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Abstract

I show that, if a term is SN for β, it remains SN when some permutation rules are added.

1 Introduction

Strong normalization (abbreviated as SN) is a property of rewriting systems that is often desired. Since about 10 years many researchers have considered the following question : If a λ-term is SN for the β-reduction, does it remain SN if some other reduction rules are added ? They are mainly interested with permutation rules they introduce to be able to delay some β-reductions in, for example, *let* $x = ...$ *in ...* constructions or in *calculi with explicit substitutions*. Here are some papers considering such permutations rules: L. Regnier [7], F Kamareddine [3], E. Moggi [5], R. Dyckhoff and S. Lengrand [2], A. J. Kfoury and J. B. Wells [4], Y. Ohta and M. Hasegawa [6], J. Esprito Santo [8] and [9].

Most of these papers show that SN is preserved by the addition of the permutation rules they introduce. But these proofs are quite long and complicated or need some restrictions to the rule. For example the rule $(M\ (\lambda x.N\ P)) \triangleright (\lambda x.(M\ N)\ P)$ is often restricted to the case when M is an abstraction (in this case it is usually called *assoc*).

I give here a very simple proof that the permutations rules preserve SN when they are added all together and with no restriction. It is done as follows. I show that every term which is typable in the system (often called system $\mathcal{D}$) of types built with $\rightarrow$ and $\wedge$ is strongly normalizing for all the rules (β and the permutation rules). Since it is well known that a term is SN for the β-rule iff it is typable in this system, the result follows.

2 Definitions and notations

Definition 2.1

- *The set of λ-terms is defined by the following grammar*

$$\mathcal{M} := x \mid \lambda x.\mathcal{M} \mid (\mathcal{M}\ \mathcal{M})$$

- *The set $\mathcal{T}$ of types is defined by the following grammar where $\mathcal{A}$ is a set of atomic constants*

$$\mathcal{T} ::= \mathcal{A} \mid \mathcal{T} \rightarrow \mathcal{T} \mid \mathcal{T} \wedge \mathcal{T}$$

- *The typing rules are the following :*

$$\frac{}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash (M\ N) : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$$

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M : B}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \wedge B}$$

Definition 2.2 *The reduction rules are the following.*

- β *:* $(\lambda x.M\ N) \triangleright M[x := N]$
- δ *:* $(\lambda y.\lambda x.M\ N) \triangleright \lambda x.(\lambda y.M\ N)$
- γ *:* $(\lambda x.M\ N\ P) \triangleright ((\lambda x.M\ P)\ N)$
- *assoc :* $(M\ (\lambda x.N\ P)) \triangleright (\lambda x.(M\ N)\ P)$

Note that, using Barendregt's convention for the names of variables, we may assume that, in γ (resp. δ, *assoc*), x is not free in P (resp. in N, in M).

Notation 2.1

- *If t is a term, $size(t)$ denotes its size and $type(t)$ the size of its type. If $t \in SN$ (i.e. every sequence of reductions starting from t is finite), $\eta(t)$ denotes the length of the longest reduction of t.*
- *In a proof by induction, IH will denote the induction hypothesis.*
- *Let σ be a substitution. We say that σ is fair if the $\sigma(x)$ for $x \in dom(\sigma)$ all have the same type (that will be denoted as $type(\sigma)$). We say that $\sigma \in SN$ if, for each $x \in dom(\sigma)$, $\sigma(x) \in SN$.*
- *Let $\sigma \in SN$ be a substitution and t be a term. We denote by $\eta(\sigma, t)$ the sum, over $x \in dom(\sigma)$, of $nb(t, x).\eta(\sigma(x))$ where $nb(t, x)$ is the number of occurrences of x in t.*
- *If $\overrightarrow{M}$ is a sequence of terms, $lg(\overrightarrow{M})$ denotes its length, $M(i)$ denotes the i-th element of the sequence and $tail(\overrightarrow{M})$ denotes $\overrightarrow{M}$ from which the first element has been deleted.*
- *Assume $t = (H\ \overrightarrow{M})$ where H is an abstraction or a variable and $lg(\overrightarrow{M}) \geq 1$.*
  - *If H is an abstraction (in this case we say that t is β-head reducible), then $M(1)$ will be denoted as $Arg[t]$ and $(R'\ tail(\overrightarrow{M}))$ will be denoted by $B[t]$ where R' is the reduct of the β-redex $(H\ Arg[t])$.*
  - *If $H = \lambda x.N$ and $lg(\overrightarrow{M}) \geq 2$ (in this case we say that t is γ-head reducible), then $(\lambda x.(N\ M(2))\ M(1)\ M(3)\ ...\ M(lg(\overrightarrow{M})))$ will be denoted by $C[t]$.*
  - *If $H = \lambda x.\lambda y.N$ (in this case we say that t is δ-head reducible), then $(\lambda y.(\lambda x.N\ M(1))\ M(2)\ ...\ M(lg(\overrightarrow{M})))$ will be denoted by $D[t]$.*
  - *If $M(i) = (\lambda x.N\ P)$, then the term $(\lambda x.(H\ M(1)\ ...\ M(i-1)\ N)\ P\ M(i+1)\ ...\ M(lg(\overrightarrow{M})))$ will be denoted by $A[t, i]$ and we say that $M(i)$ is the redex put in head position.*

3 The theorem

Theorem 3.1 *Let t be a term. Assume t is strongly normalizing for β. Then t is strongly normalizing for β, δ, γ and assoc.*

Proof This follows immediately from Theorem 3.2 and corollary 3.1 below. □

Theorem 3.2 *A term is SN for the β-rule iff it is typable in system $\mathcal{D}$.*

Proof This is a classical result. For the sake of completeness I recall here the proof of the only if direction given in [1]. Note that corollary 3.1 below actually gives the other direction. The proof is by induction on $\langle \eta(t), size(t) \rangle$.

- If $t = \lambda x\ u$. This follows immediately from the IH.

- If $t = (x\ v_1\ ...\ v_n)$. By the IH, for every j, let $x : A_j, \Gamma_j \vdash v_j : B_j$. Then $x : \bigwedge A_j \wedge (B_1, ..., B_n \to o)$, $\bigwedge \Gamma_j \vdash t : o$.

- If $t = (\lambda x.a\ b\ \overrightarrow{c})$. By the IH, $(a[x := b]\ \overrightarrow{c})$ is typable. If x occurs in a, let $A_1\ ...\ A_n$ be the types of the occurrences of b in the typing of $(a[x := b]\ \overrightarrow{c})$. Then t is typable by giving to x and b the type $A_1 \wedge ... \wedge A_n$. Otherwise, by the induction hypothesis b is typable of type B and then t is typable by giving to x the type B. □

From now on the type system is system $\mathcal{D}$ and $\triangleright$ denotes the reduction by one of the rules β, δ, γ and assoc.

Lemma 3.1 *1. The system satisfies subject reduction.*

2. If $t \triangleright t'$ then $t[x := u] \triangleright t'[x := u]$.

3. If $t' = t[x := u] \in SN$ then $t \in SN$ and $\eta(t) \leq \eta(t')$.

4. If $a \in SN$ then $(\lambda x.a\ x) \in SN$.

Proof (1) and (2) are immediate. (3) follows easily from (2). (4) is proved by induction on $\langle \eta(a), size(a) \rangle$ looking at all possible reducts of $(\lambda x.a\ x)$. □

Lemma 3.2 *Let $t = (H\ \overrightarrow{M})$ be such that H is an abstraction or a variable and $lg(\overrightarrow{M}) \geq 1$. Assume that*

1. If t is δ-head reducible (resp. γ-head reducible, β-head reducible), then $D[t] \in SN$ (resp. $C[t] \in SN$, $Arg[t], B[t] \in SN$).

2. For each i such that $M(i)$is a redex, $A[t, i] \in SN$,

Then $t \in SN$.

Proof By induction on $\eta(H) + \sum \eta(M(i))$. Show that each reduct of t is in SN. □

Theorem 3.3 *Let $t \in SN$ and $\sigma \in SN$ be a fair substitution. Then $\sigma(t) \in SN$.*

Proof By induction on $\langle type(\sigma), \eta(t), size(t), \eta(\sigma, t) \rangle$. If t is an abstraction or a variable the result is trivial. Thus assume $t = (H\ \overrightarrow{M})$ where H is an abstraction or a variable and $n = lg(\overrightarrow{M}) \geq 1$. Let $\overrightarrow{N} = \sigma(\overrightarrow{M})$.
Claim : Let $\overrightarrow{P}$ be a (strict) initial or a final sub-sequence of $\overrightarrow{N}$. Then $(z\ \overrightarrow{P}) \in SN$.
Proof : This follows immediately from Lemma 3.1 and the IH. □

We use Lemma 3.2 to show that $\sigma(t) \in SN$.

1. Assume $\sigma(t)$ is δ-head reducible. We have to show that $D[\sigma(t)] \in SN$. There are 3 cases to consider.

   (a) If t was already δ-head reducible, then $D[\sigma(t)] = \sigma(D[t])$ and the result follows from the IH.

(b) If H is a variable and $\sigma(H) = \lambda x.\lambda y.a$, then $D[\sigma(t)] = t'[z := \lambda y.(\lambda x.a\ N(1))]$ where $t' = (z\ tail(\overrightarrow{N}))$. By the claim, $t' \in SN$ and since $type(z) < type(\sigma)$ it is enough to check that $\lambda y.(\lambda x.a\ N(1)) \in SN$. But this is $\lambda y.(z'\ N(1))[z' := \lambda x.a]$. But, by the claim, $(z'\ N(1)) \in SN$ and we conclude by the IH since $type(z') < type(\sigma)$.

(c) If $H = \lambda x.z$ and $\sigma(z) = \lambda y.a$, then $D[\sigma(t)] = (\lambda y.(\lambda x.a\ N(1))\ tail(\overrightarrow{N})) = (z'\ tail(\overrightarrow{M}))[z' := \lambda y.(\lambda x.a\ N(1))]$. By the IH, it is enough to show that $(\lambda x.a\ N(1)) \in SN$. But this is $(\lambda x.z''\ N(1))[z'' := a]$ and, since $type(a) < type(\sigma)$ it is enough to show that $u = (\lambda x.z''\ N(1)) = \sigma((\lambda x.z''\ M(1))) \in SN$. But this follows from the IH since $(\lambda x.z''\ M(1))$ is (up to α-equivalence) a strict sub-term of t.

2. Assume $\sigma(t)$ is γ-head reducible. We have to show that $C[\sigma(t)] \in SN$. There are 4 cases to consider.

(a) If H is an abstraction, then $C[\sigma(t)] = \sigma(C[t])$ and the result follows immediately from the IH.

(b) H is a variable and $\sigma(H) = \lambda y.a$, then $C[\sigma(t)] = (\lambda y.(a\ N(2))\ N(1)\ N(3)\ ...\ N(n)) = (\lambda y.(a\ N(2))\ y\ N(3)\ ...\ N(n))[y := N(1)]$. Since $type(M(1) < type(H)$, it is enough, by the IH, to show that $(\lambda y.(a\ N(2))\ y\ N(3)\ ...\ N(n)) = (z\ N(3)\ ...\ N(n))[z := (\lambda y.(a\ N(2))\ y)] \in SN$. By the claim and since $type(z < type(H)$, it is enough to show that $(\lambda y.(a\ N(2))\ y)] \in SN$, i.e. (by Lemma 3.1) $(a\ N(2)) = (z'\ N(2))[z' := a] \in SN$. But this follows from the claim and the IH since $type(a) < type(H)$.

(c) H is a variable and $\sigma(H) = (\lambda y.a\ b)$, then $C[\sigma(t)] = (\lambda y.(a\ N(1))\ b\ N(2)\ ...\ N(n)) = (z\ N(2)\ ...\ N(n))[z := (\lambda y.(a\ N(1))\ b)]$. Since $type(z) < type(H)$, by the IH it is enough to show that $u = (\lambda y.(a\ N(1))\ b) \in SN$. We use Lemma 3.2.

- We first have to show that $B[u] \in SN$. But this is $(a[y := b]\ N(1))$ which is in SN since $u_1 = (a[y := b]\ \overrightarrow{N}) \in SN$ since $u_1 = \tau(t_1)$ where t_1 is the same as t but where we have given to the variable H the fresh name z, τ is the same as σ for the variables in $dom(\sigma)$ and $\tau(z) = a[y := b]$ and thus we may conclude by the IH since $\eta(\tau) < \eta(\sigma)$.

- We then have to show that, if b is a redex say $(\lambda z.b_1\ b_2)$, then $A[u, 1] = (\lambda z.(\lambda y.a\ N(1)\ b_1)\ b_2) \in SN$. Let $u_2 = \tau(t_2)$ where t_2 is the same as t but where we have given to the variable H the fresh name z, τ is the same as σ for the variables in $dom(\sigma)$ and $\tau(z) = A[\sigma(H)]$. By the IH $u_2 \in SN$. But $u_2 = (\lambda z.(\lambda y.a\ b_1)\ b_2\ \overrightarrow{N})$ and thus $u_3 = (\lambda z.(\lambda y.a\ b_1)\ b_2\ \ N(1)) \in SN$. Since u_3 reduces to $A[u, 1]$ by using twice by the γ rule, it follows that $A[u, 1] \in SN$.

(d) If H is a variable and $\sigma(H)$ is γ-head reducible, then $C[\sigma(t)] = \tau(t')$ where t' is the same as t but where we have given to the variable H the fresh name z and τ is the same as σ for the variables in $dom(\sigma)$ and $\tau(z) = C[\sigma(H)]$. The result follows then from the IH.

3. Assume that $\sigma(t)$ is β-head reducible. We have to show that $Arg[\sigma(t)] \in SN$ and that $B[\sigma(t)] \in SN$. There are 3 cases to consider.

(a) If H is an abstraction, the result follows immediately from the IH since then $Arg[\sigma(t)] = \sigma(Arg[t])$ and $B[\sigma(t)] = \sigma(B[t])$.

(b) If H is a variable and $\sigma(H) = \lambda y.v$ for some v. Then $Arg[\sigma(t)] = N(1) \in SN$ by the IH and $B[\sigma(t)] = (v[y := N(1)]\ tail(\overrightarrow{N}) = (z\ tail(\overrightarrow{N}))[z := v[y := N(1)]]$. By the claim, $(z\ tail(\overrightarrow{N})) \in SN$. By the IH, $v[y := N(1)] \in SN$ since $type(M(1)) < type(\sigma)$. Finally the IH implies that $B[\sigma(t)] \in SN$ since $type(v) < type(\sigma)$.

(c) H is a variable and $\sigma(H) = (R\ \overrightarrow{M'})$ where R is a β-redex. Then $Arg[\sigma(t)] = Arg[\sigma(H)] \in SN$ and $B[\sigma(t)] = (R'\ \overrightarrow{M'}\ \overrightarrow{N})$ where R' is the reduct of R. But then $B[\sigma(t)] = \tau(t')$ and t' is the same as t but where we have given to the variable H the fresh name z and τ is the same as σ for the variables in $dom(\sigma)$ and $\tau(z) = (R'\ \overrightarrow{M'})$. We conclude by the IH since $\eta(\tau) < \eta(\sigma)$.

4. We, finally, have to show that, for each i, $A[\sigma(t), i] \in SN$. There are again 3 cases to consider.

(a) If the redex put in head position is some $N(j)$ and $M(j)$ was already a redex. Then $A[\sigma(t), i] = \sigma(A[t, j])$ and the result follows from the IH.

(b) If the redex put in head position is some $N(j)$ and $M(j) = (x\ a)$ and $\sigma(x) = \lambda y.b$ then $A[\sigma(t), i] = \lambda y.(\sigma(H)\ N(1)\ ...\ N(j-1)\ b)\ \sigma(a)\ N(j+1)\ ...\ N(n))$. Since $type(\sigma(a)) < type(\sigma)$ it is enough, by the IH, to show that $\lambda y.(\sigma(H)\ N(1)\ ...\ N(j-1)\ b)\ y\ N(j+1)\ ...\ N(n)) = (z\ N(j+1)\ ...\ N(n))[z := \lambda y.(\sigma(H)\ N(1)\ ...\ N(j-1)\ b)\ y] \in SN$. Since $type(z) < type(\sigma)$ and, by the claim, $(z\ N(j+1)\ ...\ N(n)) \in SN$ it is enough to show $(\lambda y.(\sigma(H)\ N(1)\ ...\ N(j-1)\ b)\ y) \in SN$ i.e. (by Lemma 3.1) $u = (\sigma(H)\ N(1)\ ...\ N(j-1)\ b) \in SN$. Let $t' = (H\ \overrightarrow{M'})$ where $M'(k) = M(k)$, for $k \neq j$, $M'(j) = z'$. Since $t = t'[z := (x\ a)]$, by Lemma 3.1 and the IH, $\sigma(t') \in SN$. Since $type(b) < type(\sigma)$ it follows that $\sigma(t')[z' := b]$ and this implies $u \in SN$ since u is a sub-term of it.

(c) If, finally, H is a variable, $\sigma(H) = (H'\ \overrightarrow{M'})$ and the redex put in head position is some $M'(j)$. Then, $A[\sigma(t), i] = \tau(A[t', j])$ where t' is the same as t but where we have given to the variable H the fresh variable z and τ is the same as σ for the variables in $dom(\sigma)$ and $\tau(z) = A[\sigma(H), j]$. We conclude by the IH since $\eta(\tau) < \eta(\sigma)$.

□

Corollary 3.1 *Let t be a term typable in system $\mathcal{D}$. Then t is strongly normalizing.*

Proof By induction on $size(t)$. If t is an abstraction or a variable the result is trivial. Otherwise $t = (u\ v) = (x\ y)[x := u][y := v]$ and the result follows immediately from Theorem 3.3 and the induction hypothesis. □

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