}{2}} e^{-u} \quad \text{as } u \rightarrow +\infty,$$

we get $G(k+\frac{1}{4}) \sim G(k+\frac{1}{2})$ hence $A_k \rightarrow \sqrt{2}$, as $k \rightarrow +\infty$. \square

As a first consequence, we obtain the $L^1 \rightarrow L^1$ operator norm of Dunkl translations in dimension $N=1$.

Corollary 4.3. *Let $x \in \mathbb{R}^*$. Then τ_x is a bounded operator on $L^1(\mathbb{R}, |x|^{2k} dx)$, with $\|\tau_x\|_{L^1 \rightarrow L^1} = A_k$.*

Proof. The inequality $\|\tau_x\|_{L^1 \rightarrow L^1} \leq A_k$ follows from (5), together with (4), and it remains for us to prove the converse inequality. By symmetry, we may assume that $x > 0$. Since

$$A_k = \lim_{y \rightarrow x} \int_{\mathbb{R}} |\gamma(x, y, z)| |z|^{2k} \, dz,$$

for every $0 < \varepsilon < A_k$, there exists $0 < \eta < x$ such that, for every $y \in [x-\eta, x+\eta]$,

$$(7) \quad \int_{\mathbb{R}} |\gamma(x, y, z)| |z|^{2k} \, dz > A_k - \varepsilon.$$

Let f be a nonnegative measurable function on \mathbb{R} such that

$$\text{supp } f \subset [-x-\eta, -x+\eta] \quad \text{and} \quad \|f\|_{L^1} = \int_{\mathbb{R}} f(z) |z|^{2k} \, dz = 1.$$

Since

$$\begin{cases} \gamma(x, y, z) \geq 0 & \forall y < 0, \forall z < 0, \\ \gamma(x, y, z) \leq 0 & \forall y > 0, \forall z < 0, \end{cases}$$

we have

$$|(\tau_x f)(y)| = \int_{-x-\eta}^{-x+\eta} f(z) |\gamma(x, y, z)| |z|^{2k} dz.$$

Hence, using (4) and (7),

$$\|\tau_x f\|_{L^1} = \int_{\mathbb{R}} |(\tau_x f)(y)| |y|^{2k} dy = \int_{-x-\eta}^{-x+\eta} \left\{ \int_{\mathbb{R}} |\gamma(x, -z, -y)| |y|^{2k} dy \right\} f(z) |z|^{2k} dz$$

is bounded from below by $A_k - \varepsilon$. Consequently $\|\tau_x\|_{L^1 \rightarrow L^1} \geq A_k - \varepsilon$ and we conclude by letting $\varepsilon \rightarrow 0$. \square

Let us next compute the $L^2 \rightarrow L^2$ operator norm of Dunkl translations.

Lemma 4.4. *Let $x \in \mathbb{R}$. Then τ_x is a bounded operator on $L^2(\mathbb{R}, |x|^{2k} dx)$, with $\|\tau_x\|_{L^2 \rightarrow L^2} = 1$.*

Proof. The proof is straightforward, via the Plancherel formula, and generalizes to higher dimensions. On one hand, the inequality $\|\tau_x\|_{L^2 \rightarrow L^2} \leq 1$ follows from the estimate $|E(i\xi, x)| \leq 1$. On the other hand, let

$$f_\varepsilon(x) = \varepsilon^{k+\frac{1}{2}} f(\varepsilon x)$$

be a rescaled normalized function in $L^2(\mathbb{R}, |x|^{2k} dx)$. Then

$$\|f_\varepsilon\|_{L^2} = \|f\|_{L^2} = 1$$

while

$$\begin{aligned} \|\tau_x f_\varepsilon\|_{L^2}^2 &= \int_{\mathbb{R}} |E(i\xi, x)|^2 \varepsilon^{-2k-1} |\mathcal{D}f(\varepsilon^{-1}\xi)|^2 |\xi|^{2k} d\xi \\ &= \int_{\mathbb{R}} |E(i\varepsilon\xi, x)|^2 |\mathcal{D}f(\xi)|^2 |\xi|^{2k} d\xi \end{aligned}$$

tends to

$$\int_{\mathbb{R}} |\mathcal{D}f(\xi)|^2 |\xi|^{2k} d\xi = \|f\|_{L^2}^2 = 1$$

as $\varepsilon \rightarrow 0$. This concludes the proof of the lemma. \square

Eventually, Corollary 4.3 and Lemma 4.4 imply the following result, by interpolation and duality.

Corollary 4.5. *Let $x \in \mathbb{R}$ and $1 \leq p \leq \infty$. Then τ_x is a bounded operator on $L^p(\mathbb{R}, |x|^{2k} dx)$, with $\|\tau_x\|_{L^p \rightarrow L^p} \leq A_k^{2|1/p-1/2|}$.*

Remark 4.6. *In the product case, where $G = \mathbb{Z}_2^N$ acts on \mathbb{R}^N , we have*

$$\|\tau_x\|_{L^p \rightarrow L^p} \leq A_k^{2|\frac{1}{p}-\frac{1}{2}|N}$$

for every $x \in \mathbb{R}^N$ and $1 \leq p \leq \infty$.

5. A SUPPORT THEOREM FOR GENERALIZED TRANSLATIONS

As mentioned in the introduction, we lack information about Dunkl translations in general. In this section, we locate more precisely the support of the distribution

$$\langle \gamma_{x,y}, f \rangle = (\tau_x f)(y)$$

which is known [21] to be contained in the closed ball of radius $|x|+|y|$.

Theorem 5.1. (i) *The distribution $\gamma_{x,y}$ is supported in the spherical shell*

$$\{ z \in \mathbb{R}^N \mid ||x|-|y|| \leq |z| \leq |x|+|y| \}.$$

(ii) *If G is crystallographic, then the support of $\gamma_{x,y}$ is more precisely contained in*

$$\{ z \in \mathbb{R}^N \mid z_+ \preceq x_+ + y_+, z_+ \succcurlyeq y_+ + g_0 \cdot x_+ \text{ and } x_+ + g_0 \cdot y_+ \}.$$

Here g_0 denotes the longest element in G , which interchanges the chambers Γ_+ and $-\Gamma_+$, and \preceq the partial order on \mathbb{R}^N associated to the cone $\overline{\Gamma^+}$:

$$a \preceq b \iff b - a \in \overline{\Gamma^+}.$$

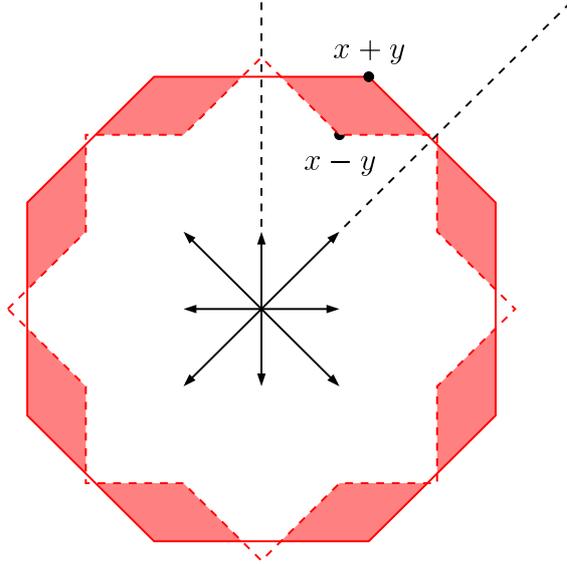


FIGURE 1. Support of $\gamma_{x,y}$ for the root system B_2

Proof. Let $h \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be an auxiliary radial function such that

$$\int_{\mathbb{R}^N} h(x) w(x) dx = 1$$

and $\text{supp } h \subset -\text{co}(G.u)$, where $u \in \Gamma_+$ is a unit vector. For every $\varepsilon > 0$ and $x, y, z \in \mathbb{R}^N$, set

$$\gamma_\varepsilon(x, y, z) = \frac{1}{c^2} \int_{\mathbb{R}^N} \mathcal{D}h(\varepsilon \xi) E(i \xi, x) E(i \xi, y) E_k(-i \xi, z) w(\xi) d\xi.$$

Firstly, according to the spherical Paley–Wiener theorem and to (2),

$$\xi \longmapsto \mathcal{D}h(\varepsilon \xi) E(i \xi, x) E(i \xi, y)$$

is an entire function on \mathbb{C}^N satisfying

$$(8) \quad \left| \mathcal{D}h(\varepsilon \xi) E(i \xi, x) E(i \xi, y) \right| \leq C_M (1 + |\xi|)^M e^{-\langle g_0 \cdot (x_+ + y_+ + \varepsilon u), (\operatorname{Im} \xi)_+ \rangle},$$

where g_0 is the longest element in G , which interchanges the chambers Γ_+ and $-\Gamma_+$. Secondly,

$$\begin{aligned} \langle \gamma_{x,y}, f \rangle &= \frac{1}{c} \int_{\mathbb{R}^N} \mathcal{D}f(\xi) E(i \xi, x) E(i \xi, y) w(\xi) d\xi \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{c} \int_{\mathbb{R}^N} \mathcal{D}h(\varepsilon \xi) \mathcal{D}f(\xi) E(i \xi, x) E(i \xi, y) w(\xi) d\xi \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} f(z) \gamma_\varepsilon(x, y, z) w(z) dz \end{aligned}$$

i.e. the distribution $\gamma_{x,y}$ is the weak limit of the measures $\gamma_\varepsilon(x, y, z) w(z) dz$. Thirdly, notice the symmetries

$$(9) \quad \gamma_\varepsilon(x, y, z) = \begin{cases} \gamma_\varepsilon(y, x, z), \\ \gamma_\varepsilon(g.x, g.y, g.z) \quad \forall g \in G \cup \{-\operatorname{Id}\}, \\ \gamma_\varepsilon(-z, y, -x) = \gamma_\varepsilon(x, -z, -y). \end{cases}$$

If G is crystallographic, we use Corollary 3.2 (actually the third version of the Paley–Wiener theorem in [9]), and deduce from (8) that the function $z \longmapsto \gamma_\varepsilon(x, y, z)$ is supported in

$$\operatorname{co}\{G \cdot (x_+ + y_+ + \varepsilon u)\} = \operatorname{co}(G.x) + \operatorname{co}(G.y) + \varepsilon \operatorname{co}(G.u).$$

Equivalently,

$$\gamma_\varepsilon(x, y, z) \neq 0 \implies z_+ \prec x_+ + y_+ + \varepsilon u.$$

Using the symmetries (9), we see that $\gamma_\varepsilon(x, y, z) \neq 0$ implies also

$$\begin{cases} -g_0.x_+ \prec -g_0.z_+ + y_+ + \varepsilon u & \text{i.e. } z_+ \succ x_+ + g_0.y_+ + \varepsilon g_0.u, \\ -g_0.y_+ \prec -g_0.z_+ + x_+ + \varepsilon u & \text{i.e. } z_+ \succ g_0.x_+ + y_+ + \varepsilon g_0.u. \end{cases}$$

The conclusion of Theorem 5.1 in the crystallographic case is obtained by letting $\varepsilon \rightarrow 0$.

If G is not crystallographic, we can only use the spherical Paley–Wiener theorem and we obtain this way that $\gamma_\varepsilon(x, y, z) \neq 0$ implies

$$\begin{cases} |z| \leq |x| + |y| + \varepsilon, \\ |x| \leq |z| + |y| + \varepsilon, \\ |y| \leq |x| + |z| + \varepsilon, \end{cases}$$

hence

$$\left| |x| - |y| \right| - \varepsilon \leq |z| \leq |x| + |y| + \varepsilon.$$

We conclude again by letting $\varepsilon \rightarrow 0$. □

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