



HAL
open science

On Landau damping

Clément Mouhot, Cédric Villani

► **To cite this version:**

| Clément Mouhot, Cédric Villani. On Landau damping. 2009. hal-00376547v3

HAL Id: hal-00376547

<https://hal.science/hal-00376547v3>

Preprint submitted on 1 Sep 2009 (v3), last revised 6 Dec 2009 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON LANDAU DAMPING
(STABLE RELEASE 2 OF September 1, 2009)

C. MOUHOT AND C. VILLANI

ABSTRACT. Going beyond the linearized study has been a longstanding problem in the theory of Landau damping. In this paper we establish Landau damping for the nonlinear Vlasov equation, for any interaction potential less singular than Coulomb. The damping phenomenon is reinterpreted in terms of transfer of regularity between kinetic and spatial variables, rather than exchanges of energy. The analysis involves new families of analytic norms, measuring regularity by comparison with solutions of the free transport equation; new functional inequalities; a control of nonlinear echoes; sharp scattering estimates; and a Newton approximation scheme. We point out the (*a priori* unexpected) critical nature of the Coulomb potential and analytic regularity, which can be seen only at the nonlinear level; in this case we derive Landau damping over finite but exponentially long times. Physical implications are discussed.

CONTENTS

| | |
|--|-----|
| 1. Introduction to Landau damping | 3 |
| 2. Main result | 13 |
| 3. Linear damping | 26 |
| 4. Analytic norms | 35 |
| 5. Scattering estimates | 69 |
| 6. Bilinear regularity and decay estimates | 77 |
| 7. Control of the time-response | 93 |
| 8. Approximation schemes | 119 |
| 9. Local in time iteration | 125 |
| 10. Global in time iteration | 129 |
| 11. Convergence in large time | 163 |
| 12. Coulomb/Newton interaction | 165 |
| 13. Expansions and counterexamples | 167 |
| 14. Beyond Landau damping | 174 |
| Appendix | 176 |
| References | 178 |

Keywords. Landau damping; plasma physics; galactic dynamics; Vlasov-Poisson equation.

AMS Subject Classification. 35B35 (35J05, 62E20, 70F15, 82C99, 85A05, 82D10, 35Q60, 76X05).

Landau damping may be the single most famous mystery of classical plasma physics. For the past sixty years it has been treated in the linear setting at various degrees of rigor; but its nonlinear version has remained elusive, since the only available results [13, 37] prove the existence of *some* damped solutions, without telling anything about their genericity.

In the present work we nearly close this gap by treating the nonlinear version of Landau damping in arbitrarily large times, under assumptions which cover both attractive and repulsive interactions, of any regularity down to Coulomb/Newton, the latter not included. Furthermore, we show that for Coulomb/Newton interaction, damping still holds over exponentially large times.

This will lead us to discover a distinctive mathematical theory of Landau damping, complete with its own functional spaces and functional inequalities. Let us make it clear that this study is not just for the sake of mathematical rigor: indeed, we shall get new insights in the physics of the problem, and identify new mathematical phenomena.

The plan of the paper is as follows.

In Section 1 we provide an introduction to Landau damping, including historical comments and a review of the existing literature. Then in Section 2, we state and comment on our main result about the “nonlinear Landau damping” over arbitrarily large times for any interaction less singular than Coulomb/Newton (Theorem 2.5); and over exponentially large times for Coulomb/Newton interaction (Theorem 2.6).

In Section 3 we provide a rather complete treatment of linear Landau damping. This section can be read independently of the rest.

In Section 4 we define the spaces of analytic functions which are used in the remainder of the paper. The careful choice of norms is one of the keys of our analysis; the complexity of the problem will naturally lead us to work with norms having up to 5 parameters. As a first application, we shall revisit linear Landau damping within this framework.

In Sections 5 to 7 we establish four types of new estimates (scattering estimates, short-term and long-term regularity extortion, echo control); these are the key sections containing in particular the physically relevant new material.

In Section 8 we adapt the Newton algorithm to the setting of the nonlinear Vlasov equation. Then in Sections 9 and 10 we establish some iterative estimates along this scheme.

From these estimates Theorem 2.5 is easily deduced in Section 11. Then in Section 12 we adapt our result to the case of Coulomb/Newton interaction and prove Theorem 2.6.

Some counterexamples and asymptotic expansions are studied in Section 13.

Final comments about the scope and range of applicability of these results are provided in Section 14.

Even though it basically proves one main result, this paper is very long. This is due partly to the intrinsic complexity and richness of the problem, partly to the need to develop an adequate functional theory from scratch, and partly to the inclusion of many remarks, comments, and explanations, as well as a few developments (most notably Subsection 4.10) which are not necessary, but could prove useful in related problems. The whole process culminates in the extremely technical iteration performed in Section 10. A short summary of our results and methods of proofs can be found in the expository paper [62].

This project started from an unlikely conjunction of discussions of the authors with various people, most notably Yan Guo, Dong Li, Freddy Bouchet and Étienne Ghys. We also got crucial inspiration from the books [10, 11] by James Binney and Scott Tremaine; and [2] by Serge Alinhac and Patrick Gérard. Warm thanks to Julien Barré, Jean Dolbeault, Thierry Gallay, Stephen Gustafson, Gregory Hammett, Donald Lynden-Bell, Michael Sigal, Éric Séré and especially Michael Kiessling for useful exchanges and references; and to Francis Filbet and Irene Gamba for providing numerical simulations. We are also grateful to Julien Barré, Patrick Bernard, Emanuele Caglioti, Yves Elskens, Michael Loss, Peter Markowich, Govind Menon, Yann Ollivier, Mario Pulvirenti, Igor Rodnianski, Yoshio Sone, Tom Spencer, and the team of the Princeton Plasma Physics Laboratory for further constructive discussions about our results. Final thanks to the Institute for Advanced Study in Princeton, for offering the second author a serene atmosphere of work and concentration during the best part of the preparation of this work.

1. INTRODUCTION TO LANDAU DAMPING

1.1. Discovery. Under adequate assumptions (collisionless regime, nonrelativistic motion, heavy ions, no magnetic field), a dilute plasma is well described by the nonlinear Vlasov–Poisson equation

$$(1.1) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_v f = 0,$$

where $f = f(t, x, v) \geq 0$ is the density of electrons in phase space (x = position, v = velocity), m is the mass of an electron, and $F = F(t, x)$ is the mean-field (self-consistent) electrostatic force:

$$(1.2) \quad F = -e E, \quad E = \nabla \Delta^{-1}(4\pi\rho).$$

Here $e > 0$ is the absolute electron charge, $E = E(t, x)$ is the electric field, and $\rho = \rho(t, x)$ is the density of charges

$$(1.3) \quad \rho = \rho_i - e \int f \, dv,$$

ρ_i being the density of charges due to ions. This model and its many variants are of tantamount importance in plasma physics [1, 6, 44, 48].

In contrast to models incorporating collisions [83], the Vlasov–Poisson equation is *time-reversible*. However, in 1946 Landau [47] stunned the physical community by predicting an irreversible behavior on the basis of this equation. This “astonishing result” (as it was called in [76]) relied on the solution of the Cauchy problem for the linearized Vlasov–Poisson equation around a spatially homogeneous Maxwellian equilibrium. Landau formally solved the equation by means of Fourier and Laplace transforms, and after a study of singularities in the complex plane, concluded that the electric field decays exponentially fast; he further studied the rate of decay as a function of the wave vector k . Landau’s computations are reproduced in [48, Section 34] or [1, Section 4.2].

An alternative argument appears in [48, Section 30]: there the thermodynamical formalism is applied to compute the amount of heat Q which is dissipated when a (small) oscillating electric field $E(t, x) = E e^{i(k \cdot x - \omega t)}$ (k a wave vector, $\omega > 0$ a frequency) is applied to a plasma whose distribution f^0 is homogeneous in space and isotropic in velocity space; the result is

$$(1.4) \quad Q = -|E|^2 \frac{\pi m e^2 \omega}{|k|^2} \phi' \left(\frac{\omega}{|k|} \right),$$

where $\phi(v_1) = \int f^0(v_1, v_2, v_3) \, dv_2 \, dv_3$. In particular, (1.4) is always positive (see the last remark in [48, Section 30]), which means that the system reacts against the perturbation, and thus possesses some “active” stabilization mechanism.

A third argument [48, Section 32] consists in studying the dispersion relation, or equivalently searching the (generalized) eigenmodes of the linearized Vlasov–Poisson equation, now with complex frequency ω . After appropriate selection, these eigenmodes are all *decaying* ($\Im \omega < 0$) as $t \rightarrow \infty$. This again suggests stability, although in a somewhat weaker sense than the computation of heat release.

The first and third arguments also apply to the *gravitational* Vlasov–Poisson equation, which is the main model for nonrelativistic galactic dynamics. This equation is similar to (1.1), but now m is the mass of a typical star (!), and f is the density of stars in phase space; moreover the first equation of (1.2) and the relation (1.3)

should be replaced by

$$(1.5) \quad F = -\mathcal{G}mE, \quad \rho = m \int f dv;$$

where \mathcal{G} is the gravitational constant, E the gravitational field, and ρ the density of mass. The books by Binney and Tremaine [10, 11] constitute excellent references about the use of the Vlasov–Poisson equation in stellar dynamics — where it is often called the “collisionless Boltzmann equation”, see footnote on [11, p. 276]. On “intermediate” time scales, the Vlasov–Poisson equation is thought to be an accurate description of very large star systems [26], which are now accessible to numerical simulations.

Since the work of Lynden-Bell [51] it has been recognized that Landau damping, and wilder collisionless relaxation processes generically dubbed “violent relaxation”, constitute a fundamental stabilizing ingredient of galactic dynamics. Without these still poorly understood mechanisms, the surprisingly short time scales for relaxation of the universe would remain unexplained.

One main difference between the electrostatic and the gravitational interactions is that in the latter case Landau damping should occur only at wavelengths smaller than the **Jeans length** [11, Section 5.2]; beyond this scale, even for very nice (say Maxwellian) velocity profiles, the Jeans instability takes over and governs planet and galaxy aggregation.¹

On the contrary, in (classical) plasma physics, Landau damping should hold at all scales under suitable assumptions on the velocity profile; and in fact one is in general not interested in scales smaller than the **Debye length**, which is roughly defined in the same way as the Jeans length.

Nowadays, not only has Landau damping become a cornerstone of plasma physics², but it has also made its way in other areas of physics (astrophysics, but also wind waves, fluids, superfluids, . . .) and even biophysics. One may consult the concise survey papers [67, 71, 81] for a discussion of its influence and some applications.

¹or at least would do, if galactic matter was smoothly distributed; in presence of “microscopic” heterogeneities, a phase transition for aggregation can occur far below this scale [42]. In the language of statistical mechanics, the Jeans length corresponds to a “spinodal point” rather than a phase transition [75].

²Ryutov [71] estimated in 1998 that “approximately every third paper on plasma physics and its applications contains a direct reference to Landau damping”.

1.2. Interpretation. True to his legend, Landau deduced the damping effect from a mathematical-style study³, without bothering to give a physical explanation of the underlying mechanism. His arguments anyway yield exact formulas, which in principle can be checked experimentally, and indeed provide good qualitative agreement with observations [52].

A first set of problems in the interpretation is related to the arrow of time. In the thermodynamic argument, the exterior field is awkwardly imposed from time $-\infty$ on; moreover, reconciling a positive energy dissipation with the reversibility of the equation is not obvious. In the dispersion argument, one has to arbitrarily impose the location of the singularities taking into account the arrow of time; at mathematical level this is equivalent to a choice of principal value:

$$\frac{1}{z - i0} = \text{p.v.} \left(\frac{1}{z} \right) + i\pi \delta_0.$$

This is not so serious, but then the spectral study requires some thinking. All in all, the most convincing argument remains Landau’s original one, since it is based only on the study of the Cauchy problem, which makes more physical sense than the study of the dispersion relation (see the remark in [10, p. 682]).

A more fundamental issue resides in the use of analytic function theory, with contour integration, singularities and residue computation, which has played a major role in the theory of the Vlasov–Poisson equation ever since Landau [48, Chapter 32] [11, Subsection 5.2.4]. Not only does this impose stringent assumptions on the data, but it helps little, if at all, to understand the underlying physical mechanism.⁴

The most popular interpretation of Landau damping considers the phenomenon from an energetic point of view, as the result of the interaction of a plasma wave with particles of nearby velocity [77, p. 18] [11, p. 412] [1, Section 4.2.3] [48, p. 127]. In a nutshell, the argument says that dominant exchanges occur with those particles which are “trapped” by the wave because their velocity is close to the wave velocity. If the distribution function is a decreasing function of $|v|$, among trapped particles more are accelerated than are decelerated, so the wave loses energy to the plasma — or the plasma surfs on the wave — and the wave is damped by the interaction.

³not completely rigorous from the mathematical point of view, but formally correct, in contrast to the previous studies by Landau’s fellow physicists — as Landau himself pointed out without mercy [47].

⁴Van Kampen [80] summarizes the conceptual problems posed to his contemporaries by Landau’s treatment, and comments on more or less clumsy attempts to resolve the apparent paradox caused by the singularity in the complex plane.

Appealing as this image may seem, to a mathematically-oriented mind it will probably make little sense at first hearing.⁵ A more down-to-Earth interpretation emerged in the fifties from the “wave packet” analysis of Van Kampen [80] and Case [14]: Landau damping would result from **phase mixing**. This phenomenon, well-known in galactic dynamics, describes the damping of oscillations occurring when a continuum is transported in phase space along an anharmonic Hamiltonian flow [11, pp. 379–380]. The mixing results from the simple fact that particles following different orbits travel at different angular⁶ speeds, so perturbations start “spiralling” (see Figure 4.27 on [11, p. 379]) and homogenize by fast spatial oscillation. From the mathematical point of view, phase mixing results in *weak convergence*; from the physical point of view, this is just the convergence of observables, defined as averages over the velocity space (this is sometimes called “convergence in the mean”).

At first sight, both points of view seem hardly compatible: Landau scenario suggests a very smooth process, while the phase mixing one involves tremendous oscillations. The coexistence of these two interpretations did generate some speculation on the nature of the damping, and on its relation to phase mixing, see e.g. [41] or [11, p. 413]. There is actually no contradiction between the two points of view; many physicists have rightly pointed out that that Landau damping should come with filamentation and oscillations of the distribution function [80, p. 962] [48, p. 141] [1, Vol. 1, pp. 223–224] [50, pp. 294–295]. Nowadays these oscillations can be visualized spectacularly thanks to deterministic numerical schemes, see e.g. [85] [36, Fig. 3] [25]. Below, we reproduce some examples provided by Filbet.

In any case, there is no definite interpretation of Landau damping: as noted by Ryutov [71, Section 9], papers devoted to the interpretation and teaching of Landau damping were still appearing regularly fifty years after its discovery; to quote just a couple of more recent examples let us mention works by Elskens and Escande [21, 23, 22].

1.3. Range of validity. The following issues are addressed in the literature [38, 41, 54, 85] and slightly controversial:

- Does Landau damping really hold for gravitational interaction? The case seems thinner in this situation than for plasma interaction, all the more that there are many instability results in the gravitational context; at present there does not seem

⁵Escande [23, Chapter 4, Footnote 6] points out some misconceptions associated with the surfer image.

⁶“Angular” here refers to action-angle variables, and applies even for straight trajectories in a periodic box.

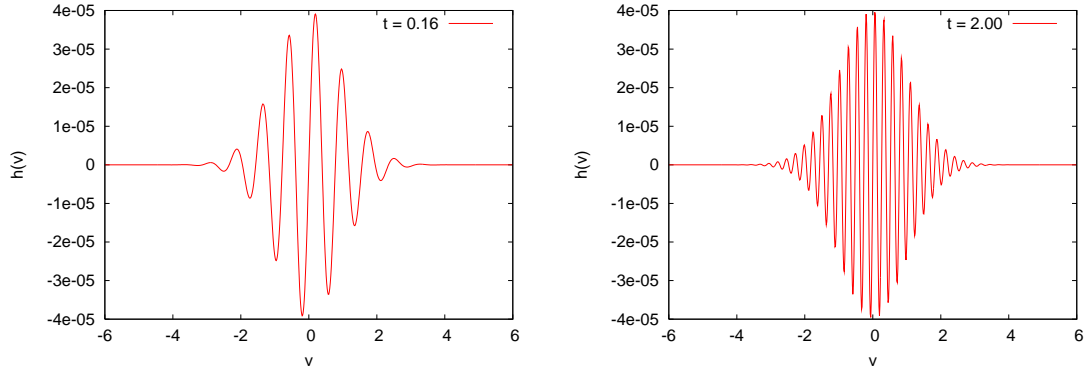


FIGURE 1. A slice of the distribution function (relative to a homogeneous equilibrium) for gravitational Landau damping, at two different times

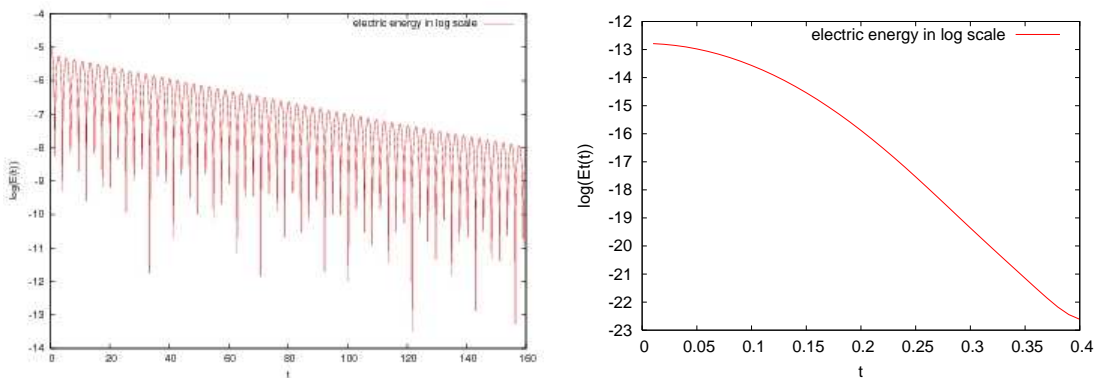


FIGURE 2. Time-evolution of the norm of the field, for electrostatic (on the left) and gravitational (on the right) interactions. Notice the fast Langmuir oscillations in the electrostatic case.

to be any consensus among mathematical physicists [69]. (Numerical evidence is not conclusive because of the difficulty of accurate simulations in very large time — even in one dimension of space.)

- Does the damping hold for unbounded systems? Counterexamples from [28, 29] show that some kind of confinement is necessary, even in the electrostatic case. More precisely, Glassey and Schaeffer show that a solution of the linearized Vlasov–Poisson equation in the whole space (linearized around a homogeneous equilibrium

f^0 of infinite mass) decays at best like $O(t^{-1})$, modulo logarithmic corrections, for $f^0(v) = c/(1 + |v|^2)$; and like $O((\log t)^{-\alpha})$ if f^0 is a Gaussian. In fact, Landau’s original calculations already indicated that the damping is *extremely* weak at large wavenumbers; see the discussion in [48, Section 32]. Of course, in the gravitational case, this is even more dramatic because of the Jeans instability.

- Does convergence hold in infinite time for the solution of the “full” nonlinear equation? This is not clear at all since there is no mechanism that would keep the distribution close to the original equilibrium *for all times*. Some authors do not believe that there is convergence as $t \rightarrow \infty$; others believe that there is convergence but argue that it should be very slow [38], say $O(1/t)$. Backus [5] notes that in general the linear and nonlinear evolution break apart after some (not very large) time, and questions the validity of the linearization. O’Neil [66] argues that relaxation holds in the “quasilinear regime” on larger time scales, when the “trapping time” (roughly proportional the inverse square root of the size of the perturbation) is much smaller than the damping time. Other speculations and arguments related to trapping appear in many sources, e.g. [57]. Kaganovich [40] argues that nonlinear effects may quantitatively affect Landau damping related phenomena by several orders of magnitude.

The so-called “quasilinear relaxation theory” [48, Section 49] [1, Section 9.1.2] [44, Chapter 10] uses second-order approximation of the Vlasov equation to predict the convergence of the spatial average of the distribution function. The procedure is most esoteric, involving averaging over statistical ensembles, and diffusion equations with discontinuous coefficients, acting only near the resonance velocity for particle-wave exchanges. Because of these discontinuities, the predicted asymptotic state is discontinuous, and collisions are invoked to restore smoothness. Linear Fokker–Planck equations⁷ in velocity space have also been used in astrophysics [51, p. 111], but only on phenomenological grounds (the *ad hoc* addition of a friction term leading to a Gaussian stationary state); and this procedure has been exported to the study of two-dimensional incompressible fluids [15, 16].

Careful numerical simulation [85] seems to show that the solution of the nonlinear Vlasov–Poisson equation does converge to a spatially homogeneous distribution, but only as long as the size of the perturbation is small enough. We shall call this phenomenon the **nonlinear Landau damping**. This terminology summarizes well the problem, still it is subject to criticism since (a) Landau himself stuck to the linear case and did not discuss the large-time convergence of the data; (b) this

⁷These equations act on some ensemble average of the distribution; they are different from the Vlasov–Landau equation.

damping is expected to hold when the regime is close to linear, but not necessarily when the nonlinear term dominates⁸; and (c) this expression is used to designate a related but different phenomenon [1, Section 10.1.3].

- Is Landau damping related to the more classical notion of stability in orbital sense? Orbital stability means that the system, slightly perturbed at initial time from an equilibrium distribution, will always remain close to this equilibrium. Even in the favorable electrostatic case, stability is not granted; the most prominent phenomenon being the Penrose instability [68] according to which a distribution with two deep bumps may be unstable. In the more subtle gravitational case, various stability and instability criteria are associated with the names of Chandrasekhar, Antonov, Goodman, Doremus, Feix, Baumann, ... [11, Section 7.4]. There is a widespread agreement (see e.g. the comments in [85]) that Landau damping and stability are related, and that Landau damping cannot be hoped for if there is no orbital stability.⁹

1.4. Conceptual problems. Summarizing, we can identify three main conceptual obstacles which make Landau damping mysterious, even sixty years after its discovery:

(i) The equation is time-reversible, yet we are looking for an irreversible behavior as $t \rightarrow +\infty$ (or $t \rightarrow -\infty$). The value of the entropy does not change in time, which physically speaking means that there is no loss of information in the distribution function. The spectacular experiment of the “plasma echo” illustrates this conservation of microscopic information [30, 53]: a plasma which is apparently back to equilibrium after an initial disturbance, will react to a second disturbance in a way that shows that it has not forgotten the first one.¹⁰ And at the linear level, if there are decaying modes, there also has to be growing modes!

(ii) When one perturbs an equilibrium, there is no mechanism forcing the system to go back to this equilibrium in large time; so there is no justification in the use of linearization to predict the large-time behavior. In the first mathematically rigorous

⁸although phase mixing might still play a crucial role in violent relaxation or other unclassified nonlinear phenomena.

⁹The question is not so absurd as it may seem. Common wisdom says, why worry about a fine damping if the system anyway goes away from the equilibrium? However, it could be the case that the system goes away in strong topology, and remains close in weak topology. Still there is no hint for such a behavior, and there is no reason to doubt the common view.

¹⁰Interestingly enough, this experiment was suggested as a way to measure the strength of irreversible phenomena going on inside a plasma, e.g. collision frequency, by measuring attenuations with respect to the predicted echo. See [74] for an interesting application and appealing figures.

study on the subject, Backus [5] had already put forward this objection in a very convincing way.

(iii) At the technical level, Landau damping rests on analyticity, and its most attractive interpretation is in terms of phase mixing. But both phenomena are *incompatible in the large-time limit*: phase mixing implies an irreversible deterioration of analyticity. For instance, it is easily checked that free transport induces an exponential growth of analytic norms as $t \rightarrow \infty$ — except if the initial datum is spatially homogeneous. In particular, the Vlasov–Poisson equation is **unstable** (in large time) in any norm incorporating velocity regularity. (Space-averaging is one of the ingredients used in the quasilinear theory to formally get rid of this instability.)

How can we respond to these issues?

One way to solve the first problem (time-reversibility) is to appeal to Van Kampen modes as in [11, p. 415]; however these are not so physical, as noticed in [10, p. 682]. A simpler conceptual solution is to invoke the notion of weak convergence: reversibility manifests itself in the conservation of the information contained in the density function; but information may be lost irreversibly in the limit when we consider weak convergence. Weak convergence only describes the long-time behavior of arbitrary observables, each of which does not contain as much information as the density function.¹¹ As a very simple illustration, consider the time-reversible evolution defined by $u(t, x) = e^{itx}u_i(x)$, and notice that it does converge weakly to 0 as $t \rightarrow \pm\infty$; this convergence is even exponentially fast if the initial datum u_i is analytic. (Our example is not chosen at random: although it is extremely simple, it may be a good illustration of what happens in phase mixing.) In a way, microscopic reversibility is compatible with macroscopic irreversibility, provided that the “microscopic regularity” is destroyed asymptotically.

Still in respect to this reversibility, it should be noted that the “dual” mechanism of **radiation**, according to which an infinite-dimensional system may lose energy towards very large scales, is relatively well understood and recognized as a crucial stability mechanism [73, 4].

The second problem only indicates that there is a large gap between the understanding of linear Landau damping, and that of the nonlinear phenomenon.

The third problem, maybe the most troubling, does not dismiss the phase mixing explanation, but suggests that we shall have to keep track of the initial time, in the

¹¹In Lynden-Bell’s appealing words [50, p.295], “a system whose density has achieved a steady state will have information about its birth still stored in the peculiar velocities of its stars.”

sense that a rigorous proof cannot be based on the propagation of some phenomenon. This situation is of course in sharp contrast with the study of dissipative systems admitting a Lyapunov functional, as do many collisional kinetic equations [83, 82]; it will require completely different mathematical techniques.

1.5. Previous mathematical results. At the linear level, the first rigorous treatments of Landau damping were performed in the sixties; see Saenz [72] for rather complete results and a review of earlier works. The theory was rediscovered and renewed at the beginning of the eighties by Degond [19], and Maslov and Fedoryuk [56]. In all these works, analytic arguments play a crucial role (for instance for analytic extension of resolvent operators), and asymptotic expansions for the electric field associated to the linearized Vlasov–Poisson equation are obtained.

Also at the linearized level, there are counterexamples by Glassey and Schaeffer [28, 29] showing that there is in general no exponential decay for the linearized Vlasov–Poisson equation without analyticity, or without confining.

In a *nonlinear* setting, the only rigorous treatments so far are those by Caglioti–Maffei [13], and later Hwang–Vélazquez [37]. Both sets of authors work in the one-dimensional torus and use fixed-point theorems and perturbative arguments to prove the *existence* of a class of solutions behaving, asymptotically as $t \rightarrow +\infty$, and in a strong sense, like perturbed solutions of free transport. Since solutions of free transport weakly converge to spatially homogeneous distributions, the solutions constructed by this “scattering” approach are indeed damped.

The weakness of these results is that they say nothing about the initial perturbations leading to such solutions, which could be very special. In other words: damped solutions do exist, but do we ever reach them?

Sparse as it may seem, this list is kind of exhaustive. On the other hand, there is a rather large mathematical literature on the orbital stability problem, due to Guo, Rein, Strauss, Wolansky and others. In this respect see for instance [33] for the plasma case, and [32] for the gravitational case; both sources contain many references on the subject. This body of works has confirmed the intuition of physicists, although with quite different methods. The gap between a formal, linear treatment and a rigorous, nonlinear one is striking: Compare the Appendix of [32] to the rest of the paper.

Our treatment of Landau damping will be performed from scratch, and will not rely on any of these results.

2. MAIN RESULT

2.1. Modelling. We shall work in adimensional units throughout the paper, in d dimensions of space and d dimensions of velocity ($d \in \mathbb{N}$).

As should be clear from our presentation in Section 1, to observe Landau damping, we need to put a restriction on the length scale; to achieve this we shall take the position space to be the d -dimensional torus of sidelength L , namely $\mathbb{T}_L^d = \mathbb{R}^d / (L\mathbb{Z})^d$. This is admittedly a bit unrealistic, but it is commonly done in plasma physics (see e.g. [6]).

In a periodic setting the Poisson equation has to be reinterpreted, since $\Delta^{-1}\rho$ is not well-defined unless $\int_{\mathbb{T}_L^d} \rho = 0$. The natural solution consists in removing the mean value of ρ , independently of any “neutrality” assumption; in galactic dynamics this is known as the **Jeans swindle**, a trick considered as efficient but logically absurd. However, in 2003 Kiessling [43] re-opened the case and acquitted Jeans, on the basis that his “swindle” can be justified by a simple limit procedure. In the present case, one may adapt Kiessling’s argument and approximate the Coulomb potential V by any cutoffted version V_κ , e.g. of Debye type (invoking screening for a plasma, or a cosmological constant for stellar systems; anyway the particular choice of approximation has no influence on the result). Then $\nabla V_\kappa * \rho$ makes sense for a periodic ρ , and moreover

$$(\nabla V_\kappa * \rho)(x) = \int_{\mathbb{R}^d} \nabla V_\kappa(x-y) \rho(y) dy = \int_{[0,L]^d} \nabla V_\kappa^{(L)}(x-y) \rho(y) dy,$$

where $V_\kappa^{(L)}(z) = \sum_{\ell \in \mathbb{Z}^d} V_\kappa(z + \ell L)$. Passing to the limit as $\kappa \rightarrow 0$ yields

$$\int_{[0,L]^d} \nabla V^{(L)}(x-y) \rho(y) dy = \int_{[0,L]^d} \nabla V^{(L)}(x-y) (\rho - \langle \rho \rangle)(y) dy = -\nabla \Delta_L^{-1}(\rho - \langle \rho \rangle),$$

where Δ_L^{-1} is the inverse Laplace operator on \mathbb{T}_L^d . (We refer to [43] for a discussion of the physics underlying this limit $\kappa \rightarrow 0$.)

More generally, we may consider any interaction potential W on \mathbb{T}_L^d , satisfying certain regularity assumptions. Then the self-consistent field will be given by

$$F = -\nabla W * \rho, \quad \rho(x) = \int f(x, v) dv,$$

where now $*$ denotes the convolution on \mathbb{T}_L^d .

In accordance with our conventions from Appendix A.3, we shall write $\widehat{W}^{(L)}(k) = \int_{\mathbb{T}_L^d} e^{-2i\pi k \cdot \frac{x}{L}} W(x) dx$. In particular, if W is the periodization of a potential $\mathbb{R}^d \rightarrow \mathbb{R}$

(still denoted W by abuse of notation), *i.e.*,

$$W(x) = W^{(L)}(x) = \sum_{\ell \in \mathbb{Z}^d} W(x + \ell L),$$

then

$$(2.1) \quad \widehat{W}^{(L)}(k) = \widehat{W}\left(\frac{k}{L}\right),$$

where $\widehat{W}(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi\xi \cdot x} W(x) dx$ is the original Fourier transform in the whole space.

2.2. Linear damping. It is well-known that Landau damping requires some stability assumptions on the unperturbed homogeneous distribution function, say $f^0(v)$. In this paper we shall use a very general assumption, expressed in terms of the Fourier transform

$$(2.2) \quad \widetilde{f}^0(\eta) = \int_{\mathbb{R}^d} e^{-2i\pi\eta \cdot v} f^0(v) dv,$$

the length L , and the interaction potential W . To state it, we define, for $t \geq 0$ and $k \in \mathbb{Z}^d$,

$$(2.3) \quad K^0(t, k) = -4\pi^2 \widehat{W}^{(L)}(k) \widetilde{f}^0\left(\frac{kt}{L}\right) \frac{|k|^2}{L^2} t;$$

and, for any $\xi \in \mathbb{C}$, we define a function \mathcal{L} *via* the following Fourier–Laplace transform of K^0 in the time variable:

$$(2.4) \quad \mathcal{L}(\xi, k) = \int_0^{+\infty} e^{2\pi\xi^* \frac{|k|}{L} t} K^0(t, k) dt,$$

where ξ^* is the complex conjugate to ξ . Our linear damping condition is expressed as follows:

(L) *There are constants $\lambda, \kappa > 0$ such that $|\widetilde{f}^0(\eta)| = O(e^{-2\pi\lambda|\eta|})$; and for any $\xi \in \mathbb{C}$ with $0 \leq \Re \xi < \lambda$,*

$$\inf_{k \in \mathbb{Z}^d} |\mathcal{L}(\xi, k) - 1| \geq \kappa.$$

We shall prove in Section 3 that **(L)** implies Landau damping. For the moment, we give two sufficient conditions for **(L)** to be satisfied. The first one can be thought of as a smallness assumption on either the length, or the potential, or the velocity

distribution. The second one is a structure assumption involving the marginals of f^0 along arbitrary wave vectors k :

$$(2.5) \quad \varphi_k(v) = \int_{\frac{k}{|k|}v+k^\perp} f^0(w) dw, \quad v \in \mathbb{R}.$$

All studies known to us use one of these assumptions, so **(L)** appears as a unifying condition for linear Landau damping around a homogeneous equilibrium.

Proposition 2.1. *Let $f^0 = f^0(v)$ be a velocity distribution such that \tilde{f}^0 decays exponentially fast at infinity, let $L > 0$ and let W be an interaction potential on \mathbb{T}_L^d , $W \in L^1(\mathbb{T}^d)$. If any one of the following two conditions is satisfied:*

(a) smallness:

$$(2.6) \quad 4\pi^2 \left(\max_{k \in \mathbb{Z}_*^d} |\widehat{W}^{(L)}(k)| \right) \left(\sup_{|\sigma|=1} \int_0^\infty |\tilde{f}^0(r\sigma)| r dr \right) < 1;$$

(b) repulsive interaction and decreasing marginals: for all $k \in \mathbb{Z}^d$,

$$(2.7) \quad \widehat{W}^{(L)}(k) \geq 0; \quad \begin{cases} v < 0 \implies \varphi'_k(v) \geq 0 \\ v > 0 \implies \varphi'_k(v) \leq 0; \end{cases}$$

then **(L)** holds true for some $\lambda, \kappa > 0$.

Remark 2.2. [48, Problem, Section 30] If f^0 is radially symmetric and positive, and $d \geq 3$, then all marginals of f^0 are decreasing functions of $|v|$. Indeed, if $\varphi(v) = \int_{\mathbb{R}^{d-1}} f(\sqrt{v^2 + |w|^2}) dw$, then after differentiation and integration by parts we find

$$\begin{cases} \varphi'(v) = -(d-3)v \int_{\mathbb{R}^{d-1}} f(\sqrt{v^2 + |w|^2}) \frac{dw}{|w|^2} & (d \geq 4) \\ \varphi'(v) = -2\pi v f(|v|) & (d = 3). \end{cases}$$

Example 2.3. Take a gravitational interaction and Maxwellian background:

$$\widehat{W}(k) = -\frac{\mathcal{G}}{\pi |k|^2}, \quad f^0(v) = \rho^0 \frac{e^{-\frac{|v|^2}{2T}}}{(2\pi T)^{d/2}}.$$

Recalling (2.1), we see that (2.6) becomes

$$(2.8) \quad L < \sqrt{\frac{\pi T}{\mathcal{G} \rho^0}} =: L_J(T, \rho^0).$$

The length L_J is the celebrated Jeans length [11, 43], so criterion (a) can be applied, all the way up to the onset of the Jeans instability.

Example 2.4. If we replace the gravitational interaction by the electrostatic interaction, the same computation yields

$$(2.9) \quad L < \sqrt{\frac{\pi T}{e^2 \rho^0}} =: L_D(T, \rho^0),$$

and now L_D is essentially the Debye length. Then criterion (a) becomes quite restrictive, but because the interaction is repulsive we can use criterion (b) as soon as f^0 is a strictly monotone function of $|v|$; this covers in particular Maxwellian distributions, *independently of the size of the box*. Criterion (b) also applies if $d \geq 3$ and f^0 has radial symmetry; or even if f^0 has two bumps but L is not too large. Then in the case of two bumps with large L , the Penrose instability will take over.

We shall show in Section 3 that **(L)** implies linear Landau damping (Theorem 3.1); then we shall prove Proposition 2.1 at the end of that section. The general ideas are close to those appearing in previous works, including Landau himself; the only novelties lie in the slightly more general assumptions, the elementary nature of the arguments, and the slightly more precise quantitative results.

2.3. Nonlinear damping. As others have done before in the study of Vlasov–Poisson [13], we shall quantify the analyticity by means of norms involving Fourier transform in both variables (also denoted with a tilde in the sequel). So we define

$$(2.10) \quad \|f\|_{\lambda, \mu} = \sup_{k, \eta} \left(e^{2\pi\lambda|\eta|} e^{2\pi\mu\frac{|k|}{L}} |\tilde{f}^{(L)}(k, \eta)| \right),$$

where k varies in \mathbb{Z}^d , $\eta \in \mathbb{R}^d$, λ, μ are positive parameters, and we recall the dependence of the Fourier transform on L (see Appendix A.3 for conventions). Now we can state our main result as follows:

Theorem 2.5 (Nonlinear Landau damping). *Let $f^0 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be an analytic velocity profile. Let $L > 0$ and $W : \mathbb{T}_L^d \rightarrow \mathbb{R}$ be an interaction potential satisfying*

$$(2.11) \quad \forall k \in \mathbb{Z}^d, \quad |\widehat{W}^{(L)}(k)| \leq \frac{C_W}{|k|^{1+\gamma}}$$

*for some constants $C_W > 0$, $\gamma > 1$. Assume that f^0 and W satisfy the stability condition **(L)** from Subsection 2.2, with some constants $\lambda, \kappa > 0$; further assume*

that, for the same parameter λ ,

$$(2.12) \quad \sup_{\eta \in \mathbb{R}^d} \left(|\tilde{f}^0(\eta)| e^{2\pi\lambda|\eta|} \right) \leq C_0, \quad \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|\nabla_v^n f^0\|_{L^1(\mathbb{R}^d)} \leq C_0 < +\infty.$$

Then for any $0 < \lambda' < \lambda$, $\beta > 0$, $0 < \mu' < \mu$, there is $\varepsilon = \varepsilon(d, L, C_W, C_0, \kappa, \lambda, \lambda', \mu, \mu', \beta, \gamma)$ with the following property: if f_i is an initial datum satisfying

$$(2.13) \quad \delta := \|f_i - f^0\|_{\lambda, \mu} + \iint_{\mathbb{T}_L^d \times \mathbb{R}^d} |f_i - f^0| e^{\beta|v|} dv dx \leq \varepsilon,$$

then

- the unique classical solution f to the nonlinear Vlasov equation

$$(2.14) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f - (\nabla W * \rho) \cdot \nabla_v f = 0, \quad \rho = \int_{\mathbb{R}^d} f dv,$$

with initial datum $f(0, \cdot) = f_i$, converges in the weak topology as $t \rightarrow \pm\infty$, with rate $O(e^{-2\pi\lambda'|t|})$, to a spatially homogeneous equilibrium $f_{\pm\infty}$;

- the density $\rho(t, x) = \int f(t, x, v) dv$ converges in the strong topology as $t \rightarrow \pm\infty$, with rate $O(e^{-2\pi\lambda'|t|})$, to the constant density

$$\rho_\infty = \frac{1}{L^d} \int_{\mathbb{R}^d} \int_{\mathbb{T}_L^d} f_i(x, v) dx dv;$$

- the space average $\langle f \rangle(t, v) = \int f(t, x, v) dx$ converges in the strong topology as $t \rightarrow \pm\infty$, with rate $O(e^{-2\pi\lambda'|t|})$, to $f_{\pm\infty}$.

More precisely, there are $C > 0$, and spatially homogeneous distributions $f_{+\infty}(v)$ and $f_{-\infty}(v)$, depending continuously on f_i and W , such that

$$(2.15) \quad \sup_{t \in \mathbb{R}} \left\| f(t, x + vt, v) - f^0(v) \right\|_{\lambda', \mu'} \leq C \delta;$$

$$\forall \eta \in \mathbb{R}^d, \quad |\tilde{f}_{\pm\infty}(\eta) - \tilde{f}^0(\eta)| \leq C \delta e^{-2\pi\lambda'|\eta|};$$

and

$$\forall (k, \eta) \in \mathbb{Z}^d \times \mathbb{R}^d, \quad \left| L^{-d} \tilde{f}^{(L)}(t, k, \eta) - \tilde{f}_{+\infty}(\eta) 1_{k=0} \right| = O(e^{-2\pi\frac{\lambda'}{L}t}) \quad \text{as } t \rightarrow +\infty;$$

$$\forall (k, \eta) \in \mathbb{Z}^d \times \mathbb{R}^d, \quad \left| L^{-d} \tilde{f}^{(L)}(t, k, \eta) - \tilde{f}_{-\infty}(\eta) 1_{k=0} \right| = O(e^{-2\pi\frac{\lambda'}{L}|t|}) \quad \text{as } t \rightarrow -\infty;$$

$$(2.16) \quad \forall r \in \mathbb{N}, \quad \left\| \rho(t, \cdot) - \rho_\infty \right\|_{C^r} = O(e^{-2\pi\frac{\lambda'}{L}|t|}) \quad \text{as } |t| \rightarrow \infty;$$

$$(2.17) \quad \forall r \in \mathbb{N}, \quad \forall \sigma > 0, \quad \left\| \langle f(t, \cdot, v) \rangle - f_{\pm\infty} \right\|_{C_\sigma^r} = O(e^{-2\pi\frac{\lambda'}{L}|t|}) \quad \text{as } t \rightarrow \pm\infty.$$

In this statement C^r stands for the usual norm on r times continuously differentiable functions, and C_σ^r involves in addition moments of order σ , namely $\|f\|_{C_\sigma^r} = \sup_{r' \leq r, v \in \mathbb{R}^d} |f^{(r')}(v)| (1 + |v|^\sigma)$.

2.4. Comments. Let us start with a list of remarks about Theorem 2.5.

- Our result covers both attractive and repulsive interactions, as long as the linear damping condition is satisfied; it fails just short of covering Newton/Coulomb potential (this would be $\gamma = 1$ in (2.11)). It may be that our method can be adapted to include this limit case; but anyway the proof clearly breaks down for $\gamma < 1$. This is a nonlinear effect, as any $\gamma \geq 0$ would work for the linearized equation. We shall see below that a slightly weaker conclusion still holds for $\gamma = 1$ (Theorem 2.6). The singularity of the interaction at short scales will anyway be the source of important technical problems.¹²

- Condition (2.12) could be replaced by

$$(2.18) \quad |\tilde{f}^0(\eta)| \leq C_0 e^{-2\pi\lambda|\eta|}, \quad \int f^0(v) e^{\beta|v|} dv \leq C_0.$$

But condition (2.12) is more general, in view of Theorem 4.20 below. For instance, $f^0(v) = 1/(1+v^2)$ in dimension $d = 1$ satisfies (2.12) but not (2.18); this distribution is commonly used in theoretical and numerical studies, see e.g. [36]. We shall also establish slightly more precise estimates under slightly more stringent conditions on f^0 , see (11.1).

- Our conditions are expressed in terms of the initial datum, which is a considerable improvement over [13, 37]. Still it is of interest to pursue the “scattering” program started in [13], e.g. in a hope of better understanding of the nonperturbative regime.

- Strictly speaking, known existence and uniqueness results for solutions of the nonlinear Vlasov–Poisson equation [7, 49] do not apply to the present setting of close-to-homogeneous analytic solutions. (The problem with [7] is that velocities are assumed to be uniformly bounded, and the problem with [49] is that the position space is the whole of \mathbb{R}^d ; in both papers these assumptions are not superficial.) However, this really is not a big deal: our proof will provide an existence theorem,

¹²In a related field, this singularity is also the reason why the Vlasov–Poisson equation is still far from being established as a mean-field limit of particle dynamics (see [34] for partial results covering much less singular interactions).

together with regularity estimates which are considerably stronger than what is needed to prove the uniqueness. We shall not come back to these issues which are rather irrelevant for our study: uniqueness only needs local in time regularity estimates, while all the difficulty in the study of Landau damping consists in getting estimates in (very) large time.

- The smallness assumption on $f_i - f^0$ is expected, for instance in view of the theoretical work of O’Neil [66], or the numerical results of [85]. We also make the standard assumption that $f_i - f^0$ is well localized.

- The norm in (2.13) cannot be weakened much. The free case ($W = 0$, no interaction) strongly suggests that the v -analyticity is mandatory to get the exponential convergence to equilibrium. Our scheme of proof also needs some dose of analyticity in x , but μ in Theorem 2.5 can be arbitrarily small.

- In any case, no convergence can be hoped for if the initial datum is only close to f^0 in the weak topology: indeed there is *instability* in the weak topology, even around a Maxwellian [13].

- Estimate (2.15) expresses the orbital “travelling stability” around f^0 ; it is much stronger than the usual orbital stability in Lebesgue norms [33, 32]. An equivalent formulation is that if $(T_t)_{t \in \mathbb{R}}$ stands for the nonlinear Vlasov evolution operator, and $(T_t^0)_{t \in \mathbb{R}}$ for the free transport operator, then in a neighborhood of a homogeneous equilibrium satisfying the stability criterion **(L)**, $T_{-t}^0 \circ T_t$ remains uniformly close to Id for all t . Note the important difference: unlike in the usual orbital stability theory, in our work we use functional spaces *which are not invariant under the free transport semigroup*. This a source of difficulty (our functional spaces are sensitive to the filamentation phenomenon), but it is also the reason for which this “analytic” orbital stability contains more information, and in particular the damping of the density.

- $f(t, \cdot)$ is *not* close to f^0 in analytic norm as $t \rightarrow \infty$, and does not converge to anything in strong topology, so the conclusion cannot be improved much. Still we shall establish more precise quantitative results, and the limit profiles $f_{\pm\infty}$ are obtained by a constructive argument.

With respect to the questions raised above, our analysis brings the following answers (assuming that the exclusion of Coulomb/Newton is only of technical nature):

(a) Convergence of the distribution f does hold for $t \rightarrow +\infty$; it is indeed based on phase mixing, and involves very fast oscillations. In this sense it is right to consider

Landau damping as a “wild” process. But on the other hand, the spatial density (and therefore the force field) converges strongly and smoothly.

(b) The space average $\langle f \rangle$ does converge in large time. However the conclusions are radically different from those of quasilinear relaxation theory, since there is no need for extra randomness, and the limiting distribution is smooth, even without collisions.

(c) Landau damping is a linear phenomenon, which survives nonlinear perturbation thanks to the structure of the Vlasov–Poisson equation. The nonlinearity manifests itself by the presence of *echoes*. Echoes were well-known to specialists of plasma physics [48, Section 35] [1, Section 12.7], but were not identified as a possible source of instability. On the one hand, we shall argue that the echo nature of the response explains, to a large extent, the stability of Landau damping over *very* large time-scales, thus answering the objection raised by Backus. But on the other hand, we shall also argue that for any interaction more singular than Coulomb/Newton, or any regularity lower than analytic, self-induced echoes might win over Landau damping.

(d) The large-time limit is in general different from the limit predicted by the linearized equation, and depends on the interaction (more precise statements will be given in Section 13); still the linearized equation, or higher-order expansions, do provide a good approximation. We shall also set up a systematic recipe for approximating the large-time limit with arbitrarily high precision as the strength of the perturbation becomes small. This justifies *a posteriori* many known computations.

(e) From the point of view of dynamical systems, the nonlinear Vlasov equation exhibits a truly remarkable behavior. It is not uncommon for a Hamiltonian system to have many, or even countably many heteroclinic orbits (there are various theories for this, a popular one being the Melnikov method); but in the present case we see that heteroclinic/homoclinic orbits¹³ are so numerous as to fill up *a whole neighborhood* of the equilibrium. This is possible only because of the infinite-dimensional nature of the system, and the possibility to work with nonequivalent norms; such a behavior has already been reported for other systems [45, 46], in relation with infinite-dimensional KAM theory. To draw a parallel with classical KAM theory, one may argue that the linearized Vlasov equation is a completely integrable, anharmonic phase mixing system, and one step of our result is the preservation of the phase mixing property under nonlinear perturbation of the interaction. (Although

¹³Here we use these words just to designate solutions connecting two distinct/equal equilibria, without any mention of stable or unstable manifolds.

there is no ergodicity in phase space, the mixing will imply an ergodic behavior for the spatial density.) In any case, the proof of Theorem 2.5 shares many features with the proof of the KAM theorem (closest to Kolmogorov’s original version, see [18] for a complete exposition).

Thus we see that three of the most famous paradoxical phenomena from classical physics: Landau damping, echoes, and KAM theorem, are intimately related. For us this was a surprise.

2.5. Main ingredients. Some of our ingredients are similar to those in [13]: in particular, the use of Fourier transform to quantify analytic regularity and to implement phase mixing. New ingredients used in our work include

- the introduction of a time-shift parameter to keep memory of the initial time (Sections 4 and 5), thus getting uniform estimates in spite of the loss of regularity in large time. We call this the **gliding regularity**: it shifts in phase space from low to high modes. Gliding regularity automatically comes with an improvement of the regularity in x , and a deterioration of the regularity in v , as time passes by.

- “finite-time scattering” *at the level of trajectories* to reduce the problem to homogenization of free flow (Section 5) *via* composition. The physical meaning is the following: when a background with gliding regularity acts on (say) a plasma, the trajectories of plasma particles are asymptotic to free transport trajectories.

- the use of carefully designed flexible analytic norms behaving well with respect to composition (Section 4). This requires care, because analytic norms are very sensitive to composition, contrary to, say, Sobolev norms.

- new functional inequalities of bilinear type, involving analytic functional spaces, integration in time and velocity variables, and evolution by free transport (Section 6). These inequalities morally mean the following: when a plasma acts (by forcing) on a smooth background of particles, the background reacts by lending a bit of its (gliding) regularity to the plasma, uniformly in time. This most subtle effect, which is at the heart of Landau’s damping, will be mathematically expressed in the formalism of analytic norms with gliding regularity.

- a new analysis of the time response associated to the Vlasov–Poisson equation (Section 7), aimed ultimately at controlling the self-induced echoes of the plasma.

- a Newton iteration scheme, solving the nonlinear evolution problem as a succession of linear ones (Section 10). Picard iteration schemes still play a role, since they are run at each step of the iteration process, to estimate the scattering operators.

It is only in the linear study of Section 3 that the length scale L will play a crucial role, *via* the stability condition **(L)**. In all the rest of the paper we shall normalize L to 1 for simplicity.

2.6. About phase mixing. A physical mechanism transferring energy from large scales to very fine scales, asymptotically in time, is sometimes called **weak turbulence**. Phase mixing provides such a mechanism, and in a way our study shows that the Vlasov–Poisson equation is subject to weak turbulence. But the phase mixing interpretation provides a more precise picture. While one often sees weak turbulence as a “cascade” from low to high Fourier modes, the relevant picture would rather be a two-dimensional figure with an interplay between spatial Fourier modes and velocity Fourier modes. More precisely, phase mixing transfers the energy from each nonzero spatial frequency k , to large velocity frequencies η , and this transfer occurs at a speed proportional to k . This picture is clear from the solution of free transport in Fourier space, and is illustrated in Fig. 3. (Note the resemblance with a shear flow.) So there is transfer of energy from one variable (here x) to another (here v); homogenization in the first variable going together with filamentation in the second one. The same mechanism may also underlie other cases of weak turbulence.

Whether ultimately the high modes are damped by some “random” microscopic process (collisions, diffusion, ...) not described by the Vlasov–Poisson equation is certainly undisputed in plasma physics [48, Section 41]¹⁴, and is the object of debate in galactic dynamics; anyway this is a different story. Some mathematical statistical theories of Euler and Vlasov–Poisson equations do postulate the existence of some small-scale coarse graining mechanism, but resulting in mixing rather than dissipation [70, 79].

2.7. Why analyticity? For Landau damping there are two basic reasons why the analyticity is interesting:

- this is the historical assumption used by Landau and made ever since;
- it is mandatory to get the exponential convergence.

While these justifications may seem rather superficial, we shall suggest a much more fundamental reason for analyticity: in the case of a Coulomb/gravitational interaction, *it is the critical regularity for which Landau damping is stronger than the “echo” instability*. The point is that the nonlinearity of the Vlasov–Poisson equation may result in a slow growth of all regularity estimates — even measured in gliding

¹⁴See [48, Problem 41]: thanks to Landau damping, collisions are expected to smooth the distribution quite efficiently; this is clearly a hypoelliptic problematic.

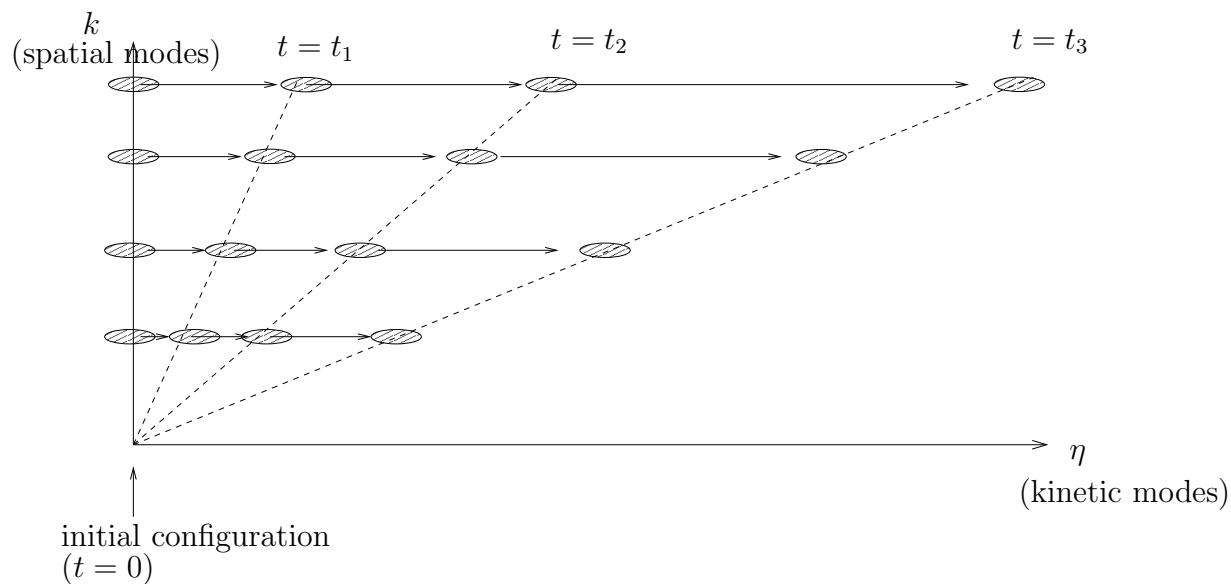


FIGURE 3. Schematic picture of the evolution of energy by free transport, or perturbation thereof; marks indicate localization of energy in phase space.

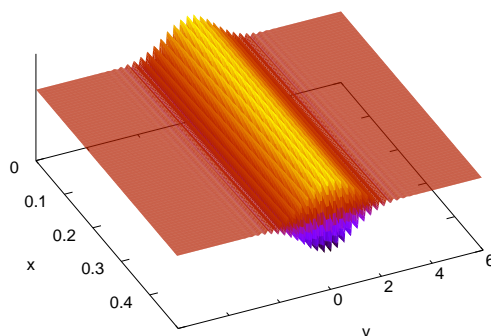


FIGURE 4. The distribution function in phase space (position, velocity) at a given time; notice how the fast oscillations in v contrast with the small variations in x .

sense. Due to the oscillatory nature of the problem, the self-interaction manifests itself not immediately, but only after some time — this is the echo phenomenon — which depends on the ratio of the frequencies involved. For a smooth (say analytic) interaction, the echoes are well separated, and the expected resulting growth is slow. But when the interaction is more singular, the echoes become closer to each other, and estimates are worse; eventually for Coulomb interaction we end up with an expected *exponential growth*, that has to be compensated by an exponential damping. This competition mechanism between nonlinear exponential “echoing” instability and linear exponential damping stability does not appear to have been noticed in previous works; in fact the regularity of the interaction potential does not play any role in linear damping. These suppositions will be explained in more detail in Section 7; it will be extremely interesting to back them up with counterexamples. For the moment we claim that unless some new stability effect is identified, there is no reason to believe in nonlinear Landau damping for, say, gravitational interaction, in any regularity class lower than analytic.

Because of the same phenomena, one may doubt the possibility of Landau damping — with any decay rate — in the C^∞ class, *even for analytic interactions*. It is all the more striking that linear damping does hold in C^∞ (with a decay rate $O(t^{-\infty})$), or even in C^r for small r .

Between C^∞ and analyticity lies the Gevrey regularity. There is no conceptual problem in adapting our methods to Gevrey regularity, provided that the potential is less singular than Coulomb (again, there will be a critical Gevrey regularity corresponding to the Sobolev regularity of the interaction). Such a study would show that Landau damping (a) is not tied to analyticity, or even quasi-analyticity, and in particular covers perturbations that are local in velocity space; (b) applies in situations where there is no analogue of spectral gap or exponential convergence: for Gevrey- α regularity ($0 < \alpha < 1$) the convergence cannot be faster than $O(e^{-\lambda t^\alpha})$.

2.8. Coulomb/Newton interaction. When $\gamma = 1$ we do not know whether the conclusions of Theorem 2.5 apply. This is precisely the most interesting case, both from the physical point of view (because it corresponds to Coulomb or Newton interaction, by far the most commonly used) and from the mathematical point of view (because it is the critical interaction).

However, the constructive nature of the proof of Theorem 2.5 makes it possible to adapt it and get a conclusion which, although weaker, is still mathematically and physically relevant: *Landau damping holds on extremely long time scales*, (almost) **exponentially large** in the size of the perturbation. This “quasi-global” proof of Landau damping is more than sufficient to answer, at least qualitatively, the

objections raised by Backus [5] on the validity of the linearization. Below is a precise statement; we use the notation (2.10).

Theorem 2.6 (Nonlinear Landau damping for Coulomb/Newton interaction). *Let $f^0 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be an analytic velocity profile. Let $L > 0$ and $W : \mathbb{T}_L^d \rightarrow \mathbb{R}$ be an interaction potential satisfying*

$$(2.19) \quad \forall k \in \mathbb{Z}^d, \quad |\widehat{W}^{(L)}(k)| \leq \frac{C_W}{|k|^2}$$

for some constant $C_W > 0$. Assume that f^0 and W satisfy the stability condition **(L)** from Subsection 2.2, with some $\lambda, \kappa > 0$; further assume that, for the same parameter λ ,

$$\sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|\nabla_v^n f^0\|_{L^1(\mathbb{R}^d)} \leq C_0 < +\infty.$$

Then for any $0 < \lambda' < \lambda$, $\beta > 0$, $0 < \mu' < \mu$ there is $\varepsilon = \varepsilon(d, L, C_W, C_0, \kappa, \lambda, \lambda', \mu, \mu', \beta)$ with the following property: if f_i is an initial datum satisfying

$$\delta := \|f_i - f^0\|_{\lambda, \mu} + \iint_{\mathbb{T}_L^d \times \mathbb{R}^d} |f_i - f^0| e^{\beta|v|} dv dx \leq \varepsilon,$$

and f is the unique classical solution to the nonlinear Vlasov equation with interaction potential W , then $\rho(t, x) = \int f(t, x, v) dv$ satisfies, for any $r \in \mathbb{N}$,

$$(2.20) \quad |t| \leq T \exp\left(\frac{K}{\delta \left(\log \frac{1}{\delta}\right)^2}\right) \implies \|\rho(t, \cdot) - \rho_\infty\|_{C^r} \leq C \delta e^{-\lambda'|t|},$$

where $\rho_\infty = L^{-d} \iint f_i dx dv$ and $T, K, C > 0$ may depend on $d, L, C_W, C_0, \kappa, \lambda, \lambda', \mu, \mu', \beta, r$.

The proof of this theorem will be sketched in Section 12; it relies directly on the fact that the Newton scheme enjoys a doubly exponential convergence rate.

Remark 2.7. It is not so common that one uses the full strength of the doubly exponential convergence in the Newton scheme: most of the time (even for the proof of the KAM theorem) it suffices to know that the convergence is faster than any exponential. The present example is all the more striking that it is not academic at all, and that it is associated with Newton's name in two completely different ways!

2.9. Further extensions. An extension which looks feasible consists of incorporating more general interactions, depending not only on the positions but also on the velocities of particles. Such cases occur notably when magnetic fields are involved, or when homogeneous equilibria are replaced by so-called BGK equilibria [9]. As we shall see, there is in principle no obstacle to adapt our methods to such situations. However, at least for BGK states, we are limited by the poor linear theory: it seems that no single BGK state is known to be linearly stable (say under perturbations of the same spatial period as the state). There is a huge literature on BGK equilibria, in large part speculative, and any progress on this issue would be valuable.

Finally, the ideas introduced in this work can be applied to two-dimensional inviscid incompressible fluids. It is well-known that the vorticity formulation of the two-dimensional Euler equations in a shear flow or in a vortex flow shares common features with the Vlasov–Poisson equations, and that a phenomenon called “inviscid damping”, similar to Landau damping could occur. In a work in progress [61] we shall develop a mathematical theory of the inviscid damping in these regimes.

3. LINEAR DAMPING

In this section we establish Landau damping for the linearized Vlasov equation. Beforehand, let us recall that the free transport equation

$$(3.1) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0$$

has a strong *mixing* property: any solution of (3.1) converges weakly in large time to a spatially homogeneous distribution equal to the space-averaging of the initial datum. Let us sketch the proof.

If f solves (3.1) in $\mathbb{T}^d \times \mathbb{R}^d$, with initial datum $f_i = f(0, \cdot)$, then $f(t, x, v) = f_i(x - vt, v)$, so the space-velocity Fourier transform of f is given by the formula

$$(3.2) \quad \tilde{f}(t, k, \eta) = \tilde{f}_i(k, \eta + kt).$$

On the other hand, if f_∞ is defined by

$$f_\infty(v) = \langle f_i(\cdot, v) \rangle = \int_{\mathbb{T}^d} f_i(x, v) dx,$$

then $\tilde{f}_\infty(k, \eta) = \tilde{f}_i(0, \eta) 1_{k=0}$. So, by Riemann’s lemma, for any fixed k, η we have

$$\left| \tilde{f}(t, k, \eta) - \tilde{f}_\infty(k, \eta) \right| \xrightarrow{|t| \rightarrow \infty} 0,$$

which shows that f converges weakly to f_∞ . The convergence holds as soon as f is merely integrable; and by (3.2), the rate of convergence is determined by the decay of $\tilde{f}_i(k, \eta)$ as $|\eta| \rightarrow \infty$, or equivalently the smoothness in the velocity variable. In particular, the convergence is exponentially fast if (and only if) $f_i(x, v)$ is analytic in v .

This argument obviously works independently of the size of the box. But when we turn to the Vlasov equation, length scales will matter, so we shall introduce a length $L > 0$, and work in $\mathbb{T}_L^d = \mathbb{R}^d / (L\mathbb{Z}^d)$. Then the length scale will appear in the Fourier transform: see Appendix A.3. (This is the only section in this paper where the scale will play a nontrivial role, so in all the rest of the paper we shall take $L = 1$.)

Any velocity distribution $f^0 = f^0(v)$ defines a stationary state for the nonlinear Vlasov equation with interaction potential W . Then the linearization of that equation around f^0 yields

$$(3.3) \quad \begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f - (\nabla W * \rho) \cdot \nabla_v f^0 = 0 \\ \rho = \int f \, dv. \end{cases}$$

Note that there is no force term in (3.3), due to the fact that f^0 does not depend on x . This equation describes what happens to a plasma density f which tries to force a stationary homogeneous background f^0 ; equivalently, it describes the *reaction* exerted by the background which is acted upon.

Theorem 3.1 (Linear Landau damping). *Let $f^0 = f^0(v)$, $L > 0$, $W : \mathbb{T}_L^d \rightarrow \mathbb{R}$ such that $\|\nabla W\|_{L^1} \leq C_W < +\infty$, and $f_i(x, v)$ such that*

(i) *Condition **(L)** from Subsection 2.2 holds for some constants $\lambda, \kappa > 0$;*

(ii) $\forall \eta \in \mathbb{R}^d, \quad |\tilde{f}^0(\eta)| \leq C_0 e^{-2\pi\lambda|\eta|},$

(iii) $\forall k \in \mathbb{Z}^d, \forall \eta \in \mathbb{R}^d, \quad |\tilde{f}_i^{(L)}(k, \eta)| \leq C_i e^{-2\pi\alpha|\eta|}$ for some constant $\alpha > 0$.

Then as $t \rightarrow +\infty$ the solution $f(t, \cdot)$ to the linearized Vlasov equation (3.3) with initial datum f_i converges weakly to $f_\infty = \langle f_i \rangle$ defined by

$$f_\infty(v) = \frac{1}{L^d} \int_{\mathbb{T}_L^d} f_i(x, v) \, dx;$$

and $\rho(x) = \int f(x, v) dv$ converges strongly to the constant

$$\rho_\infty = \frac{1}{L^d} \iint_{\mathbb{T}_L^d \times \mathbb{R}^d} f_i(x, v) dx dv.$$

More precisely, for any $\lambda' < \min\{\lambda; \alpha\}$,

$$\begin{cases} \forall r \in \mathbb{N}, & \|\rho(t, \cdot) - \rho_\infty\|_{C^r} = O(e^{-\frac{2\pi\lambda'}{L}|t|}) \\ \forall (k, \eta) \in \mathbb{Z}^d \times \mathbb{Z}^d, & |\tilde{f}^{(L)}(t, k, \eta) - \tilde{f}_\infty^{(L)}(k, \eta)| = O(e^{-\frac{2\pi\lambda'}{L}|kt|}). \end{cases}$$

Remark 3.2. Even if the initial datum is more regular than analytic, the convergence will in general not be better than exponential (except in some exceptional cases [35]). See [11, pp. 414–416] for an illustration.

Remark 3.3. The fact that the convergence is to the average of the initial datum will not survive nonlinear perturbation, as shown by the counterexamples in Subsection 13.

Remark 3.4. Dimension does not play any important role in the linear analysis. This can be attributed to the fact that only longitudinal waves occur, so everything happens “in the direction of the wave vector”. Transversal waves arise in plasma physics only when magnetic effects are taken into account [1, Chapter 5].

Remark 3.5. The proof can be adapted to the case when f^0 and f_i are only C^∞ ; then the convergence is not exponential, but still $O(t^{-\infty})$. The regularity can also be further decreased, down to $W^{s,1}$, at least for any $s > 2$; more precisely, if $f^0 \in W^{s_0,1}$ and $f_i \in W^{s_i,1}$ there will be damping with a rate $O(t^{-\kappa})$ for any $\kappa < \max\{s_0 - 2; s_i\}$. (Compare with [1, Vol. 1, p. 189].) This is independent of the regularity of the interaction.

The proof of Theorem 3.1 relies on the following elementary estimate for Volterra equations. We use the notation of Subsection 2.2.

Lemma 3.6. *Assume that **(L)** holds true for some constants $\kappa, \lambda > 0$, and let $C_0 = \sup_\eta |\tilde{f}^0(\eta)| e^{2\pi\lambda|\eta|}$, $C_W = \|W\|_{L^1(\mathbb{T}^d)}$. Then any solution $\varphi(t, k)$ of*

$$(3.4) \quad \varphi(t, k) = a(t, k) + \int_0^t K^0(t - \tau, k) \varphi(\tau, k) d\tau$$

(where K^0 is defined in (2.3)) satisfies, for any $k \in \mathbb{Z}^d$ and any $\lambda' < \lambda$,

$$\sup_{t \geq 0} \left(|\varphi(t, k)| e^{2\pi\lambda' \frac{|k|}{L} t} \right) \leq \left[1 + C_0 C_W C(\lambda, \lambda', \kappa) \right] \sup_{t \geq 0} \left(|a(t, k)| e^{2\pi\lambda \frac{|k|}{L} t} \right).$$

Remark 3.7. It has been known since the early days of the theory that the linearized Vlasov equation can be reduced to an infinite system of uncoupled Volterra equations. It is standard to solve them by Laplace transform; but, with a view to the nonlinear setting, we shall prefer a more flexible and quantitative approach.

Proof of Lemma 3.6. If $k = 0$ this is obvious since $K^0 = 0$; so we assume $k \neq 0$. Consider $\lambda' < \lambda$, and multiply (3.4) by $e^{2\pi\lambda' \frac{|k|}{L} t}$, and write

$$\Phi(t, k) = \varphi(t, k) e^{2\pi\lambda' \frac{|k|}{L} t}, \quad A(t, k) = a(t, k) e^{2\pi\lambda' \frac{|k|}{L} t};$$

so (3.4) becomes

$$(3.5) \quad \Phi(t, k) = A(t, k) + \int_0^t K^0(t - \tau, k) e^{2\pi\lambda' \frac{|k|}{L} (t - \tau)} \Phi(\tau, k) d\tau.$$

A particular case: The proof is extremely simple if we make the stronger assumption

$$\int_0^{+\infty} |K^0(\tau, k)| e^{2\pi\lambda' \frac{|k|}{L} \tau} d\tau \leq 1 - \kappa, \quad \kappa \in (0, 1) :$$

then from (3.5),

$$\begin{aligned} \sup_{0 \leq t \leq T} |\Phi(t, k)| &\leq \sup_{0 \leq t \leq T} |A(t, k)| \\ &\quad + \sup_{0 \leq t \leq T} \left(\int_0^t |K^0(t - \tau, k)| e^{2\pi\lambda' \frac{|k|}{L} (t - \tau)} d\tau \right) \sup_{0 \leq \tau \leq T} |\Phi(\tau, k)|, \end{aligned}$$

whence

$$\sup_{0 \leq \tau \leq t} |\Phi(\tau, k)| \leq \frac{\sup_{0 \leq \tau \leq t} |A(\tau, k)|}{1 - \int_0^{+\infty} |K^0(\tau, k)| e^{2\pi\lambda' \frac{|k|}{L} \tau} d\tau} \leq \frac{\sup_{0 \leq \tau \leq t} |A(\tau, k)|}{\kappa},$$

and therefore

$$\sup_{t \geq 0} \left(e^{2\pi\lambda' \frac{|k|}{L} t} |\varphi(t, k)| \right) \leq \left(\frac{1}{\kappa} \right) \sup_{t \geq 0} \left(|a(t, k)| e^{2\pi\lambda \frac{|k|}{L} t} \right).$$

The general case: To treat the general case we take the Fourier transform in the time variable, after extending K , A and Φ by 0 at negative times. (This presentation was suggested to us by Sigal, and appears to be technically simpler than the use of the Laplace transform.) Denoting the Fourier transform with a hat and recalling (2.4), we have, for $\xi = \lambda' + i\omega L/|k|$,

$$\widehat{\Phi}(\omega, k) = \widehat{A}(\omega, k) + \mathcal{L}(\xi, k) \widehat{\Phi}(\omega, k).$$

By assumption $\mathcal{L}(\xi, k) \neq 1$, so

$$\widehat{\Phi}(\omega, k) = \frac{\widehat{A}(\omega, k)}{1 - \mathcal{L}(\xi, k)}.$$

From there, it is traditional to apply the Fourier (or Laplace) inversion transform. Instead, we apply Plancherel's identity to find (for each k)

$$\|\Phi\|_{L^2(dt)} \leq \frac{\|A\|_{L^2(dt)}}{\kappa};$$

and then we plug this in the equation (3.5) to get

$$(3.6) \quad \begin{aligned} \|\Phi\|_{L^\infty(dt)} &\leq \|A\|_{L^\infty(dt)} + \|K^0 e^{2\pi\lambda' \frac{|k|}{L}t}\|_{L^2(dt)} \|\Phi\|_{L^2(dt)} \\ &\leq \|A\|_{L^\infty(dt)} + \frac{\|K^0 e^{2\pi\lambda' \frac{|k|}{L}t}\|_{L^2(dt)} \|A\|_{L^2(dt)}}{\kappa}. \end{aligned}$$

It remains to bound the second term. On the one hand,

$$(3.7) \quad \begin{aligned} \|A\|_{L^2(dt)} &= \left(\int_0^\infty |a(t, k)|^2 e^{4\pi\lambda' \frac{|k|}{L}t} dt \right)^{1/2} \\ &\leq \left(\int_0^\infty e^{-4\pi(\lambda - \lambda') \frac{|k|}{L}t} dt \right)^{1/2} \sup_{t \geq 0} \left(|a(t, k)| e^{2\pi\lambda \frac{|k|}{L}t} \right) \\ &\leq \left(\frac{L}{4\pi|k|(\lambda - \lambda')} \right)^{\frac{1}{2}} \sup_{t \geq 0} \left(|a(t, k)| e^{2\pi\lambda \frac{|k|}{L}t} \right). \end{aligned}$$

On the other hand,

$$(3.8) \quad \begin{aligned} \|K^0 e^{2\pi\lambda' \frac{|k|}{L}t}\|_{L^2(dt)} &= 4\pi^2 |\widehat{W}^{(L)}(k)| \frac{|k|^2}{L^2} \left(\int_0^\infty e^{4\pi\lambda' \frac{|k|}{L}t} \left| \widetilde{f}^0 \left(\frac{kt}{L} \right) \right|^2 t^2 dt \right)^{1/2} \\ &= 4\pi^2 |\widehat{W}^{(L)}(k)| \frac{|k|^{1/2}}{L^{1/2}} \left(\int_0^\infty e^{4\pi\lambda' u} |\widetilde{f}^0(\sigma u)|^2 u^2 du \right)^{1/2}, \end{aligned}$$

where $\sigma = k/|k|$. The estimate follows easily. (Note that the factor $|k|^{-1/2}$ in (3.7) cancels with $|k|^{1/2}$ in (3.8).)

It seems that we only used properties of the function \mathcal{L} in a strip $\Re\xi \simeq \lambda$; but this is an illusion. Indeed, we have taken the Fourier transform of Φ without checking that it belongs to $(L^1 + L^2)(dt)$, so what we have established is only an *a priori* estimate. To convert it into a rigorous result, one can use a continuity argument after replacing λ' by a parameter α which varies from $-\epsilon$ to λ' . (By the integrability of K^0 and Gronwall's lemma, φ is obviously bounded as a function of t ; so $\varphi(k, t) e^{-\epsilon|k|t/L}$ is integrable for any $\epsilon > 0$, and continuous as $\epsilon \rightarrow 0$.) Then assumption **(L)** guarantees that our bounds are uniform in the strip $0 \leq \Re\xi \leq \lambda'$, and the proof goes through. \square

Proof of Theorem 3.1. Without loss of generality we consider only $t \geq 0$. Considering (3.3) as a perturbation of free transport, we apply Duhamel's formula to get

$$(3.9) \quad f(t, x, v) = f_i(x - vt, v) + \int_0^t [(\nabla W * \rho) \cdot \nabla_v f^0](\tau, x - v(t - \tau), v) d\tau.$$

Integration in v yields

$$(3.10) \quad \rho(t, x) = \int_{\mathbb{R}^d} f_i(x - vt, v) dv + \int_0^t \int_{\mathbb{R}^d} [(\nabla W * \rho) \cdot \nabla_v f^0](\tau, x - v(t - \tau), v) dv d\tau.$$

Of course, $\int \rho(t, x) dx = \iint f_i(x, v) dx dv$.

For $k \neq 0$, taking the Fourier transform of (3.10), we obtain

$$\begin{aligned}
\widehat{\rho}^{(L)}(t, k) &= \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} f_i(x - vt, v) e^{-2i\pi \frac{k}{L} \cdot x} dv dx \\
&\quad + \int_0^t \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} [(\nabla W * \rho) \cdot \nabla_v f^0](\tau, x - v(t - \tau), v) e^{-2i\pi \frac{k}{L} \cdot x} dv dx d\tau \\
&= \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} f_i(x, v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \frac{k}{L} \cdot vt} dv dx \\
&\quad + \int_0^t \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} [(\nabla W * \rho) \cdot \nabla_v f^0](\tau, x, v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \frac{k}{L} \cdot v(t - \tau)} dv dx d\tau \\
&= \widetilde{f}_i^{(L)}\left(k, \frac{kt}{L}\right) + \int_0^t (\nabla W * \rho)^{\widetilde{(L)}}(\tau, k) \cdot \widetilde{\nabla_v f^0}\left(\frac{k(t - \tau)}{L}\right) d\tau \\
&= \widetilde{f}_i^{(L)}\left(k, \frac{kt}{L}\right) + \int_0^t \left(2i\pi \frac{k}{L} \widehat{W}^{(L)}(k) \widehat{\rho}^{(L)}(\tau, k)\right) \cdot \left(2i\pi \frac{k(t - \tau)}{L} \widetilde{f}^0\left(\frac{k(t - \tau)}{L}\right)\right) d\tau.
\end{aligned}$$

In conclusion, we have established the closed equation on $\widehat{\rho}^{(L)}$:

$$\begin{aligned}
(3.11) \quad \widehat{\rho}^{(L)}(t, k) &= \widetilde{f}_i^{(L)}\left(k, \frac{kt}{L}\right) \\
&\quad - 4\pi^2 \widehat{W}^{(L)}(k) \int_0^t \widehat{\rho}^{(L)}(\tau, k) \widetilde{f}^0\left(\frac{k(t - \tau)}{L}\right) \frac{|k|^2}{L^2} (t - \tau) d\tau.
\end{aligned}$$

Recalling (2.3), this is the same as

$$\widehat{\rho}^{(L)}(t, k) = \widetilde{f}_i^{(L)}\left(k, \frac{kt}{L}\right) + \int_0^t K^0(t - \tau, k) \widehat{\rho}^{(L)}(\tau, k) d\tau.$$

Without loss of generality, $\lambda \leq \alpha$. By Assumption **(L)** and Lemma 3.6,

$$|\widehat{\rho}^{(L)}(t, k)| \leq C_0 C_W C(\lambda, \lambda', \kappa) C_i e^{-2\pi \frac{\lambda' |k|}{L} t}.$$

In particular, for $k \neq 0$ we have

$$\forall t \geq 1, \quad |\widehat{\rho}^{(L)}(t, k)| = O\left(e^{-\frac{2\pi\lambda''}{L} t} e^{-\frac{2\pi(\lambda' - \lambda'')}{L} |k| t}\right);$$

so any Sobolev norm of $\rho - \rho_\infty$ converges to zero like $O(e^{-\frac{2\pi\lambda''}{L} t})$, where λ'' is arbitrarily close to λ' and therefore also to λ . By Sobolev embedding, the same is true for any C^r norm.

Next, we go back to (3.9) and take the Fourier transform in both variables x and v , to find

$$\begin{aligned}
\tilde{f}^{(L)}(t, k, \eta) &= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_i(x - vt, v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \eta \cdot v} dx dv \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla W * \rho)(\tau, x - v(t - \tau)) \cdot \nabla_v f^0(v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \eta \cdot v} dx dv d\tau \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_i(x, v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \frac{k}{L} \cdot vt} e^{-2i\pi \eta \cdot v} dx dv \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (\nabla W * \rho)(\tau, x) \cdot \nabla_v f^0(v) e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi \frac{k}{L} \cdot v(t - \tau)} e^{-2i\pi \eta \cdot v} dx dv d\tau \\
&= \tilde{f}_i^{(L)} \left(k, \eta + \frac{kt}{L} \right) + \int_0^t \widehat{\nabla W}^{(L)}(k) \widehat{\rho}^{(L)}(\tau, k) \cdot \widetilde{\nabla_v f^0} \left(\eta + \frac{k}{L}(t - \tau) \right) d\tau.
\end{aligned}$$

So

$$(3.12) \quad \tilde{f}^{(L)} \left(t, k, \eta - \frac{kt}{L} \right) = \tilde{f}_i^{(L)}(k, \eta) + \int_0^t \widehat{\nabla W}^{(L)}(k) \widehat{\rho}^{(L)}(\tau, k) \cdot \widetilde{\nabla_v f^0} \left(\eta - \frac{k\tau}{L} \right) d\tau.$$

In particular, for any $\eta \in \mathbb{R}^d$,

$$(3.13) \quad \tilde{f}^{(L)}(t, 0, \eta) = \tilde{f}_i^{(L)}(0, \eta);$$

in other words, $\langle f \rangle = \int f dx$ remains equal to $\langle f_i \rangle$ for all times.

On the other hand, if $k \neq 0$,

$$\begin{aligned}
(3.14) \quad \left| \tilde{f}^{(L)} \left(t, k, \eta - \frac{kt}{L} \right) \right| &\leq \left| \tilde{f}_i^{(L)}(k, \eta) \right| \\
&\quad + \int_0^t \left| \widehat{\nabla W}^{(L)}(k) \right| \left| \widehat{\rho}^{(L)}(\tau, k) \right| \left| \widetilde{\nabla_v f^0} \left(\eta - \frac{k\tau}{L} \right) \right| d\tau \\
&\leq C_i e^{-2\pi\alpha|\eta|} \\
&\quad + \int_0^t C_W C(\lambda, \lambda', \kappa) C_i e^{-2\pi\lambda' \frac{|k|}{L} \tau} \left(2\pi C_0 \left| \eta - \frac{k\tau}{L} \right| e^{-2\pi\lambda \left| \eta - \frac{k\tau}{L} \right|} \right) d\tau \\
&\leq C \left(e^{-2\pi\alpha|\eta|} + \int_0^t e^{-2\pi\lambda' \frac{|k|}{L} \tau} e^{-2\pi \frac{(\lambda' + \lambda)}{2} \left| \eta - \frac{k\tau}{L} \right|} d\tau \right),
\end{aligned}$$

where we have used $\lambda' < (\lambda' + \lambda)/2 < \lambda$, and C only depends on $C_W, C_i, \lambda, \lambda', \kappa$.
In the end,

$$\begin{aligned} \int_0^t e^{-2\pi\lambda' \frac{|k|}{L}\tau} e^{-2\pi \frac{(\lambda'+\lambda)}{2} |\eta - \frac{k\tau}{L}|} d\tau &\leq \int_0^t e^{-2\pi\lambda'|\eta|} e^{-2\pi \frac{(\lambda-\lambda')}{2} \frac{|k|}{L}\tau} d\tau \\ &\leq \frac{L}{\pi(\lambda-\lambda')} e^{-2\pi\lambda'|\eta|}. \end{aligned}$$

Plugging this back in (3.14), we obtain

$$(3.15) \quad \left| \tilde{f}^{(L)} \left(t, k, \eta - \frac{kt}{L} \right) \right| \leq C e^{-2\pi\lambda'|\eta|}.$$

In particular, for any fixed η and $k \neq 0$,

$$|\tilde{f}^{(L)}(t, k, \eta)| \leq C e^{-2\pi\lambda'|\eta + \frac{kt}{L}|} = O(e^{-2\pi \frac{\lambda'}{L}|t|}).$$

We conclude that $\tilde{f}^{(L)}$ converges pointwise, exponentially fast, to the Fourier transform of $\langle f_i \rangle$. \square

We close this section by proving Proposition 2.1.

Proof of Proposition 2.1. First assume (a). Since \tilde{f}^0 decreases exponentially fast, we can find $\lambda, \kappa > 0$ such that

$$4\pi^2 \max_{|\sigma|=1} |\widehat{W}^{(L)}(k)| \sup_{|\sigma|=1} \int_0^\infty |\tilde{f}^0(r\sigma)| r e^{2\pi\lambda r} dr \leq 1 - \kappa.$$

Performing the change of variables $kt/L = r\sigma$ inside the integral, we deduce

$$\int_0^\infty 4\pi^2 |\widehat{W}^{(L)}(k)| \left| \tilde{f}^0 \left(\frac{kt}{L} \right) \right| \frac{|k|^2 t}{L^2} e^{2\pi\lambda \frac{|k|}{L}t} dt \leq 1 - \kappa,$$

and this obviously implies **(L)**.

In case (b) the reasoning is more subtle. First we note that

$$\begin{aligned}
K^0(t, k) &= -4\pi^2 \widehat{W}(k) \int_{\mathbb{R}^d} f^0(v) e^{-2i\pi \frac{kt}{L} \cdot v} \frac{|k|^2}{L^2} t dv \\
&= -4\pi^2 \widehat{W}(k) \int_{\mathbb{R}} \varphi_k(v) e^{-2i\pi \frac{|k|}{L} tv} \frac{|k|^2}{L^2} t dv \\
&= -4\pi^2 \frac{|k|^2 \widehat{W}(k) t}{L^2} \int_{\mathbb{R}} \left(\frac{2i\pi |k| t}{L} \right)^{-1} \varphi'_k(v) e^{-2i\pi \frac{|k|}{L} tv} dv \\
&= 2i\pi \frac{|k| \widehat{W}(k)}{L} \int_{\mathbb{R}} \varphi'_k(v) e^{-2i\pi \frac{|k|}{L} tv} dv.
\end{aligned}$$

Then, for $\xi = \gamma + i\omega$, using the formula

$$\int_0^\infty e^{-st} e^{i\omega t} dt = \frac{s + i\omega}{s^2 + \omega^2},$$

we get from (2.4)

$$\mathcal{L}(\xi, k) = \widehat{W}(k) \int_{\mathbb{R}} \varphi'_k(v) \left[\frac{(v + \omega) - i\gamma}{(v + \omega)^2 + \gamma^2} \right] dv.$$

(To be rigorous, one may first establish this formula for $\gamma < 0$, and then use analyticity to derive it for $\gamma \in [0, \lambda)$.)

As $\gamma \rightarrow 0$, this expression approaches, uniformly in k and ω ,

$$\widehat{W}(k) \int \frac{\varphi'_k}{v + \omega + i0} dv = \widehat{W}(k) \text{ p.v. } \left(\int_{\mathbb{R}} \frac{\varphi'_k(v)}{v + \omega} dv \right) - i\pi \widehat{W}(k) \varphi'_k(\omega)$$

(Plemelj formula for the Cauchy transform). So the imaginary part vanishes only in the limit $\widehat{W}(k) \rightarrow 0$ (but then also the real part approaches 0), or if $|\omega| \rightarrow \infty$ (but then also the real part approaches 0), or $\omega = 0$; but then in the limit

$$\mathcal{L}(0, k) = \widehat{W}(k) \int_{\mathbb{R}} \frac{\varphi'_k}{v} dv \leq 0,$$

so even in this case \mathcal{L} cannot approach 1. Case (b) of Proposition 2.1 follows. \square

4. ANALYTIC NORMS

In this section we introduce some functional spaces of analytic functions on \mathbb{R}^d , $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, and most importantly $\mathbb{T}^d \times \mathbb{R}^d$. (Changing the sidelength of the torus will only result in some changes in the constants.) Then we establish a number of

functional inequalities which will be crucial in the subsequent analysis. At the end of this section we shall reformulate the linear study in this new setting.

Throughout the whole section d is a positive integer. Working with analytic functions will force us to be careful with combinatorial issues, and proofs will at times involve summation over many indices.

4.1. Single-variable analytic norms. Here “single-variable” means that the variable lives in either \mathbb{R}^d or \mathbb{T}^d , but d may be greater than 1. Among many possible families of norms for analytic functions, two will be of particular interest for us; they will be denoted by $\mathcal{C}^{\lambda;p}$ and $\mathcal{F}^{\lambda;p}$. The $\mathcal{C}^{\lambda;p}$ norms are defined for functions on \mathbb{R}^d or \mathbb{T}^d , while the $\mathcal{F}^{\lambda;p}$ norms are defined only for \mathbb{T}^d (although we could easily cook up a variant in \mathbb{R}^d). We shall write \mathbb{N}_0^d for the set of d -tuples of integers (the subscript being here to insist that 0 is allowed). If $n \in \mathbb{N}_0^d$ and $\lambda \geq 0$ we shall write $\lambda^n = \lambda^{|n|}$. Conventions about Fourier transform and multidimensional differential calculus are gathered in the Appendix.

Definition 4.1 (One-variable analytic norms). For any $p \in [1, \infty]$ and $\lambda \geq 0$, we define

$$(4.1) \quad \|f\|_{\mathcal{C}^{\lambda;p}} := \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|f^{(n)}\|_{L^p}; \quad \|f\|_{\mathcal{F}^{\lambda;p}} := \left(\sum_{k \in \mathbb{Z}^d} e^{2\pi\lambda p|k|} |\widehat{f}(k)|^p \right)^{1/p};$$

the latter expression standing for $\sup_k (e^{2\pi\lambda|k|} |\widehat{f}(k)|)$ if $p = \infty$. We further write

$$(4.2) \quad \mathcal{C}^{\lambda,\infty} = \mathcal{C}^\lambda, \quad \mathcal{F}^{\lambda,1} = \mathcal{F}^\lambda.$$

Remark 4.2. The parameter λ can be interpreted as a radius of convergence.

Remark 4.3. The norms \mathcal{C}^λ and \mathcal{F}^λ are of particular interest because they are algebra norms.

We shall sometimes abbreviate $\|\cdot\|_{\mathcal{C}^{\lambda;p}}$ or $\|\cdot\|_{\mathcal{F}^{\lambda;p}}$ into $\|\cdot\|_{\lambda;p}$ when no confusion is possible, or when the statement works for either.

The norms in (4.1) extend to vector-valued functions in a natural way: if f is valued in \mathbb{R}^d or \mathbb{T}^d or \mathbb{Z}^d , define $f^{(n)} = (f_1^{(n)}, \dots, f_d^{(n)})$, $\widehat{f}(k) = (\widehat{f}_1(k), \dots, \widehat{f}_d(k))$; then the formulas in (4.1) make sense provided that we choose a norm on \mathbb{R}^d or \mathbb{T}^d or \mathbb{Z}^d . Which norm we choose will depend on the context; the choice will always be done in such a way to get the duality right in the inequality $|a \cdot b| \leq \|a\| \|b\|_*$. For instance if f is valued in \mathbb{Z}^d and g in \mathbb{T}^d , and we have to estimate $f \cdot g$, we may norm

\mathbb{Z}^d by $|k| = \sum |k_i|$ and \mathbb{T}^d by $|x| = \sup |x_i|$.¹⁵ This will not pose any problem, and the reader can forget about this issue; we shall just make remarks about it whenever needed. For the rest of this section, we shall focus on scalar-valued functions for simplicity of exposition.

Next, we define “homogeneous” analytic seminorms by removing the zero-order term. We write $\mathbb{N}_*^d = \mathbb{N}_0^d \setminus \{0\}$, $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$.

Definition 4.4 (One-variable homogeneous analytic seminorms). For $p \in [1, \infty]$ and $\lambda \geq 0$ we write

$$\|f\|_{\dot{\mathcal{C}}^{\lambda;p}} = \sum_{n \in \mathbb{N}_*^d} \frac{\lambda^n}{n!} \|f^{(n)}\|_{L^p}; \quad \|f\|_{\dot{\mathcal{F}}^{\lambda;p}} = \left(\sum_{k \in \mathbb{Z}_*^d} e^{2\pi\lambda p|k|} |\widehat{f}(k)|^p \right)^{1/p}.$$

It is interesting to note that affine functions $x \mapsto a \cdot x + b$ can be included in $\dot{\mathcal{C}}^\lambda = \dot{\mathcal{C}}^{\lambda;\infty}$, even though they are unbounded; in particular $\|a \cdot x + b\|_{\dot{\mathcal{C}}^\lambda} = \lambda |a|$. On the other hand, linear forms $x \mapsto a \cdot x$ do not naturally belong to $\dot{\mathcal{F}}^\lambda$, because their Fourier expansion is not even summable (it decays like $1/k$).

The spaces $\mathcal{C}^{\lambda;p}$ and $\mathcal{F}^{\lambda;p}$ enjoy remarkable properties, summarized in Propositions 4.5, 4.8 and 4.10 below. Some of these properties are well-known, other not so.

Proposition 4.5 (Algebra property). (i) For any $\lambda \geq 0$, and $p, q, r \in [1, +\infty]$ such that $1/p + 1/q = 1/r$, we have

$$\|fg\|_{\mathcal{C}^{\lambda;r}} \leq \|f\|_{\mathcal{C}^{\lambda;p}} \|g\|_{\mathcal{C}^{\lambda;q}}.$$

(ii) For any $\lambda \geq 0$, and $p, q, r \in [1, +\infty]$ such that $1/p + 1/q = 1/r + 1$, we have

$$\|fg\|_{\mathcal{F}^{\lambda;r}} \leq \|f\|_{\mathcal{F}^{\lambda;p}} \|g\|_{\mathcal{F}^{\lambda;q}}.$$

(iii) As a consequence, for any $\lambda \geq 0$, $\mathcal{C}^\lambda = \mathcal{C}^{\lambda;\infty}$ and $\mathcal{F}^\lambda = \mathcal{F}^{\lambda;1}$ are normed algebras: for either space,

$$\|fg\|_\lambda \leq \|f\|_\lambda \|g\|_\lambda.$$

In particular, $\|f^n\|_\lambda \leq \|f\|_\lambda^n$ for any $n \in \mathbb{N}_0$, and $\|e^f\|_\lambda \leq e^{\|f\|_\lambda}$.

Remark 4.6. Ultimately, property (iii) relies on the fact that L^∞ and L^1 are normed algebras for the multiplication and convolution, respectively.

¹⁵Of course all norms are equivalent, still the choice is not innocent when the estimates are iterated infinitely many times; an advantage of the supremum norm on \mathbb{R}^d is that it has the algebra property.

Remark 4.7. It follows from the Fourier inversion formula and Proposition 4.5 that $\|f\|_{\mathcal{C}^\lambda} \leq \|f\|_{\mathcal{F}^\lambda}$ (and $\|f\|_{\dot{\mathcal{C}}^\lambda} \leq \|f\|_{\dot{\mathcal{F}}^\lambda}$); this is a special case of Proposition 4.8 (iv) below. The reverse inequality does not hold, because $\|f\|_\infty$ does not control $\|\widehat{f}\|_{L^1}$.

Analytic norms are very sensitive to composition; think that if $a > 0$ then $\|f \circ (a \text{Id})\|_{\mathcal{C}^{\lambda;p}} = a^{-d/p} \|f\|_{\mathcal{C}^{a\lambda;p}}$; so we typically lose on the functional space. This is a major difference with more traditional norms used in partial differential equations theory, such as Hölder or Sobolev norms, for which composition may affect constants but not regularity indices. The next proposition controls the loss of regularity implied by composition.

Proposition 4.8 (Composition inequality). *(i) For any $\lambda > 0$ and any $p \in [1, +\infty]$,*

$$\|f \circ H\|_{\mathcal{C}^{\lambda;p}} \leq \|(\det \nabla H)^{-1}\|_\infty^{1/p} \|f\|_{\mathcal{C}^{\nu;p}}, \quad \nu = \|H\|_{\dot{\mathcal{C}}^\lambda},$$

where H is possibly unbounded.

(ii) For any $\lambda > 0$, any $p \in [1, \infty]$ and any $a > 0$,

$$\left\| f \circ (a \text{Id} + G) \right\|_{\mathcal{C}^{\lambda;p}} \leq a^{-d/p} \|f\|_{\mathcal{C}^{a\lambda+\nu;p}}, \quad \nu = \|G\|_{\mathcal{C}^\lambda}.$$

(iii) For any $\lambda > 0$,

$$\left\| f \circ (\text{Id} + G) \right\|_{\mathcal{F}^\lambda} \leq \|f\|_{\mathcal{F}^{\lambda+\nu}}, \quad \nu = \|G\|_{\dot{\mathcal{F}}^\lambda}.$$

(iv) For any $\lambda > 0$ and any $a > 0$,

$$\left\| f \circ (a \text{Id} + G) \right\|_{\mathcal{C}^\lambda} \leq \|f\|_{\mathcal{F}^{a\lambda+\nu}}, \quad \nu = \|G\|_{\dot{\mathcal{C}}^\lambda}.$$

Remark 4.9. Inequality (iv), with \mathcal{C} on the left and \mathcal{F} on the right, will be most useful. The reverse inequality is not likely to hold, in view of Remark 4.7.

The last property of interest for us is the control of the loss of regularity involved by differentiation.

Proposition 4.10 (Control of gradients). *For any $\bar{\lambda} > \lambda$, any $p \in [1, +\infty]$, we have*

$$(4.3) \quad \|\nabla f\|_{\mathcal{C}^{\lambda;p}} \leq \left(\frac{1}{\lambda e \log(\bar{\lambda}/\lambda)} \right) \|f\|_{\dot{\mathcal{C}}^{\bar{\lambda};p}};$$

$$(4.4) \quad \|\nabla f\|_{\mathcal{F}^{\lambda;p}} \leq \left(\frac{1}{2\pi e (\bar{\lambda} - \lambda)} \right) \|f\|_{\dot{\mathcal{F}}^{\bar{\lambda};p}}.$$

The proofs of Propositions 4.5 to 4.10 will be preparations for the more complicated situations considered in the sequel.

Proof of Proposition 4.5. (i) Denoting by $\|\cdot\|_{\lambda;p}$ the norm of $\mathcal{C}^{\lambda;p}$, using the multi-dimensional Leibniz formula from Appendix A.2, we have

$$\begin{aligned}
\|fg\|_{\lambda;r} &= \sum_{\ell \in \mathbb{N}_0^d} \|(fg)^{(\ell)}\|_{L^r} \frac{\lambda^\ell}{\ell!} \leq \sum_{\ell \in \mathbb{N}_0^d} \sum_{m \leq \ell} \binom{\ell}{m} \|f^{(m)} g^{(\ell-m)}\|_{L^r} \frac{\lambda^\ell}{\ell!} \\
&\leq \sum_{\ell \in \mathbb{N}_0^d} \sum_{m \leq \ell} \binom{\ell}{m} \|f^{(m)}\|_{L^p} \|g^{(\ell-m)}\|_{L^q} \frac{\lambda^\ell}{\ell!} \\
&= \sum_{\ell} \sum_m \frac{\|f^{(m)}\|_{L^p} \lambda^m}{m!} \frac{\|g^{(\ell-m)}\|_{L^q} \lambda^{\ell-m}}{(\ell-m)!} \\
&= \|f\|_{\lambda;p} \|g\|_{\lambda;q}.
\end{aligned}$$

(ii) Denoting now by $\|\cdot\|_{\lambda;p}$ the norm of $\mathcal{F}^{\lambda;p}$, and applying Young's convolution inequality, we get

$$\begin{aligned}
\|fg\|_{\lambda;r} &= \left(\sum_k |\widehat{fg}(k)|^r e^{2\pi\lambda r|k|} \right)^{1/r} \leq \left(\sum_k \left(\sum_\ell |\widehat{f}(\ell)| |\widehat{g}(k-\ell)| e^{2\pi\lambda|k-\ell|} e^{2\pi\lambda|\ell|} \right)^r \right)^{1/r} \\
&\leq \left(\sum_k |\widehat{f}(k)|^p e^{2\pi\lambda p|k|} \right)^{\frac{1}{p}} \left(\sum_\ell |\widehat{g}(\ell)|^q e^{2\pi\lambda q|\ell|} \right)^{\frac{1}{q}}.
\end{aligned}$$

□

Proof of Proposition 4.8. Case (i). We use the (multi-dimensional) Faà di Bruno formula:

$$(f \circ H)^{(n)} = \sum_{\sum_{j=1}^n j m_j = n} \frac{n!}{m_1! \dots m_n!} (f^{(m_1+\dots+m_n)} \circ H) \prod_{j=1}^n \left(\frac{H^{(j)}}{j!} \right)^{m_j};$$

so

$$\|(f \circ H)^{(n)}\|_{L^p} \leq \sum_{\sum_{j=1}^n j m_j = n} \frac{n!}{m_1! \dots m_n!} \|f^{(m_1+\dots+m_n)} \circ H\|_{L^p} \prod_{j=1}^n \left\| \frac{H^{(j)}}{j!} \right\|_{\infty}^{m_j};$$

thus

$$\begin{aligned} \sum_{n \geq 1} \frac{\lambda^n}{n!} \|(f \circ H)^{(n)}\|_{L^p} &\leq \|(\det \nabla H)^{-1}\|_\infty^{1/p} \left(\sum_{k=1}^{+\infty} \|f^{(k)}\|_{L^p} \right. \\ &\quad \left. \sum_{\sum_{j=1}^n j m_j = n, \sum_{j=1}^n m_j = k} \frac{\lambda^n}{m_1! \dots m_n!} \prod_{j=1}^n \left\| \frac{H^{(j)}}{j!} \right\|_\infty^{m_j} \right) \\ &= \|(\det \nabla H)^{-1}\|_\infty^{1/p} \left(\sum_{k \geq 1} \|f^{(k)}\|_{L^p} \frac{1}{k!} \left(\sum_{|\ell| \geq 1} \frac{\lambda^\ell}{\ell!} \|H^{(\ell)}\|_\infty \right)^k \right), \end{aligned}$$

where the last step follows from the multidimensional binomial formula.

Case (ii). We decompose $h(x) := f(ax + G(x))$ as

$$h(x) = \sum_{n \in \mathbb{N}_0^d} \frac{(f^{(n)})(ax)}{n!} G(x)^n$$

and we apply ∇^k :

$$\nabla^k h(x) = \sum_{k_1+k_2=k, \in \mathbb{N}_0^d} \sum_{n \in \mathbb{N}_0^d} \frac{k! a^{k_1}}{k_1! k_2! n!} (\nabla^{k_1+n} f)(ax) (\nabla^{k_2} (G^n))(x).$$

Then we take the L^p norm, multiply by $\lambda^k/k!$ and sum over k :

$$\begin{aligned} \|h\|_{\mathcal{C}^{\lambda;p}} &\leq |a|^{-d/p} \sum_{k_1, k_2, n \geq 0} \frac{\lambda^{k_1+k_2} |a|^{k_1}}{k_1! k_2! n!} \|\nabla^{k_1+n} f\|_{L^p} \|\nabla^{k_2} (G^n)\|_\infty \\ &= |a|^{-d/p} \sum_{k_1, n \geq 0} \frac{\lambda^{k_1} |a|^{k_1}}{k_1! n!} \|\nabla^{k_1+n} f\|_{L^p} \|G^n\|_{\mathcal{C}^\lambda} \\ &\leq |a|^{-d/p} \sum_{k_1, n \geq 0} \frac{\lambda^{k_1} |a|^{k_1}}{k_1! n!} \|\nabla^{k_1+n} f\|_{L^p} \|G\|_{\mathcal{C}^\lambda}^n \\ &= |a|^{-d/p} \sum_{m \geq 0} \frac{(a \lambda + \|G\|_{\mathcal{C}^\lambda})^m}{m!} \|\nabla^m f\|_{L^p}, \end{aligned}$$

where Proposition 4.5 (iii) was used in the but-to-last step.

Case (iii). In this case we write, with $G_0 = \widehat{G}(0)$,

$$h(x) = f(x + G(x)) = \sum_k \widehat{f}(k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot G_0} e^{2i\pi k \cdot (G(x) - G_0)};$$

so

$$\widehat{h}(\ell) = \sum_k \widehat{f}(k) e^{2i\pi k \cdot G_0} [e^{2i\pi k \cdot (G - G_0)}]^\wedge(\ell - k).$$

Then (using again Proposition 4.5)

$$\begin{aligned} \sum_\ell |\widehat{h}(\ell)| e^{2\pi\lambda|\ell|} &\leq \sum_k \sum_\ell |\widehat{f}(k)| e^{2\pi\lambda|k|} e^{2\pi\lambda|\ell-k|} \left| [e^{2i\pi k \cdot (G - G_0)}]^\wedge(\ell - k) \right| \\ &= \sum_k |\widehat{f}(k)| e^{2\pi\lambda|k|} \left\| e^{2i\pi k \cdot (G - G_0)} \right\|_\lambda \\ &\leq \sum_k |\widehat{f}(k)| e^{2\pi\lambda|k|} e^{\|2\pi k \cdot (G - G_0)\|_\lambda} \\ &\leq \sum_k |\widehat{f}(k)| e^{2\pi\lambda|k|} e^{2\pi|k| \|G - G_0\|_\lambda} \\ &= \|f\|_{\lambda + \|G - G_0\|_\lambda} = \|f\|_{\lambda + \nu}, \quad \nu = \|G\|_{\dot{c}^\lambda}. \end{aligned}$$

Case (iv). We actually have the more precise result

$$(4.5) \quad \|f \circ H\|_{c^\lambda} \leq \sum |\widehat{f}(k)| e^{2\pi|k| \|H\|_{c^\lambda}}.$$

Writing $f \circ H = \sum \widehat{f}(k) e^{2i\pi k \cdot H}$, we see that (4.5) follows from

$$(4.6) \quad \|e^{ih}\|_{c^\lambda} \leq e^{\|h\|_{c^\lambda}}.$$

To prove (4.6), let P_n be the polynomial in the variables X_m ($m \leq n$) defined by the identity $(e^f)^{(n)} = P_n((f^{(m)})_{m \leq n}) e^f$; this polynomial (which can be made more explicit from the Faà di Bruno formula) has nonnegative coefficients, so $\|(e^{if})^{(n)}\|_\infty \leq P_n(\|(f^{(m)})_{m \leq n}\|)$. The conclusion will follow from the identity (between formal series!)

$$(4.7) \quad 1 + \sum_{n \in \mathbb{N}_*^d} \frac{\lambda^n}{n!} P_n((X_m)_{m \leq n}) = \exp \left(\sum_{k \in \mathbb{N}_*^d} \frac{\lambda^k}{k!} X_k \right).$$

To prove (4.7), it is sufficient to note that the left-hand side is the expansion of e^g in powers of λ at 0, where $g(\lambda) = \sum_{k \in \mathbb{N}_*^d} \frac{\lambda^k}{k!} X_k$. \square

Proof of Proposition 4.10. (a) Writing $\|\cdot\|_{\lambda;p} = \|\cdot\|_{\mathcal{C}^{\lambda;p}}$, we have

$$\|\partial_i f\|_{\lambda;p} = \sum_n \frac{\lambda^n}{n!} \|\partial_x^n \partial_i f\|_{L^p},$$

where $\partial_i = \partial/\partial x_i$. If 1_i is the d -uple of integers with 1 in position i , then $(n+1_i)! \leq (|n|+1)n!$, so

$$\|\partial_i f\|_{\lambda;p} \leq \sup_n \left(\frac{(|n|+1)\lambda^n}{\bar{\lambda}^{n+1}} \right) \sum_{|m|\geq 1} \frac{\bar{\lambda}^m}{m!} \|\nabla^m f\|_{L^p},$$

and the proof of (4.3) follows easily.

(b) Writing $\|\cdot\|_{\lambda;p} = \|\cdot\|_{\mathcal{F}^{\lambda;p}}$, we have

$$\begin{aligned} \|\partial_i f\|_{\lambda;p} &= \left(\sum_k |k_i|^p |\widehat{f}(k)|^p e^{2\pi\lambda p|k|} \right)^{1/p} \\ &\leq \left[\sup_{k \in \mathbb{Z}^d} \left(|k| e^{2\pi(\lambda-\bar{\lambda})|k|} \right) \right] \left(\sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^p e^{2\pi\bar{\lambda}p|k|} \right)^{1/p}, \end{aligned}$$

and (4.4) follows. \square

4.2. Analytic norms in two variables. To estimate solutions and trajectories of kinetic equations we will work on the phase space $\mathbb{T}_x^d \times \mathbb{R}_v^d$, and use three parameters: λ (*gliding* analytic regularity); μ (analytic regularity in x); and τ (time-shift along the free transport semigroup). The regularity quantified by λ is said to be *gliding* because for $\tau = 0$ this is an analytic regularity in v , but as τ grows the regularity is progressively transferred from velocity to spatial modes, according to the evolution by free transport. This catch is crucial to our analysis: indeed, the solution of a transport equation like free transport or Vlasov *cannot* be uniformly analytic¹⁶ in v as time goes by — except of course if it is spatially homogeneous. Instead, the best we can do is compare the solution at time τ to the solution of free transport at the same time — a kind of scattering point of view.

The parameters λ, μ will be nonnegative; τ will vary in \mathbb{R} , but often be restricted to \mathbb{R}_+ , just because we shall work in positive time. When τ is not specified, this means $\tau = 0$. Sometimes we shall abuse notation by writing $\|f(x, v)\|$ instead of $\|f\|$, to stress the dependence of f on the two variables.

¹⁶By this we mean of course that some *norm* or seminorm quantifying the degree of analytic smoothness in v will remain uniformly bounded.

Putting aside the time-shift for a moment, we may generalize the norms \mathcal{C}^λ and \mathcal{F}^λ in an obvious way:

Definition 4.11 (Two-variables analytic norms). For any $\lambda, \mu \geq 0$, we define

$$(4.8) \quad \|f\|_{\mathcal{C}^{\lambda,\mu}} = \sum_{m \in \mathbb{N}_0^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \frac{\mu^m}{m!} \left\| \nabla_x^m \nabla_v^n f \right\|_{L^\infty(\mathbb{T}_x^d \times \mathbb{R}_v^d)};$$

$$(4.9) \quad \|f\|_{\mathcal{F}^{\lambda,\mu}} = \sum_{k \in \mathbb{Z}^d} \int_{\eta \in \mathbb{R}^d} |\tilde{f}(k, \eta)| e^{2\pi\lambda|\eta|} e^{2\pi\mu|k|} d\eta.$$

Of course one might also introduce variants based on L^p or ℓ^p norms (with two additional parameters p, q , since one can make different choices for the space and velocity variables).

The norm (4.9) is better adapted to the periodic nature of the problem, and is very well suited to estimate solutions of kinetic equations (with fast decay as $|v| \rightarrow \infty$); but in the sequel we shall also have to estimate characteristics (trajectories) which are unbounded functions of v . We could hope to play with two different families of norms, but this would entail considerable technical difficulties. Instead, we shall mix the two recipes to get the following *hybrid norm*:

Definition 4.12 (Hybrid analytic norms). For any $\lambda, \mu \geq 0$, let

$$(4.10) \quad \|f\|_{\mathcal{Z}^{\lambda,\mu}} = \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| \widehat{\nabla_v^n f}(\ell, v) \right\|_{L^\infty(\mathbb{R}_v^d)}.$$

More generally, for any $p \in [1, \infty]$ we define

$$(4.11) \quad \|f\|_{\mathcal{Z}^{\lambda,\mu;p}} = \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| \widehat{\nabla_v^n f}(\ell, v) \right\|_{L^p(\mathbb{R}_v^d)}.$$

Now let us introduce the time-shift τ . We denote by $(S_\tau^0)_{\tau \geq 0}$ the geodesic semigroup: $(S_\tau^0)(x, v) = (x + v\tau, v)$. Recall that the backward free transport semigroup is defined by $(f \circ S_\tau^0)_{\tau \geq 0}$, and the forward semigroup by $(f \circ S_{-\tau}^0)_{\tau \geq 0}$.

Definition 4.13 (Time-shift pure and hybrid analytic norms).

$$(4.12) \quad \|f\|_{\mathcal{C}_\tau^{\lambda,\mu}} = \|f \circ S_\tau^0\|_{\mathcal{C}^{\lambda,\mu}} = \sum_{m \in \mathbb{N}_0^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \frac{\mu^m}{m!} \left\| \nabla_x^m (\nabla_v + \tau \nabla_x)^n f \right\|_{L^\infty(\mathbb{T}_x^d \times \mathbb{R}_v^d)};$$

$$(4.13) \quad \|f\|_{\mathcal{F}_\tau^{\lambda,\mu}} = \|f \circ S_\tau^0\|_{\mathcal{F}^{\lambda,\mu}} = \sum_{k \in \mathbb{Z}^d} \int_{\eta \in \mathbb{R}^d} |\tilde{f}(k, \eta)| e^{2\pi\lambda|k\tau+\eta|} e^{2\pi\mu|k|} d\eta;$$

$$(4.14) \quad \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} = \|f \circ S_\tau^0\|_{\mathcal{Z}^{\lambda,\mu}} = \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| (\nabla_v + 2i\pi\tau\ell)^n \widehat{f}(\ell, v) \right\|_{L^\infty(\mathbb{R}_v^d)};$$

$$(4.15) \quad \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} = \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| (\nabla_v + 2i\pi\tau\ell)^n \widehat{f}(\ell, v) \right\|_{L^p(\mathbb{R}_v^d)}.$$

This choice of norms is one of the cornerstones of our analysis: first, because of their hybrid nature, they will connect well to both periodic (in x) estimates on the force field, and uniform (in v) estimates on the “scattering transforms” studied in Section 5. Secondly, they are well-behaved with respect to the properties of free transport, allowing to keep track of the initial time without needing ridiculous (and inaccessible) amounts of regularity in x as time goes by. Thirdly, they will satisfy the algebra property (for $p = \infty$), the composition inequality and the gradient inequality (for any $p \in [1, \infty]$). Before going on with the proof of these properties, we note the following alternative representations.

Proposition 4.14. *The norm $\mathcal{Z}_\tau^{\lambda,\mu;p}$ admits the alternative representations:*

$$(4.16) \quad \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} = \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| \nabla_v^n (\widehat{f}(\ell, v) e^{2i\pi\tau\ell \cdot v}) \right\|_{L^p(\mathbb{R}_v^d)};$$

$$(4.17) \quad \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} = \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \left\| (\nabla_v + \tau \nabla_x)^n f \right\|_{\mu;p},$$

where

$$(4.18) \quad \|g\|_{\mu;p} = \sum_{\ell \in \mathbb{Z}^d} e^{2\pi\mu|\ell|} \left\| \widehat{g}(\ell, v) \right\|_{L^p(\mathbb{R}_v^d)}.$$

4.3. Relations between functional spaces. The next propositions are easily checked.

Proposition 4.15. *With the notation from Subsection 4.2, for any $\tau \in \mathbb{R}$,*

(i) *if f is a function only of x then*

$$\|f\|_{\mathcal{C}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{C}^{\lambda|\tau|+\mu}}, \quad \|f\|_{\mathcal{F}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{F}^{\lambda|\tau|+\mu}};$$

(ii) if f is a function only of v then

$$\|f\|_{\mathcal{C}_\tau^{\lambda,\mu;p}} = \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} = \|f\|_{\mathcal{C}^{\lambda;p}}, \quad \|f\|_{\mathcal{F}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{F}^\lambda};$$

(iii) for any function $f = f(x, v)$, if $\langle \cdot \rangle$ stands for spatial average then

$$\|\langle f \rangle\|_{\mathcal{C}^{\lambda;p}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}};$$

(iv) for any function $f = f(x, v)$,

$$\left\| \int_{\mathbb{R}^d} f \, dv \right\|_{\mathcal{F}^{\lambda|\tau|+\mu}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;1}}.$$

Remark 4.16. Note, in Proposition 4.15, how the regularity in x is improved by the time-shift.

Proof of Proposition 4.15. Only (iv) requires some explanations. Let $\rho(x) = \int f(x, v) \, dv$. Then for any $k \in \mathbb{Z}^d$,

$$\widehat{\rho}(k) = \int_{\mathbb{R}^d} \widehat{f}(k, v) \, dv;$$

so for any $n \in \mathbb{N}_0^d$,

$$\begin{aligned} (2i\pi tk)^n \widehat{\rho}(k) &= \int (2i\pi tk)^n \widehat{f}(k, v) \, dv \\ &= \int (\nabla_v + 2i\pi tk)^n \widehat{f}(k, v) \, dv. \end{aligned}$$

Recalling the conventions from Appendix A.1 we deduce

$$\begin{aligned} \sum_{k,n} e^{2\pi\mu|k|} \frac{|2\pi\lambda tk|^n}{n!} |\widehat{\rho}(k)| &\leq \sum_{k,n} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \int |(\nabla_v + 2i\pi tk)^n \widehat{f}(k, v)| \, dv \\ &= \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;1}}. \end{aligned}$$

□

Proposition 4.17. *With the notation from Subsection 4.2,*

$$\lambda \leq \lambda', \quad \mu \leq \mu' \implies \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda',\mu'}}.$$

Moreover, for $\tau, \bar{\tau} \in \mathbb{R}$, and any $p \in [1, \infty]$,

$$(4.19) \quad \|f\|_{\mathcal{Z}_{\bar{\tau}}^{\lambda,\mu;p}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu+\lambda|\bar{\tau}-\tau|;p}}.$$

Remark 4.18. Note carefully that the spaces $\mathcal{Z}_\tau^{\lambda,\mu}$ are **not** ordered with respect to the parameter τ , which cannot be thought of as a regularity index. We could dispense with this parameter if we were working in time $O(1)$; but (4.19) is of course of absolutely no use in the study of large time behavior. This means that errors on the exponent τ should remain somehow small, in order to be controllable by small losses on the exponent μ .

Finally we state an easy proposition which follows from the time-invariance of the free transport equation:

Proposition 4.19. *For any $X \in \{\mathcal{C}, \mathcal{F}, \mathcal{Z}\}$, and any $t, \tau \in \mathbb{R}$,*

$$\|f \circ S_t^0\|_{X_\tau^{\lambda,\mu}} = \|f\|_{X_{t+\tau}^{\lambda,\mu}}.$$

Now we shall see that the hybrid norms, and certain variants thereof, enjoy properties rather similar to those of the single-variable analytic norms studied before. This will not be very pleasant, and the reader who would like to connect to physical problems can go directly to Subsection 4.11.

4.4. Injections. In this section we relate $\mathcal{Z}_\tau^{\lambda,\mu;p}$ norms to more standard norms entirely based on Fourier space. In the next theorem we write

$$(4.20) \quad \|f\|_{\mathcal{Y}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{F}_\tau^{\lambda,\mu;\infty}} = \sup_{k \in \mathbb{Z}^d} \sup_{\eta \in \mathbb{R}^d} e^{2\pi\mu|k|} e^{2\pi\lambda|\eta+k\tau|} |\tilde{f}(k, \eta)|.$$

Theorem 4.20 (Injections between analytic spaces). *(i) If $\lambda, \mu \geq 0$ and $\tau \in \mathbb{R}$ then*

$$(4.21) \quad \|f\|_{\mathcal{Y}_\tau^{\lambda,\mu}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;1}}.$$

(ii) If $0 < \lambda < \bar{\lambda}$, $0 < \mu < \bar{\mu} \leq M$, $\tau \in \mathbb{R}$, then

$$(4.22) \quad \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq \frac{C(d, \bar{\mu})}{(\bar{\lambda} - \lambda)^d (\bar{\mu} - \mu)^d} \|f\|_{\mathcal{Y}_\tau^{\bar{\lambda}, \bar{\mu}}}.$$

(iii) If $0 < \lambda < \bar{\lambda} \leq \Lambda$, $0 < \mu < \bar{\mu} \leq M$, $b \leq \beta \leq B$, then there is $C = C(\Lambda, M, b, B, d)$ such that

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;1}} \leq C^{\frac{1}{\min\{\bar{\lambda}-\lambda; \bar{\mu}-\mu\}}} \left(\|f\|_{\mathcal{Y}_\tau^{\bar{\lambda}, \bar{\mu}}} + \max \left\{ \left(\iint |f(x, v)| e^{\beta|v|} dv dx \right); \left(\iint |f(x, v)| e^{\beta|v|} dv dx \right)^2 \right\} \right).$$

Remark 4.21. The combination of (ii) and (iii), plus elementary Lebesgue interpolation, enables to control all norms $\mathcal{Z}_\tau^{\lambda, \mu; p}$, $1 \leq p \leq \infty$.

Proof of Theorem 4.20. By the invariance under the action of free transport, it is sufficient to do the proof for $\tau = 0$.

By integration by parts in the Fourier transform formula, we have

$$\tilde{f}(k, \eta) = \int \widehat{f}(k, v) e^{-2i\pi\eta \cdot v} dv = \int \nabla_v^m \widehat{f}(k, v) \frac{e^{-2i\pi\eta \cdot v}}{(2i\pi\eta)^m} dv.$$

So

$$|\tilde{f}(k, \eta)| \leq \frac{1}{(2\pi|\eta|)^m} \int |\nabla_v^m \widehat{f}(k, v)| dv;$$

and therefore

$$\begin{aligned} e^{2\pi\mu|k|} e^{2\pi\lambda|\eta|} |\tilde{f}(k, \eta)| &\leq e^{2\pi\mu|k|} \sum_n \frac{(2\pi\lambda)^n}{n!} |\eta|^n |\tilde{f}(k, \eta)| \\ &\leq e^{2\pi\mu|k|} \sum_n \frac{\lambda^n}{n!} \int |\nabla_v^n \widehat{f}(k, v)| dv. \end{aligned}$$

This establishes (i).

Next, by differentiating the identity

$$\widehat{f}(k, v) = \int \tilde{f}(k, \eta) e^{2i\pi\eta \cdot v} d\eta,$$

we get

$$(4.23) \quad \nabla_v^m \widehat{f}(k, v) = \int \tilde{f}(k, \eta) (2i\pi\eta)^m e^{2i\pi\eta \cdot v} d\eta.$$

Then we deduce (ii) by writing

$$\begin{aligned} \sum_{k, m} e^{2\pi\mu|k|} \frac{\lambda^m}{m!} \|\nabla_v^m \widehat{f}(k, v)\|_{L^\infty(dv)} \\ \leq \sum_k e^{2\pi\mu|k|} \int e^{2\pi\lambda|\eta|} |\tilde{f}(k, \eta)| d\eta \\ \leq \left(\sum_k e^{-2\pi(\bar{\mu}-\mu)|k|} \right) \left(\int e^{-2\pi(\bar{\lambda}-\lambda)|\eta|} d\eta \right) \left(\sup_{k, \eta} e^{2\pi\bar{\lambda}|\eta|} e^{2\pi\bar{\mu}|k|} |\tilde{f}(k, \eta)| \right). \end{aligned}$$

The proof of (iii) is the most tricky. We start again from (4.23), but now we integrate by parts in the η variable:

$$(4.24) \quad \nabla_v^m \widehat{f}(k, v) = (-1)^q \int \nabla_\eta^q [\widetilde{f}(k, \eta) (2i\pi\eta)^m] \frac{e^{2i\pi\eta \cdot v}}{(2i\pi v)^q} dv,$$

where $q = q(v)$ is a multi-index to be chosen.

We split \mathbb{R}_v^d into 2^d disjoint regions $\Delta(i_1, \dots, i_n)$, where the i_j are distinct indices in $\{1, \dots, d\}$:

$$\Delta(I) = \left\{ v \in \mathbb{R}^d; |v_i| \geq 1 \ \forall i \in I, \quad |v_i| < 1 \ \forall i \notin I \right\}.$$

If $v \in \Delta(i_1, \dots, i_n)$ we apply (4.24) with the multi-index q defined by $q_j = 2$ if $j \in \{i_1, \dots, i_n\}$, $q_j = 0$ otherwise. This gives

$$\int_{\Delta(i_1, \dots, i_n)} |\nabla_v^m \widehat{f}(k, v)| dv \leq \left(\frac{1}{(2\pi)^{2n}} \int_{\Delta(i_1, \dots, i_n)} \frac{dv_{i_1} \dots dv_{i_n}}{|v_{i_1}|^2 \dots |v_{i_n}|^2} \right) \sup_{k, \eta} \left| \nabla_\eta^q [\widetilde{f}(k, \eta) (2i\pi\eta)^m] \right|.$$

Summing up all pieces and using the Leibniz formula, we get

$$\int |\nabla_v^m \widehat{f}(k, v)| dv \leq C(d)(1 + m^{2d}) \sup_{k, \eta} \sup_{|q| \leq 2d} |\nabla_\eta^q \widetilde{f}(k, \eta)| |2\pi\eta|^{m-q}.$$

At this point we apply Lemma 4.22 below with

$$\varepsilon = \frac{1}{4} \min \left\{ \frac{\bar{\lambda} - \lambda}{\bar{\lambda}}; \frac{\bar{\mu} - \mu}{\bar{\mu}} \right\},$$

and we get, for $q \leq 2d$,

$$\begin{aligned} |\nabla_\eta^q \widetilde{f}(k, \eta)| &\leq C(d)^{\max\{\frac{\bar{\lambda}}{\bar{\lambda}-\lambda}; \frac{\bar{\mu}}{\bar{\mu}-\mu}\}} K(b, B) e^{-2\pi \frac{\lambda+\bar{\lambda}}{2} |\eta|} \\ &\left(\sup_\eta e^{2\pi \bar{\lambda} |\eta|} |\widetilde{f}(k, \eta)| \right)^{1-\varepsilon} \max \left\{ \left(\sup_{\ell, \eta} \frac{\beta^\ell \|\nabla_\eta^\ell \widetilde{f}\|_\infty}{\ell!} \right)^\varepsilon; \left(\sup_{\ell, \eta} \frac{\beta^\ell \|\nabla_\eta^\ell \widetilde{f}\|_\infty}{\ell!} \right)^{2\varepsilon} \right\}. \end{aligned}$$

Of course,

$$\begin{aligned} \frac{\beta^\ell |\nabla_\eta^\ell \widetilde{f}(k, \eta)|}{\ell!} &\leq (2\pi\beta)^\ell \int_{\mathbb{R}^d} |\widehat{f}(k, v)| \frac{|v|^\ell}{\ell!} dv \\ &\leq \int_{\mathbb{R}^d} |f(x, v)| (2\pi\beta)^\ell \frac{|v|^\ell}{\ell!} dv \leq \int_{\mathbb{R}^d} |f(x, v)| e^{2\pi\beta|v|} dv. \end{aligned}$$

So, all in all,

$$\begin{aligned} & \sum_{k,m} e^{2\pi\mu|k|} \frac{\lambda^m}{m!} \int |\nabla_v^m \widehat{f}(k, v)| dv \\ & \leq \sum_{|q| \leq 2d} C(d, \Lambda, M, b, B)^{\frac{1}{\min\{\bar{\lambda}-\lambda; \bar{\mu}-\mu\}}} \\ & \sup_{\eta \in \mathbb{R}^d} \left(e^{-2\pi \frac{\lambda+\bar{\lambda}}{2} |\eta|} \sum_m \frac{\lambda^m (1+m)^{2d} |2\pi\eta|^{m-q}}{m!} \right) \left(\sum_k e^{-2\pi(\bar{\mu}(1-\varepsilon)-\mu)|k|} \right) \\ & \left(\sup_{k, \eta} e^{2\pi\bar{\mu}|k|} e^{2\pi\bar{\lambda}|\eta|} |\widetilde{f}(k, \eta)| \right)^{1-\varepsilon} \max \left\{ \left(\int_{\mathbb{R}^d} |f(x, v)| e^{\beta|v|} dv \right)^\varepsilon ; \left(\int_{\mathbb{R}^d} |f(x, v)| e^{\beta|v|} dv \right)^{2\varepsilon} \right\}. \end{aligned}$$

Since

$$\sum_m \frac{\lambda^m (1+m)^{2d} |2\pi\eta|^{m-q}}{m!} \leq C(q, \Lambda) e^{2\pi \frac{\lambda+\bar{\lambda}}{2} |\eta|}$$

and

$$\sum_k e^{-2\pi(\bar{\mu}(1-\varepsilon)-\mu)|k|} \leq \sum_k e^{-\pi(\bar{\mu}-\mu)|k|} \leq C/(\bar{\mu}-\mu)^d,$$

we easily end up with the desired result. \square

Lemma 4.22. *Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$, and let $\alpha > 0$, $A \geq 1$, $q \in \mathbb{N}_0^d$. Let β such that $0 < b \leq \beta \leq B$. If $|f(x)| \leq A e^{-\alpha|x|}$ for all x , then for any $\varepsilon \in (0, 1/4)$ one has*

$$|\nabla^q f(x)| \leq C(q, d)^{\frac{1}{\varepsilon}} K(b, B) A^{1-\varepsilon} e^{-(1-2\varepsilon)\alpha|x|}$$

$$\sup_{r \in \mathbb{N}_0^d} \max \left\{ \left(\beta^r \frac{\|\nabla^r f\|_\infty}{r!} \right)^\varepsilon ; \left(\beta^r \frac{\|\nabla^r f\|_\infty}{r!} \right)^{2\varepsilon} \right\}.$$

Remark 4.23. One may conjecture that the optimal constant is in fact polynomial in $1/\varepsilon$; then the constants in Theorem 4.20 (iii) can be improved accordingly. Mironescu communicated to us a derivation of polynomial bounds for the optimal constant in the related interpolation inequality

$$\|f^{(k)}\|_{L^\infty(\mathbb{R})} \leq C(k) \|f\|_{L^1(\mathbb{R})}^{1/(k+2)} \|f^{(k+1)}\|_{L^\infty(\mathbb{R})}^{(k+1)/(k+2)},$$

based on a real interpolation method.

Proof of Lemma 4.22. Let us first see f as a function of x_1 , and treat $x' = (x_2, \dots, x_d)$ as a parameter. Thus the assumption is $|f(x_1, x')| \leq (A e^{-\alpha|x'|}) e^{-\alpha|x_1|}$. By a more or less standard interpolation inequality [20, Lemma A.1],

$$(4.25) \quad |\partial_1 f(x_1, x')| \leq 2\sqrt{A e^{-\alpha|x'|}} \sqrt{e^{-\alpha|x_1|}} \|\partial_1^2 f(x_1, x')\|_{\infty}^{\frac{1}{2}} = 2\sqrt{A e^{-\alpha|x'|}} \sqrt{\|\partial_1^2 f\|_{\infty}}.$$

Let C_{q_1, r_1} be the optimal constant (not smaller than 1) such that

$$(4.26) \quad |\partial_1^{q_1} f(x_1, x')| \leq C_{q_1, r_1} (A e^{-\alpha|x'|})^{1 - \frac{q_1}{r_1}} \|\partial_1^{r_1} f(x_1, x')\|_{\infty}^{\frac{q_1}{r_1}}.$$

By iterating (4.25), we find $C_{q_1, r_1} \leq 2\sqrt{C_{q_1-1, r_1} C_{q_1+1, r_1}}$. It follows by induction that

$$C_{q, r} \leq 2^{q(r-q)}.$$

Next, using (4.26) and interpolating according to the second variable x_2 as in (4.25), we get

$$\begin{aligned} |\partial_2^{q_2} \partial_1^{q_1} f(x)| &\leq C_{q_2, r_2} \left(C_{q_1, r_1} (A e^{-\alpha|x'|})^{1 - \frac{q_1}{r_1}} \|\partial_1^{r_1} f\|_{\infty}^{\frac{q_1}{r_1}} \right)^{1 - \frac{q_2}{r_2}} \|\partial_2^{r_2} \partial_1^{q_1} f\|_{\infty}^{\frac{q_2}{r_2}} \\ &\leq C_{q_1, r_1} C_{q_2, r_2} (A e^{-\alpha|x'|})^{(1 - \frac{q_1}{r_1})(1 - \frac{q_2}{r_2})} \|\partial_1^{r_1} f\|_{\infty}^{\frac{q_1}{r_1}(1 - \frac{q_2}{r_2})} \|\partial_2^{r_2} \partial_1^{q_1} f\|_{\infty}^{\frac{q_2}{r_2}}. \end{aligned}$$

We repeat this until we get

$$(4.27) \quad |\nabla^q f(x)| \leq (C_{q_1, r_1} \dots C_{q_d, r_d}) (A e^{-\alpha|x'|})^{(1 - \frac{q_1}{r_1}) \dots (1 - \frac{q_d}{r_d})} \|\partial_1^{r_1} f\|_{\infty}^{\frac{q_1}{r_1}(1 - \frac{q_2}{r_2}) \dots (1 - \frac{q_d}{r_d})} \|\partial_1^{q_1} \partial_2^{q_2} \dots \partial_{d-1}^{q_{d-1}} \partial_d^{r_d} f\|_{\infty}^{\frac{q_d}{r_d}}.$$

Choose r_i ($1 \leq i \leq d$) in such a way that

$$\frac{\varepsilon}{d} \leq \frac{q_i}{r_i} \leq \frac{2\varepsilon}{d};$$

this is always possible for $\varepsilon < d/4$. Then $C_{q_i, r_i} \leq (2^{dq_i^2})^{1/\varepsilon}$, and (4.27) implies

$$|\nabla^q f(x)| \leq (2^{d|q|^2})^{1/\varepsilon} (A e^{-\alpha|x'|})^{1-\varepsilon} \max_{s \leq r+q} \{ \|\nabla^s f\|_{\infty}^{\varepsilon}; \|\nabla^s f\|_{\infty}^{2\varepsilon} \}.$$

Then, since $2(r+q)\varepsilon \leq 3dq$ we have, by a crude application of Stirling's formula (in quantitative form), for $s \leq r+q$,

$$\begin{aligned} \|\nabla^s f\|_{\infty}^{\varepsilon} &\leq \left(\frac{\beta^s \|\nabla^s f\|_{\infty}}{s!} \right)^{\varepsilon} \left(\frac{s!}{\beta^s} \right)^{\varepsilon} \\ &\leq \left(\sup_n \frac{\beta^n \|\nabla^n f\|_{\infty}}{n!} \right)^{\varepsilon} C(\beta, q, d) \varepsilon^{-3dq}, \end{aligned}$$

and the result follows easily. \square

4.5. Algebra property in two variables. In this section we only consider the norms $\mathcal{Z}_\tau^{\lambda,\mu;p}$; but similar results would hold true for the two-variables \mathcal{C} and \mathcal{F} spaces, and could be proven with the same method as those used for the one-variable spaces \mathcal{F}^λ and \mathcal{C}^λ respectively (note that the Leibniz formula still applies because ∇_x and $(\nabla_v + \tau\nabla_x)$ commute).

Proposition 4.24. (i) For any $\lambda, \mu \geq 0$, $\tau \in \mathbb{R}$ and $p, q, r \in [1, +\infty]$ such that $1/p + 1/q = 1/r$, we have

$$\|fg\|_{\mathcal{Z}_\tau^{\lambda,\mu;r}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \|g\|_{\mathcal{Z}_\tau^{\lambda,\mu;q}}.$$

(ii) As a consequence, $\mathcal{Z}_\tau^{\lambda,\mu} = \mathcal{Z}_\tau^{\lambda,\mu;\infty}$ is a normed algebra:

$$\|fg\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \|g\|_{\mathcal{Z}_\tau^{\lambda,\mu}}.$$

In particular, $\|f^n\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}}^n$ for any $n \in \mathbb{N}_0$, and $\|e^f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq e^{\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}}}$.

Proof of Proposition 4.24. First we note that (with the notation (4.18)) $\|\cdot\|_{\mu;r}$ satisfies the “ (p, q, r) property”: whenever $p, q, r \in [1, +\infty]$ satisfy $1/p + 1/q = 1/r$, we have

$$\begin{aligned} \|fg\|_{\mu;r} &= \sum_{\ell \in \mathbb{Z}^d} e^{2\pi\mu|\ell|} \|\widehat{fg}(\ell, \cdot)\|_{L^r(\mathbb{R}_v^d)} \\ &= \sum_{\ell \in \mathbb{Z}^d} e^{2\pi\mu|\ell|} \left\| \sum_k \widehat{f}(k, \cdot) \widehat{g}(\ell - k, \cdot) \right\|_{L^r(\mathbb{R}_v^d)} \\ &\leq \sum_{\ell \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} e^{2\pi\mu|k|} e^{2\pi\mu|\ell-k|} \|\widehat{f}(k, \cdot)\|_{L^p(\mathbb{R}_v^d)} \|\widehat{g}(\ell - k, \cdot)\|_{L^q(\mathbb{R}_v^d)} \\ &= \|f\|_{\mu;p} \|g\|_{\mu;q}. \end{aligned}$$

Next, we write

$$\begin{aligned}
\|fg\|_{\mathcal{Z}_\tau^{\lambda,\mu;r}} &= \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|(\nabla_v + \tau \nabla_x)^n(fg)\|_{\mu;r} \\
&= \sum_n \frac{\lambda^n}{n!} \left\| \sum_{m \leq n} \binom{n}{m} (\nabla_v + \tau \nabla_x)^m f (\nabla_v + \tau \nabla_x)^{n-m} g \right\|_{\mu;r} \\
&\leq \sum_n \frac{\lambda^n}{n!} \sum_{m \leq n} \binom{n}{m} \|(\nabla_v + \tau \nabla_x)^m f\|_{\mu;p} \|(\nabla_v + \tau \nabla_x)^{n-m} g\|_{\mu;q} \\
&= \left(\sum_m \frac{\lambda^m}{m!} \|(\nabla_v + \tau \nabla_x)^m f\|_{\mu;p} \right) \left(\sum_\ell \frac{\lambda^\ell}{\ell!} \|(\nabla_v + \tau \nabla_x)^\ell g\|_{\mu;q} \right) \\
&= \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \|g\|_{\mathcal{Z}_\tau^{\lambda,\mu;q}}.
\end{aligned}$$

(We could also reduce to $\tau = 0$ by means of Proposition 4.19.) □

4.6. Composition inequality.

Proposition 4.25 (Composition inequality in two variables). *For any $\lambda, \mu \geq 0$ and any $p \in [1, \infty]$, $\tau \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$,*

$$(4.28) \quad \left\| f(x + bv + X(x, v), av + V(x, v)) \right\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \leq |a|^{-d/p} \|f\|_{\mathcal{Z}_\sigma^{\alpha,\beta;p}},$$

where

$$(4.29) \quad \alpha = \lambda|a| + \|V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}, \quad \beta = \mu + \lambda|b + \tau - a\sigma| + \|X - \sigma V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}.$$

Remark 4.26. The norms in (4.29) for X and V have to be based on L^∞ , not just any L^p . Also note: the fact that the second argument of f has the form $av + V$ (and not $av + cx + V$) is related to Remark 4.7.

Proof of Proposition 4.25. The proof is a combination of the arguments in Proposition 4.8. In a first step, we do it for the case $\tau = \sigma = 0$, and we write $\|\cdot\|_{\lambda,\mu;p} = \|\cdot\|_{\mathcal{Z}_0^{\lambda,\mu;p}}$.

From the expansion $f(x, v) = \sum \widehat{f}(k, v) e^{2i\pi k \cdot x}$ we deduce

$$\begin{aligned} h(x, v) &:= f\left(x + bv + X(x, v), av + V(x, v)\right) \\ &= \sum_k \widehat{f}(k, av + V) e^{2i\pi k \cdot (x + bv + X)} \\ &= \sum_k \sum_m \nabla_v^m \widehat{f}(k, av) \cdot \frac{V^m}{m!} e^{2i\pi k \cdot x} e^{2i\pi k \cdot bv} e^{2i\pi k \cdot X}. \end{aligned}$$

Taking the Fourier transform in x , we see that for any $\ell \in \mathbb{Z}^d$,

$$\widehat{h}(\ell, v) = \sum_k \sum_m \nabla_v^m \widehat{f}(k, av) e^{2i\pi k \cdot bv} \sum_j \frac{(V^m \widehat{f})(j)}{m!} (e^{2i\pi k \cdot X}) \widehat{f}(\ell - k - j).$$

Differentiating n times *via* the Leibniz formula (here applied to a product of four functions), we get

$$\begin{aligned} \nabla_v^n \widehat{h}(\ell, v) &= \sum_{k, m, j} \sum_{n_1 + n_2 + n_3 + n_4 = n} \frac{n! a^{n_1}}{n_1! n_2! n_3! n_4!} \nabla_v^{m+n_1} \widehat{f}(k, av) \\ &\quad \frac{\nabla_v^{n_2} (V^m \widehat{f})(j)}{m!} \nabla_v^{n_3} (e^{2i\pi k \cdot X}) \widehat{f}(\ell - k - j) (2i\pi b k)^{n_4} e^{2i\pi k \cdot bv}. \end{aligned}$$

Multiplying by $\lambda^n e^{2\pi\mu|\ell|}/n!$ and summing over n and ℓ , taking L^p norms and using $\|fg\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^\infty}$, we finally obtain

$$\begin{aligned}
\|h\|_{\lambda,\mu} &\leq |a|^{-d/p} \sum_{k,j,\ell \in \mathbb{Z}_0^d; m,n,n_1+n_2+n_3+n_4=n \geq 0} \frac{\lambda^n e^{2\pi\mu|\ell|} |a|^{n_1}}{n_1! n_2! n_3! n_4!} \|\nabla_v^{m+n_1} \widehat{f}(k, \cdot)\|_{L^p} \left\| \frac{\nabla_v^{n_2} (V^m) \widehat{\cdot}(j)}{m!} \right\|_\infty \\
&\quad \left\| \nabla_v^{n_3} (e^{2i\pi k \cdot X}) \widehat{\cdot}(\ell - k - j) \right\|_\infty (2\pi|b| |k|)^{n_4} \\
&= |a|^{-d/p} \sum_{k,j,\ell \in \mathbb{Z}_0^d, m,n_1,n_2,n_3,n_4 \geq 0} \frac{\lambda^{n_1+n_2+n_3+n_4} e^{2\pi\mu|k|} e^{2\pi\mu|j|} e^{2\pi\mu|\ell-k-j|} |a|^{n_1}}{n_1! n_2! n_3! n_4!} \|\nabla_v^{m+n_1} \widehat{f}(k, \cdot)\|_{L^p} \\
&\quad \left\| \frac{\nabla_v^{n_2} (V^m) \widehat{\cdot}(j)}{m!} \right\|_\infty \left\| \nabla_v^{n_3} (e^{2i\pi k \cdot X}) \widehat{\cdot}(\ell - k - j) \right\|_\infty (2\pi|b| |k|)^{n_4} \\
&\leq |a|^{-d/p} \sum_{k,n_1,m} \frac{\lambda^{n_1} |a|^{n_1}}{n_1!} \|\nabla_v^{n_1+m} \widehat{f}(k, \cdot)\|_{L^p} e^{2\pi\mu|k|} \left(\frac{1}{m!} \sum_{n_2,j} \frac{\lambda^{n_2}}{n_2!} e^{2\pi\mu|j|} \|\nabla_v^{n_2} (V^m) \widehat{\cdot}(j)\|_\infty \right) \\
&\quad \left(\sum_{n_3,h} \frac{\lambda^{n_3}}{n_3!} e^{2\pi\mu|h|} \|\nabla_v^{n_3} (e^{2i\pi k \cdot X}) \widehat{\cdot}(h)\|_\infty \right) \left(\sum_{n_4} \frac{(2\pi\lambda|b||k|)^{n_4}}{n_4!} \right) \\
&= |a|^{-d/p} \sum_{k,p,m} \frac{(\lambda|a|)^{n_1}}{n_1!} e^{2\pi\mu|k|} \|\nabla_v^{n_1+m} \widehat{f}(k, \cdot)\|_{L^p} \frac{\|V^m\|_{\lambda,\mu}}{m!} \|e^{2i\pi k \cdot X}\|_{\lambda,\mu} e^{2\pi\lambda|b||k|} \\
&\leq |a|^{-d/p} \sum_{k,n_1,m} \frac{(\lambda|a|)^{n_1}}{n_1!} e^{2\pi(\mu+\lambda|b|)|k|} \|\nabla_v^{n_1+m} \widehat{f}(k, \cdot)\|_{L^p} \frac{\|V\|_{\lambda,\mu}^m}{m!} e^{2\pi|k|\|X\|_{\lambda,\mu}} \\
&= |a|^{-d/p} \sum_{k,n} \frac{1}{n!} (\lambda|a| + \|V\|_{\lambda,\mu})^n \|\nabla_v^n \widehat{f}(k, \cdot)\|_{L^{\overline{p}}} e^{2\pi|k|(\mu+\lambda|b|+\|X\|_{\lambda,\mu})} \\
&= |a|^{-d/p} \|f\|_{\lambda|a|+\|V\|_{\lambda,\mu}, \mu+\lambda|b|+\|X\|_{\lambda,\mu}}.
\end{aligned}$$

Now we generalize this to arbitrary values of σ and τ : by Proposition 4.19,

$$\begin{aligned}
& \left\| f\left(x + bv + X(x, v), av + V(x, v)\right) \right\|_{\mathcal{Z}_\tau^{\lambda, \mu; p}} \\
&= \left\| f\left(x + v(b + \tau) + X(x + v\tau, v), av + V(x + v\tau, v)\right) \right\|_{\mathcal{Z}^{\lambda, \mu; p}} \\
&= \left\| f \circ S_\sigma^0 \circ S_{-\sigma}^0\left(x + v(b + \tau) + X(x + v\tau, v), av + V(x + v\tau, v)\right) \right\|_{\mathcal{Z}^{\lambda, \mu; p}} \\
&= \left\| (f \circ S_\sigma^0)\left(x + v(b + \tau - a\sigma) + (X - \sigma V)(x + v\tau, v), av + V(x + v\tau, v)\right) \right\|_{\mathcal{Z}^{\lambda, \mu; p}} \\
&= \left\| (f \circ S_\sigma^0)\left(x + v(b + \tau - a\sigma) + Y(x, v), av + W(x, v)\right) \right\|_{\mathcal{Z}^{\lambda, \mu; p}},
\end{aligned}$$

where

$$W(x, v) = V \circ S_\tau^0(x, v), \quad Y(x, v) = (X - \sigma V) \circ S_\tau^0(x, v).$$

Applying the result for $\tau = 0$, we deduce that the norm of $h(x, v) = f(x + bv + X(x, v), av + V(x, v))$ in $\mathcal{Z}_\tau^{\lambda, \mu}$ is bounded by

$$\|f \circ S_\sigma^0\|_{\mathcal{Z}^{\alpha, \beta; p}} = \|f\|_{\mathcal{Z}_\sigma^{\alpha, \beta; p}},$$

where

$$\alpha = \lambda|a| + \|V \circ S_\tau^0\|_{\mathcal{Z}^{\lambda, \mu}} = |a|\lambda + \|V\|_{\mathcal{Z}_\tau^{\lambda, \mu}},$$

and

$$\beta = \mu + \lambda|b + \tau - a\sigma| + \|(X - \sigma V) \circ S_\tau^0\|_{\mathcal{Z}^{\lambda, \mu}} = \mu + \lambda|b + \tau - a\sigma| + \|X - \sigma V\|_{\mathcal{Z}_\tau^{\lambda, \mu}}.$$

This establishes the desired bound. \square

4.7. Gradient inequality. In the next proposition we shall write

$$(4.30) \quad \|f\|_{\dot{\mathcal{Z}}_\tau^{\lambda, \mu}} = \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} \left\| (\nabla_v + 2i\pi\tau\ell)^n \widehat{f}(\ell, v) \right\|_{L^\infty(\mathbb{R}_v^d)}.$$

This is again a homogeneous (in the x variable) seminorm.

Proposition 4.27. *For $\bar{\lambda} > \lambda \geq 0$, $\bar{\mu} > \mu \geq 0$, we have the functional inequalities*

$$\begin{aligned}
\|\nabla_x f\|_{\mathcal{Z}_\tau^{\lambda, \mu; p}} &\leq \frac{C(d)}{(\bar{\mu} - \mu)} \|f\|_{\dot{\mathcal{Z}}_\tau^{\lambda, \bar{\mu}; p}}; \\
\|(\nabla_v + \tau\nabla_x)f\|_{\mathcal{Z}_\tau^{\lambda, \mu; p}} &\leq \frac{C(d)}{\lambda \log(\bar{\lambda}/\lambda)} \|f\|_{\mathcal{Z}_\tau^{\bar{\lambda}, \mu; p}}.
\end{aligned}$$

In particular, for $\tau \geq 0$ we have

$$\|\nabla_v f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \leq C(d) \left[\left(\frac{1}{\lambda \log(\bar{\lambda}/\lambda)} \right) \|f\|_{\mathcal{Z}_\tau^{\bar{\lambda},\bar{\mu};p}} + \left(\frac{\tau}{(\bar{\mu} - \mu)} \right) \|f\|_{\dot{\mathcal{Z}}_\tau^{\bar{\lambda},\bar{\mu};p}} \right].$$

The proof is similar to the proof of Proposition 4.10; the constant $C(d)$ arises in the choice of norm on \mathbb{R}^d . As a consequence, if $1 < \bar{\lambda}/\lambda \leq 2$, we have e.g. the bound

$$\|\nabla f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \leq C(d) \left(\frac{1}{\bar{\lambda} - \lambda} + \frac{1 + \tau}{\bar{\mu} - \mu} \right) \|f\|_{\mathcal{Z}_\tau^{\bar{\lambda},\bar{\mu};p}}.$$

4.8. Inversion. From the composition inequality follows an inversion estimate.

Proposition 4.28 (Inversion inequality). *(i) Let $\lambda, \mu \geq 0$, $\tau \in \mathbb{R}$, and $F : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{T}^d \times \mathbb{R}^d$. Then there is $\varepsilon = \varepsilon(d)$ such that if F satisfies*

$$\|\nabla(F - \text{Id})\|_{\mathcal{Z}_\tau^{\lambda',\mu'}} \leq \varepsilon(d),$$

where

$$\lambda' = \lambda + 2\|F - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,\mu}}, \quad \mu' = \mu + 2(1 + |\tau|)\|F - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,\mu}},$$

then F is invertible and

$$(4.31) \quad \|F^{-1} - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq 2\|F - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,\mu}}.$$

(ii) More generally, if F and G are functions $\mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{T}^d \times \mathbb{R}^d$ such that

$$(4.32) \quad \|\nabla(F - \text{Id})\|_{\mathcal{Z}_\tau^{\lambda',\mu'}} \leq \varepsilon(d),$$

where

$$\lambda' = \lambda + 2\|F - G\|_{\mathcal{Z}_\tau^{\lambda,\mu}}, \quad \mu' = \mu + 2(1 + |\tau|)\|F - G\|_{\mathcal{Z}_\tau^{\lambda,\mu}},$$

then F is invertible and

$$(4.33) \quad \|F^{-1} \circ G - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leq 2\|F - G\|_{\mathcal{Z}_\tau^{\lambda,\mu}}.$$

Remark 4.29. As could be expected, the conditions become very stringent as τ becomes large: basically, $F - \text{Id}$ (or $F - G$ in case (ii)) should be of order $o(1/\tau)$ for Proposition 4.28 to be applicable.

Remark 4.30. By Proposition 4.27, a sufficient condition for (4.32) to hold is that there be λ'', μ'' such that $\lambda \leq \lambda'' \leq 2\lambda$, $\mu \leq \mu''$, and

$$\|F - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda'',\mu''}} \leq \frac{\varepsilon'(d)}{1 + \tau} \min\{\lambda'' - \lambda'; \mu'' - \mu'\}.$$

However, this condition is in practice hard to fulfill.

Proof of Proposition 4.28. We prove only (ii), of which (i) is a particular case. Let $f = F - \text{Id}$, $h = F^{-1} \circ G - \text{Id}$, $g = G - \text{Id}$, so that $\text{Id} + g = (\text{Id} + f) \circ (\text{Id} + h)$, or equivalently

$$h = g - f \circ (\text{Id} + h).$$

So h is a

$$\Phi : Z \longmapsto g - f \circ (\text{Id} + Z).$$

Note that $\Phi(0) = g - f$. If Φ is $(1/2)$ -Lipschitz on the ball $B(0, 2\|f - g\|)$ in $\mathcal{Z}_\tau^{\lambda, \mu}$, then (4.33) will follow by fixed point iteration as in Theorem A.2.

So let Z, \tilde{Z} be given with

$$\|Z\|_{\mathcal{Z}_\tau^{\lambda, \mu}}, \|\tilde{Z}\|_{\mathcal{Z}_\tau^{\lambda, \mu}} \leq 2\|f - g\|_{\mathcal{Z}_\tau^{\lambda, \mu}}.$$

We have

$$\begin{aligned} \Phi(Z) - \Phi(\tilde{Z}) &= f(\text{Id} + \tilde{Z}) - f(\text{Id} + Z) \\ &= \left(\int_0^1 \nabla f(\text{Id} + (1 - \theta)Z + \theta\tilde{Z}) d\theta \right) \cdot (\tilde{Z} - Z). \end{aligned}$$

By Proposition 4.24,

$$\|\Phi(Z) - \Phi(\tilde{Z})\|_{\mathcal{Z}_\tau^{\lambda, \mu}} \leq \left(\int_0^1 \left\| \nabla f(\text{Id} + (1 - \theta)Z + \theta\tilde{Z}) \right\|_{\mathcal{Z}_\tau^{\lambda, \mu}} d\theta \right) \|\tilde{Z} - Z\|_{\mathcal{Z}_\tau^{\lambda, \mu}}.$$

For any $\theta \in [0, 1]$, by Proposition 4.25,

$$\left\| \nabla f(\text{Id} + (1 - \theta)Z + \theta\tilde{Z}) \right\|_{\mathcal{Z}_\tau^{\lambda, \mu}} \leq \|\nabla f\|_{\mathcal{Z}_\tau^{\hat{\lambda}, \hat{\mu}}},$$

where

$$\hat{\lambda} = \lambda + \max\{\|Z\|; \|\tilde{Z}\|\} \leq \lambda + 2\|f - g\|_{\mathcal{Z}_\tau^{\lambda, \mu}}$$

and (writing $Z = (Z_x, Z_v)$, $\tilde{Z} = (\tilde{Z}_x, \tilde{Z}_v)$)

$$\hat{\mu} = \mu + \max\{\|Z_x - \tau Z_v\|; \|\tilde{Z}_x - \tau \tilde{Z}_v\|\} \leq \mu + 2(1 + |\tau|)\|f - g\|_{\mathcal{Z}_\tau^{\lambda, \mu}}.$$

If F and G satisfy the assumptions of Proposition 4.28, we deduce that

$$\|\Phi\|_{\text{Lip}(B(0,2))} \leq C(d)\varepsilon(d),$$

and this is bounded above by $1/2$ if $\varepsilon(d)$ is small enough. \square

4.9. Sobolev corrections. We shall need to quantify Sobolev regularity corrections to the analytic regularity, in the x variable.

Definition 4.31 (Hybrid analytic norms with Sobolev corrections). For $\lambda, \mu, \gamma \geq 0$, $\tau \in \mathbb{R}$, $p \in [1, \infty]$, we define

$$\begin{aligned} \|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} &= \sum_{\ell \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|\ell|} (1 + |\ell|)^\gamma \|(\nabla_v + 2i\pi\tau\ell)^n \widehat{f}(\ell, v)\|_{L^p(\mathbb{R}_v^d)}; \\ \|f\|_{\mathcal{F}^{\lambda,\gamma}} &= \sum_{k \in \mathbb{Z}^d} e^{2\pi\lambda|k|} (1 + |k|)^\gamma |\widehat{f}(k)|. \end{aligned}$$

Proposition 4.32. *Let $\lambda, \mu, \gamma \geq 0$, $\tau \in \mathbb{R}$ and $p \in [1, +\infty]$. We have the following functional inequalities:*

- (i) $\|f\|_{\mathcal{Z}_{t+\tau}^{\lambda,(\mu,\gamma);p}} = \|f \circ S_t^0\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}}$;
- (ii) $1/p + 1/q = 1/r \implies \|fg\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);r}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} \|g\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);q}}$ and therefore in particular $\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)} = \mathcal{Z}_\tau^{\lambda,(\mu,\gamma);\infty}$ is a normed algebra;
- (iii) If f depends only on x then $\|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}} = \|f\|_{\mathcal{F}^{\lambda|\tau|+\mu,\gamma}}$;
- (iv) $\|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} \leq \|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu+\lambda|\tau-\bar{\tau}|,\gamma);p}}$;
- (v) for any $\sigma \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$, $p \in [1, \infty]$,

$$\left\| f\left(x + bv + X(x, v), av + V(x, v)\right) \right\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} \leq |a|^{-d/p} \|f\|_{\mathcal{Z}_\sigma^{\alpha,(\beta,\gamma);p}},$$

where $\alpha = \lambda|a| + \|V\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}}$ and $\beta = \mu + \lambda|b + \tau - a\sigma| + \|X - \sigma V\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}}$.

(vi) *Gradient inequality:*

$$\begin{aligned} \|\nabla_x f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} &\leq \frac{C(d)}{\bar{\mu} - \mu} \|f\|_{\mathcal{Z}_\tau^{\lambda,(\bar{\mu},\gamma);p}}, \\ \|\nabla f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} &\leq C(d) \left(\frac{1}{\bar{\lambda} - \lambda} + \frac{1 + \tau}{\bar{\mu} - \mu} \right) \|f\|_{\mathcal{Z}_\tau^{\bar{\lambda},(\bar{\mu},\gamma);p}}. \end{aligned}$$

(vii) *Inversion:* If F and G are functions $\mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{T}^d \times \mathbb{R}^d$ such that

$$\|\nabla(F - \text{Id})\|_{\mathcal{Z}_\tau^{\lambda',(\mu',\gamma)}} \leq \varepsilon(d),$$

where

$$\lambda' = \lambda + 2\|F - G\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}}, \quad \mu' = \mu + 2(1 + |\tau|) \|F - G\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}},$$

then

$$(4.34) \quad \|F^{-1} \circ G - \text{Id}\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}} \leq 2 \|F - G\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma)}}.$$

Proof of Proposition 4.32. The proofs are the same as for the “plain” hybrid norms; the only noticeable point is that for the proof of (ii) we use, in addition to $e^{2\pi\lambda|k|} \leq e^{2\pi\lambda|k-\ell|} e^{2\pi\lambda|\ell|}$, the inequality

$$(1 + |k|)^\gamma \leq (1 + |k - \ell|)^\gamma (1 + |\ell|)^\gamma.$$

□

Remark 4.33. Of course, some of the estimates in Proposition 4.32 can be “improved” by taking advantage of γ ; e.g. for $\gamma \geq 1$ we have

$$\|\nabla_x f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} \leq C(d) \|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}}.$$

4.10. Two-shift hybrid norms. Our norms can be further refined by using two (or more) different time shifts. While this can be avoided for the proof of our main result, it played a crucial role in a preliminary version of this paper, and has independent interest because it provides more flexibility. We shall illustrate this in Section 6.

Definition 4.34 (Two-shift hybrid analytic norms with Sobolev correction). Let $\lambda, \mu, \gamma \geq 0$, $\tau \in \mathbb{R}$ and $p \in [1, +\infty]$. We define the space $\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}$ by

$$(4.35) \quad \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} = \sum_k \sum_{n,n' \in \mathbb{N}_0^d} \frac{e^{2\pi\mu|k|} (1 + |k|)^\gamma}{n! n'!} \left\| [\lambda(\nabla_v + 2i\pi\tau k)]^n [\lambda'(\nabla_v + 2i\pi\tau' k)]^{n'} \widehat{g}(k, v) \right\|_{L^p(dv)}.$$

As before, the case $p = \infty$ will be simply denoted by $\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}$, and the case $\tau = \tau' = 0$ by $\mathcal{Z}^{(\lambda,\lambda'),(\mu,\gamma);p}$.

Remark 4.35. If $\tau = \tau'$ we can apply the formula

$$\sum_{n+n'=m} \binom{m}{n} \lambda^n (\lambda')^{n'} = (\lambda + \lambda')^m$$

to see that the above space reduces to $\mathcal{Z}_\tau^{\lambda+\lambda',(\mu,\gamma);p}$. Further note that if in (4.35) we had exchanged $\sum_{n+n'=m}$ with the L^p norm, by the formula

$$\sum_{n+n'=m} \binom{m}{n} (\lambda \nabla_v + 2i\pi\lambda\tau k)^n (\lambda' \nabla_v + 2i\pi\lambda'\tau' k)^{n'} = ((\lambda + \lambda') \nabla_v + 2i\pi(\lambda\tau + \lambda'\tau'))^m$$

the resulting space would be just $\mathcal{Z}_{\bar{\tau}}^{\bar{\lambda},(\mu,\gamma);p}$, with

$$\bar{\lambda} = \lambda + \lambda'; \quad \bar{\tau} = \frac{\lambda\tau + \lambda'\tau'}{\lambda + \lambda'}.$$

These two-shift hybrid norms do not in general reduce to the previous single-time-shift hybrid norms. Even if their interpretation in terms of the transport semigroup is not so easy, we have

$$(4.36) \quad \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} = \sum_{n,n' \in \mathbb{N}_0^d} \frac{1}{n!n'!} \left\| (\lambda \nabla_v)^n S_{\tau-\tau'}^* (\lambda' \nabla_v)^{n'} S_{\tau'}^* f \right\|_{\mu;p}$$

where $S_{\tau}^* f = f \circ S_{\tau}^0$ is the pullback of f by the characteristics of the free transport; or equivalently the evolution of f by the backward free transport semigroup.

Variants of the estimates which we established for the hybrid norms remain valid (although with a more tricky proof) for the two-shift hybrid norms. Let us establish

Proposition 4.36. *Let $\lambda, \lambda', \mu, \gamma \geq 0$, $\tau, \tau' \in \mathbb{R}$ and $p \in [1, +\infty]$. We have the following functional inequalities:*

$$(i) \quad \|f\|_{\mathcal{Z}_{(t+\tau, t+\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} = \left\| f \circ S_t^0 \right\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}};$$

(ii) *For any $p, q, r \in [1, +\infty]$ such that $1/p + 1/q = 1/r$, we have*

$$\|fg\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);r}} \leq \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} \|g\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);q}}$$

and therefore in particular the spaces $\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}$ are normed algebras;

(iii) *If f depends only on x then*

$$\|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}} = \|f\|_{\mathcal{F}^{\lambda|\tau|+\lambda'|\tau'|+\mu,\gamma}};$$

(iv) *Change of time-shifts:*

$$\|f\|_{\mathcal{Z}_{(\bar{\tau},\bar{\tau}')}^{(\lambda,\lambda'),(\mu,\gamma);p}} \leq \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu+\lambda|\tau-\bar{\tau}|+\lambda'|\tau'-\bar{\tau}'|,\gamma);p}}$$

and in particular

$$\|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} \leq \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu+(\lambda|\tau|+\lambda'|\tau'|),\gamma);p}};$$

(v) *For any $\sigma, \sigma' \in \mathbb{R}$, $a \neq 0$, $b \in \mathbb{R}$, $p \in [1, \infty]$, we have*

$$\left\| f \left(x + bv + X(x, v), av + V(x, v) \right) \right\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma);p}} \leq |a|^{-d/p} \|f\|_{\mathcal{Z}_{(\sigma,\sigma')}^{(\alpha,\alpha'),\beta;p}},$$

with

$$\alpha = \lambda|a| + \|V\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}}, \quad \alpha' = \lambda'|a|,$$

$$\beta = \mu + (|a|\lambda + \|V\|) |\tau + b - \sigma| + |a|\lambda' |\tau' + b - \sigma'| + \left\| X - \left(\frac{b + \tau}{a} \right) V \right\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}}$$

or symmetrically with

$$\alpha = \lambda|a|, \quad \alpha' = \lambda'|a| + \|V\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}},$$

$$\beta = \mu + \lambda|b + \tau - a\sigma| + \left(\lambda' + \frac{\|V\|}{|a|} \right) |b + \tau' - a\sigma'| + \left\| X - \left(\frac{b + \tau'}{a} \right) V \right\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),(\mu,\gamma)}}.$$

Proof of Proposition 4.36. For simplicity we do the proof for $\gamma = 0$; the general case is completely similar.

(i). This comes readily from (4.36):

$$\begin{aligned} \|f \circ S_t^0\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} &= \sum_{n,n' \in \mathbb{N}_0^d} \frac{1}{n! n'} \left\| (\lambda \nabla_v)^n S_{\tau-\tau'}^* (\lambda' \nabla_v)^{n'} S_{\tau'}^* S_t^* f \right\|_{\mu;p} \\ &= \sum_{n,n' \in \mathbb{N}_0^d} \frac{1}{n! n'} \left\| (\lambda \nabla_v)^n S_{(t+\tau)-(t+\tau')}^* (\lambda' \nabla_v)^{n'} S_{t+\tau'}^* f \right\|_{\mu;p} \\ &= \|f\|_{\mathcal{Z}_{(t+\tau,t+\tau')}^{(\lambda,\lambda'),\mu;p}}. \end{aligned}$$

(ii). Let $D := \lambda(\tau \nabla_x + \nabla_v)$ and $D' := \lambda'(\tau' \nabla_x + \nabla_v)$. Then

$$\|f g\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;r}} = \sum_{n,n' \in \mathbb{N}_0^d} \frac{1}{n! n'} \left\| D^n (D')^{n'} (f g) \right\|_{\mu;r}.$$

Thus

$$\begin{aligned} \|f g\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;r}} &\leq \sum_{n,n',m,m' \in \mathbb{N}_0^d} \frac{1}{n! m! n'! m'!} \left\| (D^n (D')^{n'} f) (D^m (D')^{m'} g) \right\|_{\mu;r} \\ &\leq \sum_{n,n',m,m' \in \mathbb{N}_0^d} \frac{1}{n! n'! m! m'!} \left\| (D^n (D')^{n'} f) \right\|_{\mu;p} \left\| (D^m (D')^{m'} g) \right\|_{\mu;q} \\ &= \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} \|g\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;q}}. \end{aligned}$$

(iii). If f depends only on x , then we have

$$\begin{aligned}
\|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu}} &= \sum_{n,n' \in \mathbb{N}_0^d} \frac{1}{n!n'} (|\tau|\lambda)^n (|\tau'|\lambda')^{n'} \|\nabla_x^{n+n'} f\|_{\mathcal{F}^\mu} \\
&= \sum_{n \in \mathbb{N}_0^d} \frac{(|\tau|\lambda + |\tau'|\lambda')^n}{n!} \|\nabla_x^n f\|_{\mathcal{F}^\mu} \\
&= \sum_{n \in \mathbb{N}_0^d} \sum_{k \in \mathbb{Z}^d} e^{2\pi\mu|k|} \frac{(|\tau|\lambda + |\tau'|\lambda')^n}{n!} (2\pi|k|)^n |\widehat{f}(k)| \\
&= \sum_{k \in \mathbb{Z}^d} e^{2\pi\mu|k| + 2\pi(|\tau|\lambda + |\tau'|\lambda')|k|} |\widehat{f}(k)|
\end{aligned}$$

which is the desired result.

(iv). Let $D := \lambda(\tau\nabla_x + \nabla_v)$, $D' := \lambda'(\tau'\nabla_x + \nabla_v)$, $\bar{D} := \lambda(\bar{\tau}\nabla_x + \nabla_v)$, and $\bar{D}' := \lambda'(\bar{\tau}'\nabla_x + \nabla_v)$. Then

$$\begin{aligned}
\|f\|_{\mathcal{Z}_{(\bar{\tau},\bar{\tau}')^{(\lambda,\lambda'),\mu;r}} &= \sum_{n,n' \in \mathbb{N}_0^d} \frac{\lambda^n (\lambda')^{n'}}{n!n'} \|\bar{D}^n (\bar{D}')^{n'} f\|_{\mu;r} \\
&= \sum_{k \in \mathbb{Z}^d} e^{2\pi\mu|k|} \sum_{n,n',m,m' \in \mathbb{N}_0^d} \frac{\lambda^n (\lambda')^{n'} (2\pi\lambda|\tau - \bar{\tau}||k|)^m (2\pi\lambda'|\tau' - \bar{\tau}'||k|)^{m'}}{n!n'm!m'} \left\| (D^n (\widehat{D}')^{n'} f)(k, \cdot) \right\|_{L_v^r} \\
&= \sum_{k \in \mathbb{Z}^d} e^{2\pi(\mu + \lambda|\tau - \bar{\tau}| + \lambda'|\tau' - \bar{\tau}'|)|k|} \sum_{n,n' \in \mathbb{N}_0^d} \frac{\lambda^n (\lambda')^{n'}}{n!n'} \left\| (D^n (\widehat{D}')^{n'} f)(k, \cdot) \right\|_{L_v^r} \\
&= \sum_{n,n' \in \mathbb{N}_0^d} \frac{\lambda^n (\lambda')^{n'}}{n!n'} \|D^n (D')^{n'} f\|_{\mu + \lambda|\tau - \bar{\tau}| + \lambda'|\tau' - \bar{\tau}'|;p}
\end{aligned}$$

which is the desired result.

(v). The proof of this point is more involved, so we shall progress step by step.

Step 1. We first consider the (most important) case $b = 0$, $a = 1$, $\tau' = 0$ and $\sigma = \sigma' = 0$. We still write $D := \lambda(\tau\nabla_x + \nabla_v)$ and $D' := \lambda'(\tau'\nabla_x + \nabla_v) = \lambda'\nabla_v$.

We wish to estimate the $\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}$ norm of

$$h(x, v) := f(x + X, v + V).$$

In contrast with the proof of Proposition 4.25, we write a Taylor expansion in *both* variables x and v :

$$h(x, v) = \sum_{r, s \in \mathbb{N}_0^d} \frac{1}{r! s!} (\nabla_x^r \nabla_v^s f)(x, v) X^r(x, v) V^s(x, v).$$

Here are some guidelines for the reader: (a) all indices arising from polynomial expansions and Leibniz formulae (whatever the number of variables) will run on \mathbb{N}_0^d and come with the inverse of their factorial, whereas (b) all indices arising from Fourier decomposition and Fourier convolution formulae will run on \mathbb{Z}^d and come with no factorial.

Applying the operator $D^m (D')^n$ for $m, n \in \mathbb{N}_0^d$ yields

$$\begin{aligned} \frac{1}{m! n!} D^m (D')^n h &= \sum_{r, s, m_1+m_2+m_3=m, n_1+n_2+n_3=n, \in \mathbb{N}_0^d} \frac{1}{r! s! m_1! m_2! m_3! n_1! n_2! n_3!} \\ &\quad (D^{m_1} (D')^{n_1} \nabla_x^r \nabla_v^s f) (D^{m_2} (D')^{n_2} X^r) (D^{m_3} (D')^{n_3} V^s). \end{aligned}$$

Taking Fourier transform, we get

$$\begin{aligned} \frac{1}{m! n!} (D^m (D')^n h) \widehat{h}(k, v) &= \sum_{k_1+k_2+k_3=k, \in \mathbb{Z}_0^d} \sum_{r, s, m_1+m_2+m_3=m, n_1+n_2+n_3=n, \in \mathbb{N}_0^d} \frac{1}{r! s! m_1! m_2! m_3! n_1! n_2! n_3!} \\ &\quad (D^{m_1} (D')^{n_1} \nabla_x^r \nabla_v^s f) \widehat{h}(k_1, v) (D^{m_2} (D')^{n_2} X^r) \widehat{h}(k_2, v) (D^{m_3} (D')^{n_3} V^s) \widehat{h}(k_3, v). \end{aligned}$$

Then we take the L_v^p norm, multiply by $\lambda^m (\lambda')^n e^{2\pi\mu|k|}$ and sum over $k \in \mathbb{Z}_0^d$ and $m, n \in \mathbb{N}_0^d$:

$$\begin{aligned} \|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} &\leq \sum_{k_1, k_2, k_3 \in \mathbb{Z}_0^d} \sum_{r, s, m_1, m_2, m_3, n_1, n_2, n_3 \in \mathbb{N}_0^d} \frac{\lambda^{m_1+m_2+m_3} (\lambda')^{n_1+n_2+n_3}}{r! s! m_1! m_2! m_3! n_1! n_2! n_3!} \\ &\quad e^{2\pi\mu|k_1|} \left\| (D^{m_1} (D')^{n_1} \nabla_x^r \nabla_v^s f) \widehat{h}(k_1, v) \right\|_{L_v^p} e^{2\pi\mu|k_2|} \left\| (D^{m_2} (D')^{n_2} X^r) \widehat{h}(k_2, v) \right\|_{L_v^\infty} \\ &\quad e^{2\pi\mu|k_3|} \left\| (D^{m_3} (D')^{n_3} V^s) \widehat{h}(k_3, v) \right\|_{L_v^\infty}. \end{aligned}$$

By computing the sums over the indices k_2, m_2, n_2 and k_3, m_3, n_3 we get:

$$\|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{r,s,m_1,n_1 \in \mathbb{N}_0^d} \frac{1}{r! s! m_1! n_1!} e^{2\pi\mu|k_1|} \left\| (D^{m_1} (D')^{n_1} \nabla_x^r \nabla_v^s f) \widehat{(\cdot)}(k_1, v) \right\|_{L_v^p} \|X^r\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}} \|V^s\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}.$$

Then we use the algebra property for $\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}$:

$$\|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{r,s,m_1,n_1 \in \mathbb{N}_0^d} \frac{1}{r! s! m_1! n_1!} e^{2\pi\mu|k_1|} \left\| (D^{m_1} (D')^{n_1} \nabla_x^r \nabla_v^s f) \widehat{(\cdot)}(k_1, v) \right\|_{L_v^p} \|X\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^r \|V\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^s.$$

Since ∇_x commutes with D and D' , we can get out the factor $(2\pi|k_1|)^r$ created by the operator ∇_x^r in the norm:

$$\|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{r,s,m_1,n_1 \in \mathbb{N}_0^d} \frac{1}{r! s! m_1! n_1!} e^{2\pi\mu|k_1|} (2\pi|k_1|)^r \left\| (D^{m_1} (D')^{n_1} \nabla_v^s f) \widehat{(\cdot)}(k_1, v) \right\|_{L_v^p} \|X\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^r \|V\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^s.$$

Summation over $r \in \mathbb{N}_0^d$ yields

$$\|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{s,m_1,n_1 \in \mathbb{N}_0^d} \frac{1}{s! m_1! n_1!} e^{2\pi(\mu+\|X\|)|k_1|} \left\| (D^{m_1} \widehat{(D')^{n_1}} \nabla_v^s f)(k_1, v) \right\|_{L_v^p} \|V\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^s$$

where $\|X\|$ is a shorthand for $\|X\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}$. Since $D' = \lambda' \nabla_v$, this is the same as

$$\|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{s,m_1,n_1 \in \mathbb{N}_0^d} \frac{1}{s! m_1! n_1!} e^{2\pi(\mu+\|X\|)|k_1|} (\lambda')^{n_1} \left\| (D^{m_1} \widehat{\nabla_v^{n_1+s}} f)(k_1, v) \right\|_{L_v^p} \|V\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}^s.$$

But

$$\sum_{n_1+s=\ell, \ell \in \mathbb{N}_0^d} \frac{\ell!}{n_1! s!} (\lambda')^{n_1} \|V\|^s = (\lambda' + \|V\|)^\ell,$$

where $\|V\|$ is a shorthand for $\|V\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu}}$; so

$$\begin{aligned} \|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} &\leq \sum_{k_1 \in \mathbb{Z}_0^d} \sum_{m_1, \ell \in \mathbb{N}_0^d} \frac{1}{m_1! \ell!} e^{2\pi(\mu + \|X\|)|k_1|} (\lambda' + \|V\|)^\ell \|(D^{m_1} \nabla_v^\ell f)^\wedge(k_1, v)\|_{L_v^p} \\ &= \|h\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'),\mu;p}} \leq \|f\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda,\lambda'+\|V\|,\mu+\|X\|;p)}. \end{aligned}$$

Step 2. Let us relax the assumption $\tau' = 0$. We start again from $h(x, v) := f(x + X, v + V)$, and we want to estimate the $\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}$ norm of h . By (i),

$$\|h\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} = \|h \circ S_{\tau'}^0\|_{\mathcal{Z}_{(\tau-\tau',0)}^{(\lambda,\lambda'),\mu;p}}.$$

But

$$\begin{aligned} h \circ S_{\tau'}^0 &= f\left(x + \tau'v + X(x + \tau'v, v), v + V(x + \tau'v, v)\right) \\ &= (f \circ S_{\tau'}^0)\left(x + (X - \tau'V)(x + \tau'v, v), v + V(x + \tau'v, v)\right). \end{aligned}$$

So by applying Step 1, we get

$$\|h\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} \leq \|f \circ S_{\tau'}^0\|_{\mathcal{Z}_{(\tau-\tau',0)}^{(\alpha,\alpha'),\beta;p}}$$

with

$$\alpha = \lambda, \quad \alpha' = \lambda' + \|V \circ S_{\tau'}^0\|_{\mathcal{Z}_{(\tau-\tau',0)}^{(\lambda,\lambda'),\mu}}, \quad \beta = \mu + \|(X - \tau'V) \circ S_{\tau'}^0\|_{\mathcal{Z}_{(\tau-\tau',0)}^{(\lambda,\lambda'),\mu}}.$$

Using (i) again, we get

$$\|h\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} \leq \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'+\|V\|,\mu+\|X-\tau'V\|;p}}$$

where $\|V\| = \|V\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu}}$ and $\|X - \tau'V\| = \|X - \tau'V\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu}}$.

Of course the same proof would yield symmetrically:

$$\|h\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda,\lambda'),\mu;p}} \leq \|f\|_{\mathcal{Z}_{(\tau,\tau')}^{(\lambda+\|V\|,\lambda'),\mu+\|X-\tau'V\|;p}}.$$

Step 3. To treat the general case, we define $f_a(x, v) = f(x, av)$; then

$$(S_b^* f_a) \left(x + \left(X - \frac{bV}{a} \right), v + \frac{V}{a} \right) = f(x + bv + X, av + V).$$

So

$$\begin{aligned}
\left\| f(x + bv + X, av + V) \right\|_{\mathcal{Z}_{(\tau, \tau')}^{(\lambda, \lambda'), \mu; p}} &\leq \|S_b^* f_a\|_{\mathcal{Z}_{(\tau, \tau')}^{(\lambda + \frac{V}{a}, \lambda'), \mu + \|X - \frac{b}{a}V - \tau \frac{V}{a}\|; p}} \\
&= \|f_a\|_{\mathcal{Z}_{(\tau + b, \tau' + b)}^{(\lambda + \frac{\|V\|}{a}, \lambda'), \mu + \|X - \frac{b}{a}V - \frac{\tau}{a}V\|; p}} \\
&= |a|^{-d/p} \|f\|_{\mathcal{Z}_{(\frac{\tau + b}{a}, \frac{\tau' + b}{a})}^{(|a|\lambda + \|V\|, |a|\lambda'), \mu + \|X - \frac{b}{a}V - \frac{\tau}{a}V\|; p}} \\
&\leq |a|^{-d/p} \|f\|_{\mathcal{Z}_{(\sigma, \sigma')}^{(\alpha, \alpha'), \beta; p}},
\end{aligned}$$

where $\alpha, \alpha', \beta, \sigma, \sigma'$ are as in the statement of the proposition; the last step follows from (iv). (Note: With respect to Proposition 4.25, we have less room for translation along the semigroup, and the change of the final time-shifts results in an error in the coefficient μ which also depends on the norm of V .) \square

4.11. Measuring solutions of kinetic equations in large time. As we already discussed at length, even for the simplest kinetic equation, namely free transport, we cannot hope to have uniform in time regularity estimates in the velocity variable: rather, because of filamentation, we may have $\|\nabla_v f(t, \cdot)\| = O(t)$, $\|\nabla_v^2 f(t, \cdot)\| = O(t^2)$, etc. For analytic norms we may at best hope for an exponential growth.

But the invariance of the “gliding” norms $\mathcal{Z}_\tau^{\lambda, \mu}$ under free transport (Proposition 4.19) makes it possible to look for uniform estimates such as

$$(4.37) \quad \|f(\tau, \cdot)\|_{\mathcal{Z}_\tau^{\lambda, \mu}} = O(1) \quad \text{as } \tau \rightarrow +\infty.$$

Of course, by Proposition 4.27, (4.37) implies

$$(4.38) \quad \|\nabla_v f(\tau, \cdot)\|_{\mathcal{Z}_\tau^{\lambda', \mu'}} = O(\tau),$$

and nothing better as far as the asymptotic behavior of $\nabla_v f$ is concerned; but (4.37) is much more precise than (4.38). For instance it implies $\|(\nabla_v + \tau \nabla_x) f(\tau, \cdot)\|_{\mathcal{Z}_\tau^{\lambda', \mu'}} = O(1)$ for $\lambda' < \lambda$, $\mu' < \mu$.

Another way to get rid of filamentation is to *average* over the spatial variable x , a common sense procedure which has already been used in physics [48, Section 49]. Think that, if f evolves according to free transport, or even according to the linearized Vlasov equation (3.3), then its space-average

$$(4.39) \quad \langle f \rangle(\tau, v) := \int_{\mathbb{T}^d} f(\tau, x, v) dx$$

is time-invariant. (We used this infinite number of conservation laws to determine the long-time behavior in Theorem 3.1.)

The bound (4.37) easily implies a bound on the space average: indeed,

$$(4.40) \quad \|\langle f \rangle(\tau, \cdot)\|_{C^\lambda} = \|\langle f \rangle(\tau, \cdot)\|_{\mathcal{Z}_\tau^{\lambda, \mu}} \leq \|f(\tau, \cdot)\|_{\mathcal{Z}_\tau^{\lambda, \mu}} = O(1) \quad \text{as } \tau \rightarrow \infty;$$

and in particular, for $\lambda' < \lambda$,

$$(4.41) \quad \|\langle \nabla_v f \rangle(\tau, \cdot)\|_{C^{\lambda'}} = O(1) \quad \text{as } \tau \rightarrow \infty.$$

Again, (4.37) contains a lot more information than (4.41).

Remark 4.37. The idea to estimate solutions of a nonlinear equation by comparison to some unperturbed (reversible) linear dynamics is already present in the definition of Bourgain spaces $X^{s,b}$ [12]. The analogy stops here, since time is a dummy variable in $X^{s,b}$ spaces, while in $\mathcal{Z}_t^{\lambda, \mu}$ spaces it is frozen and appears as a parameter, on which we shall play later.

4.12. Linear damping revisited. As a simple illustration of the functional analysis introduced in this section, let us recast the linear damping (Theorem 3.1) in this language. This will be the first step for the study of the nonlinear damping. For simplicity we set $L = 1$.

Theorem 4.38 (Linear Landau damping again). *Let $f^0 = f^0(v)$, $W : \mathbb{T}^d \rightarrow \mathbb{R}$ such that $\|\nabla W\|_{L^1} \leq C_W$, and $f_i(x, v)$ such that*

- (i) Condition **(L)** from Subsection 2.2 holds for some constants $C_0, \lambda, \kappa > 0$;
- (ii) $\|f^0\|_{C^{\lambda;1}} \leq C_0$;
- (iii) $\|f_i\|_{\mathcal{Z}^{\lambda, \mu;1}} \leq \delta$ for some $\mu > 0$;

Then for any $\lambda' < \lambda$ and $\mu' < \mu$, the solution of the linearized Vlasov equation (3.3) satisfies

$$(4.42) \quad \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{\mathcal{Z}_t^{\lambda', \mu';1}} \leq C \delta,$$

for some constant $C = C(d, C_W, C_0, \lambda, \lambda', \mu, \mu', \kappa)$. In particular, $\rho = \int f dv$ satisfies

$$(4.43) \quad \sup_{t \in \mathbb{R}} \|\rho(t, \cdot)\|_{\mathcal{F}^{\lambda'|t+\mu'}} \leq C \delta.$$

As a consequence, as $|t| \rightarrow \infty$, ρ converges strongly to $\rho_\infty = \iint f_i(x, v) dx dv$, and f converges weakly to $\langle f_i \rangle = \int f_i dx$, at rate $O(e^{-\lambda''|t|})$ for any $\lambda'' < \lambda'$.

If moreover $\|f^0\|_{C^{\lambda;p}} \leq C_0$ and $\|f_i\|_{\mathcal{Z}^{\lambda, \mu;p}} \leq \delta$ for all p in some interval $[1, \bar{p}]$, then (4.42) can be reinforced into

$$(4.44) \quad \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{\mathcal{Z}_t^{\lambda', \mu';p}} \leq C \delta, \quad 1 \leq p \leq \bar{p}.$$

Remark 4.39. The notions of weak and convergence are the same as those in Theorem 3.1. With respect to that statement, we have added an extra analyticity assumption in the x variable; it is likely though that a slightly more careful analysis would allow $\mu = \mu' = 0$.

Proof of Theorem 4.38. Without loss of generality we restrict our attention to $t \geq 0$. Although (4.43) follows from (4.42) by Proposition 4.15, we shall establish (4.43) first, and deduce (4.42) thanks to the equation. We shall write C for various constants depending only on the parameters in the statement of the theorem.

As in the proof of Theorem 3.1, we have

$$\widehat{\rho}(t, k) = \widetilde{f}_i(k, kt) + \int_0^t K^0(t - \tau, k) \widehat{\rho}(\tau, k) d\tau$$

for any $t \geq 0$, $k \in \mathbb{Z}^d$. By Lemma 3.6, for any $\lambda' < \lambda$, $\mu' < \mu$,

$$\begin{aligned} & \sup_{t \geq 0} \left(\sum_k |\widehat{\rho}(t, k)| e^{2\pi(\lambda't + \mu')|k|} \right) \\ & \leq C(\lambda, \lambda', \kappa) \left(\sum_k e^{-2\pi(\mu - \mu')|k|} \right) \sup_{t \geq 0} \sup_{k \in \mathbb{Z}^d} |\widetilde{f}_i(k, kt)| e^{2\pi(\lambda't + \mu)|k|} \\ & \leq C(\lambda, \lambda', \kappa, \mu, \mu') \sup_{t \geq 0} \left(\sum_{k \in \mathbb{Z}^d} |\widetilde{f}_i(k, kt)| e^{2\pi(\lambda t + \mu)|k|} \right). \end{aligned}$$

Equivalently,

$$(4.45) \quad \sup_{t \geq 0} \|\rho(t, \cdot)\|_{\mathcal{F}^{\lambda't + \mu'}} \leq C \sup_{t \geq 0} \left\| \int f_i \circ S_{-t}^0 dv \right\|_{\mathcal{F}^{\lambda t + \mu}}.$$

By Propositions 4.15 and 4.19,

$$\left\| \int f_i \circ S_{-t}^0 dv \right\|_{\mathcal{F}^{\lambda t + \mu}} \leq \|f_i \circ S_{-t}^0\|_{\mathcal{Z}_t^{\lambda, \mu; 1}} = \|f_i\|_{\mathcal{Z}_0^{\lambda, \mu; 1}} \leq \delta.$$

This and (4.45) imply (4.43).

To deduce (4.42), we first write

$$f(t, \cdot) = f_i \circ S_{-t}^0 + \int_0^t \left((\nabla W * \rho_\tau) \circ S_{-(t-\tau)}^0 \right) \cdot \nabla_v f^0 d\tau,$$

where $\rho_\tau = \rho(\tau, \cdot)$. Then for any $\lambda'' < \lambda'$ we have, by Propositions 4.24 and 4.15, for all $t \geq 0$,

(4.46)

$$\begin{aligned} \|f\|_{\mathcal{Z}_t^{\lambda'', \mu'; 1}} &\leq \|f_i \circ S_{-t}^0\|_{\mathcal{Z}_t^{\lambda'', \mu; 1}} + \int_0^t \left\| (\nabla W * \rho_\tau) \circ S_{-(t-\tau)}^0 \right\|_{\mathcal{Z}_t^{\lambda'', \mu'; \infty}} \|\nabla_v f^0\|_{\mathcal{Z}_t^{\lambda'', \mu; 1}} d\tau \\ &= \|f_i\|_{\mathcal{Z}^{\lambda'', \mu; 1}} + \left(\int_0^t \|\nabla W * \rho_\tau\|_{\mathcal{F}^{\lambda'' \tau + \mu'}} d\tau \right) \|\nabla_v f^0\|_{\mathcal{C}^{\lambda''; 1}}. \end{aligned}$$

Since $\widehat{\nabla W}(0) = 0$, we have, for any $\tau \geq 0$,

$$\begin{aligned} \|\nabla W * \rho_\tau\|_{\mathcal{F}^{\lambda'' \tau + \mu}} &\leq e^{-2\pi(\lambda'' - \lambda')\tau} \|\nabla W * \rho_\tau\|_{\mathcal{F}^{\lambda' \tau + \mu'}} \\ &\leq \|\nabla W\|_{L^1} e^{-2\pi(\lambda'' - \lambda')\tau} \|\rho_\tau\|_{\mathcal{F}^{\lambda' \tau + \mu'}} \\ &\leq C_W C \delta e^{-2\pi(\lambda'' - \lambda')\tau}, \end{aligned}$$

in particular

$$(4.47) \quad \int_0^t \|\nabla W * \rho_\tau\|_{\mathcal{F}^{\lambda'' + \mu'}} \leq \frac{C \delta}{\lambda'' - \lambda'}.$$

Also, by Proposition 4.10, for $1 < \lambda'/\lambda'' \leq 2$ we have

$$(4.48) \quad \|\nabla_v f^0\|_{\mathcal{C}^{\lambda''; 1}} \leq \frac{C}{\lambda - \lambda''} \|f^0\|_{\mathcal{C}^{\lambda; 1}} \leq \frac{C C_0}{\lambda - \lambda''}.$$

Plugging (4.47) and (4.48) in (4.46), we deduce (4.42). The end of the proof is an easy exercise if one recalls that $\langle f(t, \cdot) \rangle = \langle f_i \rangle$ for all t . \square

5. SCATTERING ESTIMATES

Let be given a small time-dependent force field, denoted by $\varepsilon F(t, x)$, on $\mathbb{T}^d \times \mathbb{R}^d$, whose analytic regularity *improves* linearly in time. (Think of εF as the force created by a damped density.) This force field perturbs the trajectories $S_{\tau, t}^0$ of the free transport (τ the initial time, t the current time) into trajectories $S_{\tau, t}$. The goal of this section is to get an estimate on the maps $\Omega_{t, \tau} = S_{t, \tau} \circ S_{\tau, t}^0$ (so that $S_{t, \tau} = \Omega_{t, \tau} \circ S_{t, \tau}^0$). These bounds should be in an analytic class about as good as F , with a loss of analyticity depending on ε ; they should also be (for $0 \leq \tau \leq t$)

- uniform in $t \geq \tau$;
- small as $\tau \rightarrow \infty$;
- small as $\tau \rightarrow t$.

We shall informally say that $\Omega_{t,\tau}$ is a **scattering** transform, even though this terminology is usually reserved for the asymptotic regime $t \rightarrow \pm\infty$.

Remark 5.1. The order of composition of the free semigroup and perturbed semigroup is dictated by the need to get uniformity as $t \rightarrow \infty$. If we had defined, say, $\Lambda_{t,\tau} = S_{\tau,t}^0 \circ S_{t,\tau}$, so that $S_{t,\tau} = S_{t,\tau}^0 \circ \Lambda_{t,\tau}$, and if the force was, say, supported in $0 \leq t \leq 1$, we would get (denoting $S_{t,\tau} = (X_{t,\tau}, V_{t,\tau})$)

$$\Lambda_{t,0}(x, v) = \left(X_{1,0}(x - v(t-1), v) + tV_{1,0}(x - v(t-1), v), V_{1,0}(x - v(t-1), v) \right),$$

which does not converge to anything as $t \rightarrow \infty$.

5.1. Formal expansion. Before stating the main result, we sketch a heuristic perturbation study. Let us write a formal expansion of $V_{0,t}(x, v)$ as a perturbation series:

$$V_{0,t}(x, v) = v + \varepsilon v^{(1)}(t, x, v) + \varepsilon^2 v^{(2)}(t, x, v) + \dots$$

Then we deduce

$$X_{0,t}(x, v) = x + vt + \varepsilon \int_0^t v^{(1)}(s, x, v) ds + \varepsilon^2 \int_0^t v^{(2)}(s, x, v) ds + \dots,$$

with $v^{(i)}(t=0) = 0$.

So

$$\frac{\partial^2 X_{0,t}}{\partial t^2} = \varepsilon \frac{\partial v^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial v^{(2)}}{\partial t} + \dots$$

On the other hand,

$$\begin{aligned} \varepsilon F(t, X_{0,t}) &= \varepsilon \sum_k \widehat{F}(t, k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot vt} e^{2i\pi k \cdot [\varepsilon \int_0^t v^{(1)} ds + \varepsilon^2 \int_0^t v^{(2)} ds + \dots]} \\ &= \varepsilon \sum_k \widehat{F}(t, k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot vt} \left[1 + 2i\pi \varepsilon k \cdot \int_0^t v^{(1)} ds + 2i\pi \varepsilon^2 k \cdot \int_0^t v^{(2)} ds \right. \\ &\quad \left. - (2\pi)^2 \varepsilon^2 \left(k \cdot \int_0^t v^{(1)} ds \right)^2 + \dots \right]. \end{aligned}$$

By successive identification,

$$\begin{aligned} \frac{\partial v^{(1)}}{\partial t} &= \sum_k \widehat{F}(t, k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot vt}, \\ \frac{\partial v^{(2)}}{\partial t} &= \sum_k \widehat{F}(t, k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot vt} 2i\pi k \cdot \int_0^t v^{(1)} ds; \end{aligned}$$

$$\frac{\partial v^{(3)}}{\partial t} = \sum_k \widehat{F}(t, k) e^{2i\pi k \cdot x} e^{2i\pi k \cdot vt} \left[2i\pi k \cdot \int_0^t v^{(2)} ds - (2\pi)^2 \varepsilon^2 \left(k \cdot \int_0^t v^{(1)} ds \right)^2 \right],$$

etc.

In particular notice that $\left| \frac{\partial v^{(1)}}{\partial t} \right| \leq \sum_k |\widehat{F}(t, k)|$, so

$$\begin{aligned} \int_0^\infty \left| \frac{\partial v^{(1)}}{\partial t} \right| dt &\leq \int_0^\infty \sum_k |\widehat{F}(t, k)| dt \leq \int_0^\infty \sum_k |\widehat{F}(t, k)| e^{2\pi\mu t} e^{-2\pi\mu t} dt \\ &\leq C_F \int_0^\infty e^{-2\pi\mu t} dt = \frac{C_F}{2\pi\mu}. \end{aligned}$$

So, under our uniform analyticity assumptions we expect $V_{0,t}(x, v)$ to be a *uniformly bounded* analytic perturbation of v .

5.2. Main result. On \mathbb{T}_x^d we consider the dynamical system

$$\frac{d^2 X}{dt^2} = \varepsilon F(t, X);$$

its phase space is $\mathbb{T}^d \times \mathbb{R}^d$. Although this system is reversible we shall only consider $t \geq 0$.

For any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ and any two times $\tau, t \in \mathbb{R}_+$, let $S_{\tau,t}$ be the transform mapping the state of the system at time τ , to the state of the system at time t . In more precise terms, $S_{\tau,t}$ is described by the equations

$$S_{\tau,t}(x, v) = (X_{\tau,t}(x, v), V_{\tau,t}(x, v));$$

$$X_{\tau,\tau}(x, v) = x, \quad V_{\tau,\tau}(x, v) = v;$$

$$(5.1) \quad \frac{d}{dt} X_{\tau,t}(x, v) = V_{\tau,t}(x, v), \quad \frac{d}{dt} V_{\tau,t}(x, v) = \varepsilon F(t, X_{\tau,t}(x, v)).$$

From the definition we have the composition identity

$$(5.2) \quad S_{t_2, t_3} \circ S_{t_1, t_2} = S_{t_1, t_3};$$

in particular $S_{t,\tau}$ is the inverse of $S_{\tau,t}$.

We also write $S_{\tau,t}^0$ for the same transform in the case of the free dynamics ($\varepsilon = 0$); in this case there is an explicit expression:

$$(5.3) \quad S_{\tau,t}^0(x, v) = (x + v(t - \tau), v),$$

where $x + v(t - \tau)$ is evaluated modulo \mathbb{Z}^d . Finally, we define the “scattering transforms associated with εF ”:

$$(5.4) \quad \Omega_{t,\tau} = S_{t,\tau} \circ S_{\tau,t}^0.$$

(There is no simple semigroup property for the transforms $\Omega_{t,\tau}$.)

In this section we establish the following estimates:

Theorem 5.2 (Analytic estimates on scattering transforms in hybrid norms). *Let $\varepsilon > 0$ and let $F = F(t, x)$ on $\mathbb{R}_+ \times \mathbb{T}^d$ satisfy*

$$(5.5) \quad \widehat{F}(t, 0) = 0, \quad \sup_{t \geq 0} \left(\|F(t, \cdot)\|_{\mathcal{F}^{\lambda t + \mu}} + \|\nabla_x F(t, \cdot)\|_{\mathcal{F}^{\lambda t + \mu}} \right) \leq C_F$$

for some parameters $\lambda, \mu > 0$ and $C_F > 0$. Let $t \geq \tau \geq 0$, and let

$$\Omega_{t,\tau} = (\Omega X_{t,\tau}, \Omega V_{t,\tau})$$

be the scattering transforms associated with εF . Let $0 \leq \lambda' < \lambda$, $0 \leq \mu' < \mu$ and $\tau' \geq 0$ be such that

$$(5.6) \quad \lambda'(\tau' - \tau) \leq \frac{(\mu - \mu')}{2}.$$

Let

$$\begin{cases} R_1(\tau, t) = C_F e^{-2\pi(\lambda - \lambda')\tau} \min \{ (t - \tau); (2\pi(\lambda - \lambda'))^{-1} \}; \\ R_2(\tau, t) = C_F e^{-2\pi(\lambda - \lambda')\tau} \min \{ (t - \tau)^2/2; (2\pi(\lambda - \lambda'))^{-2} \}. \end{cases}$$

Assume that

$$(5.7) \quad \forall 0 \leq \tau \leq t, \quad \varepsilon R_2(\tau, t) \leq \frac{(\mu - \mu')}{4},$$

and

$$(5.8) \quad \varepsilon C_F \leq \frac{4\pi^2(\lambda - \lambda')^2}{2}.$$

Then

$$(5.9) \quad \forall 0 \leq \tau \leq t, \quad \|\Omega X_{t,\tau} - \text{Id}\|_{\mathcal{Z}_{\tau'}^{\lambda', \mu'}} \leq 2\varepsilon R_2(\tau, t)$$

and

$$(5.10) \quad \forall 0 \leq \tau \leq t, \quad \|\Omega V_{t,\tau} - \text{Id}\|_{\mathcal{Z}_{\tau'}^{\lambda', \mu'}} \leq \varepsilon R_1(\tau, t).$$

Remark 5.3. Note that the parameter ε plays no other role than to serve the intuition by recalling the perturbative nature of the result.

Proof of Theorem 5.2. For a start, let us make the ansatz

$$S_{t,\tau}(x, v) = \left(x - v(t - \tau) + \varepsilon Z_{t,\tau}(x, v), v + \varepsilon \partial_\tau Z_{t,\tau}(x, v) \right),$$

with

$$Z_{t,t}(x, v) = 0, \quad \partial_\tau Z_{t,\tau} \Big|_{\tau=t} (x, v) = 0.$$

Then it is easily checked that

$$\Omega_{t,\tau} - \text{Id} = \varepsilon (Z, \partial_\tau Z) \circ S_{t-\tau}^0;$$

in particular

$$\|\Omega_{t,\tau} - \text{Id}\|_{\mathcal{Z}_{\tau'}^{\lambda', \mu'}} = \varepsilon \|(Z, \partial_\tau Z)\|_{\mathcal{Z}_{t+\tau'-\tau}^{\lambda', \mu'}}.$$

To estimate this we shall use a fixed point argument based on the equation for $S_{t,\tau}$, namely

$$\frac{d^2 X_{t,\tau}}{d\tau^2} = \varepsilon F(\tau, X_{t,\tau}),$$

or equivalently

$$\frac{d^2 Z_{t,\tau}}{d\tau^2} = F\left(\tau, x - v(t - \tau) + \varepsilon Z_{t,\tau}\right).$$

So let us fix t and define

$$\Psi : (W_{t,\tau})_{0 \leq \tau \leq t} \longmapsto (Z_{t,\tau})_{0 \leq \tau \leq t}$$

such that $(Z_{t,\tau})_{0 \leq \tau \leq t}$ is the solution of

$$(5.11) \quad \begin{cases} \frac{\partial^2 Z_{t,\tau}}{\partial \tau^2} = F\left(\tau, x - v(t - \tau) + \varepsilon W_{t,\tau}\right) \\ Z_{t,t} = 0, \quad (\partial_\tau Z_{t,\tau}) \Big|_{\tau=t} = 0. \end{cases}$$

What we are after is an estimate of the fixed point of Ψ . We do this in two steps.

Step 1. Estimate of $\Psi(0)$. Let $Z^0 = \Psi(0)$. By integration of (5.11) (for $W = 0$) we have

$$Z_{t,\tau}^0 = \int_\tau^t (s - \tau) F(s, x - v(t - s)) ds.$$

Let σ such that $\lambda'\sigma \leq (\mu - \mu')/2$. We apply the $\mathcal{Z}_{t+\sigma}^{\lambda', \mu'}$ norm and use Proposition 4.19:

$$\|Z_{t,\tau}^0\|_{\mathcal{Z}_{t+\sigma}^{\lambda', \mu'}} \leq \int_\tau^t (s - \tau) \|F(s, \cdot)\|_{\mathcal{Z}_{s+\sigma}^{\lambda', \mu'}} ds = \int_\tau^t (s - \tau) \|F(s, \cdot)\|_{\mathcal{F}^{\lambda's + \lambda'\sigma + \mu'}} ds.$$

Of course $\lambda'\sigma + \mu' \leq \mu$, so in particular

$$\lambda' s + \lambda' \sigma + \mu' \leq -(\lambda - \lambda') s + \lambda s + \mu.$$

Combining this with the assumption $\widehat{F}(s, 0) = 0$ yields

$$\begin{aligned} \|F(s, \cdot)\|_{\mathcal{F}^{\lambda' s + \lambda' \sigma + \mu'}} &\leq \|F(s, \cdot)\|_{\mathcal{F}^{\lambda s + \mu}} e^{-2\pi(\lambda - \lambda') s} \\ &\leq C_F e^{-2\pi(\lambda - \lambda') s}. \end{aligned}$$

So

$$\begin{aligned} \|Z_{t,\tau}^0\|_{\mathcal{Z}_{t+\sigma}^{\lambda',\mu'}} &\leq C_F \int_{\tau}^t (s - \tau) e^{-2\pi(\lambda - \lambda') s} ds \\ &\leq C_F e^{-2\pi(\lambda - \lambda') \tau} \min \left\{ \frac{(t - \tau)^2}{2}; \frac{1}{(2\pi(\lambda - \lambda'))^2} \right\} \leq R_2(\tau, t). \end{aligned}$$

With t still fixed, we define the norm

$$(5.12) \quad \left\| \left\| (Z_{t,\tau})_{0 \leq \tau \leq t} \right\| \right\| := \sup \left\{ \frac{\|Z_{t,\tau}\|_{\mathcal{Z}_{t+\sigma}^{\lambda',\mu'}}}{R_2(\tau, t)}; \quad 0 \leq \tau \leq t; \sigma + t \geq 0; \lambda'\sigma \leq \frac{\mu - \mu'}{2} \right\}.$$

The above estimates show that $\left\| \Psi(0) \right\| \leq 1$. (We can assume $t + \sigma \geq 0$ since $t + (\tau' - \tau) \geq t - \tau \geq 0$, and we aim at finally choosing $\sigma = \tau' - \tau$.)

Step 2. Lipschitz constant of Ψ . We shall prove that under our assumptions, Ψ is 1/2-Lipschitz on the ball $B(0, 2)$ in the norm $\left\| \cdot \right\|$. Let $W, \widetilde{W} \in B(0, 2)$, and $Z = \Psi(W)$, $\widetilde{Z} = \Psi(\widetilde{W})$. By solving the differential inequality for $Z - \widetilde{Z}$ we get

$$\begin{aligned} Z_{t,\tau} - \widetilde{Z}_{t,\tau} &= \varepsilon \left[\int_0^1 \int_{\tau}^t (s - \tau) \nabla_x F \left(s, x - v(t - s) \right. \right. \\ &\quad \left. \left. + \varepsilon \left(\theta W_{t,s} + (1 - \theta) \widetilde{W}_{t,s} \right) \right) ds d\theta \right] \cdot \left(W_{t,s} - \widetilde{W}_{t,s} \right). \end{aligned}$$

We divide by $R_2(\tau, t)$, take the \mathcal{Z} norm, and note that $R_2(s, t) \leq R_2(\tau, t)$; we get

$$\left\| \left\| (Z_{t,\tau} - \widetilde{Z}_{t,\tau})_{0 \leq \tau \leq t} \right\| \right\| \leq \varepsilon \left\| \left\| (W_{t,s} - \widetilde{W}_{t,s})_{0 \leq s \leq t} \right\| \right\| A(t)$$

with

$$A(t) = \sup_{\sigma, \tau} \int_0^1 \int_{\tau}^t (s-\tau) \left\| \nabla_x F \left(s, x-v(t-s) + \varepsilon \left(\theta W_{t,s} + (1-\theta) \widetilde{W}_{t,s} \right) \right) \right\|_{Z_{t+\sigma}^{\lambda', \mu'}} ds d\theta.$$

By Proposition 4.25 (composition inequality),

$$\begin{aligned} A(t) &\leq \int_{\tau}^t (s-\tau) \left\| \nabla_x F(s, \cdot) \right\|_{Z_{s+\sigma}^{\lambda', \mu' + e(t,s,\sigma)}} ds \\ &= \int_{\tau}^t (s-\tau) \left\| \nabla_x F(s, \cdot) \right\|_{\mathcal{F}^{\lambda's + \lambda'\sigma + \mu' + e(t,s,\sigma)}} ds, \end{aligned}$$

with

$$e(t, s, \sigma) := \varepsilon \left\| \theta W_{t,s} + (1-\theta) \widetilde{W}_{t,s} \right\|_{Z_{t+\sigma}^{\lambda', \mu'}} \leq 2\varepsilon R_2(s, t) \leq 2\varepsilon R_2(\tau, t).$$

Using (5.7), we get

$$\begin{aligned} \lambda's + \lambda'\sigma + \mu' + e(t, s, \sigma) &\leq \lambda's + \lambda'\sigma + \mu' + 2\varepsilon R_2(\tau, t) \\ &\leq \lambda's + \mu = (\lambda s + \mu) - (\lambda - \lambda')s. \end{aligned}$$

Using again the bound on $\nabla_x F$ and the assumption $\widehat{F}(s, 0) = 0$, we deduce

$$A(t) \leq \sup_{\tau} \int_{\tau}^t (s-\tau) C_F e^{-2\pi(\lambda - \lambda')s} ds \leq R_2(0, t) \leq \frac{C_F}{4\pi^2 (\lambda - \lambda')^2}.$$

Using (5.8), we conclude that

$$\left\| \left\| \left(Z_{t,\tau} - \widetilde{Z}_{t,\tau} \right)_{0 \leq \tau \leq t} \right\| \right\| \leq \frac{1}{2} \left\| \left\| \left(W_{t,s} - \widetilde{W}_{t,s} \right)_{0 \leq s \leq t} \right\| \right\|.$$

So Ψ is 1/2-Lipschitz on $B(0, 2)$, and we can conclude the proof of (5.9) by applying Theorem A.2 and choosing $\sigma = \tau' - \tau$.

It remains to control the velocity component of Ω , *i.e.*, establish (5.10); this will follow from the control of the position component. Indeed, if we write $Q_{t,\tau} = \varepsilon^{-1}(\Omega V_{t,\tau} - \text{Id})(x, v)$, we have

$$Q_{t,\tau} = \int_{\tau}^t F \left(s, x - v(t-s) + \varepsilon W_{t,s} \right) ds$$

so we can estimate as before

$$\|Q_{t,\tau}\|_{Z_{t+(\tau'-\tau)}^{\lambda', \mu'}} \leq \int_{\tau}^t \|F(s, \cdot)\|_{\mathcal{F}^{\lambda's + \lambda'(\tau'-\tau) + \mu' + e(t,s,\tau'-\tau)}} ds$$

to get

$$\|Q_{t,\tau}\|_{\mathcal{Z}_{t+(\tau'-\tau)}^{\lambda',\mu'}} \leq \int_{\tau}^t C_F e^{-2\pi(\lambda-\lambda')s} ds \leq R_1(\tau, t).$$

Thus the proof is complete. \square

The proof of Theorem 5.2 is easily adapted to include Sobolev corrections. It is important to note that the scattering transforms are smooth, uniformly in time, not just in gliding regularity ($\tau' = 0$ is admissible in (5.6)). This is why the estimate also resists additional time-shifts in the norm. The following theorem illustrates this; its proof is similar to that of Theorem 5.2.

Theorem 5.4 (Analytic estimates on scattering transforms in two-shift hybrid norms). *Let $\varepsilon > 0$ and let $F = F(t, x)$ on $\mathbb{R}_+ \times \mathbb{T}^d$ satisfy*

$$(5.13) \quad \widehat{F}(t, 0) = 0, \quad \sup_{t \geq 0} \left(\|F(t, \cdot)\|_{\mathcal{F}^{\lambda t + \mu, \gamma}} + \|\nabla_x F(t, \cdot)\|_{\mathcal{F}^{\lambda t + \mu, \gamma}} \right) \leq C_F$$

for some parameters $\lambda, \mu > 0$ and $C_F > 0$. Let $t \geq \tau \geq 0$, and let

$$\Omega_{t,\tau} = (\Omega X_{t,\tau}, \Omega V_{t,\tau})$$

be the scattering transforms associated with εF . Let $0 \leq \lambda' + \lambda'' < \lambda$, $0 \leq \mu' < \mu$ and $\tau', \tau'' \geq 0$ be such that

$$(5.14) \quad \lambda'(\tau' - \tau) + \lambda''(\tau'' - \tau) \leq \frac{(\mu - \mu')}{2}.$$

Let

$$\begin{cases} R_1^b(\tau, t) = C_F e^{-2\pi(\lambda - \lambda' - \lambda'')\tau} \min \{(t - \tau); (2\pi(\lambda - \lambda' - \lambda''))^{-1}\}; \\ R_2^b(\tau, t) = C_F e^{-2\pi(\lambda - \lambda' - \lambda'')\tau} \min \{(t - \tau)^2/2; (2\pi(\lambda - \lambda' - \lambda''))^{-2}\}. \end{cases}$$

Assume that

$$(5.15) \quad \varepsilon R_2(\tau, t) \leq \frac{(\mu - \mu')}{4},$$

and

$$(5.16) \quad \varepsilon C_F \leq \frac{4\pi^2 (\lambda - \lambda' - \lambda'')^2}{2}.$$

Then

$$\|\Omega X_{t,\tau} - \text{Id}\|_{\mathcal{Z}_{(\tau', \tau'')}^{(\lambda', \lambda''), (\mu', \gamma)}} \leq 2\varepsilon R_2(\tau, t)$$

and

$$\|\Omega V_{t,\tau} - \text{Id}\|_{\mathcal{Z}_{(\tau',\tau'')}^{(\lambda',\lambda''),(\mu',\gamma)}} \leq \varepsilon R_1(\tau, t).$$

While Theorem 5.2 (and 5.4) provides a good idea of the scattering estimates associated with gliding regularity, it will not be sufficient for later purposes, and we shall have to establish variants thereof. The motivation for these variants will come only in the course of the next three sections, so we postpone these extensions to Section 10.

Remark 5.5. Loss and Bernard independently suggested to compare the estimates in the present section with the Nekhoroshev theorem in dynamical systems theory [64, 65]. The latter theorem roughly states that for a perturbation of a completely integrable system, trajectories remain close to those of the unperturbed system for a time growing exponentially in the inverse of the size of the perturbation (unlike KAM theory, this result is not global in time; but it is more general in the sense that it also applies outside invariant tori). The fact that our scattering estimates are better (global in time) can probably be attributed to the exponential decay of the force field; in the end, it will demonstrate that the nonlinear retroaction has a positive effect for Landau damping. It might be of interest to explore this more in detail.

6. BILINEAR REGULARITY AND DECAY ESTIMATES

6.1. Presentation. To introduce this crucial section, let us reproduce and improve a key computation from Section 3. Let G be a function of v , and R a time-dependent function of x with $\widehat{R}(0) = 0$; both G and R may be vector-valued. (Think of $G(v)$ as $\nabla_v f(v)$ and of $R(\tau, x)$ as $\nabla W * \rho(\tau, x)$.) Let further

$$\sigma(t, x) = \int_0^t \int_{\mathbb{R}^d} G(v) \cdot R(\tau, x - v(t - \tau)) \, dv \, d\tau.$$

Then

$$\begin{aligned} \widehat{\sigma}(t, k) &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} G(v) \cdot R(\tau, x - v(t - \tau)) e^{-2i\pi k \cdot x} \, dv \, dx \, d\tau \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} G(v) \cdot R(\tau, x) e^{-2i\pi k \cdot x} e^{-2i\pi k \cdot v(t - \tau)} \, dv \, dx \, d\tau \\ &= \int_0^t \widetilde{G}(k(t - \tau)) \cdot \widehat{R}(\tau, k) \, d\tau. \end{aligned}$$

Let us assume that G has a “high” gliding analytic regularity $\bar{\lambda}$, and estimate σ in regularity λt , with $\lambda < \bar{\lambda}$. Let $\alpha = \alpha(t, \tau)$ satisfy

$$0 \leq \alpha(t, \tau) \leq (\bar{\lambda} - \lambda)(t - \tau);$$

then

$$\begin{aligned} \|\sigma(t)\|_{\mathcal{F}^{\lambda t}} &\leq \sum_{k \neq 0} \int_0^t e^{2\pi\lambda t|k|} |\tilde{G}(k(t - \tau))| |\hat{R}(\tau, k)| d\tau \\ &\leq \int_0^t \left(\sup_{k \neq 0} e^{2\pi[\lambda(t - \tau) + \alpha]|k|} |\tilde{G}(k(t - \tau))| \right) \left(\sum_k e^{2\pi(\lambda\tau - \alpha)|k|} |\hat{R}(\tau, k)| \right) d\tau \\ &\leq \left(\sup_{\eta} e^{2\pi\bar{\lambda}|\eta|} |\tilde{G}(\eta)| \right) \left(\sup_{0 \leq \tau \leq t} \|R(\tau, \cdot)\|_{\mathcal{F}^{\lambda\tau - \alpha}} \right) \int_0^t e^{-2\pi[(\bar{\lambda} - \lambda)(t - \tau) - \alpha]} d\tau, \end{aligned}$$

where we have used

$$k \neq 0 \implies 2\pi(\lambda(t - \tau) + \alpha)|k| \leq 2\pi\bar{\lambda}|k|(t - \tau) - ((\bar{\lambda} - \lambda)(t - \tau) - \alpha).$$

Let us choose

$$\alpha(t, \tau) = \frac{(\bar{\lambda} - \lambda)}{2} \min\{1; t - \tau\};$$

then

$$\int_0^t e^{-2\pi[(\bar{\lambda} - \lambda)(t - \tau) - \alpha]} d\tau \leq \int_0^t e^{-\pi(\bar{\lambda} - \lambda)(t - \tau)} d\tau \leq \frac{1}{\pi(\bar{\lambda} - \lambda)}.$$

So in the end

$$\|\sigma(t)\|_{\mathcal{F}^{\lambda t}} \leq \frac{\|G\|_{\mathcal{X}^{\bar{\lambda}}}}{\pi(\bar{\lambda} - \lambda)} \sup_{0 \leq \tau \leq t} \|R(\tau)\|_{\mathcal{F}^{\lambda\tau - \alpha(t, \tau)}},$$

where $\|G\|_{\mathcal{X}^{\bar{\lambda}}} = \sup_{\eta} (e^{2\pi\bar{\lambda}|\eta|} |\tilde{G}(\eta)|)$.

In the preceding computation there are three important things to notice, which lie at the heart of Landau damping:

- The natural index of analytic regularity of σ in x increases linearly in time: this is an automatic consequence of the gliding regularity, already observed in Section 4.
- A bit $\alpha(t, \tau)$ of analytic regularity of G was transferred from G to R , however not more than a fraction of $(\bar{\lambda} - \lambda)(t - \tau)$. We call this the **regularity extortion**: if f forces \bar{f} , it satisfies an equation of the form $\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v \bar{f} = S$, then \bar{f} will give away some (gliding) smoothness to $\rho = \int f dv$.

- The combination of higher regularity of G and the assumption $\widehat{R}(0) = 0$ has been converted into a time decay, so that the time-integral is bounded, uniformly as $t \rightarrow \infty$. Thus there is **decay by regularity**.

The main goal of this section is to establish quantitative variants of these effects in some general situations when G is not only a function of v and R not only a function of t, x . Note that we shall have to work with regularity indices depending on t and τ !

Regularity extortion is related to velocity averaging regularity, well-known in kinetic theory [39]; what is unusual though is that we are working in analytic regularity, and in large time, while velocity averaging regularity is mainly a short-time effect. In fact we shall study two distinct mechanisms for the extortion: the first one will be well suited for short times ($t - \tau$ small), and will be crucial later to get rid of small deteriorations in the functional spaces due to composition; the second one will be well adapted to large times ($t - \tau \rightarrow \infty$) and will ensure convergence of the time integrals.

The estimates in this section lead to a serious twist on the popular view on Landau damping, according to which the waves gives energy to the particles that it forces; instead, the picture here is that the wave gains regularity from the background, and regularity is converted into decay.

For the sake of pedagogy, we shall first establish the basic, simple bilinear estimate, and then discuss the various mechanisms once at a time.

6.2. Basic bilinear estimate.

Proposition 6.1 (Basic bilinear estimate in gliding regularity). *Let $G = G(\tau, x, v)$, $R = R(\tau, x, v)$,*

$$\beta(\tau, x) = \int_{\mathbb{R}^d} (G \cdot R)(\tau, x - v(t - \tau), v) dv,$$

$$\sigma(t, x) = \int_0^t \beta(\tau, x) d\tau.$$

Then

$$(6.1) \quad \|\beta(\tau, \cdot)\|_{\mathcal{F}^{\lambda t + \mu}} \leq \|G\|_{\mathcal{Z}_\tau^{\lambda, \mu; 1}} \|R\|_{\mathcal{Z}_\tau^{\lambda, \mu}};$$

and

$$(6.2) \quad \|\sigma(t, \cdot)\|_{\mathcal{F}^{\lambda t + \mu}} \leq \int_0^t \|G\|_{\mathcal{Z}_\tau^{\lambda, \mu; 1}} \|R\|_{\mathcal{Z}_\tau^{\lambda, \mu}} d\tau.$$

Proof of Proposition 6.1. Obviously (6.2) follows from (6.1). To prove (6.1) we apply successively Propositions 4.15, 4.19 and 4.24:

$$\begin{aligned} \|\beta(\tau, \cdot)\|_{\mathcal{F}^{\lambda t + \mu}} &\leq \left\| \int_{\mathbb{R}^d} (G \cdot R) \circ S_{\tau-t}^0 dv \right\|_{\mathcal{F}^{\lambda t + \mu}} \\ &\leq \left\| (G \cdot R) \circ S_{\tau-t}^0 \right\|_{\mathcal{Z}_t^{\lambda, \mu; 1}} \\ &= \|G \cdot R\|_{\mathcal{Z}_\tau^{\lambda, \mu; 1}} \leq \|G\|_{\mathcal{Z}_\tau^{\lambda, \mu; 1}} \|R\|_{\mathcal{Z}_\tau^{\lambda, \mu}}. \end{aligned}$$

□

6.3. Short-term regularity extortion by time cheating.

Proposition 6.2 (Short-term regularity extortion). *Let $G = G(x, v)$, $R = R(x, v)$, and*

$$\beta(x) = \int_{\mathbb{R}^d} (G \cdot R)(x - v(t - \tau), v) dv.$$

Then for any $\lambda, \mu, t \geq 0$ and any $b > -1$, we have

$$(6.3) \quad \|\beta\|_{\mathcal{F}^{\lambda t + \mu}} \leq \|G\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda(1+b), \mu; 1}} \|R\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda(1+b), \mu; \infty}}.$$

Remark 6.3. If R only depends on t, x , then the norm of R in the right-hand side of (6.3) is $\|R\|_{\mathcal{F}^\nu}$ with

$$\nu = \lambda(1+b) \left| \tau - \frac{bt}{1+b} \right| + \mu = (\lambda\tau + \mu) - b(t - \tau),$$

as soon as $\tau \geq bt/(1+b)$. This is where something has been gained with respect to Proposition 6.1. Even if R is not a function of t, x alone, but rather a function of t, x composed with a function depending on all the variables, this gain will be preserved through the composition inequality.

Proof of Proposition 6.2. The proof presented here may not be the most direct, but has the advantage to explicitly rely on commutators involving ∇_v, ∇_x and the transport semigroup, all of them classically related to hypoelliptic regularity and velocity averaging. Separating the different components of R and G , we may assume that both are scalar-valued.

Let $S = S_{\tau-t}^0$, so that $R \circ S(x, v) = R(x - v(t - \tau), v)$. By direct computation,

$$(6.4) \quad t\nabla_x(R \circ S) = (t\nabla_x R) \circ S = \left[((\tau - b(t - \tau))\nabla_x + (1+b)\nabla_v)R \right] \circ S - (1+b)\nabla_v(R \circ S).$$

Let

$$D = D_{\tau,t,b} := (\tau - b(t - \tau))\nabla_x + (1 + b)\nabla_v.$$

Then (6.4) becomes

$$(6.5) \quad t\nabla_x(R \circ S) = (DR) \circ S - (1 + b)\nabla_v(R \circ S).$$

Since ∇_x commutes with ∇_v and D , and with the composition by S as well, we deduce from (6.5) that

$$\begin{aligned} t\partial_{x_i} \left[(1 + b)^k \nabla_v^k ((D^\ell R) \circ S) \right] &= (1 + b)^k \nabla_v^k (t\partial_{x_i} (D^\ell R) \circ S) \\ &= (1 + b)^k \nabla_v^k ((D^{\ell+1_i} R) \circ S) - (1 + b)^k \nabla_v^k \left((1 + b) \partial_{v_i} (D^\ell R \circ S) \right) \\ &= [(1 + b)\nabla_v]^k ((D^{\ell+1_i} R) \circ S) - [(1 + b)\nabla_v]^{k+1_i} ((D^\ell R) \circ S). \end{aligned}$$

So by induction,

$$(6.6) \quad (t\nabla_x)^n (R \circ S) = \sum_{m \leq n} \binom{n}{m} [-(1 + b)\nabla_v]^m ((D^{n-m} R) \circ S).$$

Applying this formula with R replaced by $G \cdot R$ and integrating in v yields

$$\begin{aligned} (t\nabla_x)^n \int_{\mathbb{R}^d} (G \cdot R) \circ S_{\tau-t}^0 dv &= \int_{\mathbb{R}^d} D^n (G \cdot R) \circ S_{\tau-t}^0 dv \\ &= \int_{\mathbb{R}^d} D^n (G \cdot R) dv. \end{aligned}$$

It follows by taking Fourier transform that

$$\begin{aligned} (2i\pi tk)^n \widehat{\beta}(k) &= \int_{\mathbb{R}^d} [D^n (G \cdot R)] \widehat{dv} \\ &= \int_{\mathbb{R}^d} \left((1 + b)\nabla_v + 2i\pi(\tau - b(t - \tau))k \right)^n (G \cdot R) \widehat{(k, v)} dv, \end{aligned}$$

whence

$$\begin{aligned} &\sum_{k,n} e^{2\pi\mu|k|} \frac{|2\pi\lambda tk|^n}{n!} |\widehat{\beta}(k)| \\ &\leq \sum_{k,n} e^{2\pi\mu|k|} \frac{(\lambda(1 + b))^n}{n!} \left\| \left[\nabla_v + 2i\pi \left(\tau - \frac{bt}{1 + b} \right) k \right]^n (G \cdot R) \widehat{(k, v)} \right\|_{L^1(dv)} \\ &= \|G \cdot R\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda(1+b), \mu; 1}}, \end{aligned}$$

and the conclusion follows by Proposition 4.24. \square

Let us conclude this subsection with some comments on Proposition 6.2. When we wish to apply it, what constraints on $b(t, \tau)$ (assumed to be nonnegative to fix the ideas) does this presuppose? First, b should be small, so that $\lambda(1+b) \leq \bar{\lambda}$ given. But most importantly, we have estimated G_τ in a norm $\mathcal{Z}_{\tau'}$ instead of \mathcal{Z}_τ (this is the time cheating), where $|\tau' - \tau| = bt/(1+b)$. To compensate for this discrepancy, we may apply (4.19), but for this to work $bt/(1+b)$ should be small, otherwise we would lose a large index of analyticity in x — or at best we would inherit an undesirable exponentially growing constant. So all we are allowed is $b(t, \tau) = O(1/(1+t))$. This is not enough to get the time-decay which would lead to Landau damping. Indeed, if $R = R(x)$ with $\widehat{R}(0) = 0$, then

$$\|R\|_{\mathcal{Z}_{\tau - bt/(1+b)}^{\lambda(1+b), \mu}} = \|R\|_{\mathcal{F}^{\lambda\tau + \mu - \lambda b(t-\tau)}} \leq e^{-\lambda b(t-\tau)} \|R\|_{\mathcal{F}^{\lambda\tau + \mu}};$$

so we gain a coefficient $e^{-\lambda b(t-\tau)}$, but then

$$\int_0^t e^{-\lambda b(t-\tau)} d\tau \geq \int_0^t e^{-\lambda \varepsilon (\frac{t-\tau}{t})} d\tau = \left(\frac{1 - e^{-\lambda \varepsilon}}{\lambda \varepsilon} \right) t,$$

which of course diverges in large time.

To summarize: Proposition 6.2 is helpful when $(t - \tau) = O(1)$, which will already be very useful; but for long-time estimates we need another, complementary mechanism.

6.4. Long-term regularity extortion. To search for the extra decay, let us refine the computation of the beginning of this section. Assume that $G_\tau = \nabla_v g_\tau$, where $(g_\tau)_{\tau \geq 0}$ solves a transport-like equation, so $\widetilde{G}(\tau, k, \eta) = 2i\pi\eta \widetilde{g}(\tau, k, \eta)$, and

$$|\widetilde{G}(\tau, k, \eta)| \lesssim 2\pi|\eta| e^{-2\pi\bar{\mu}|k|} e^{-2\pi\bar{\lambda}|\eta+k\tau|}.$$

Up to slightly increasing $\bar{\lambda}$ and $\bar{\mu}$, we may assume

$$(6.7) \quad |\widetilde{G}(\tau, k, \eta)| \lesssim (1 + \tau) e^{-2\pi\bar{\mu}|k|} e^{-2\pi\bar{\lambda}|\eta+k\tau|}.$$

Let then $\rho(\tau, x) = \int f(\tau, x, v) dv$, where also f solves a transport equation, but has a lower analytic regularity; and $R = \nabla W * \rho$. Assuming $|\widehat{\nabla W}(k)| = O(|k|^{-\gamma})$ for some $\gamma \geq 0$, we have

$$(6.8) \quad |\widehat{R}(\tau, k)| \lesssim \frac{e^{-2\pi(\lambda\tau + \mu)|k|} 1_{k \neq 0}}{1 + |k|^\gamma}.$$

Let again

$$\sigma(t, x) = \int_0^t \int_{\mathbb{R}^d} G(\tau, x - v(t - \tau), v) \cdot R(\tau, x - v(t - \tau)) \, dv \, d\tau.$$

As $t \rightarrow +\infty$, G in the integrand of σ oscillates wildly in phase space, so it is not clear that it will help at all. But let us compute:

$$\begin{aligned} \widehat{\sigma}(t, k) &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} G(\tau, x - v(t - \tau), v) \cdot R(\tau, x - v(t - \tau)) e^{-2i\pi k \cdot x} \, dx \, dv \, d\tau \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} G(\tau, x, v) \cdot R(\tau, x) e^{-2i\pi k \cdot x} e^{-2i\pi k \cdot v(t - \tau)} \, dx \, dv \, d\tau \\ &= \int_0^t \int_{\mathbb{R}^d} \widehat{G \cdot R}(\tau, k, v) e^{-2i\pi k \cdot v(t - \tau)} \, dv \, d\tau \\ &= \int_0^t \int_{\mathbb{R}^d} \sum_{\ell} \widehat{G}(\tau, \ell, v) \cdot \widehat{R}(\tau, k - \ell) e^{-2i\pi k \cdot v(t - \tau)} \, dv \, d\tau \\ &= \int_0^t \sum_{\ell} \widetilde{G}(\tau, \ell, k(t - \tau)) \cdot \widehat{R}(\tau, k - \ell) \, d\tau. \end{aligned}$$

At this level, the difference with respect to the beginning of this section lies in the fact that there is a summation over $\ell \in \mathbb{Z}^d$, instead of just choosing $\ell = 0$. Note that $\widehat{\sigma}(t, 0) = \int_0^t \iint G(\tau, x, v) \cdot R(\tau, x) \, dx \, dv \, d\tau = 0$, because G is a v -gradient.

From (6.7) and (6.8) we deduce

$$\begin{aligned} &\sum_k e^{2\pi(\lambda t + \mu)|k|} |\widehat{\sigma}(t, k)| \\ &\lesssim \int_0^t (1 + \tau) \sum_{\ell \neq k, k \neq 0} e^{2\pi\mu|k|} e^{2\pi\lambda t|k|} e^{-2\pi\bar{\mu}|\ell|} e^{-2\pi\bar{\lambda}|k(t - \tau) + \ell\tau|} e^{-2\pi\mu|k - \ell|} \frac{e^{-2\pi\lambda\tau|k - \ell|}}{1 + |k - \ell|^\gamma}. \end{aligned}$$

Using the inequalities

$$e^{-2\pi\mu|k - \ell|} e^{2\pi\mu|k|} e^{-2\pi\bar{\mu}|\ell|} \leq e^{-2\pi(\bar{\mu} - \mu)|\ell|}$$

and

$$e^{-2\pi\lambda\tau|k - \ell|} e^{2\pi\lambda t|k|} e^{-2\pi\bar{\lambda}|k(t - \tau) + \ell\tau|} \leq e^{-2\pi(\bar{\lambda} - \lambda)|k(t - \tau) + \ell\tau|},$$

we end up with

$$\|\sigma(t)\|_{\mathcal{F}^{\lambda t + \mu}} \lesssim \sum_{k \neq 0, \ell \neq k} \frac{e^{-2\pi(\bar{\mu} - \mu)|\ell|}}{1 + |k - \ell|^\gamma} \int_0^t e^{-2\pi(\bar{\lambda} - \lambda)|k(t - \tau) + \ell\tau|} (1 + \tau) \, d\tau.$$

If it were not for the negative exponential, the time-integral would be $O(t^2)$ as $t \rightarrow \infty$. The exponential helps only a bit: its argument vanishes e.g. for $d = 1$, $k > 0$, $\ell < 0$ and $\tau = (k/(k + |\ell|))t$. Thus we have the essentially optimal bounds

$$(6.9) \quad \int_0^t e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} d\tau \leq \frac{1}{\pi(\bar{\lambda}-\lambda)|k-\ell|}$$

and

$$(6.10) \quad \int_0^t e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} \tau d\tau \leq \frac{1}{2\pi^2(\bar{\lambda}-\lambda)^2|k-\ell|^2} + \left(\frac{1}{\pi(\bar{\lambda}-\lambda)} \right) \frac{|k|t}{|k-\ell|}.$$

From this computation we conclude that:

- The higher regularity of G has allowed to reduce the time-integral thanks to a factor $e^{-\alpha|k(t-\tau)+\ell\tau|}$, but this factor is not small when τ/t is equal to $k/(k-\ell)$. As discussed in the next section, this reflects an important physical phenomenon called (plasma) echo.

- If we had (in “gliding” norm) $\|G_\tau\| = O(1)$ this would ensure a uniform bound on the integral, as soon as $\gamma > 0$, thanks to (6.9) and

$$\sum_{k,\ell} \frac{e^{-\alpha|\ell|}}{(1+|k-\ell|)^{1+\gamma}} < +\infty.$$

- But G_τ is a velocity-gradient, so — unless of course G depends only on v — $\|G_\tau\|$ diverges like $O(\tau)$ as $\tau \rightarrow \infty$, which implies a divergence of our bounds in large time, as can be seen from (6.10). If $\gamma \leq 1$ this comes with a divergence in the k variable, since in this case $\sum_{k,\ell} \frac{e^{-\alpha|\ell||k|}}{(1+|k-\ell|)^{1+\gamma}} = +\infty$. (The Coulomb case corresponds to $\gamma = 1$, so in this respect it has a borderline divergence.)

For the moment let us adapt the previous computation to the formalism of hybrid norms, still assuming $R = R(\tau, x)$. Note that we shall only estimate the *seminorm* of σ , not the full norm.

Proposition 6.4 (Long-term extortion and decay). *Let $G = G(\tau, x, v)$, $R = R(\tau, x)$, and*

$$\sigma(t, x) = \int_0^t \int_{\mathbb{R}^d} G(\tau, x - v(t-\tau), v) \cdot R(\tau, x - v(t-\tau)) dv d\tau.$$

Then for any $\bar{\lambda} > \lambda > 0$, $\bar{\mu} > \mu > 0$, $\gamma > 0$,

$$(6.11) \quad \|\sigma(t, \cdot)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t K(t, \tau) \|R_\tau\|_{\mathcal{F}^{\lambda\tau + \mu, \gamma}} d\tau,$$

where

$$(6.12) \quad K(t, \tau) = e^{-2\pi(\bar{\lambda}-\lambda)(t-\tau)} \left\| \int_{\mathbb{T}^d} G(\tau, x, \cdot) dx \right\|_{\mathcal{C}^{\bar{\lambda};1}} + \left(\sup_{0 \leq \tau \leq t} \frac{\|G_\tau\|_{\mathcal{Z}_\tau^{\bar{\lambda}, \bar{\mu};1}}}{1 + \tau} \right) K_1(t, \tau),$$

and

$$(6.13) \quad K_1(t, \tau) = (1 + \tau) \sup_{k \neq 0, \ell \neq 0} \frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{-2\pi(\bar{\mu}-\mu)|\ell|}}{1 + |k - \ell|^\gamma}.$$

Remark 6.5. Notice the sharp contrast with Proposition 6.2 as far as the time shifts are concerned: now G_τ is measured in a norm with time-shift τ .

Remark 6.6. It is essential in (6.12) to separate the contribution of $\widehat{G}(\tau, 0, v)$ from the rest. Indeed, if we removed the restriction $\ell \neq 0$ in (6.16) the kernel K_1 would be too large to be correctly controlled in large time. What makes this separation reasonable is that, although in cases of application $G(\tau, x, v)$ is expected to grow like $O(\tau)$ in large time, the spatial average $\int G(\tau, x, v) dx$ is expected to be bounded.

Proof of Proposition 6.4. As in the computation opening this subsection, we have

$$\widehat{\sigma}(t, k) = \int_0^t \sum_{\ell \neq k} \left(\int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right) \widehat{R}(\tau, k - \ell) d\tau.$$

Discarding the contribution of $k = 0$, we have

$$\begin{aligned} \|\sigma(t, \cdot)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} &\leq \sum_{k \neq 0} e^{2\pi\lambda t|k|} e^{2\pi\mu|k|} \int_0^t \sum_{\ell \neq k} \left| \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| |\widehat{R}(\tau, k - \ell)| d\tau \\ &\leq \sum_{\ell \neq k \neq 0} \int_0^t e^{2\pi\bar{\mu}|\ell|} e^{2\pi\mu|k-\ell|} e^{-2\pi(\bar{\mu}-\mu)|\ell|} e^{2\pi\lambda\tau|k-\ell|} e^{2\pi\bar{\lambda}|k(t-\tau)+\ell\tau|} e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} \\ &\quad \left| \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| |\widehat{R}(\tau, k - \ell)| d\tau. \end{aligned}$$

Next,

$$\begin{aligned}
& e^{2\pi\bar{\lambda}|k(t-\tau)+\ell\tau|} \left| \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| \\
&= \sum_{n \in \mathbb{N}_0^d} \frac{(2\pi\bar{\lambda})^n}{n!} |k(t-\tau) + \ell\tau|^n \left| \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| \\
&= \sum_{n \in \mathbb{N}_0^d} \frac{\bar{\lambda}^n}{n!} \left| \int_{\mathbb{R}^d} (2i\pi)^n (k(t-\tau) + \ell\tau)^n \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right|.
\end{aligned}$$

Since $2i\pi(k(t-\tau) + \ell\tau) e^{-2i\pi k \cdot v(t-\tau)} = (-\nabla_v + 2i\pi\ell\tau) e^{-2i\pi k \cdot v(t-\tau)}$, the v -integral coincides with

$$\begin{aligned}
& \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) (-\nabla_v + 2i\pi\ell\tau)^n e^{-2i\pi k \cdot v(t-\tau)} dv \\
&= \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\ell\tau)^n \widehat{G}(\tau, \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv.
\end{aligned}$$

Reporting this in the estimate of σ , we conclude in the end that

$$\begin{aligned}
\|\sigma(t)\|_{\dot{X}^{\lambda t + \mu}} &\leq \int_0^t \left(\inf_{\ell \neq k \neq 0} e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{-2\pi(\bar{\mu}-\mu)|\ell|} \right) \\
&\left(\sum_{k \in \mathbb{Z}^d} |\widehat{R}(\tau, k)| e^{2\pi(\lambda\tau + \mu)|k|} \right) \left(\sum_{\ell \in \mathbb{Z}^d, n \in \mathbb{N}_0^d} e^{2\pi\bar{\mu}|\ell|} \frac{\bar{\lambda}^n}{n!} \left\| (\nabla_v + 2i\pi\ell\tau)^n \widehat{G}(\tau, \ell, v) \right\|_{L^1(dv)} \right) d\tau.
\end{aligned}$$

The conclusion follows easily. \square

6.5. Extension to the velocity dependence. Even though we will not need it in the proof of Theorem 2.5, it is a natural question whether Proposition 6.4 extends to the case when R also depends on v : this problem arises immediately if R is a density composed by a scattering, or if the interaction has a velocity dependence.

In this situation we did not manage to find a relevant inequality in the setting of the “plain” hybrid norms from Section 4; instead, we will be able to provide such an estimate in the setting of the *two-shift* hybrid norms from Subsection 4.10:

Proposition 6.7 (Long-term extortion with velocity dependence). *Let $G = G(\tau, x, v)$, $R = R(\tau, x, v)$, and*

$$\sigma(t, x) = \int_0^t \int_{\mathbb{R}^d} G(\tau, x - v(t-\tau), v) \cdot R(\tau, x - v(t-\tau), v) dv d\tau.$$

Then for any $\bar{\lambda} > \lambda > 0$, $\bar{\mu} > \mu > 0$, $\gamma > 0$,

$$(6.14) \quad \|\sigma(t, \cdot)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t K(t, \tau) \|R_\tau\|_{\mathcal{Z}_{(\tau, 0)}^{(\lambda, \bar{\lambda} - \lambda), (\mu, \gamma)}} d\tau,$$

where

$$(6.15) \quad K(t, \tau) = e^{-2\pi(\bar{\lambda} - \lambda)(t - \tau)} \left\| \int_{\mathbb{T}^d} G(\tau, x, \cdot) dx \right\|_{\mathcal{C}^{\bar{\lambda}, 1}} + \left(\sup_{0 \leq \tau \leq t} \frac{\|G_\tau\|_{\mathcal{Z}_{\tau}^{\bar{\lambda}, \bar{\mu}, 1}}}{1 + \tau} \right) K_1(t, \tau),$$

and

$$(6.16) \quad K_1(t, \tau) = (1 + \tau) \sup_{k \neq 0, \ell \neq 0} \frac{e^{-2\pi(\bar{\lambda} - \lambda)|k(t - \tau) + \ell\tau|} e^{-2\pi(\bar{\mu} - \mu)|\ell|}}{1 + |k - \ell|^\gamma}.$$

Remark 6.8. If $R = R(\tau, x)$ then in view of Proposition 4.32 the norm of R inside (6.14) is just $\|R_\tau\|_{\mathcal{F}^{\lambda\tau + \mu, \gamma}}$, and we are back to Proposition 6.4.

Proof of Proposition 6.7. Let us first assume that $\int G(\tau, x, v) dx = 0$. We start by computing the Fourier transform of σ :

$$\begin{aligned} \widehat{\sigma}(t, k) &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} (G \cdot R)(\tau, x, v) e^{-2i\pi k \cdot x} e^{-2i\pi k \cdot v(t - \tau)} dx dv d\tau \\ &= \int_0^t \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{R}^d} \widehat{G}(\tau, \ell, v) \cdot \widehat{R}(\tau, k - \ell, v) e^{-2i\pi k \cdot v(t - \tau)} dv d\tau. \end{aligned}$$

Taking into account only nonzero values of k , we have

$$\begin{aligned} \|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} &= \sum_{k \neq 0} e^{2\pi(\lambda t + \mu)|k|} \left| \int_0^t \sum_{\ell \neq 0} \int_{\mathbb{R}^d} \widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t - \tau)} dv \right| d\tau \\ &\leq \int_0^t \sum_{k \neq 0, \ell \neq 0} e^{2\pi(\lambda t + \mu)|k|} \left| \int_{\mathbb{R}^d} \widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t - \tau)} dv \right| d\tau \\ &= \int_0^t \sum_{k, \ell, n} e^{2\pi\mu|k|} \frac{(2\pi\lambda|k|t)^n}{n!} \left| \int_{\mathbb{R}^d} \widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t - \tau)} dv \right| d\tau \\ &= \int_0^t \sum_{k, \ell, n} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \left| \int_{\mathbb{R}^d} (2i\pi k t)^n \widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t - \tau)} dv \right| d\tau. \end{aligned}$$

(Needless to say, it is crucial to have the v -integral inside the modulus, not outside.)

Next, we use

$$(2i\pi kt)^n e^{-2i\pi k \cdot v(t-\tau)} = (-\nabla_v + 2i\pi k\tau)^n e^{-2i\pi k \cdot v(t-\tau)}.$$

After integration by parts in the v -integral, this results in
(6.17)

$$\|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t \sum_{k, \ell, n} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \left| \int_{\mathbb{R}^d} (\nabla_v + 2i\pi k\tau)^n (\widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v)) e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau.$$

By the Leibniz-type formula from Lemma A.1, with $a = 2i\pi\ell\tau$ and $b = 2i\pi(k-\ell)\tau$,

$$\begin{aligned} & (\nabla_v + 2i\pi k\tau)^n (\widehat{G}(\ell, v) \cdot \widehat{R}(k - \ell, v)) \\ &= \sum_m \binom{n}{m} (\nabla_v + 2i\pi\tau\ell)^m \widehat{G}(\ell, v) \cdot (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}(k - \ell, v). \end{aligned}$$

Plugging this in (6.17) we get

$$\begin{aligned} \|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} &\leq \int_0^t \sum_{k, \ell, n} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \\ &\left| \sum_m \binom{n}{m} \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau\ell)^m \widehat{G}_\tau(\ell, v) \cdot (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau. \end{aligned}$$

If now we exchange the v -integral and the modulus, we shall be back to the old estimate from Proposition 6.1. Instead, we insert (inside the k, ℓ summation, but outside the m summation)

$$1 = e^{-2\pi(\bar{\lambda} - \lambda)|k(t-\tau) + \ell\tau|} \sum_{q \in \mathbb{N}_0^d} \frac{\left| 2i\pi(\bar{\lambda} - \lambda)(k(t-\tau) + \ell\tau) \right|^q}{q!}.$$

Thus we get

$$\begin{aligned} \|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} &\leq \int_0^t \sum_{k, n, \ell, q} e^{2\pi\mu|k|} e^{-2\pi(\bar{\lambda} - \lambda)|k(t-\tau) + \ell\tau|} \frac{\lambda^n (\bar{\lambda} - \lambda)^q}{n! q!} \\ &\left| \sum_m \binom{n}{m} \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau\ell)^m \widehat{G}_\tau(\ell, v) \cdot (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) \right. \\ &\quad \left. (2i\pi(k(t-\tau) + \ell\tau))^q e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau. \end{aligned}$$

The v -integral above is equal to

$$\begin{aligned}
& \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau\ell)^m \widehat{G}_\tau(\ell, v) \cdot (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) (-\nabla_v + 2i\pi\tau\ell)^q e^{-2i\pi k \cdot v(t-\tau)} dv \\
&= \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau\ell)^q \left[(\nabla_v + 2i\pi\tau\ell)^m \widehat{G}_\tau(\ell, v) \cdot (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) \right] e^{-2i\pi k \cdot v(t-\tau)} dv \\
&= \sum_{r \leq q} \binom{q}{r} \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \cdot \nabla_v^{q-r} (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv,
\end{aligned}$$

where the last step follows from Lemma A.1 again, this time with $a = 2i\pi\tau\ell$ and $b = 0$. We deduce

$$\begin{aligned}
\|\sigma(t)\|_{\dot{\mathcal{X}}^{\lambda t + \mu}} &\leq \int_0^t \sum_{k, n, \ell, q} e^{2\pi\mu|k|} e^{-2\pi(\bar{\lambda} - \lambda)|k(t-\tau) + \ell\tau|} \frac{\lambda^n (\bar{\lambda} - \lambda)^q}{n! q!} \left| \sum_{m, r} \binom{n}{m} \binom{q}{r} \right. \\
&\quad \left. \int (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \cdot \nabla_v^{q-r} (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau \\
&\leq \int_0^t \sum_{k, n, \ell, q, m, r} e^{-2\pi(\bar{\lambda} - \lambda)|k(t-\tau) + \ell\tau|} e^{2\pi\mu|k|} \frac{\lambda^n (\bar{\lambda} - \lambda)^q}{m! (n-m)! r! (q-r)!} \\
&\quad \left\| (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \right\|_{L^1(dv)} \left\| \nabla_v^{q-r} (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) \right\|_{L^\infty(dv)} d\tau \\
&\leq \int_0^t \sum_{k, n, \ell, q, m, r} e^{-2\pi(\bar{\lambda} - \lambda)|k(t-\tau) + \ell\tau|} e^{-2\pi(\bar{\mu} - \mu)|\ell|} e^{2\pi\bar{\mu}|\ell|} \frac{e^{2\pi\mu|k-\ell|}}{1 + |k - \ell|^\gamma} (1 + |k - \ell|^\gamma) \\
&\quad \frac{\lambda^m \lambda^{n-m} (\bar{\lambda} - \lambda)^r (\bar{\lambda} - \lambda)^{q-r}}{m! (n-m)! r! (q-r)!} \left\| (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \right\|_{L^1(dv)} \\
&\quad \left\| \nabla_v^{q-r} (\nabla_v + 2i\pi\tau(k - \ell))^{n-m} \widehat{R}_\tau(k - \ell, v) \right\|_{L^\infty(dv)} d\tau,
\end{aligned}$$

and therefore

$$\begin{aligned}
\|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} &\leq \int_0^t \sup_{k,\ell} \left(\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{-2\pi(\bar{\mu}-\mu)|\ell|}}{1+|k-\ell|^\gamma} \right) \\
&\quad \sum_{k,p,\ell,s,m,r} \frac{\lambda^m \lambda^p (\bar{\lambda}-\lambda)^r (\bar{\lambda}-\lambda)^s}{m! p! r! s!} e^{2\pi\bar{\mu}|\ell|} e^{2\pi\mu|k-\ell|} (1+|k-\ell|^\gamma) \left\| (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \right\|_{L^1(dv)} \\
&\quad \left\| \nabla_v^s (\nabla_v + 2i\pi\tau(k-\ell))^p \widehat{R}_\tau(k-\ell, v) \right\|_{L^\infty(dv)} d\tau \\
&= \int_0^t \frac{K_1(t, \tau)}{1+\tau} \left(\sum_{m,r,\ell} \frac{e^{2\pi\bar{\mu}|\ell|} \lambda^m (\bar{\lambda}-\lambda)^r}{m! r!} \left\| (\nabla_v + 2i\pi\tau\ell)^{m+r} \widehat{G}_\tau(\ell, v) \right\|_{L^1(dv)} \right) \\
&\quad \left(\sum_{p,\ell,s} e^{2\pi\mu|k-\ell|} (1+|k-\ell|^\gamma) \frac{\lambda^p (\bar{\lambda}-\lambda)^s}{p! s!} \left\| \nabla_v^s (\nabla_v + 2i\pi\tau(k-\ell))^p \widehat{R}_\tau(k-\ell, v) \right\|_{L^\infty(dv)} \right) d\tau \\
&= \int_0^t \frac{K_1(t, \tau)}{1+\tau} \|G_\tau\|_{\mathcal{Z}_\tau^{\bar{\lambda}, \bar{\mu}, 1}} \|R_\tau\|_{\mathcal{Z}_{(\tau,0)}^{(\lambda, \bar{\lambda}-\lambda), (\mu, \gamma)}} d\tau,
\end{aligned}$$

and the estimate is complete.

It remains to include the contribution of $\int G(\tau, x, v) dx$, *i.e.*, the term $\ell = 0$. This is done in just the same way, noting that

$$\sup_{k \neq 0} \frac{e^{-2\pi(\bar{\lambda}-\lambda)|k|(t-\tau)}}{1+|k|^\gamma} \leq e^{-2\pi(\bar{\lambda}-\lambda)(t-\tau)}.$$

(For this term we do not need to take advantage of the parameter γ .) \square

6.6. Combined regularity extortion and decay estimate. Now we shall *combine* the two mechanisms appearing in Propositions 6.2 and 6.4 (or 6.7).

Theorem 6.9 (Combined regularity transfer and regularity-to-decay). *Let $G = G(\tau, x, v)$, $R = R(\tau, x, v)$, and*

$$\sigma(t, x) = \int_0^t \int_{\mathbb{R}^d} G(\tau, x - v(t-\tau), v) \cdot R(\tau, x - v(t-\tau), v) dv d\tau.$$

Let $\lambda, \bar{\lambda}, \mu, \bar{\mu}$, $\mu' = \mu'(t, \tau)$, such that $\bar{\lambda} > \lambda > 0$, $\bar{\mu} \geq \mu'(\tau, t) > \mu > 0$ and $b = b(t, \tau) \geq 0$ such that

$$(6.18) \quad \forall 0 \leq \tau \leq t, \quad b \leq \frac{1}{2t} \left(\frac{\bar{\mu} - \mu}{\bar{\lambda} - \lambda} \right).$$

Then

$$\|\sigma(t, \cdot)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t K_0^G(t, \tau) \|R_\tau\|_{\mathcal{Z}^{((1+b)\lambda, \bar{\lambda}-\lambda), \mu'}(\tau - \frac{bt}{1+b}, 0)}} d\tau + \int_0^t K_1^G(t, \tau) \|R_\tau\|_{\mathcal{Z}^{((1+b)\lambda, \bar{\lambda}-\lambda), (\mu', \gamma)}(\tau - \frac{bt}{1+b}, 0)}} d\tau$$

where

$$(6.19) \quad K_0^G(t, \tau) = e^{-2\pi(\frac{\bar{\lambda}-\lambda}{2})(t-\tau)} \left\| \int_{\mathbb{R}^d} G(\tau, x, \cdot) dx \right\|_{\mathcal{C}^{\bar{\lambda}+b\lambda; 1}}$$

and

$$(6.20) \quad K_1^G(t, \tau) = \left(\sup_{0 \leq \tau \leq t} \frac{\|G_\tau\|_{\mathcal{Z}^{\bar{\lambda}+b\lambda, \bar{\mu}}(\tau - \frac{bt}{1+b}, 0)}}{1 + \tau} \right) K_1(t, \tau),$$

and

$$(6.21) \quad K_1(t, \tau) = (1 + \tau) \sup_{k \neq 0, \ell \neq 0} e^{-2\pi(\frac{\bar{\mu}-\mu}{2})|\ell|} \left(\frac{e^{-2\pi(\mu'-\mu)|k-\ell|}}{1 + |k-\ell|^\gamma} \right) e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|}.$$

In particular, if R does not depend on v , we have

$$\|\sigma(t, \cdot)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t K_0^G(t, \tau) \|R_\tau\|_{\mathcal{F}^\nu} d\tau + \int_0^t K_1^G(t, \tau) \|R_\tau\|_{\mathcal{F}^{\nu, \gamma}} d\tau,$$

where

$$\nu = (1 + b)\lambda \left| \tau - \frac{bt}{1+b} \right| + \mu'.$$

Proof of Theorem 6.9. The proof is basically obtained by reproducing in Fourier space the proof of Proposition 6.2 and mixing it with the proof of Proposition 6.7. Let us first assume $\widehat{G}(\tau, 0, v) = 0$, and write again

$$\|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t + \mu}} \leq \int_0^t \sum_{k, \ell, n} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \left| \int_{\mathbb{R}^d} (2i\pi kt)^n \widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k - \ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau;$$

but now we write

$$\begin{aligned} (2i\pi kt)^n e^{-2i\pi k \cdot v(t-\tau)} &= (1 + b)^n \left(\frac{2i\pi kt}{1 + b} \right)^n e^{-2i\pi k \cdot v(t-\tau)} \\ &= (1 + b)^n (-\nabla_v + 2i\pi k \tau')^n e^{-2i\pi k \cdot v(t-\tau)}, \end{aligned}$$

where

$$\tau' = \tau - \frac{bt}{1+b}.$$

Integrating by parts in the v -integral and applying Lemma A.1, this gives

$$\begin{aligned}
\|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} &\leq \int_0^t \sum_{k,\ell,n} e^{2\pi\mu|k|} \frac{(1+b)^n}{n!} \lambda^n \\
&\quad \left| \int_{\mathbb{R}^d} (\nabla_v + 2i\pi k\tau')^n [\widehat{G}_\tau(\ell, v) \cdot \widehat{R}_\tau(k-\ell, v)] e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau \\
&= \int_0^t \sum_{k,\ell,n} e^{2\pi\mu|k|} \frac{(1+b)^n}{n!} \lambda^n \\
&\quad \left| \sum_m \binom{n}{m} \int_{\mathbb{R}^d} (\nabla_v + 2i\pi\tau'\ell)^m \widehat{G}_\tau(\ell, v) \cdot (\nabla_v + 2i\pi\tau'(k-\ell))^{n-m} \widehat{R}_\tau(k-\ell, v) e^{-2i\pi k \cdot v(t-\tau)} dv \right| d\tau.
\end{aligned}$$

Now we insert

$$1 = e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau'|} \sum_{q \in \mathbb{N}_0^d} \frac{\left| 2i\pi(\bar{\lambda}-\lambda)(\tau'\ell + k(t-\tau)) \right|^q}{q!}.$$

Reasoning as in the proof of Proposition 6.7, we get

$$\begin{aligned}
\|\sigma(t)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} &\leq \int_0^t \sup_{k,\ell} \left(\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau'|} e^{-2\pi(\bar{\mu}-\mu)|\ell|} e^{-2\pi(\mu'-\mu)|k-\ell|}}{1 + |k-\ell|^\gamma} \right) \\
&\quad \sum_{k,p,\ell,s,m,r} \frac{((1+b)\lambda)^m ((1+b)\lambda)^p (\bar{\lambda}-\lambda)^r (\bar{\lambda}-\lambda)^s}{m! p! r! s!} e^{2\pi\bar{\mu}|\ell|} e^{2\pi\mu'|k-\ell|} (1 + |k-\ell|^\gamma) \\
&\quad \left\| (\nabla_v + 2i\pi\tau'\ell)^{m+r} \widehat{G}_\tau(\ell, v) \right\|_{L^1(dv)} \left\| \nabla_v^s (\nabla_v + 2i\pi\tau'(k-\ell))^p \widehat{R}_\tau(k-\ell, v) \right\|_{L^\infty(dv)} d\tau \\
&\leq \int_0^t \frac{K'_1(t, \tau)}{1 + \tau} \|G_\tau\|_{\mathcal{Z}^{\left(\frac{1+b}{\tau-\frac{bt}{1+b}}, \bar{\lambda}-\lambda, \bar{\mu}; 1\right)}} \|R_\tau\|_{\mathcal{Z}^{\left(\frac{(1+b)\lambda, \bar{\lambda}-\lambda, (\mu', \gamma)}{\left(\tau-\frac{bt}{1+b}, 0\right)}\right)}} d\tau,
\end{aligned}$$

where

$$K'_1(t, \tau) = (1 + \tau) \sup_{k,\ell} \left(\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau'|} e^{-2\pi(\bar{\mu}-\mu)|\ell|} e^{-2\pi(\mu'-\mu)|k-\ell|}}{1 + |k-\ell|^\gamma} \right).$$

Moreover,

$$\begin{aligned} e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau'|} e^{-2\pi(\bar{\mu}-\mu)|\ell} &\leq e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{2\pi(\bar{\lambda}-\lambda)|\tau'-\tau||\ell|} e^{-2\pi(\bar{\mu}-\mu)|\ell} \\ &\leq e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{-2\pi((\bar{\mu}-\mu)-(\bar{\lambda}-\lambda)|\tau'-\tau||\ell|)} \\ &\leq e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|} e^{-2\pi\left(\frac{\bar{\mu}-\mu}{2}\right)|\ell|} \end{aligned}$$

by using assumption (6.18). We conclude that

$$\|\sigma\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} \leq \int_0^t \frac{K_1(t, \tau)}{1+\tau} \|G_\tau\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{\bar{\lambda}+b\lambda, \bar{\mu}; 1}} \|R_\tau\|_{\mathcal{Z}_{\left(\tau-\frac{bt}{1+b}, 0\right)}^{((1+b)\lambda, \bar{\lambda}-\lambda), (\mu', \gamma)}}$$

Finally, the contribution of $\ell = 0$ is handled separately, as in Proposition 6.7, and yields the first term in (6.19); this estimate is insensitive to γ . \square

7. CONTROL OF THE TIME-RESPONSE

To motivate this section, let us start from (3.3), but now assume that f^0 depends on t, x, v and that there is an extra source term S , decaying in time. Thus the equation is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - (\nabla W * \rho) \cdot \nabla_v f^0 = S,$$

and the equation for the density ρ , as in the proof of Theorem 3.1, is

(7.1)

$$\begin{aligned} \rho(t, x) &= \int_{\mathbb{R}^d} f_i(x-vt, v) dv + \int_0^t \int_{\mathbb{R}^d} \nabla_v f^0(\tau, x-v(t-\tau), v) \cdot (\nabla W * \rho)(\tau, x-v(t-\tau)) dv d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^d} S(\tau, x-v(t-\tau), v) dv d\tau. \end{aligned}$$

Hopefully we may apply Theorem 6.9 to deduce from (7.1) an integral inequality on $\varphi(t) := \|\rho(t)\|_{\mathcal{F}^{\lambda t+\mu}}$, which will look like

$$(7.2) \quad \varphi(t) \leq A + c \int_0^t K(t, \tau) \varphi(\tau) d\tau,$$

where A is the contribution of the initial datum and the source term, and $K(t, \tau)$ a kernel looking like, say, (6.21).

Then from (7.2) how do we proceed? Assume for a start that a smallness condition of the form (a) in Proposition 2.1 is satisfied. Then the simple and natural way, as in Section 3, would be to write

$$\varphi(t) \leq A + c \left(\int_0^t K(t, \tau) d\tau \right) \left(\sup_{0 \leq \tau \leq t} \varphi(\tau) \right),$$

and deduce

$$(7.3) \quad \varphi(t) \leq \frac{A}{\left(1 - c \int_0^t K(t, \tau) d\tau\right)}$$

(assuming of course the denominator to be positive). However, if K is given by (6.21), it is easily seen that $\int_0^t K(t, \tau) d\tau \geq \kappa t$ as $t \rightarrow \infty$, where $\kappa > 0$; then (7.3) is useless. In fact (7.2) *does not* prevent φ from going to $+\infty$ as $t \rightarrow \infty$. Nevertheless, its growth may be controlled under certain assumptions, as we shall see in this section. Before embarking on cumbersome calculations, we shall start with a qualitative discussion.

7.1. Qualitative discussion. The kernel K in (6.21) depends on the choice of $\mu' = \mu(t, \tau)$. How large $\mu' - \mu$ can be depends in turn on the amount of regularization offered by the convolution with the interaction ∇W . We shall distinguish two cases.

7.1.1. *Analytic interaction.* If ∇W is analytic, there is $\sigma > 0$ such that

$$\forall \nu \geq 0, \quad \|\rho * \nabla W\|_{\mathcal{F}^{\nu+\sigma}} \leq C \|\rho\|_{\mathcal{F}^\nu};$$

then in (6.21) we can afford to choose, say, $\mu' - \mu = \sigma/2$, and $\gamma = 0$. Thus K is bounded by

$$(7.4) \quad \overline{K}^{(\alpha)}(t, \tau) = (1 + \tau) \sup_{k \neq 0, \ell \neq 0} e^{-\alpha|\ell|} e^{-\alpha|k-\ell|} e^{-\alpha|k(t-\tau)+\ell\tau|},$$

where $\alpha = \frac{1}{2} \min\{\overline{\lambda} - \lambda; \overline{\mu} - \mu; \sigma\}$. Let us work in dimension $d = 1$. The goal is to estimate solutions of

$$(7.5) \quad \varphi(t) \leq a + c \int_0^t \overline{K}^{(\alpha)}(t, \tau) \varphi(\tau) d\tau.$$

Whenever τ/t is a rational number distinct from 0 or 1, there are $k, \ell \in \mathbb{Z}$ such that $|k(t - \tau) + \ell\tau| = 0$, and the size of $\overline{K}^{(\alpha)}(t, \tau)$ mainly depends on the minimum

admissible values of k and $k - \ell$. Looking at values of τ/t of the form $1/(n + 1)$ or $n/(n + 1)$ suggests the approximation

$$(7.6) \quad \overline{K}^{(\alpha)} \lesssim (1 + \tau) \min \left\{ e^{-\alpha(\frac{\tau}{t-\tau})} e^{-2\alpha}; e^{-2\alpha(\frac{t-\tau}{\tau})} e^{-\alpha} \right\},$$

which seems to be an exact upper bound for $\tau \geq t/2$. But this estimate is terrible: the time-integral of the right-hand side is much larger than the integral of $\overline{K}^{(\alpha)}$. In fact, the fast variation and “wiggling” behavior of $\overline{K}^{(\alpha)}$ are essential to get decent estimates.

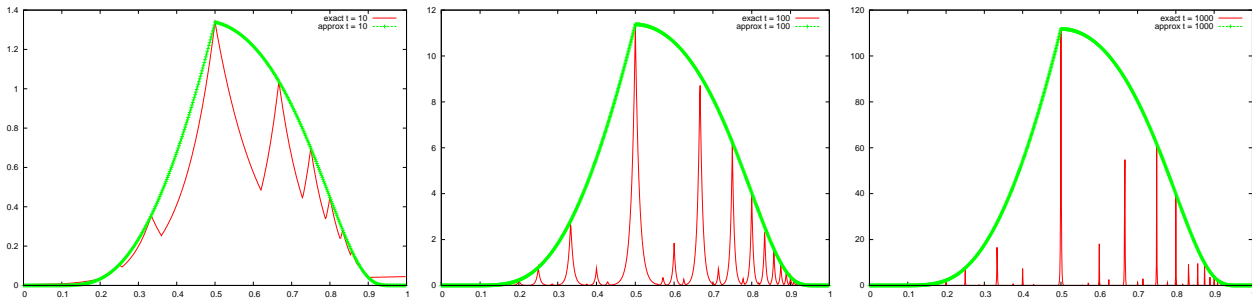


FIGURE 5. the kernel $\overline{K}^{(\alpha)}(t, \tau)$, together with the approximate upper bound in (7.6), for $\alpha = 0.5$ and $t = 10, t = 100, t = 1000$.

To get a better feeling for $\overline{K}^{(\alpha)}$, let us only retain the term in $k = 1, \ell = -1$; this looks reasonable since we have an exponential decay as k or ℓ go to infinity (anyway, throwing away all other terms can only improve the estimates). So we look at $\tilde{K}^{(\alpha)}(t, \tau) = (1 + \tau) e^{-3\alpha} e^{-\alpha|t-2\tau|}$. Let us time-rescale by setting $k_t(\theta) = t \tilde{K}^{(\alpha)}(t, t\theta)$ for $\theta \in [0, 1]$ (the t factor because $d\tau = t d\theta$); then it is not hard to see that

$$\frac{k_t}{t} \longrightarrow \frac{e^{-3\alpha}}{2\alpha} \delta_{\frac{1}{2}}$$

This suggests the following baby model for (7.5):

$$(7.7) \quad \varphi(t) \leq a + ct \varphi\left(\frac{t}{2}\right).$$

The important point in (7.7) is that, although the kernel has total mass $O(t)$, this mass is located far from the endpoint $\tau = t$; this is what prevents the fast growth of φ . Compare with the inequality $\varphi(t) \leq a + ct \varphi(t)$, which implies no restriction at all on φ .

To be slightly more quantitative, let us look for a power series $\Phi(t) = \sum_k a_k t^k$ achieving equality in (7.7). This yields $a_0 = a$, $a_{k+1} = c a_k 2^{-k}$, so

$$(7.8) \quad \Phi(t) = a \sum_{k=0}^{\infty} \frac{c^k t^k}{2^{k(k-1)/2}}.$$

The function Φ exhibits a truly remarkable behavior: it grows faster than any polynomial, but slower than any fractional exponential $\exp(ct^\nu)$, $\nu \in (0, 1)$; essentially it behaves like $A^{(\log t)^2}$ (as can also be seen directly from (7.7)). One may conjecture that solutions of (7.5) exhibit a similar kind of growth.

7.1.2. Sobolev interaction. If ∇W only has Sobolev regularity, we cannot afford in (6.21) to take $\mu'(t, \tau)$ larger than $\mu + \eta(t - \tau)/t$ (because the amount of regularity transferred in the bilinear estimates is only $O((t - \tau)/t)$, recall the discussion at the end of Subsection 6.3). On the other hand, we have $\gamma > 0$ such that

$$\forall \nu \geq 0, \quad \|\nabla W * \rho\|_{\mathcal{F}^{\nu, \gamma}} \leq C \|\rho\|_{\mathcal{F}^\nu},$$

and then we can choose this γ in (6.21). So K in (6.21) will be controlled by

$$(7.9) \quad K^{(\alpha), \gamma}(t, \tau) = (1 + \tau) \sup_{k \neq 0, \ell \neq 0} \frac{e^{-\alpha|\ell|} e^{-\alpha(\frac{t-\tau}{t})|k-\ell|} e^{-\alpha|k(t-\tau)+\ell\tau|}}{1 + |k - \ell|^\gamma},$$

where $\alpha = \frac{1}{2} \min\{\bar{\lambda} - \lambda; \bar{\mu} - \mu; \eta\}$. The equation we are considering now is

$$(7.10) \quad \varphi(t) \leq a + \int_0^t K^{(\alpha), \gamma}(t, \tau) \varphi(\tau) d\tau.$$

For, say, $\tau \leq t/2$, we have $K^{(\alpha)} \leq \bar{K}^{(\alpha/2)}$, and the discussion is similar to that in 7.1.1. But when τ approaches t , the term $\exp(-\alpha(\frac{t-\tau}{t})|k-\ell|)$ hardly helps. Keeping only $k > 0$ and $\ell = -1$ (because of the exponential decay in ℓ) leads to consider the kernel

$$\check{K}^{(\alpha)}(t, \tau) = (1 + \tau) \sup_{k \neq 0} \frac{e^{-\alpha|kt-(k+1)\tau|}}{1 + (k+1)^\gamma}.$$

Once again we perform a time-rescaling, setting $\check{k}_t(\theta) = t \check{K}^{(\alpha)}(t, t\theta)$, and let $t \rightarrow \infty$. In this limit each exponential $\exp(-\alpha|kt - (k+1)\tau|)$ becomes localized in a neighborhood of size $O(1/kt)$ around $\theta = k/(k+1)$, and contributes a Dirac mass at $\theta = k/(k+1)$, with amplitude $2/(\alpha(k+1))$;

$$\frac{\check{k}_t}{t} \xrightarrow{t \rightarrow \infty} \frac{2}{\alpha} \sum_k \frac{1}{1 + (k+1)^\gamma} \frac{k}{(k+1)^2} \delta_{1 - \frac{1}{k+1}}.$$

This leads us to the following baby model for (7.10):

$$(7.11) \quad \varphi(t) \leq a + ct \sum_{k \geq 1} \frac{1}{k^{1+\gamma}} \varphi \left(\left(1 - \frac{1}{k}\right) t \right).$$

If we search for $\sum a_n t^n$ achieving equality, this yields

$$a_0 = a, \quad a_{n+1} = c \left(\sum_{k \geq 1} \frac{1}{k^{1+\gamma}} \left(1 - \frac{1}{k}\right)^n \right) a_n.$$

To estimate the behavior of the \sum_k above, we compare it with

$$\begin{aligned} \int_1^\infty \frac{1}{t^{1+\gamma}} \left(1 - \frac{1}{t}\right)^n dt &= \int_0^1 u^{\gamma-1} (1-u)^n du = B(\gamma, n+1) \quad (\text{Beta function}) \\ &= \frac{\Gamma(\gamma) \Gamma(n+1)}{\Gamma(n+\gamma+1)} = O\left(\frac{1}{n^\gamma}\right). \end{aligned}$$

All in all, we may expect φ in (7.10) to behave qualitatively like

$$\Phi(t) = a \sum_{n \geq 0} \frac{c^n t^n}{(n!)^\gamma}.$$

Notice that Φ is subexponential for $\gamma > 1$ (it grows essentially like the fractional exponential $\exp(t^{1/\gamma})$) and exponential for $\gamma = 1$.

7.1.3. Physical implications. The kernel K in (6.21) takes into account possible “resonances” between spatial modes, whose control is crucial in the study of the nonlinear Vlasov equation. Then the growth of solutions to (7.2) hopefully provides qualitative information about the disturbance by resonances; this depends on the index γ of (Sobolev) regularity of ∇W .

- When $\gamma > 1$, we expect a subexponential growth of the disturbance, which should be absorbed by the exponential damping. For this reason, one may now — and only now — believe in nonlinear Landau damping for such potentials. The case $\gamma = 1$ seems to be critical and requires a more careful discussion; as a matter of fact, this is by far the most important case, since it corresponds to the electrostatic/gravitational potential used in physics!

- If we work in a setting of functions that are less smooth than analytic, the decay due to phase mixing will not be as good as exponential; but rather $O(t^{-\infty})$ for C^∞ data, or $\exp(-t^\beta)$ for Gevrey- β regularity, etc. On the other hand, the heuristics

from 7.1.2 suggest a disturbance growing like $\exp(t^{1/\gamma})$. So one may doubt the reality of the nonlinear Landau damping below Gevrey- $(1/\gamma)$ regularity!

- Even in the extremely favorable case of an analytic interaction, one may doubt the possibility of nonlinear Landau damping in C^∞ , since Φ in (7.8) grows faster than any fixed power of t . This is all the more interesting that linear Landau damping does hold in C^∞ , and even in C^k (*modulo* decay conditions at large velocities) for k as low as $1 + \varepsilon$; recall Remark 3.5.

The kind of resonances which we are chasing is at the basis of the **plasma echo** phenomenon [30]: a plasma which has been excited by a first pulse and has subsequently damped, will react twice to a second pulse: first at the instant of the excitation, and then at a later time which is determined by the frequencies of the two pulses, as well as the time interval between them. The question which we are eventually asking is *whether self-induced echoes in the plasma slowly destroy the damping*. Our conjecture is that this highly depends on the regularity of the interaction and of the distribution function: for analytic interaction the echoes are well-separated and not very dangerous; but if the interaction is very rough, they become almost instantaneous. Our analysis suggests that for Coulomb interaction, and data which are not analytic, Landau damping may be drowned in a slow accumulation of such echoes.

7.2. Exponential moments of the kernel. Now we start to estimate the kernel $K^{(\alpha),\gamma}$ from (7.9), without any approximation this time. The first step consists in estimating exponential moments.

Proposition 7.1 (Exponential moments of the kernel). *Let $\gamma \in [1, \infty]$ be given. For any $\alpha \in (0, 1)$, let $K^{(\alpha),\gamma}$ be defined by (7.9) for $\gamma < \infty$, and by (7.4) if $\gamma = \infty$. Then for any $\gamma < \infty$ there is $\bar{\alpha} = \bar{\alpha}(\gamma) > 0$ such that if $\alpha \leq \bar{\alpha}$ and $\varepsilon \in (0, 1)$, then for any $t > 0$,*

$$\begin{aligned} & e^{-\varepsilon t} \int_0^t K^{(\alpha),\gamma}(t, \tau) e^{\varepsilon \tau} d\tau \\ & \leq C \left(\frac{1}{\alpha \varepsilon^\gamma t^{\gamma-1}} + \frac{\ln \frac{1}{\alpha}}{\alpha \varepsilon^\gamma t^\gamma} + \frac{1}{\alpha^2 \varepsilon^{1+\gamma} t^{1+\gamma}} + \left(\frac{1}{\alpha^3} + \frac{\ln \frac{1}{\alpha}}{\alpha^2 \varepsilon} \right) e^{-\frac{\varepsilon t}{4}} + \frac{e^{-\frac{\alpha t}{2}}}{\alpha^3} \right), \end{aligned}$$

where $C = C(\gamma)$. In particular,

- If $\gamma > 1$ and $\varepsilon \leq \alpha$, then $e^{-\varepsilon t} \int_0^t K^{(\alpha),\gamma}(t, \tau) e^{\varepsilon \tau} d\tau \leq \frac{C(\gamma)}{\alpha^3 \varepsilon^{1+\gamma} t^{\gamma-1}}$;

- If $\gamma = 1$ then $e^{-\varepsilon t} \int_0^t K^{(\alpha),\gamma}(t, \tau) e^{\varepsilon \tau} d\tau \leq \frac{C}{\alpha^3} \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2 t} \right)$.

Additionally,

- If $\gamma = \infty$, then $e^{-\varepsilon t} \int_0^t K^{(\alpha),\gamma}(t, \tau) e^{\varepsilon \tau} d\tau \leq C \left(\frac{1}{\alpha^4} + \frac{\ln \frac{1}{\alpha}}{\alpha^3 \varepsilon} \right) e^{-\frac{\varepsilon t}{4}} + \frac{e^{-\frac{\alpha t}{2}}}{\alpha^3}$.

Proof of Proposition 7.1. To simplify notation we shall not recall the dependence of K on γ . We first assume $\gamma < \infty$, and consider $\tau \leq t/2$, which is the favorable case. We write

$$K^{(\alpha)}(t, \tau) \leq (1 + \tau) \sup_{k \neq 0} \sup_{\ell} e^{-\alpha|\ell|} e^{-\frac{\alpha}{2}|k-\ell|} e^{-\alpha|k(t-\tau)+\ell\tau|}.$$

Since we got rid of the condition $\ell \neq 0$, the right-hand side is now a nonincreasing function of d . (To see this, pick up a nonzero component of k , and recall our norm conventions from Appendix A.1.) So we assume $d = 1$. By symmetry we may also assume $k > 0$.

Explicit computations yield

$$\int_0^{t/2} e^{-\alpha|k(t-\tau)+\ell\tau|} (1+\tau) d\tau \leq \begin{cases} \frac{1}{\alpha(\ell-k)} + \frac{1}{\alpha^2(\ell-k)^2} & \text{if } \ell > k \\ e^{-\alpha kt} \left(\frac{t}{2} + \frac{t^2}{8} \right) & \text{if } \ell = k \\ \frac{e^{-\alpha(\frac{k+\ell}{2})t}}{\alpha|k-\ell|} \left(1 + \frac{t}{2} \right) & \text{if } -k \leq \ell < k \\ \left(\frac{2}{\alpha|k-\ell|} + \frac{2kt}{\alpha|k-\ell|^2} + \frac{1}{\alpha^2|k-\ell|^2} \right) & \text{if } \ell < -k. \end{cases}$$

In all cases,

$$\int_0^{t/2} e^{-\alpha|k(t-\tau)+\ell\tau|} (1+\tau) d\tau \leq \left(\frac{3}{\alpha|k-\ell|} + \frac{1}{\alpha^2|k-\ell|^2} + \frac{2t}{\alpha|k-\ell|} \right) 1_{k \neq \ell} + e^{-\alpha kt} \left(\frac{t}{2} + \frac{t^2}{8} \right) 1_{\ell=k}.$$

So

$$\begin{aligned}
& e^{-\varepsilon t} \int_0^{t/2} e^{-\alpha|k(t-\tau)+\ell\tau|} (1+\tau) e^{\varepsilon\tau} d\tau \\
& \leq e^{-\frac{\varepsilon t}{2}} \left(\frac{3}{\alpha|k-\ell|} + \frac{1}{\alpha^2|k-\ell|^2} + \frac{2t}{\alpha|k-\ell|} \right) 1_{k \neq \ell} + e^{-\alpha kt} \left(\frac{t}{2} + \frac{t^2}{8} \right) 1_{\ell=k} \\
& \leq e^{-\frac{\varepsilon t}{4}} \left(\frac{3}{\alpha|k-\ell|} + \frac{1}{\alpha^2|k-\ell|^2} + \frac{8z}{\alpha\varepsilon|k-\ell|} \right) 1_{k \neq \ell} + e^{-\frac{t\alpha}{2}} \left(\frac{z}{\alpha} + \frac{8z^2}{\alpha^2} \right) 1_{\ell=k},
\end{aligned}$$

where $z = \sup(xe^{-x}) = e^{-1}$. Then

$$\begin{aligned}
& e^{\varepsilon t} \int_0^{t/2} K^{(\alpha)}(t, \tau) e^{\varepsilon\tau} d\tau \\
& \leq e^{-\frac{\varepsilon t}{4}} \sum_{k \neq 0} \sum_{\ell \neq k} e^{-\alpha|\ell|} e^{-\frac{\alpha}{2}|k-\ell|} \left(\frac{3}{\alpha|k-\ell|} + \frac{1}{\alpha^2|k-\ell|^2} + \frac{8z}{\alpha\varepsilon|k-\ell|} \right) \\
& \quad + e^{-\frac{t\alpha}{2}} \sum_{\ell} e^{-\alpha|\ell|} \left(\frac{z}{\alpha} + \frac{8z^2}{\alpha^2} \right).
\end{aligned}$$

Using the bounds (for $\alpha \sim 0^+$)

$$\sum_{\ell} e^{-\alpha\ell} = O\left(\frac{1}{\alpha}\right), \quad \sum_{\ell} \frac{e^{-\alpha\ell}}{\ell} = O\left(\ln \frac{1}{\alpha}\right), \quad \sum_{\ell} \frac{e^{-\alpha\ell}}{\ell^2} = O(1),$$

we end up, for $\alpha \leq 1/4$, with a bound like

$$\begin{aligned}
& C e^{-\frac{\varepsilon t}{4}} \left(\frac{1}{\alpha^2} \ln \frac{1}{\alpha} + \frac{1}{\alpha^3} + \frac{1}{\alpha^2\varepsilon} \ln \frac{1}{\alpha} \right) + C e^{-\frac{\alpha t}{2}} \left(\frac{1}{\alpha^2} + \frac{1}{\alpha^3} \right) \\
& \leq C \left[e^{-\frac{\varepsilon t}{4}} \left(\frac{1}{\alpha^3} + \frac{1}{\alpha^2\varepsilon} \ln \frac{1}{\alpha} \right) + \frac{e^{-\frac{\alpha t}{2}}}{\alpha^3} \right].
\end{aligned}$$

(Note that the last term is $O(t^{-3})$, so it is anyway negligible in front of the other terms if $\gamma \leq 4$; in this case the restriction $\varepsilon \leq \alpha$ can be dispensed with.)

• Next we turn to the more delicate contribution of $\tau \geq t/2$. For this case we write

$$(7.12) \quad K^{(\alpha)}(t, \tau) \leq (1+\tau) \sup_{\ell \neq 0} e^{-\alpha|\ell|} \sup_k \frac{e^{-\alpha|k(t-\tau)+\ell\tau|}}{1+|k-\ell|^\gamma},$$

and the upper bound is a nonincreasing function of d , so we assume $d = 1$. Without loss of generality we restrict the supremum to $\ell > 0$.

The function $x \mapsto (1 + |x - \ell|^\gamma)^{-1} e^{-\alpha|x(t-\tau)+\ell\tau|}$ is decreasing for $x \geq \ell$, increasing for $x \leq -\ell\tau/(t - \tau)$; and on the interval $[-\frac{\ell\tau}{t-\tau}, \ell]$ its logarithmic derivative goes from

$$\left(-\alpha + \frac{\frac{\gamma}{\ell t}}{1 + \left(\frac{(t-\tau)}{\ell t}\right)^\gamma} \right) (t - \tau) \quad \text{to} \quad -\alpha(t - \tau).$$

So if $t \geq \gamma/\alpha$ there is a unique maximum at $x = -\ell\tau/(t - \tau)$, and the supremum in (7.12) is achieved for k equal to either the lower integer part, or the upper integer part of $-\ell\tau/(t - \tau)$. Thus a given integer k occurs in the supremum only for some times τ satisfying $k - 1 < -\ell\tau/(t - \tau) < k + 1$. Since only negative values of k occur, let us change the sign so that k is nonnegative. The equation

$$k - 1 < \frac{\ell\tau}{t - \tau} < k + 1$$

is equivalent to

$$\left(\frac{k - 1}{k + \ell - 1} \right) t < \tau < \left(\frac{k + 1}{k + \ell + 1} \right) t.$$

Moreover, $\tau > t/2$ implies $k \geq \ell$. Thus, for $t \geq \gamma/\alpha$ we have

$$(7.13) \quad e^{-\varepsilon t} \int_{t/2}^t K^{(\alpha)}(t, \tau) e^{\varepsilon\tau} d\tau \leq e^{-\varepsilon t} \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \int_{\left(\frac{k-1}{k+\ell-1}\right)t}^{\left(\frac{k+1}{k+\ell+1}\right)t} (1 + \tau) \frac{e^{-\alpha|k(t-\tau)-\ell\tau|} e^{\varepsilon\tau}}{1 + (k + \ell)^\gamma} d\tau.$$

For $t \leq \gamma/\alpha$ we have the trivial bound

$$e^{-\varepsilon t} \int_{t/2}^t K^{(\alpha)}(t, \tau) e^{\varepsilon\tau} d\tau \leq \frac{\gamma}{2\alpha};$$

so in the sequel we shall just focus on the estimate of (7.13).

To evaluate the integral in the right-hand side of (7.13), we separate according to whether τ is smaller or larger than $kt/(k + \ell)$; we use trivial bounds for $e^{\varepsilon\tau}$ inside the integral, and in the end we get the explicit bounds

$$e^{-\varepsilon t} \int_{\left(\frac{k-1}{k+\ell-1}\right)t}^{\left(\frac{k}{k+\ell}\right)t} (1 + \tau) e^{-\alpha|k(t-\tau)-\ell\tau|} e^{\varepsilon\tau} d\tau \leq e^{-\frac{\varepsilon\ell t}{k+\ell}} \left[\frac{1}{\alpha(k + \ell)} + \frac{kt}{\alpha(k + \ell)^2} \right],$$

$$e^{-\varepsilon t} \int_{\left(\frac{k}{k+\ell}\right)t}^{\left(\frac{k+1}{k+\ell+1}\right)t} (1+\tau) e^{-\alpha|k(t-\tau)-\ell\tau|} e^{\varepsilon\tau} d\tau \leq e^{-\frac{\varepsilon\ell t}{k+\ell+1}} \left[\frac{1}{\alpha(k+\ell)} + \frac{kt}{\alpha(k+\ell)^2} + \frac{1}{\alpha^2(k+\ell)^2} \right].$$

All in all, there is a numeric constant C such that (7.13) is bounded above by

$$(7.14) \quad C \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \left(\frac{1}{\alpha^2(k+\ell)^{2+\gamma}} + \frac{1}{\alpha(k+\ell)^{1+\gamma}} + \frac{kt}{\alpha(k+\ell)^{2+\gamma}} \right) e^{-\frac{\varepsilon\ell t}{k+\ell}},$$

together with an additional similar term where $e^{-\varepsilon\ell t/(k+\ell)}$ is replaced by $e^{-\varepsilon\ell t/(k+\ell+1)}$, and which will satisfy similar estimates.

We consider separately the three contributions in the right-hand side of (7.14). The first one is

$$\frac{1}{\alpha^2} \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \frac{e^{-\frac{\varepsilon\ell t}{k+\ell}}}{(k+\ell)^{2+\gamma}}.$$

To evaluate the behavior of this sum, we compare it to the two-dimensional integral

$$I(t) = \frac{1}{\alpha^2} \int_1^\infty e^{-\alpha x} \int_x^\infty \frac{e^{-\frac{\varepsilon x t}{x+y}}}{(x+y)^{2+\gamma}} dy dx.$$

We change variables $(x, y) \rightarrow (x, u)$, where $u(x, y) = \varepsilon x t / (x + y)$. This has Jacobian determinant $(dx dy) / (dx du) = (\varepsilon x t) / u^2$, and we find

$$I(t) = \frac{1}{\alpha^2 \varepsilon^{1+\gamma} t^{1+\gamma}} \int_1^\infty \frac{e^{-\alpha x}}{x^{1+\gamma}} dx \int_0^{\varepsilon t/2} e^{-u} u^\gamma du = O\left(\frac{1}{\alpha^2 \varepsilon^{1+\gamma} t^{1+\gamma}}\right).$$

The same computation for the second integral in the right-hand side of (7.14) yields

$$\frac{1}{\alpha \varepsilon^\gamma t^\gamma} \int_1^\infty \frac{e^{-\alpha x}}{x^\gamma} dx \int_0^{\varepsilon t/2} e^{-u} u^{\gamma-1} du = O\left(\frac{\ln \frac{1}{\alpha}}{\alpha \varepsilon^\gamma t^\gamma}\right).$$

(The logarithmic factor arises only for $\gamma = 1$.)

The third exponential in the right-hand side of (7.14) is the worse. It yields a contribution

$$(7.15) \quad \frac{t}{\alpha} \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \frac{e^{-\frac{\varepsilon\ell t}{k+\ell}} k}{(k+\ell)^{2+\gamma}}.$$

We compare this with the integral

$$\frac{t}{\alpha} \int_1^\infty e^{-\alpha x} \int_x^\infty \frac{e^{-\frac{\varepsilon x t}{x+y}} y}{(x+y)^{2+\gamma}} dx dy,$$

and the same change of variables as before equates this with

$$\begin{aligned} \frac{1}{\alpha \varepsilon^\gamma t^{\gamma-1}} \int_1^\infty \frac{e^{-\alpha x}}{x^\gamma} dx \int_0^{\varepsilon t/2} e^{-u} u^{\gamma-1} du - \frac{1}{\alpha \varepsilon^{1+\gamma} t^\gamma} \int_1^\infty \frac{e^{-\alpha x}}{x^\gamma} dx \int_0^{\varepsilon t/2} e^{-u} u^\gamma du \\ = O\left(\frac{\ln \frac{1}{\alpha}}{\alpha \varepsilon^\gamma t^{\gamma-1}}\right). \end{aligned}$$

(Again the logarithmic factor arises only for $\gamma = 1$.)

The proof of Proposition 7.1 in the case $\gamma < \infty$ follows by collecting all these bounds and keeping only the worse.

We now consider the case $\gamma = \infty$ (that is, for analytic interaction). The analysis for $\tau \leq t/2$ is just the same as in the case $\gamma < \infty$. On the other hand, for $\tau \geq t/2$, the analysis is much more favorable and, as we shall see, almost similar to the symmetric case $\tau \leq t/2$. We first write

$$\begin{aligned} K^{(\alpha)} &\leq (1 + \tau) \sup_{\ell \neq 0} \sup_k e^{-\alpha|\ell|} e^{-\alpha|k-\ell|} e^{-\alpha|k(t-\tau)+\ell\tau|} \\ &\leq (1 + \tau) \sup_{\ell \neq 0} \sup_k e^{-\alpha|k|/2} e^{-\alpha|k-\ell|/2} e^{-\alpha|k(t-\tau)+\ell\tau|}. \end{aligned}$$

Next we note that for $\ell \neq 0$ and $t \geq 1$,

$$\begin{aligned} \int_{t/2}^t e^{-\alpha|k(t-\tau)+\ell\tau|} \tau d\tau \\ \leq \int_{t/2}^t |\ell| \tau e^{-\alpha|k(t-\tau)+\ell\tau|} d\tau \\ \leq \int_{t/2}^t (|k(t-\tau) + \ell\tau| + |k(t-\tau)|) e^{-\alpha|k(t-\tau)+\ell\tau|} d\tau \\ \leq \frac{2z}{\alpha} \int_{t/2}^t e^{-\frac{\alpha}{2}|k(t-\tau)+\ell\tau|} d\tau + |k| \int_{t/2}^t (t-\tau) e^{-\alpha|k(t-\tau)+\ell\tau|} d\tau. \end{aligned}$$

In particular,

$$\begin{aligned} e^{-\alpha|k|/2} e^{-\alpha|k-\ell|/2} \int_{t/2}^t e^{-\alpha|k(t-\tau)+\ell\tau|} \tau d\tau \\ \leq \frac{6z}{\alpha} e^{-\alpha|k|/4} e^{-\alpha|k-\ell|/2} \int_{t/2}^t e^{-\frac{\alpha}{2}|k(t-\tau)+\ell\tau|} [1 + (t-\tau)] d\tau. \end{aligned}$$

Exchanging k and ℓ on one hand, τ and $t - \tau$ on the other, brings us back to the computations in the case $\tau \leq t/2$, up to a factor $O(1/\alpha)$. The result follows immediately. \square

Remark 7.2. The fact that these estimates boil down to a one-dimensional analysis is not surprising: the interaction between waves is strongest when their wave vectors are parallel; so higher dimension should neither help, nor harm.

Remark 7.3. It is disappointing that the case $\gamma = 1$ yields no decay as $t \rightarrow \infty$, even logarithmic. However, our bounds are essentially optimal. For $\ell = 1$, the exponential $\exp(-\alpha((t - \tau)/\tau)|k + \ell|)$ is of order 1 (not small) when $\tau \simeq kt/(k + 1)$. A more precise computation, using the identities

$$\frac{k + 1}{k + \ell + 1} - \frac{k}{k + \ell} = \frac{\ell}{(k + \ell)(k + \ell + 1)}, \quad \frac{k}{k + \ell} - \frac{k - 1}{k + \ell - 1} = \frac{\ell}{(k + \ell)(k + \ell - 1)},$$

yields a dominant term like

$$\frac{2 e^{-\frac{\varepsilon \ell t}{k + \ell}} t}{\alpha(k + \ell)} \left(\frac{k}{k + \ell} \right) \left[\left(1 - e^{-\frac{(\alpha(k + \ell) + \varepsilon) \ell t}{(k + \ell)(k + \ell - 1)}} \right) + \left(1 - e^{-\frac{(\alpha(k + \ell) - \varepsilon) \ell t}{(k + \ell)(k + \ell + 1)}} \right) \right]$$

for the time-integral between $(k - 1)/(k + \ell - 1)$ to $(k + 1)/(k + \ell + 1)$. But then, discarding negligible error terms, we are led to estimate

$$t \sum_{\ell \geq 1} e^{-\alpha \ell} \sum_{k \geq \ell} e^{-\frac{\varepsilon \ell t}{k + \ell}} \frac{k}{(k + \ell)^3} (1 - e^{-\frac{\alpha \ell t}{k + \ell}}),$$

which in turn leads to the integral

$$t \int_1^\infty e^{-\alpha x} \int_x^\infty e^{-\frac{\varepsilon x t}{x + y}} \frac{y}{(x + y)^3} (1 - e^{-\frac{\alpha x t}{x + y}}) dx dy.$$

To get the asymptotics $t \rightarrow \infty$ we may replace $y/(x + y)^3$ by $1/(x + y)^2$, and obtain after change of variables

$$\left(\int_1^\infty \frac{e^{-\alpha x}}{x} dx \right) \left(\frac{1}{\varepsilon} \int_0^{\varepsilon t/2} e^{-u} du - \frac{1}{\varepsilon + \alpha} \int_0^{(\varepsilon + \alpha)t/2} e^{-u} du \right) \\ \xrightarrow{t \rightarrow \infty} \frac{\alpha}{\varepsilon(\varepsilon + \alpha)} \left(\int_1^\infty \frac{e^{-\alpha x}}{x} dx \right);$$

so there is still no decay.

Remark 7.4. It is not easy to catch (say numerically) the behavior of (7.15), because it comes as a superposition of exponentially decaying modes; any truncation in k would lead to a radically different time-asymptotics. This is probably the reason why plasma echoes were apparently never considered as a possible source of instability (this eventuality is not mentioned e.g. in [1, Section 12.7], although the nonlinear nature of the echo phenomenon is clearly pointed out).

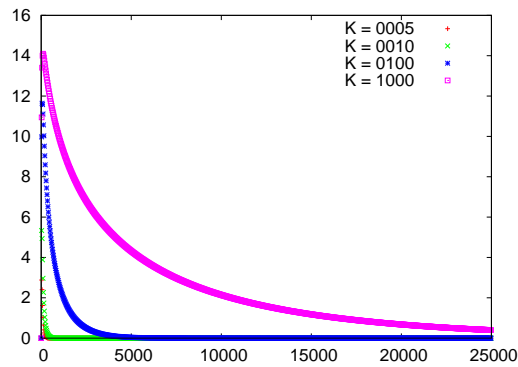


FIGURE 6. The function (7.15) truncated at $k = 5$, $k = 10$, $k = 100$, $k = 1000$. The decay is slower and slower, but still exponential; however, the maximum value (on the extreme left of the graph, because it occurs with a much faster rate than the decay) slowly increases with the truncation parameter.

From Proposition 7.1 we deduce L^2 exponential bounds:

Corollary 7.5 (L^2 exponential moments of the kernel). *With the same notation as in Proposition 7.1,*

$$(7.16) \quad e^{-2\epsilon t} \int_0^t e^{2\epsilon\tau} K_1^{(\alpha),\gamma}(t, \tau)^2 d\tau \leq \begin{cases} \frac{C(\gamma)}{\alpha^4 \epsilon^{1+2\gamma} t^{2(\gamma-1)}} & \text{if } \gamma > 1 \\ C \left(\frac{1}{\alpha^3 \epsilon^2} + \frac{1}{\alpha^2 \epsilon^3 t} \right) & \text{if } \gamma = 1 \\ \frac{C e^{-\epsilon t/4}}{\alpha^4 \epsilon} & \text{if } \gamma = \infty. \end{cases}$$

Proof of Corollary 7.5. This follows easily from Proposition 7.1 and the obvious bound

$$K_1^{(\alpha),\gamma}(t, \tau)^2 \leq C(1+t) K_1^{(2\alpha),2\gamma}(t, \tau).$$

□

7.3. Dual exponential moments.

Proposition 7.6. *With the same notation as in Proposition 7.1, for any $\gamma \geq 1$ we have*

$$(7.17) \quad \sup_{\tau \geq 0} e^{\varepsilon\tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha),\gamma}(t, \tau) dt \leq C(\gamma) \left(\frac{1}{\alpha^2 \varepsilon} + \frac{\ln \frac{1}{\alpha}}{\alpha \varepsilon^\gamma} \right).$$

Remark 7.7. The corresponding computation for the baby model considered before is

$$\begin{aligned} e^{\varepsilon\tau} \left(\frac{1+\tau}{\alpha} \right) \sum_{k \geq 1} \frac{e^{-\varepsilon \left(\frac{k+1}{k} \right) \tau}}{k^{1+\gamma}} &\simeq \left(\frac{1+\tau}{\alpha} \right) \int_1^{\infty} \frac{e^{-\varepsilon\tau/x}}{x^{1+\gamma}} dx \\ &= \left(\frac{1+\tau}{\tau^\gamma} \right) \left(\frac{1}{\alpha \varepsilon^\gamma} \right) \int_0^{\varepsilon\tau} e^{-u} u^{\gamma-1} du. \end{aligned}$$

So we expect the dependence upon ε in (7.17) to be sharp for $\gamma \rightarrow 1$.

Proof of Proposition 7.6. We first reduce to $d = 1$, and split the integral as

$$\begin{aligned} e^{\varepsilon\tau} \int_{\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha),\gamma}(t, \tau) dt &= e^{\varepsilon\tau} \int_{2\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha),\gamma}(t, \tau) dt + e^{\varepsilon\tau} \int_{\tau}^{2\tau} e^{-\varepsilon t} K^{(\alpha),\gamma}(t, \tau) dt \\ &=: I_1 + I_2. \end{aligned}$$

The first term I_1 is easy: for $2\tau \leq t \leq +\infty$ we have

$$K^{(\alpha),\gamma}(t, \tau) \leq (1+\tau) \sum_{k>1, \ell \neq 0} e^{-\alpha|\ell| - \frac{\alpha}{2}|k-\ell|} \leq \frac{C(1+\tau)}{\alpha^2},$$

and thus

$$e^{\varepsilon\tau} \int_{2\tau}^{\infty} e^{-\varepsilon t} K^{(\alpha),\gamma}(t, \tau) dt \leq \frac{C(1+\tau)}{\alpha^2} e^{-\varepsilon\tau} \leq \frac{C}{\varepsilon \alpha^2}.$$

We treat the second term I_2 as in the proof of Proposition 7.1:

$$\begin{aligned}
& e^{\varepsilon\tau} \int_{\tau}^{2\tau} K^{(\alpha),\gamma}(t, \tau) e^{-\varepsilon t} dt \\
& \leq e^{\varepsilon\tau} (1 + \tau) \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \int_{\left(\frac{k+\ell+1}{k+1}\right)\tau}^{\left(\frac{k+\ell-1}{k-1}\right)\tau} \frac{e^{-\alpha|k(t-\tau)-\ell\tau|}}{1 + (k + \ell)^\gamma} e^{-\varepsilon t} dt \\
& \leq (1 + \tau) \sum_{\ell \geq 1} e^{-\alpha\ell} \sum_{k \geq \ell} \frac{e^{-\varepsilon\frac{\ell}{k}\tau}}{k^\gamma} \left(\frac{2}{k\alpha} \right).
\end{aligned}$$

We compare this with

$$\begin{aligned}
& \frac{2(1 + \tau)}{\alpha} \int_1^\infty e^{-\alpha x} \int_x^\infty \frac{e^{-\varepsilon\frac{x}{y}\tau}}{y^{1+\gamma}} dy dx \\
& = \frac{2}{\alpha\varepsilon^\gamma} \left(\frac{1 + \tau}{\tau^\gamma} \right) \int_1^\infty \frac{e^{-\alpha x}}{x^\gamma} \int_0^{\varepsilon\tau} e^{-u} u^{\gamma-1} du dx \\
& \leq \frac{C \ln(1/\alpha)}{\alpha\varepsilon^\gamma},
\end{aligned}$$

where we used the change of variables $u = \varepsilon x\tau/y$. The desired conclusion follows. Note that as before the term $\ln(1/\alpha)$ only occurs when $\gamma = 1$, and that for $\gamma > 1$, one could improve the estimate above into a time decay of the form $O(\tau^{-(\gamma-1)})$. \square

7.4. Growth control. To state the main result of this section we shall write $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$; and if a sequence of functions $\Phi(k, t)$ ($k \in \mathbb{Z}_*^d$, $t \in \mathbb{R}$) is given, then $\|\Phi(t)\|_\lambda = \sum_k e^{2\pi\lambda|k|} |\Phi(k, t)|$. We shall use $K(s)\Phi(t)$ as a shorthand for $(K(k, s)\Phi(k, t))_{k \in \mathbb{Z}_*^d}$, etc.

Theorem 7.8 (Growth control *via* integral inequalities). *Let $f^0 = f^0(v)$ and $W = W(x)$ satisfy*

$$\sup_{\eta \in \mathbb{R}^d} |\tilde{f}^0(\eta)| e^{2\pi\bar{\lambda}|\eta|} \leq C_0, \quad \sum_{k \in \mathbb{Z}_*^d} |\widehat{W}(k)| \leq C_W, \quad \sup_{k \in \mathbb{Z}_*^d} |k| |\widehat{W}(k)| \leq C_W,$$

*together with condition **(L)** from Subsection 2.2 with constants $\bar{\lambda}, \kappa > 0$. Let $A \geq 0$, $\mu \geq 0$, $\lambda \in (0, \lambda^*]$ with $0 < \lambda^* < \bar{\lambda}$. Let $(\Phi(k, t))_{k \in \mathbb{Z}_*^d, t \geq 0}$ be a continuous function*

of $t \geq 0$, valued in $\mathbb{C}^{\mathbb{Z}^d}$, such that

$$(7.18) \quad \forall t \geq 0, \quad \left\| \Phi(t) - \int_0^t K^0(t, \tau) \Phi(\tau) d\tau \right\|_{\lambda t + \mu} \\ \leq A + \int_0^t \left[K_0(t, \tau) + K_1(t, \tau) + \frac{c_0}{(1 + \tau)^m} \right] \|\Phi(\tau)\|_{\lambda \tau + \mu} d\tau,$$

where $c_0 \geq 0$, $m > 1$ and $K_0(t, \tau)$, $K_1(t, \tau)$ are nonnegative kernels. Let $\varphi(t) = \|\Phi(t)\|_{\lambda t + \mu}$. Then

(i) Assume $1 < \gamma < \infty$ and $K_1 = cK^{(\alpha), \gamma}$ for some $c > 0$, $\alpha \in (0, \bar{\alpha}(\gamma))$, where $K^{(\alpha), \gamma}$ is defined by (7.9), and $\bar{\alpha}(\gamma)$ appears in Proposition 7.1. Then there are positive constants C and χ , depending only on $\gamma, \lambda^*, \bar{\lambda}, \kappa, C_0, C_W, m$, uniform as $\gamma \rightarrow 1$, such that if

$$(7.19) \quad \sup_{t \geq 0} \int_0^t K_0(t, \tau) d\tau \leq \chi$$

and

$$(7.20) \quad c_0 + \sup_{t \geq 0} \int_0^t K_0(t, \tau)^2 d\tau + \sup_{\tau \geq 0} \int_\tau^\infty K_0(t, \tau) dt \leq 1,$$

then for any $\varepsilon \in (0, \alpha)$,

$$(7.21) \quad \forall t \geq 0, \quad \varphi(t) \leq C A e^{cT} \left(1 + \frac{c}{\alpha \varepsilon^{3/2}} \right) e^{C c (1+T^2)} e^{\varepsilon t},$$

where

$$(7.22) \quad T = C \max \left\{ \left(\frac{c + c^2}{\alpha^5 \varepsilon^{2+\gamma}} \right)^{\frac{1}{\gamma-1}} ; \left(\frac{c_0^2}{\varepsilon} \right)^{\frac{1}{2m-1}} \right\}.$$

(ii) Assume $K_1 = c\bar{K}^{(\alpha)}$ for some $c > 0$, $\alpha \in (0, \bar{\alpha})$, where $\bar{K}^{(\alpha)}$ is defined by (7.4) (case “ $\gamma = +\infty$ ” of analytic interactions), and $\bar{\alpha}$ appears in Proposition 7.1. Then the same result as above holds for $\varepsilon \in (0, \alpha)$ with

$$T = \max \left\{ \frac{4}{\varepsilon} \log \left(\frac{\alpha^6 \varepsilon^3}{2 c^2 C} \right) ; \left(\frac{c_0^2}{\varepsilon} \right)^{\frac{1}{2m-1}} \right\}.$$

(iii) Assume $K_1 = \sum_{1 \leq i \leq N} c_i K^{(\alpha_i),1}$ for some $\alpha_i \in (0, \bar{\alpha}(1))$, where $\bar{\alpha}(1)$ appears in Proposition 7.1; then there is a numeric constant $\Gamma > 0$ such that whenever

$$1 \geq \varepsilon \geq \Gamma \sum_{i=1}^N \frac{c_i}{\alpha_i^3},$$

one has, with the same notation as in (i),

$$(7.23) \quad \forall t \geq 0, \quad \varphi(t) \leq C A \frac{e^{CT}}{\sqrt{\varepsilon}} e^{Cc(1+T^2)} e^{\varepsilon t},$$

where

$$c = \sum_{i=1}^N c_i, \quad T = C \max \left\{ \frac{1}{\varepsilon^2} \left(\sum_{i=1}^N \frac{c_i}{\alpha_i^3} \right); \left(\frac{c_0^2}{\varepsilon} \right)^{\frac{1}{2m-1}} \right\}.$$

(iii') Assume $K_1 = \sum_{1 \leq i \leq N} c_i K^{(\alpha_i),1}$ for some $\alpha_i \in (0, \bar{\alpha}(1))$, where $\bar{\alpha}(1)$ appears in Proposition 7.1; then there is a numeric constant $\Gamma' > 0$ such that whenever

$$1 \geq \varepsilon \geq \Gamma' \max \left\{ \left(\sum_{i=1}^N \frac{c_i}{\alpha_i^3} \right); \left(\sum_{i=1}^N \frac{c_i}{\alpha_i^3} \right)^{1/2} \right\},$$

one has, with the same notation as in (i),

$$(7.24) \quad \forall t \geq 0, \quad \varphi(t) \leq C A \frac{e^{\varepsilon t}}{\sqrt{\varepsilon}}.$$

Remark 7.9. Other estimates can be made, depending on the restrictions on γ and ε ; also in case (ii) one could quantify the degree of analyticity of the interaction. But the important point to note is that for $\gamma = 1$ we are not allowed to choose ε arbitrarily small.

Remark 7.10. Let apart the term $c_0/(1+\tau)^m$ which will appear as a technical correction, there are three different kernels appearing in Theorem 7.8: the kernel K^0 , which is associated with the linearized Landau damping; the kernel K_1 , describing nonlinear echoes (due to interaction between differing Fourier modes); and the kernel K_0 , describing the instantaneous response (due to interaction between identical Fourier modes).

We shall first prove Theorem 7.8 assuming

$$(7.25) \quad c_0 = 0$$

and

$$(7.26) \quad \int_0^\infty \sup_k |K^0(k, t)| e^{2\pi\bar{\lambda}|k|t} dt \leq 1 - \kappa, \quad \kappa \in (0, 1),$$

which is a reinforcement of condition **(L)**. Under these assumptions the proof of Theorem 7.8 is much simpler, and its conclusion can be substantially simplified too: χ depends only on κ ; condition (7.20) on K_0 can be dropped; and the factor $e^{CT}(1 + c/(\alpha \varepsilon^{3/2}))$ in (7.21) can be omitted. If $\widehat{W} \leq 0$ (as for gravitational interaction) and $\widehat{f}^0 \geq 0$ (as for Maxwellian background), these additional assumptions do not constitute a loss of generality, since (7.26) becomes essentially equivalent to **(L)**, and for c_0 small enough the term $c_0(1 + \tau)^{-m}$ can be incorporated inside K_0 .

Proof of Theorem 7.8 under (7.20) and (7.26). We have

$$(7.27) \quad \varphi(t) \leq A + \int_0^t \left(|K^0|(t, \tau) + K_0(t, \tau) + K_1(t, \tau) \right) \varphi(\tau) d\tau,$$

where $|K^0(t)| = \sup_k |K^0(k, t)|$. We shall estimate φ by a maximum principle argument. Let $\psi(t) = B e^{\varepsilon t}$, where B will be chosen later. If ψ satisfies, for some $T \geq 0$,

$$(7.28) \quad \begin{cases} \varphi(t) < \psi(t) & \text{for } 0 \leq t \leq T, \\ \psi(t) \geq A + \int_0^t \left(|K^0|(t, \tau) + K_0(t, \tau) + K_1(t, \tau) \right) \psi(\tau) d\tau & \text{for } t \geq T, \end{cases}$$

then $u(t) := \psi(t) - \varphi(t)$ is positive for $t \leq T$, and satisfies $u(t) \geq \int_0^t K(t, \tau) u(\tau) d\tau$ for $t \geq T$, with $K = |K^0| + K_0 + K_1 > 0$; this prevents u from vanishing at later times, so $u \geq 0$ and $\varphi \leq \psi$. Thus it is sufficient to establish (7.28).

Case (i): By Proposition 7.1, and since $\int (|K^0| + K_0) d\tau \leq 1 - \kappa/2$ (for $\chi \leq \kappa/2$),

$$(7.29) \quad \begin{aligned} A + \int_0^t [|K^0|(t, \tau) + K_0(t, \tau)] \psi(\tau) d\tau + c \int_0^t K^{(\alpha), \gamma}(t, \tau) \psi(\tau) d\tau \\ \leq A + \left[\left(1 - \frac{\kappa}{2} \right) + \frac{cC(\gamma)}{\alpha^3 \varepsilon^{2+\gamma} t^{\gamma-1}} \right] B e^{\varepsilon t}. \end{aligned}$$

For $t \geq T := (4cC/(\alpha^3 \varepsilon^{2+\gamma} \kappa))^{1/(\gamma-1)}$, this is bounded above by $A + (1 - \kappa/4) B e^{\varepsilon t}$, which in turn is bounded by $B e^{\varepsilon t}$ as soon as $B \geq 4A/\kappa$.

On the other hand, from the inequality

$$\varphi(t) \leq A + \left(1 - \frac{\kappa}{2}\right) \sup_{0 \leq \tau \leq t} \varphi(\tau) + c(1+t) \int_0^t \varphi(\tau) d\tau$$

we deduce

$$\varphi(t) \leq \left(\frac{2A}{\kappa}\right) (1+t) e^{\frac{2c}{\kappa}(t+\frac{t^2}{2})}$$

In particular, if

$$\frac{4A}{\kappa} e^{c'(T+T^2)} \leq B$$

with $c' = c'(c, \kappa)$ large enough, then for $0 \leq t \leq T$ we have $\varphi(t) \leq \psi(t)/2$, and (7.28) holds.

Case (ii): $K_1 = c \overline{K}^{(\alpha)}$. The computation is similar, replacing the right-hand side in (7.29) by

$$A + \left[\left(1 - \frac{\kappa}{2}\right) + \frac{cC}{\alpha^4 \varepsilon} e^{-\varepsilon t/4} \right] B e^{\varepsilon t},$$

according to the last estimate in Proposition 7.1.

Case (iii)-(iii'): $K_1 = \sum c_i K^{(\alpha_i)}$. We use the same reasoning, replacing the right-hand side in (7.29) by

$$A + \left[\left(1 - \frac{\kappa}{2}\right) + C \left(\sum_{i=1}^N \frac{c_i}{\alpha_i^3 \varepsilon} + \sum_{i=1}^N \frac{c_i}{\alpha_i^3 \varepsilon^2 t} \right) \right] B e^{\varepsilon t}.$$

To conclude the proof, we may first impose a lower bound on ε to ensure

$$(7.30) \quad C \sum_{i=1}^N \frac{c_i}{\alpha_i^3 \varepsilon} \leq \frac{\kappa}{8},$$

and then choose t large enough to guarantee

$$(7.31) \quad C \sum_{i=1}^N \frac{c_i}{\alpha_i^3 \varepsilon^2 t} \leq \frac{\kappa}{8};$$

this yields (iii). Alternatively, one may ensure both (7.30) and (7.31) by using just a lower bound on ε , and this leads to (iii'). \square

Proof of Theorem 7.8 in the general case. We only treat (i), since the reasoning for the other cases is rather similar; and we only establish the conclusion as an *a priori* estimate, skipping the continuity/approximation argument needed to turn it into a rigorous estimate. Then the proof is done in three steps.

Step 1: *Crude pointwise bounds.* From (7.18) we have

$$\begin{aligned}
(7.32) \quad \varphi(t) &= \sum_{k \in \mathbb{Z}_*^d} |\Phi(k, t)| e^{2\pi(\lambda t + \mu)|k|} \\
&\leq A + \sum_k \int_0^t |K^0(k, t - \tau)| e^{2\pi(\lambda t + \mu)|k|} |\Phi(t, \tau)| d\tau \\
&\quad + \int_0^1 \left[K_0(t, \tau) + K_1(t, \tau) + \frac{c_0}{(1 + \tau)^m} \right] \varphi(\tau) d\tau \\
&\leq A + \int_0^t \left[\left(\sup_k |K^0(k, t - \tau)| e^{2\pi\lambda(t - \tau)|k|} \right) \right. \\
&\quad \left. + K_1(t, \tau) + K_0(t, \tau) + \frac{c_0}{(1 + \tau)^m} \right] \varphi(\tau) d\tau.
\end{aligned}$$

We note that for any $k \in \mathbb{Z}_*^d$ and $t \geq 0$,

$$\begin{aligned}
|K^0(k, t - \tau)| e^{2\pi\lambda|k|(t - \tau)} &\leq 4\pi^2 |\widehat{W}(k)| C_0 e^{-2\pi(\bar{\lambda} - \lambda)|k|t} |k|^2 t \\
&\leq \frac{C C_0}{\bar{\lambda} - \lambda} \left(\sup_{k \neq 0} |k| |\widehat{W}(k)| \right) \leq \frac{C C_0 C_W}{\bar{\lambda} - \lambda},
\end{aligned}$$

where (here as below) C stands for a numeric constant which may change from line to line. Assuming $\int K_0(t, \tau) d\tau \leq 1/2$, we deduce from (7.32)

$$\varphi(t) \leq A + \frac{1}{2} \left(\sup_{0 \leq \tau \leq t} \varphi(\tau) \right) + C \int_0^t \left(\frac{C_0 C_W}{\bar{\lambda} - \lambda} + c(1 + t) + \frac{c_0}{(1 + \tau)^m} \right) \varphi(\tau) d\tau,$$

and by Gronwall's lemma

$$(7.33) \quad \varphi(t) \leq 2A e^{C \left(\frac{C_0 C_W}{\bar{\lambda} - \lambda} t + C(t + t^2) + c_0 C_m \right)},$$

where $C_m = \int_0^\infty (1 + \tau)^{-m} d\tau$.

Step 2: *L^2 bound.* This is the step where the smallness assumption will be most important. For all $k \in \mathbb{Z}_*^d$, $t \geq 0$, we define

$$(7.34) \quad \Psi_k(t) = e^{-\varepsilon t} \Phi(k, t) e^{2\pi(\lambda t + \mu)|k|},$$

$$(7.35) \quad \mathcal{K}_k^0(t) = e^{-\varepsilon t} K^0(k, t) e^{2\pi(\lambda t + \mu)|k|},$$

$$(7.36) \quad \begin{aligned} R_k(t) &= e^{-\varepsilon t} \left(\Phi(k, t) - \int_0^t K^0(k, t - \tau) \Phi(k, \tau) d\tau \right) e^{2\pi(\lambda t + \mu)|k|} \\ &= (\Psi_k - \Psi_k * \mathcal{K}_k^0)(t), \end{aligned}$$

and we extend all these functions by 0 for negative values of t . Taking Fourier transform in the time variable yields $\widehat{R}_k = (1 - \widehat{\mathcal{K}}_k^0) \widehat{\Psi}_k$; since condition **(L)** implies $|1 - \widehat{\mathcal{K}}_k^0| \geq \kappa$, we deduce $\|\widehat{\Psi}_k\|_{L^2} \leq \kappa^{-1} \|\widehat{R}_k\|_{L^2}$, *i. e.*,

$$(7.37) \quad \|\Psi_k\|_{L^2(dt)} \leq \frac{\|R_k\|_{L^2(dt)}}{\kappa}.$$

(Note: In the absence of any smoothness bound on K^0 , which anyway is discontinuous at $t = 0$, this L^2 bound is about the best we can hope to deduce from the Fourier identity, at least without further work.) Plugging (7.37) into (7.36), we deduce

$$(7.38) \quad \forall k \in \mathbb{Z}_*^d, \quad \|\Psi_k - R_k\|_{L^2(dt)} \leq \frac{\|\mathcal{K}_k^0\|_{L^1(dt)}}{\kappa} \|R_k\|_{L^2(dt)}.$$

Then

$$(7.39) \quad \begin{aligned} \|\varphi(t) e^{-\varepsilon t}\|_{L^2(dt)} &= \left\| \sum_k |\Psi_k| \right\|_{L^2(dt)} \\ &\leq \left\| \sum_k |R_k| \right\|_{L^2(dt)} + \sum_k \|R_k - \Psi_k\|_{L^2(dt)} \\ &\leq \left\| \sum_k |R_k| \right\|_{L^2(dt)} \left(1 + \frac{1}{\kappa} \sum_{\ell \in \mathbb{Z}_*^d} \|\mathcal{K}_\ell^0\|_{L^1(dt)} \right). \end{aligned}$$

(Note: We bounded $\|R_\ell\|$ by $\|\sum_k |R_k|\|$, which seems very crude; but the decay of \mathcal{K}_k^0 as a function of k will save us.) Next, we note that

$$\begin{aligned} \|\mathcal{K}_k^0\|_{L^1(dt)} &\leq 4\pi^2 |\widehat{W}(k)| \int_0^\infty C_0 e^{-2\pi(\bar{\lambda} - \lambda)|k|t} |k|^2 t dt \\ &\leq 4\pi^2 |\widehat{W}(k)| \frac{C_0}{(\bar{\lambda} - \lambda)^2}, \end{aligned}$$

so

$$\sum_k \|\mathcal{K}_k^0\|_{L^1(dt)} \leq 4\pi^2 \left(\sum_k |\widehat{W}(k)| \right) \frac{C_0}{(\bar{\lambda} - \lambda)^2}.$$

Plugging this in (7.39) and using (7.18) again, we obtain

$$\begin{aligned}
(7.40) \quad & \|\varphi(t) e^{-\varepsilon t}\|_{L^2(dt)} \leq \left(1 + \frac{C C_0 C_W}{\kappa (\bar{\lambda} - \lambda)^2}\right) \left\| \sum_k |R_k| \right\|_{L^2(dt)} \\
& \leq \left(1 + \frac{C C_0 C_W}{\kappa (\bar{\lambda} - \lambda)^2}\right) \left\{ \int_0^\infty e^{-2\varepsilon t} \left(A + \int_0^t \left[K_1 + K_0 + \frac{c_0}{(1+\tau)^m} \right] \varphi(\tau) d\tau \right)^2 dt \right\}^{\frac{1}{2}}.
\end{aligned}$$

We separate this (by Minkowski's inequality) into various contributions which we estimate separately. First, of course

$$(7.41) \quad \left(\int_0^\infty e^{-2\varepsilon t} A^2 dt \right)^{\frac{1}{2}} \leq \frac{A}{\sqrt{2\varepsilon}}.$$

Next, for any $T \geq 1$, by Step 1 and $\int_0^t K_1(t, \tau) d\tau \leq Cc(1+t)/\alpha$,

$$\begin{aligned}
(7.42) \quad & \left\{ \int_0^T e^{-2\varepsilon t} \left(\int_0^t K_1(t, \tau) \varphi(\tau) d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq \left[\sup_{0 \leq t \leq T} \varphi(t) \right] \left(\int_0^T e^{-2\varepsilon t} \left(\int_0^t K_1(t, \tau) d\tau \right)^2 dt \right)^{\frac{1}{2}} \\
& \leq C A e^{C \left[\frac{c_0 C_W}{\bar{\lambda} - \lambda} T + c(T+T^2) \right]} \frac{c}{\alpha} \left(\int_0^\infty e^{-2\varepsilon t} (1+t)^2 dt \right)^{\frac{1}{2}} \\
& \leq C A \frac{c}{\alpha \varepsilon^{3/2}} e^{C \left[\frac{c_0 C_W}{\bar{\lambda} - \lambda} T + c(T+T^2) \right]}.
\end{aligned}$$

Invoking Jensen and Fubini, we also have

(7.43)

$$\begin{aligned}
& \left\{ \int_T^\infty e^{-2\varepsilon t} \left(\int_0^t K_1(t, \tau) \varphi(\tau) d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq \left\{ \int_T^\infty \left(\int_0^t K_1(t, \tau) e^{-\varepsilon(t-\tau)} e^{-\varepsilon\tau} \varphi(\tau) d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq \left\{ \int_T^\infty \left(\int_0^t K_1(t, \tau) e^{-\varepsilon(t-\tau)} d\tau \right) \left(\int_0^t K_1(t, \tau) e^{-\varepsilon(t-\tau)} e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau \right) dt \right\}^{\frac{1}{2}} \\
& \leq \left(\sup_{t \geq T} \int_0^t e^{-\varepsilon t} K_1(t, \tau) e^{\varepsilon\tau} d\tau \right)^{\frac{1}{2}} \left(\int_T^\infty \int_0^t K_1(t, \tau) e^{-\varepsilon(t-\tau)} e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau dt \right)^{\frac{1}{2}} \\
& = \left(\sup_{t \geq T} \int_0^t e^{-\varepsilon t} K_1(t, \tau) e^{\varepsilon\tau} d\tau \right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\max\{\tau; T\}}^{+\infty} K_1(t, \tau) e^{-\varepsilon(t-\tau)} e^{-2\varepsilon\tau} \varphi(\tau)^2 dt d\tau \right)^{\frac{1}{2}} \\
& \leq \left(\sup_{t \geq T} \int_0^t e^{-\varepsilon t} K_1(t, \tau) e^{\varepsilon\tau} d\tau \right)^{\frac{1}{2}} \left(\sup_{\tau \geq 0} \int_\tau^\infty e^{\varepsilon\tau} K_1(t, \tau) e^{-\varepsilon t} dt \right)^{\frac{1}{2}} \left(\int_0^\infty e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

(Basically we copied the proof of Young's inequality.) Similarly,

(7.44)

$$\begin{aligned}
& \left\{ \int_0^\infty e^{-2\varepsilon t} \left(\int_0^t K_0(t, \tau) \varphi(\tau) d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq \left(\sup_{t \geq 0} \int_0^t e^{-\varepsilon t} K_0(t, \tau) e^{\varepsilon\tau} d\tau \right)^{\frac{1}{2}} \left(\sup_{\tau \geq 0} \int_\tau^\infty e^{\varepsilon\tau} K_0(t, \tau) e^{-\varepsilon t} dt \right)^{\frac{1}{2}} \left(\int_0^\infty e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau \right)^{\frac{1}{2}} \\
& \leq \left(\sup_{t \geq 0} \int_0^t K_0(t, \tau) d\tau \right)^{\frac{1}{2}} \left(\sup_{\tau \geq 0} \int_\tau^\infty K_0(t, \tau) dt \right)^{\frac{1}{2}} \left(\int_0^\infty e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

The last term is also split, this time according to $\tau \leq T$ or $\tau > T$:

$$\begin{aligned}
(7.45) \quad & \left\{ \int_0^\infty e^{-2\varepsilon t} \left(\int_0^T \frac{c_0 \varphi(\tau)}{(1+\tau)^m} d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq c_0 \left(\sup_{0 \leq \tau \leq T} \varphi(\tau) \right) \left\{ \int_0^\infty e^{-2\varepsilon t} \left(\int_0^T \frac{d\tau}{(1+\tau)^m} \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq c_0 \frac{C A}{\sqrt{\varepsilon}} e^{C \left[\left(\frac{c_0 C_W}{\bar{\lambda} - \lambda} \right) T + c(T+T^2) \right]} C_m,
\end{aligned}$$

and

$$\begin{aligned}
(7.46) \quad & \left\{ \int_0^\infty e^{-2\varepsilon t} \left(\int_T^t \frac{c_0 \varphi(\tau)}{(1+\tau)^m} d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq c_0 \left\{ \int_0^\infty \left(\int_T^t e^{-\varepsilon(t-\tau)} \frac{e^{-\varepsilon\tau} \varphi(\tau)}{(1+\tau)^m} d\tau \right)^2 dt \right\}^{\frac{1}{2}} \\
& \leq c_0 \left\{ \int_0^\infty \left(\int_T^t \frac{e^{-2\varepsilon(t-\tau)}}{(1+\tau)^{2m}} d\tau \right) \left(\int_T^t e^{-2\varepsilon\tau} \varphi(\tau)^2 d\tau \right) dt \right\}^{\frac{1}{2}} \\
& \leq c_0 \left(\int_0^\infty e^{-2\varepsilon t} \varphi(t)^2 dt \right)^{\frac{1}{2}} \left(\int_0^\infty \int_T^t \frac{e^{-2\varepsilon(t-\tau)}}{(1+\tau)^{2m}} d\tau dt \right)^{\frac{1}{2}} \\
& = c_0 \left(\int_0^\infty e^{-2\varepsilon t} \varphi(t)^2 dt \right)^{\frac{1}{2}} \left(\int_T^\infty \frac{1}{(1+\tau)^{2m}} \left(\int_\tau^\infty e^{-2\varepsilon(t-\tau)} dt \right) d\tau \right)^{\frac{1}{2}} \\
& = c_0 \left(\int_0^\infty e^{-2\varepsilon t} \varphi(t)^2 dt \right)^{\frac{1}{2}} \left(\int_T^\infty \frac{d\tau}{(1+\tau)^{2m}} \right)^{\frac{1}{2}} \left(\int_0^\infty e^{-2\varepsilon s} ds \right)^{\frac{1}{2}} \\
& = \frac{C_{2m}^{1/2} c_0}{\sqrt{\varepsilon} T^{m-1/2}} \left(\int_0^\infty e^{-2\varepsilon t} \varphi(t)^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Gathering estimates (7.41) to (7.46), we deduce from (7.40)

$$\begin{aligned}
(7.47) \quad & \|\varphi(t) e^{-\varepsilon t}\|_{L^2(dt)} \leq \left(1 + \frac{C C_0 C_W}{\kappa(\bar{\lambda} - \lambda)} \right) \frac{C A}{\sqrt{\varepsilon}} \left[1 + \left(\frac{c}{\alpha \varepsilon} + c_0 C_m \right) \right] e^{C \left[\frac{c_0 C_W}{\bar{\lambda} - \lambda} T + c(T+T^2) \right]} \\
& \quad + a \|\varphi(t) e^{-\varepsilon t}\|_{L^2(dt)},
\end{aligned}$$

where

$$a = \left(1 + \frac{C C_0 C_W}{\kappa(\bar{\lambda} - \lambda)}\right) \left[\left(\sup_{t \geq T} \int_0^t e^{-\varepsilon t} K_1(t, \tau) e^{\varepsilon \tau} d\tau \right)^{\frac{1}{2}} \left(\sup_{\tau \geq 0} \int_{\tau}^{\infty} e^{\varepsilon \tau} K_1(t, \tau) e^{-\varepsilon t} dt \right)^{\frac{1}{2}} \right. \\ \left. + \left(\sup_{t \geq 0} \int_0^t K_0(t, \tau) d\tau \right)^{\frac{1}{2}} \left(\sup_{\tau \geq 0} \int_{\tau}^{\infty} K_0(t, \tau) dt \right)^{\frac{1}{2}} + \frac{C_{2m}^{1/2} c_0}{\sqrt{\varepsilon} T^{m-1/2}} \right].$$

Using Propositions 7.1 (case $\gamma > 1$) and 7.6(i), as well as assumptions (7.19) and (7.20), we see that $a \leq 1/2$ for χ small enough and T satisfying (7.22). Then from (7.47) follows

$$\|\varphi(t) e^{-\varepsilon t}\|_{L^2(dt)} \leq \left(1 + \frac{C C_0 C_W}{\kappa(\bar{\lambda} - \lambda)}\right) \frac{C A}{\sqrt{\varepsilon}} \left[1 + \left(\frac{c}{\alpha \varepsilon} + c_0 C_m\right)\right] e^{C \left[\frac{C_0 C_W}{\bar{\lambda} - \lambda} T + c(T+T^2)\right]}.$$

Step 3: Refined pointwise bounds. Let us use (7.18) a third time, now for $t \geq T$:

(7.48)

$$e^{-\varepsilon t} \varphi(t) \leq A e^{-\varepsilon t} + \int_0^t \left(\sup_k |K^0(k, t - \tau)| e^{2\pi\lambda(t-\tau)|k|} \right) \varphi(\tau) e^{-\varepsilon \tau} d\tau \\ + \int_0^t \left[K_0(t, \tau) + \frac{c_0}{(1 + \tau)^m} \right] \varphi(\tau) e^{-\varepsilon \tau} d\tau \\ + \int_0^t (e^{-\varepsilon t} K_1(t, \tau) e^{\varepsilon \tau}) \varphi(\tau) e^{-\varepsilon \tau} d\tau \\ \leq A e^{-\varepsilon t} + \left[\left(\int_0^t \left(\sup_{k \in \mathbb{Z}_*^d} |K^0(k, t - \tau)| e^{2\pi\lambda(t-\tau)|k|} \right)^2 d\tau \right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_0^t K_0(t, \tau)^2 d\tau \right)^{\frac{1}{2}} + \left(\int_0^{\infty} \frac{c_0^2}{(1 + \tau)^{2m}} d\tau \right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_0^t e^{-2\varepsilon t} K_1(t, \tau)^2 e^{2\varepsilon \tau} d\tau \right)^{\frac{1}{2}} \right] \left(\int_0^{\infty} \varphi(\tau)^2 e^{-2\varepsilon \tau} d\tau \right)^{\frac{1}{2}}.$$

We note that, for any $k \in \mathbb{Z}_*^d$,

$$\begin{aligned}
\left(|K^0(k, t)| e^{2\pi\lambda|k|t}\right)^2 &\leq 16 \pi^4 |\widehat{W}(k)|^2 |\widetilde{f}^0(kt)|^2 |k|^4 t^2 e^{4\pi\lambda|k|t} \\
&\leq C C_0^2 |\widehat{W}(k)|^2 e^{-4\pi(\bar{\lambda}-\lambda)|k|t} |k|^4 t^2 \\
&\leq \frac{C C_0^2}{(\bar{\lambda}-\lambda)^2} |\widehat{W}(k)|^2 e^{-2\pi(\bar{\lambda}-\lambda)|k|t} |k|^2 \\
&\leq \frac{C C_0^2}{(\bar{\lambda}-\lambda)^2} C_W^2 e^{-2\pi(\bar{\lambda}-\lambda)|k|t} \\
&\leq \frac{C C_0^2}{(\bar{\lambda}-\lambda)^2} C_W^2 e^{-2\pi(\bar{\lambda}-\lambda)t};
\end{aligned}$$

so

$$\int_0^t \left(\sup_{k \in \mathbb{Z}_*^d} |K^0(k, t-\tau)| e^{2\pi\lambda(t-\tau)|k|} \right)^2 d\tau \leq \frac{C C_0^2 C_W^2}{(\bar{\lambda}-\lambda)^3}.$$

Then the conclusion follows from (7.48), Corollary 7.5, conditions (7.22) and (7.20), and Step 2. \square

Remark 7.11. Theorem 7.8 leads to enormous constants, and it is legitimate to ask about their sharpness, say with respect to the dependence in ε . In the case $\gamma = \infty$, we find a constant of order $O(e^{C/\varepsilon})$. The expected behavior is given by (7.8); so the behavior of the optimal constant might be well captured by

$$\sup_{k,t} \frac{c^k t^k e^{-\varepsilon t}}{2^{k^2}} = \sup_k \left(\frac{c k}{2^k \varepsilon} \right)^k,$$

which is roughly like $(\ln 1/\varepsilon)^{\ln 1/\varepsilon}$; this is much smaller than our exponential bound, but still grows faster than any inverse power of $1/\varepsilon$. As for the case $\gamma < \infty$, we expect the constant to be roughly of the order of

$$\sup_t (e^{(ct)^{1/\gamma}} e^{-\varepsilon t}) \simeq \exp\left(\varepsilon^{-\frac{1}{\gamma-1}}\right).$$

Our bound is roughly like $\exp(\varepsilon^{-(4+2\gamma)/(\gamma-1)})$; this is worse, but displays the expected behavior as an exponential of an inverse power of ε , with a power that diverges like $O((1-\gamma)^{-1})$ as $\gamma \rightarrow 1$.

8. APPROXIMATION SCHEMES

Having defined a functional setting (Section 4) and identified several mathematical/physical mechanisms (Sections 5 to 7), we are prepared to fight Landau damping problem. For that we need an approximation scheme solving the nonlinear Vlasov equation. The problem is not to prove the existence of solutions (this is much easier), but to devise the scheme in such a way that it leads to relevant estimates for our study.

Because analytic norms are so sensitive to composition, and because kinetic equations act by composition with characteristics (trajectories in phase space), we expect an unavoidable loss of regularity; this seems to dismiss the classical Picard iteration. Instead, a **Newton iteration** will provide more flexibility in the regularity indices, while at the same time yielding an extremely fast rate of convergence (something like $O(\varepsilon^{2^n})$) which will be most welcome to absorb the gigantic constants coming from Theorem 7.8.

8.1. The natural Newton scheme. Let us adapt the abstract Newton scheme to an abstract evolution equation in the form

$$\frac{\partial f}{\partial t} = Q(f),$$

around a stationary solution f^0 (so $Q(f^0) = 0$). Write the Cauchy problem with initial datum $f_i \simeq f^0$ in the form

$$\Phi(f) := \left(\partial_t f - Q(f), f(0, \cdot) \right) = (0, f_i).$$

Starting from f^0 , the Newton iteration consists in solving inductively $\Phi(f^{n-1}) + \Phi'(f^{n-1}) \cdot (f^n - f^{n-1}) = 0$ for $n \geq 1$. More explicitly, writing $h^n = f^n - f^{n-1}$, we should solve

$$\begin{cases} \partial_t h^1 = Q'(f^0) \cdot h^1 \\ h^1(0, \cdot) = f_i - f^0 \end{cases}$$

$$\forall n \geq 1, \quad \begin{cases} \partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} - [\partial_t f^n - Q(f^n)] \\ h^{n+1}(0, \cdot) = 0. \end{cases}$$

By induction, for $n \geq 1$ this is the same as

$$\begin{cases} \partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} + [Q(f^{n-1} + h^n) - Q(f^{n-1}) - Q'(f^{n-1}) \cdot h^n] \\ h^{n+1}(0, \cdot) = 0. \end{cases}$$

This is easily applied to the nonlinear Vlasov equation, for which the nonlinearity is quadratic. So we define the **natural Newton scheme for the nonlinear Vlasov equation** as follows:

$$\begin{aligned}
& f^0 = f^0(v) \quad \text{is given (homogeneous stationary state)} \\
& f^n = f^0 + h^1 + \dots + h^n, \quad \text{where} \\
(8.1) \quad & \begin{cases} \partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0 \\ h^1(0, \cdot) = f_i - f^0 \end{cases} \\
(8.2) \quad & \\
\forall n \geq 1, \quad & \begin{cases} \partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + F[f^n] \cdot \nabla_v h^{n+1} + F[h^{n+1}] \cdot \nabla_v f^n = -F[h^n] \cdot \nabla_v h^n \\ h^{n+1}(0, \cdot) = 0. \end{cases}
\end{aligned}$$

Here $F[f]$ is the force field created by the particle distribution f , namely

$$(8.3) \quad F[f](t, x) = - \iint_{\mathbb{T}^d \times \mathbb{R}^d} \nabla W(x - y) f(t, y, w) dy dw.$$

Note also that all the $\rho^n = \int h^n dv$ for $n \geq 1$ have zero spatial average.

8.2. Battle plan. The treatment of (8.1) was performed in Subsection 4.12. Now the problem is to handle all equations appearing in (8.2). This is much more complicated, because for $n \geq 1$ the background density f^n depends on t and x , instead of just v ; as a consequence,

(a) Equation (8.2) cannot be considered as a perturbation of free transport, because of the presence of $\nabla_v h^{n+1}$ in the left-hand side;

(b) The reaction term $F[h^{n+1}] \cdot \nabla_v f^n$ no longer has the simple product structure (function of x) \times (function of v), so it becomes harder to get hands on the homogenization phenomenon;

(c) Because of spatial inhomogeneities, echoes will appear; they are all the more dangerous that $\nabla_v f^n$ “filaments” and becomes unbounded as $t \rightarrow \infty$, even in gliding regularity.

The estimates in Sections 5 to 7 have been designed precisely to overcome these problems; however we still have a few conceptual difficulties to solve before applying these tools.

Recall the discussion in Subsection 4.11: the natural strategy is to propagate the bound

$$(8.4) \quad \sup_{\tau \geq 0} \|f_\tau\|_{\mathcal{Z}_\tau^{\lambda, \mu; 1}} < +\infty$$

along the scheme; this estimate contains in particular two crucial pieces of information:

- a control of $\rho_\tau = \int f_\tau dv$ in $\mathcal{F}^{\lambda+\mu}$ norm;
- a control of $\langle f_\tau \rangle = \int f_\tau dx$ in $\mathcal{C}^{\lambda; 1}$ norm.

So the plan would be to try to get inductively estimates of each h^n in a norm like the one in (8.4), in such a way that h^n is extremely small as $n \rightarrow \infty$, and allowing a slight deterioration of the indices λ, μ as $n \rightarrow \infty$. Let us try to see how this would work: assuming

$$\forall 0 \leq k \leq n, \quad \sup_{\tau \geq 0} \|h_\tau^k\|_{\mathcal{Z}_\tau^{\lambda_k, \mu_k; 1}} \leq \delta_k,$$

we should try to bound h_τ^{n+1} . To “solve” (8.2), we apply the classical method of characteristics: as in Section 5 we define $(X_{\tau, t}^n, V_{\tau, t}^n)$ as the solution of

$$\begin{cases} \frac{d}{dt} X_{\tau, t}^n(x, v) = V_{\tau, t}^n(x, v), & \frac{d}{dt} V_{\tau, t}^n(x, v) = F[f^n](t, X_{\tau, t}^n(x, v)) \\ X_{\tau, \tau}^n(x, v) = x, & V_{\tau, \tau}^n(x, v) = v. \end{cases}$$

Then (8.2) is equivalent to

$$(8.5) \quad \frac{d}{dt} h^{n+1}(t, X_{0, t}^n, V_{0, t}^n(x, v)) = \Sigma^{n+1}(t, X_{0, t}^n(x, v), V_{0, t}^n(x, v)),$$

where

$$(8.6) \quad \Sigma^{n+1}(t, x, v) = -F[h^{n+1}] \cdot \nabla_v f^n - F[h^n] \cdot \nabla_v h^n.$$

Integrating (8.5) in time and recalling that $h^{n+1}(0, \cdot) = 0$, we get

$$h^{n+1}(t, X_{0, t}^n(x, v), V_{0, t}^n(x, v)) = \int_0^t \Sigma^{n+1}(\tau, X_{0, \tau}^n(x, v), V_{0, \tau}^n(x, v)) d\tau.$$

Composing with $(X_{t, 0}^n, V_{t, 0}^n)$ and using (5.2) yields

$$h^{n+1}(t, x, v) = \int_0^t \Sigma^{n+1}(\tau, X_{t, \tau}^n(x, v), V_{t, \tau}^n(x, v)) d\tau.$$

We rewrite this using the “scattering transforms”

$$\Omega_{t, \tau}^n(x, v) = (X_{t, \tau}^n, V_{t, \tau}^n)(x + v(t - \tau), v) = S_{t, \tau}^n \circ S_{\tau, t}^0;$$

then we finally obtain

$$\begin{aligned}
 (8.7) \quad h^{n+1}(t, x, v) &= \int_0^t (\Sigma_\tau^{n+1} \circ \Omega_{t,\tau}^n)(x - v(t - \tau), v) d\tau \\
 &= - \int_0^t \left[\left(F[h_\tau^{n+1}] \circ \Omega_{t,\tau}^n \right) \cdot \left((\nabla_v f_\tau^n) \circ \Omega_{t,\tau}^n \right) \right] (x - v(t - \tau), v) d\tau \\
 &\quad - \int_0^t \left[\left(F[h_\tau^n] \circ \Omega_{t,\tau}^n \right) \cdot \left((\nabla_v h_\tau^n) \circ \Omega_{t,\tau}^n \right) \right] (x - v(t - \tau), v) d\tau.
 \end{aligned}$$

Since the unknown h^{n+1} appears on both sides of (8.7), we need to get a self-consistent estimate. For this we have little choice but to integrate in v and get an integral equation on $\rho[h^{n+1}] = \int h^n dv$, namely

$$\begin{aligned}
 (8.8) \quad \rho[h^{n+1}](t, x) &= \int_0^t \int \left[\left((\rho[h_\tau^{n+1}] * \nabla W) \circ \Omega_{t,\tau}^n \right) \cdot G_{\tau,t}^n \right] \circ S_{\tau-t}^0(x, v) dv d\tau \\
 &\quad + (\text{stuff from stage } n),
 \end{aligned}$$

where $G_{\tau,t}^n = \nabla_v f_\tau^n \circ \Omega_{t,\tau}^n$. By induction hypothesis $G_{\tau,t}^n$ is smooth with regularity indices roughly equal to λ_n, μ_n ; so if we accept to lose just a bit more on the regularity we may hope to apply the long-term regularity extortion and decay estimates from Section 6, and then time-response estimates of Section 7, and get the desired damping.

However, we are facing a major problem: composition of $\rho[h_\tau^{n+1}] * \nabla W$ by $\Omega_{t,\tau}^n$ implies a loss of regularity in the right-hand side with respect to the left-hand side, which is of course unacceptable if one wants a closed estimate. The short-term regularity extortion from Section 6 remedies this, but the price to pay is that G^n should now be estimated at time $\tau' = \tau - bt/(1+b)$ instead of τ , and with index of gliding analytic regularity roughly equal to $\lambda_n(1+b)$ rather than λ_n . Now the catch is that the error induced by composition by Ω^n depends on *the whole distribution* f^n , not just h^n ; thus, if the parameter b should control this error it should stay of order 1 as $n \rightarrow \infty$, instead of converging to 0.

So it seems we are sentenced to lose a fixed amount of regularity (or rather of radius of convergence) in the transition from stage n to stage $n+1$; this is reminiscent of the ‘‘Nash–Moser syndrom’’ [3]. The strategy introduced by Nash [63] to remedy such a problem (in his case arising in the construction of C^∞ isometric imbeddings) consisted in combining a Newton scheme with regularization; his method was later developed by Moser [59] for the C^∞ KAM theorem (see [60, pp. 19–21] for some interesting historical comments). The Nash–Moser technique is arguably the most

powerful perturbation technique known to this day. However, despite significant effort, we were unable to set up any relevant regularization procedure (in gliding regularity, of course) which could be used in Nash–Moser style, because of three serious problems:

- The convergence of the Nash–Moser scheme is no longer as fast as that of the “raw” Newton iteration; instead, it is determined by the regularity of the data, and the resulting rates would be unlikely to be fast enough to win over the gigantic constants coming from Section 7.
- Analytic regularization in the v variable is extremely costly, especially if we wish to keep a good localization in velocity space, as the one appearing in Theorem 4.20(iii), that is exponential integrability in v ; then the uncertainty principle basically forces us to pay $O(e^{C/\varepsilon^2})$, where ε is the strength of the regularization.
- Regularization comes with an increase of amplitude (there is as usual a trade-off between size and regularity); if we regularize before composition by Ω^n , this will devastate the estimates, because the analytic regularity of $f \circ g$ depends not only on the regularity of f and g , but also on the amplitude of $g - \text{Id}$.

Fortunately, it turned out that a “raw” Newton scheme could be used; but this required to give up the natural estimate (8.4), and replace it by the *pair* of estimates

$$(8.9) \quad \begin{cases} \sup_{\tau \geq 0} \|\rho_\tau\|_{\mathcal{F}^{\lambda\tau+\mu}} < +\infty; \\ \sup_{t \geq \tau \geq 0} \|f_\tau \circ \Omega_{t,\tau}\|_{\mathcal{Z}^{\bar{\lambda}(1+b), \bar{\mu}; 1}}^{\tau - \frac{bt}{1+b}} < +\infty. \end{cases}$$

Here $b = b(t)$ takes the form $\text{const.}/(1+t)$, and is kept fixed all along the scheme; moreover λ, μ will be slightly larger than $\bar{\lambda}, \bar{\mu}$, so that none of the two estimates in (8.9) implies the other one. Note carefully that there are now two times (t, τ) explicitly involved, so this is much more complex than (8.4). Let us explain why this strategy is nonetheless workable.

First, the density $\rho^n = \int f^n dv$ determines the characteristics at stage n , and therefore the associated scattering Ω^n . If ρ_τ^n is bounded in $\mathcal{F}^{\lambda_n\tau+\mu_n}$, then by Theorem 5.2 we can estimate $\Omega_{t,\tau}^n$ in $\mathcal{Z}_{\tau'}^{\lambda_n', \mu_n'}$, as soon as (essentially) $\lambda_n' \tau' + \mu_n' \leq \lambda_n \tau + \mu_n$, $\lambda_n' < \lambda_n$, and these bounds are uniform in t .

Of course, we cannot apply this theorem in the present context, because $\bar{\lambda}_n(1+b)$ is not bounded above by λ_n . However, for large times t we may afford $\bar{\lambda}_n(1+b(t)) < \lambda_n$, while $\bar{\lambda}_n(1+b)(\tau - bt/(1+b)) \leq \lambda_n\tau$ for all times; this will be sufficient to repeat the arguments in Section 5, getting uniform estimates *in a regularity which depends*

on t . (The constants are uniform in t ; but the index of regularity goes down with t .) We can also do this while preserving the other good properties from Theorem 5.2, namely exponential decrease in τ , and vanishing near $\tau = t$.

Figure 7 below summarizes schematically the way we choose and estimate the gliding regularity indices.

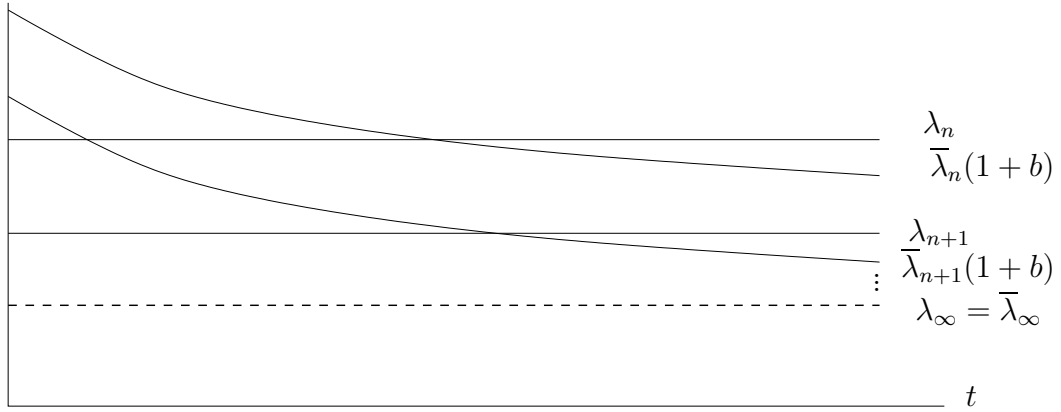


FIGURE 7. Indices of gliding regularity appearing throughout our Newton scheme, respectively in the norm of $\rho[h_\tau]$ and in the norm of $h_\tau \circ \Omega_{t,\tau}$, plotted as functions of t

Besides being uniform in t , our bounds need to be uniform in n . For this we shall have to *stratify* all our estimates, that is decompose $\rho[f^n] = \rho[h^1] + \dots + \rho[h^n]$, and consider separately the influence of each term in the equations for characteristics. This can work only if the scheme converges very fast.

Once we have estimates on $\Omega_{t,\tau}^n$ in a time-varying regularity, we can work with the kinetic equation to derive estimates on $h_\tau^n \circ \Omega_{t,\tau}^n$; and then on all $h_\tau^k \circ \Omega_{t,\tau}^n$, also in a norm of time-varying regularity. We can also estimate their spatial average, in a norm $\mathcal{C}^{\bar{\lambda}(1+b);1}$; thanks to the exponential convergence of the scattering transform as $\tau \rightarrow \infty$ these estimates will turn out to be uniform in τ . Then we can use all this information, in conjunction with Theorem 6.9, to get an integral inequality on the norm of $\rho[h_\tau^{n+1}]$ in $\mathcal{F}^{\lambda+\mu}$, where λ and μ are only slightly smaller than λ_n and μ_n .

Then we can go through the response estimates of Section 7; for potentials which are just a bit better behaved than Coulomb, this gives us an arbitrarily small loss in the decay rate, at the price of a huge constant which will eventually be wiped out by the fast convergence of the scheme. So we have an estimate on $\rho[h^{n+1}]$, and we are in

business to continue the iteration. (To ensure the propagation of the linear damping condition, or equivalently of the smallness of K_0 in Theorem 7.8, throughout the scheme, we shall have to stratify the estimates once more.)

Remark 8.1. Any reformulation of the nonlinear Vlasov equation yields a distinctive Newton iteration. An interesting scheme arises when we *first* rewrite the equation in integral form *via* composition by characteristics, and *then* run the Newton algorithm with the function

$$\Phi(f) = \left(f(t, X_{0,t}(x, v), V_{0,t}(x, v)) - f_i(x, v) \right)_{t \geq 0}.$$

We will not pursue in this direction.

9. LOCAL IN TIME ITERATION

Before working out the core of the proof of Theorem 2.5 in Section 10, we shall need a short-time estimate, which will act as an “initial regularity layer” for the Newton scheme. (This will give us room later to allow the regularity index to depend on t .) So we run the whole scheme once in this section, and another time in the next section.

Short-time estimates in the analytic class are not new for the nonlinear Vlasov equation: see in particular the work of Benachour [8] on Vlasov–Poisson. His arguments can probably be adapted for our purpose; also the Cauchy–Kowalevskaya method could certainly be applied. We shall provide here an alternative method, based on the analytic function spaces from Section 4, but not needing the apparatus from Sections 5 to 7. Unlike the more sophisticated estimates which will be performed in Section 10, these ones are “almost” Eulerian (the only characteristics are those of free transport). The main tool is the

Lemma 9.1. *Let f be an analytic function, $\lambda(t) = \lambda - Kt$, $\mu(t) = \mu - Kt$; let $T > 0$ be so small that $\lambda(t), \mu(t) > 0$ for $0 \leq t \leq T$. Then for any $\tau \in [0, T]$ and any $p \geq 1$,*

$$(9.1) \quad \left. \frac{d^+}{dt} \right|_{t=\tau} \|f\|_{\mathcal{Z}_\tau^{\lambda(t), \mu(t); p}} \leq -\frac{K}{1+\tau} \|\nabla f\|_{\mathcal{Z}_\tau^{\lambda(\tau), \mu(\tau); p}},$$

where (d^+/dt) stands for the upper right derivative.

Remark 9.2. Time-differentiating Lebesgue integrability exponents is common practice in certain areas of analysis; see e.g. [31]. Time-differentiation with respect to regularity exponents is less common; however, as pointed out to us by Strain, Lemma 9.1 is strongly reminiscent of a method recently used by Chemin [17] to derive local

analytic regularity bounds for the Navier–Stokes equation. We expect that similar ideas can be applied to more general situations of Cauchy–Kowalevskaya type, especially for first-order equations, and maybe this has already been done.

Proof of Lemma 9.1. For notational simplicity, let us assume $d = 1$. The left-hand side of (9.1) is

$$\begin{aligned}
& \sum_{n,k} e^{2\pi\mu(\tau)|k|} 2\pi\dot{\mu}(\tau) |k| \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{f}(k, v) \right\|_{L^p(dv)} \\
& \quad + \sum_{n,k} e^{2\pi\mu(\tau)|k|} \dot{\lambda}(\tau) \frac{\lambda^{n-1}(\tau)}{(n-1)!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{f}(k, v) \right\|_{L^p(dv)} \\
& \leq -K \sum_{n,k} e^{2\pi\mu(\tau)|k|} 2\pi |k| \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{f}(k, v) \right\|_{L^p(dv)} \\
& \quad - K \sum_{n,k} e^{2\pi\mu(\tau)|k|} \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^{n+1} \widehat{f}(k, v) \right\|_{L^p(dv)} \\
& \leq -K \sum_{n,k} e^{2\pi\mu(\tau)|k|} \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{\nabla_x f}(k, v) \right\|_{L^p(dv)} \\
& \quad + \frac{K\tau}{1+\tau} \sum_{n,k} e^{2\pi\mu(\tau)|k|} \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{\nabla_x f}(k, v) \right\|_{L^p(dv)} \\
& \quad - \frac{K}{1+\tau} \sum_{n,k} e^{2\pi\mu(\tau)|k|} \frac{\lambda^n(\tau)}{n!} \left\| (\nabla_v + 2i\pi k\tau)^n \widehat{\nabla_v f}(k, v) \right\|_{L^p(dv)},
\end{aligned}$$

where in the last step we used $\|(\nabla_v + 2i\pi k\tau)h\| \geq (1/(1+\tau))(\|\nabla_v h\| - \tau\|2i\pi kh\|)$. The conclusion follows. \square

Now let us see how to propagate estimates through the Newton scheme described in Section 10. The first stage of the iteration (h^1 in the notation of (8.1)) was considered in Subsection 4.12, so we only need to care about higher orders. For any $k \geq 1$ we solve $\partial_t h^{k+1} + v \cdot \nabla_x h^{k+1} = \widetilde{\Sigma}^{k+1}$, where

$$\widetilde{\Sigma}^{k+1} = -\left(F[h^{k+1}] \cdot \nabla_v f^k + F[f^k] \cdot \nabla_v h^{k+1} + F[h^k] \cdot \nabla_v h^k\right)$$

(note the difference with (8.5)–(8.6)). Recall that $f^k = f^0 + h^1 + \dots + h^k$. We define $\lambda_k(t) = \lambda_k - 2Kt$, $\mu_k(t) = \mu_k - Kt$, where $(\lambda_k)_{k \in \mathbb{N}}$, $(\mu_k)_{k \in \mathbb{N}}$ are decreasing sequences of positive numbers.

We assume inductively that at stage n of the iteration, we have constructed $(\lambda_k)_{k \leq n}$, $(\mu_k)_{k \leq n}$, $(\delta_k)_{k \leq n}$ such that

$$\forall k \leq n, \quad \sup_{0 \leq t \leq T} \|h^k(t, \cdot)\|_{\mathcal{Z}_t^{\lambda_k(t), \mu_k(t); 1}} \leq \delta_k,$$

for some fixed $T > 0$. The issue is to construct λ_{n+1} , μ_{n+1} and δ_{n+1} so that the induction hypothesis is satisfied at stage $n+1$.

At $t=0$, $h^{n+1} = 0$. Then we estimate the time-derivative of $\|h^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}(t), \mu_{n+1}(t); 1}}$. Let us first pretend that the regularity indices λ_{n+1} and μ_{n+1} do not depend on t ; then $h^{n+1}(t) = \int_0^t \tilde{\Sigma}^{n+1} \circ S_{-(t-\tau)}^0 d\tau$, so by Proposition 4.19

$$\begin{aligned} \|h^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} &\leq \int_0^t \|\tilde{\Sigma}_\tau^{n+1} \circ S_{-(t-\tau)}^0\|_{\mathcal{Z}_\tau^{\lambda_{n+1}, \mu_{n+1}; 1}} d\tau \\ &\leq \int_0^t \|\tilde{\Sigma}_\tau^{n+1}\|_{\mathcal{Z}_\tau^{\lambda_{n+1}, \mu_{n+1}; 1}} d\tau, \end{aligned}$$

and thus

$$\frac{d^+}{dt} \|h^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} \leq \|\tilde{\Sigma}_t^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}}.$$

Finally, according to Lemma 9.1, to this estimate we should add a negative multiple of the norm of ∇h^{n+1} to take into account the time-dependence of λ_{n+1} , μ_{n+1} .

All in all, after application of Proposition 4.24, we get

$$\begin{aligned} \frac{d^+}{dt} \|h^{n+1}(t, \cdot)\|_{\mathcal{Z}_t^{\lambda_{n+1}(t), \mu_{n+1}(t); 1}} &\leq \|F[h_t^{n+1}]\|_{\mathcal{F}^{\lambda_{n+1}t + \mu_{n+1}}} \|\nabla_v f_t^n\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} \\ &\quad + \|F[f_t^n]\|_{\mathcal{F}^{\lambda_{n+1}t + \mu_{n+1}}} \|\nabla_v h_t^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} \\ &\quad + \|F[h_t^n]\|_{\mathcal{F}^{\lambda_{n+1}t + \mu_{n+1}}} \|\nabla_v h_t^n\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} \\ &\quad - K \|\nabla_x h_t^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} - K \|\nabla_v h_t^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}}, \end{aligned}$$

where $K > 0$, t is sufficiently small, and all exponents λ_{n+1} and μ_{n+1} in the right-hand side actually depend on t .

From Proposition 4.15 (iv) we easily get $\|F[h]\|_{\mathcal{F}^{\lambda t + \mu}} \leq C \|\nabla h\|_{\mathcal{Z}_t^{\lambda, \mu; 1}}$. Moreover, by Proposition 4.10,

$$\|\nabla f^n\|_{\mathcal{Z}_t^{\lambda_{n+1}, \mu_{n+1}; 1}} \leq \sum_{k \leq n} \|\nabla h^k\|_{\mathcal{Z}_t^{\lambda_{k+1}, \mu_{k+1}; 1}} \leq C \sum_{k \leq n} \frac{\|h^k\|_{\mathcal{Z}_t^{\lambda_{k+1}, \mu_{k+1}; 1}}}{\min\{\lambda_k - \lambda_{n+1}; \mu_k - \mu_{n+1}\}}.$$

We end up with the bound

$$\begin{aligned} & \frac{d^+}{dt} \|h^{n+1}(t, \cdot)\|_{\mathcal{Z}_t^{\lambda_{n+1}(t), \mu_{n+1}(t); 1}} \\ & \leq \left[C \left(\sum_{k \leq n} \frac{\delta_k}{\min\{\lambda_k - \lambda_{n+1}; \mu_k - \mu_{n+1}\}} \right) - K \right] \|\nabla h^{n+1}\|_{\mathcal{Z}_t^{\lambda_{n+1}(t), \mu_{n+1}(t); 1}} \\ & \qquad \qquad \qquad + \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}; \mu_n - \mu_{n+1}\}}. \end{aligned}$$

We conclude that if

$$(9.2) \quad \sum_{k \leq n} \frac{\delta_k}{\min\{\lambda_k - \lambda_{n+1}; \mu_k - \mu_{n+1}\}} \leq \frac{K}{C},$$

then we may choose

$$(9.3) \quad \delta_{n+1} = \frac{\delta_n^2}{\min\{\lambda_n - \lambda_{n+1}; \mu_n - \mu_{n+1}\}}.$$

This is our first encounter with the principle of ‘‘stratification’’ of errors, which will be crucial in the next section: to control the error at stage $n+1$, we use not only the smallness of the error from stage n , but also an information about all previous errors; namely the fact that the convergence of the size of the error is much faster than the convergence of the regularity loss. Let us see how this works. We choose $\lambda_k - \lambda_{k+1} = \mu_k - \mu_{k+1} = \Lambda/k^2$, where $\Lambda > 0$ is arbitrarily small. Then for $k \leq n$, $\lambda_k - \lambda_{n+1} \geq \Lambda/k^2$, and therefore $\delta_{n+1} \leq \delta_n^2 n^2 / \Lambda$. The problem is to check

$$(9.4) \quad \sum_{n=1}^{\infty} n^2 \delta_n < +\infty.$$

Indeed, then we can choose K large enough for (9.2) to be satisfied, and then T small enough that, say $\lambda^* - 2KT \geq \lambda^\sharp$, $\mu^* - KT \geq \mu^\sharp$, where $\lambda^\sharp < \lambda^*$, $\mu^\sharp < \mu^*$ have been fixed in advance.

If $\delta_1 = \delta$, the general term in the series of (9.4) is

$$n^2 \frac{\delta^{2^n}}{\Lambda^n} (2^2)^{2^{n-1}} (3^2)^{2^{n-2}} (4^2)^{2^{n-2}} \dots ((n-1)^2)^2 n^2.$$

To prove the convergence for δ small enough, we assume by induction that $\delta_n \leq z^{a^n}$, where a is fixed in the interval $(1, 2)$ (say $a = 1.5$); and we claim that this condition

propagates if $z > 0$ is small enough. Indeed,

$$\delta_{n+1} \leq \frac{z^{2a^n}}{\Lambda} n^2 \leq z^{a^{n+1}} \left(\frac{z^{(2-a)a^n} n^2}{\Lambda} \right),$$

and this is bounded above by $z^{a^{n+1}}$ if z is so small that

$$\forall n \in \mathbb{N}, \quad z^{(2-a)a^n} \leq \frac{\Lambda}{n^2}.$$

This concludes the iteration argument. Note that the convergence is still extremely fast — in $O(z^{a^n})$ for any $a < 2$. (Of course, when a approaches 2, the constants become huge, and the restriction on the size of the perturbation becomes more and more stringent.)

10. GLOBAL IN TIME ITERATION

Now let us implement the scheme described in Section 8, with some technical modifications. If f is a given kinetic distribution, we write $\rho[f] = \int f dv$ and $F[f] = -\nabla W * \rho[f]$. We let

$$(10.1) \quad f^n = f^0 + h^1 + \dots + h^n,$$

where the h^k are defined by the natural Newton scheme introduced in Section 8. As in Section 5 we define $\Omega_{t,\tau}^k$ as the scattering from time t to time τ , generated by the force field $F[f^k] = -\nabla W * \rho[f^k]$. (Note that $\Omega^0 = \text{Id}$.)

10.1. The statement of the induction. We shall fix $\bar{p} \in [1, \infty]$ and make the following assumptions:

- *Regularity of the background:* there are $\lambda > 0$ and $C_0 > 0$ such that

$$\forall p \in [1, \bar{p}], \quad \|f^0\|_{C^{\lambda;p}} \leq C_0.$$

- *Linear damping condition:* The stability condition **(L)** from Subsection 2.2 holds with parameters λ (the same as above) and $\kappa > 0$.

- *Regularity of the interaction:* There are $\gamma > 1$ and $C_F > 0$ such that for any $\nu > 0$,

$$(10.2) \quad \|\nabla W * \rho\|_{\mathcal{F}^{\nu,\gamma}} \leq C_F \|\rho\|_{\dot{\mathcal{F}}^\nu}.$$

• *Initial layer of regularity:* Having chosen $\lambda^\# < \lambda$, $\mu^\# < \mu$, we assume that for all $p \in [1, \bar{p}]$

$$(10.3) \quad \forall k \geq 1, \quad \sup_{0 \leq t \leq T} \left(\|h_t^k\|_{\mathcal{Z}^{\lambda^\#, \mu^\#; p}} + \|\rho[h_t^k]\|_{\mathcal{F}^{\mu^\#}} \right) \leq \zeta_k,$$

where T is *some* positive time, and ζ_k converges to zero extremely fast: $\zeta_k = O(z_I^{a_I^k})$, $z_I \leq C\delta < 1$, $1 < a_I < 2$ (a_I chosen in advance, arbitrarily close to 2).

• *Smallness of the solution of the linearized equation:* Given $\lambda_1 < \lambda^\#$, $\mu_1 < \mu^\#$, we assume

$$(10.4) \quad \forall p \in [1, \bar{p}], \quad \begin{cases} \sup_{\tau \geq 0} \|\rho[h_\tau^1]\|_{\mathcal{F}^{\lambda_1 \tau + \mu_1}} \leq \delta_1 \\ \sup_{t \geq \tau \geq 0} \|h_\tau^1\|_{\mathcal{Z}^{\lambda_1(1+b), \mu_1; p}} \leq \delta_1, \end{cases}$$

where $\delta_1 \leq C\delta$.

Then we prove the following induction: for any $n \geq 1$,

$$(10.5) \quad \forall k \in \{1, \dots, n\}, \quad \forall p \in [1, \bar{p}], \quad \begin{cases} \sup_{\tau \geq 0} \|\rho[h_\tau^k]\|_{\mathcal{F}^{\lambda_k \tau + \mu_k}} \leq \delta_k \\ \sup_{t \geq \tau \geq 0} \|h_\tau^k \circ \Omega_{t, \tau}^{k-1}\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}} \leq \delta_k, \end{cases}$$

where

• $(\delta_k)_{k \in \mathbb{N}}$ is a sequence satisfying $0 < C_F \zeta_k \leq \delta_k$, and $\delta_k = O(z^{a^k})$, $z < z_I$, $1 < a < a_I$ (a arbitrarily close to a_I),

• (λ_k, μ_k) are decreasing to $(\lambda_\infty, \mu_\infty)$, where $(\lambda_\infty, \mu_\infty)$ are arbitrarily close to (λ_1, μ_1) ; in particular we impose

$$(10.6) \quad \lambda^\# - \lambda_\infty \leq \min\left\{1; \frac{\lambda_\infty}{2}\right\}, \quad \mu^\# - \mu_\infty \leq \min\left\{1; \frac{\mu_\infty}{2}\right\}.$$

• T is some small positive time in (10.3); we impose

$$(10.7) \quad \lambda^\# T \leq \frac{\mu^\# - \mu_1}{2}.$$

• $b = b(t) = \frac{B}{1+t}$, where $B \in (0, T)$ is a (small) constant.

10.2. Preparatory remarks. We shall propagate the following “primary” controls on the density and distribution:

$$(10.8) \quad (\mathbf{E}_\rho^n) \quad \forall k \in \{1, \dots, n\}, \quad \sup_{\tau \geq 0} \|\rho[h_\tau^k]\|_{\mathcal{F}^{\lambda_k \tau + \mu_k}} \leq \delta_k$$

and

$$(10.9) \quad (\mathbf{E}_h^n) \quad \forall k \in \{1, \dots, n\}, \quad \forall p \in [1, \bar{p}], \quad \sup_{t \geq \tau \geq 0} \left\| h_\tau^k \circ \Omega_{t, \tau}^{k-1} \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}} \leq \delta_k.$$

(We denote by δ_k various small constants which appear at stage k , not necessarily equal but all of the same order.)

Estimate (\mathbf{E}_ρ^n) obviously implies, *via* (10.2), up to a multiplicative constant,

$$(10.10) \quad (\tilde{\mathbf{E}}_\rho^n) \quad \forall k \in \{1, \dots, n\}, \quad \sup_{\tau \geq 0} \|F[h_\tau^k]\|_{\mathcal{F}^{\lambda_k \tau + \mu_k, \gamma}} \leq \delta_k.$$

Before we can go from there to stage $n+1$, we need an additional set of estimates on the scattering maps $(\Omega^k)_{k=1, \dots, n}$, which will be used to

- (1) update the control on $\Omega_{t, \tau}^k - \text{Id}$
- (2) establish the needed control along the characteristics for the background $(\nabla_v f_\tau^n) \circ \Omega_{t, \tau}^n$ (same index for the distribution and the scattering);
- (3) update some technical controls allowing to exchange (asymptotically) gradient and composition by $\Omega_{t, \tau}^k$; this will be crucial to handle the contribution of the zero mode of the background after composition by characteristics.

This set of scattering estimates falls into three categories. The first group expresses the closeness of Ω^k to Id :

$$(10.11) \quad (\mathbf{E}_\Omega^n) \quad \forall k \in \{1, \dots, n\}, \quad \begin{cases} \sup_{t \geq \tau \geq 0} \left\| \Omega^k X_{t, \tau} - \text{Id} \right\|_{\mathcal{Z}^{\lambda_k^*(1+b), (\mu_k^*, \gamma)}} \leq 2 \mathcal{R}_2^k(\tau, t), \\ \sup_{t \geq \tau \geq 0} \left\| \Omega^k V_{t, \tau} - \text{Id} \right\|_{\mathcal{Z}^{\lambda_k^*(1+b), (\mu_k^*, \gamma)}} \leq \mathcal{R}_1^k(\tau, t), \end{cases}$$

with $\lambda_k > \lambda_k^* > \lambda_{k+1}$, $\mu_k > \mu_k^* > \mu_{k+1}$, and

$$(10.12) \quad \begin{cases} \mathcal{R}_1^k(\tau, t) = \left(\sum_{j=1}^k \frac{\delta_j e^{-2\pi(\lambda_j - \lambda_j^*)\tau}}{2\pi(\lambda_j - \lambda_j^*)} \right) \min \{ (t - \tau); 1 \} \\ \mathcal{R}_2^k(\tau, t) = \left(\sum_{j=1}^k \frac{\delta_j e^{-2\pi(\lambda_j - \lambda_j^*)\tau}}{(2\pi(\lambda_j - \lambda_j^*))^2} \right) \min \left\{ \frac{(t - \tau)^2}{2}; 1 \right\}. \end{cases}$$

The second group of estimates expresses the fact that $\Omega^n - \Omega^k$ is very small when k is large:

$$(10.13) \quad (\tilde{\mathbf{E}}_{\Omega}^n) \quad \forall k \in \{0, \dots, n-1\}, \quad \begin{cases} \sup_{t \geq \tau \geq 0} \left\| \Omega^n X_{t,\tau} - \Omega^k X_{t,\tau} \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), (\mu_n^*, \gamma)}_{\tau - \frac{bt}{1+b}}} \leq 2 \mathcal{R}_2^{k,n}(\tau, t), \\ \sup_{t \geq \tau \geq 0} \left\| \Omega^n V_{t,\tau} - \Omega^k V_{t,\tau} \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), (\mu_n^*, \gamma)}_{\tau - \frac{bt}{1+b}}} \leq \mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t), \\ \sup_{t \geq \tau \geq 0} \left\| (\Omega_{t,\tau}^k)^{-1} \circ \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*}_{\tau - \frac{bt}{1+b}}} \leq 4(\mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t)), \end{cases}$$

with

$$(10.14) \quad \begin{cases} \mathcal{R}_1^{k,n}(\tau, t) = \left(\sum_{j=k+1}^n \frac{\delta_j e^{-2\pi(\lambda_j - \lambda_j^*)\tau}}{2\pi(\lambda_j - \lambda_j^*)} \right) \min \{ (t - \tau); 1 \} \\ \mathcal{R}_2^{k,n}(\tau, t) = \left(\sum_{j=k+1}^n \frac{\delta_j e^{-2\pi(\lambda_j - \lambda_j^*)\tau}}{(2\pi(\lambda_j - \lambda_j^*))^2} \right) \min \left\{ \frac{(t - \tau)^2}{2}; 1 \right\} \end{cases}$$

(Choosing $k = 0$ brings us back to the previous estimates (\mathbf{E}_{Ω}^n)).

The last group of estimates expresses the fact that the differential of the scattering is uniformly close to the identity (in a way which is more precise than what would follow from the first group of estimates):

$$(10.15) \quad (\mathbf{E}_{\nabla \Omega}^n) \quad \forall k = 1, \dots, n, \quad \begin{cases} \sup_{t \geq \tau \geq 0} \left\| \nabla \Omega^k X_{t,\tau} - I \right\|_{\mathcal{Z}^{\lambda_k^*(1+b), \mu_k^*}_{\tau - \frac{bt}{1+b}}} \leq 2 \mathcal{R}_2^k(\tau, t), \\ \sup_{t \geq \tau \geq 0} \left\| \nabla \Omega^k V_{t,\tau} - I \right\|_{\mathcal{Z}^{\lambda_k^*(1+b), \mu_k^*}_{\tau - \frac{bt}{1+b}}} \leq \mathcal{R}_1^k(\tau, t) + \mathcal{R}_2^k(\tau, t), \end{cases}$$

where the gradient ∇ denotes either ∇_x or ∇_v , and I is the identity matrix.

All these estimates are very robust and can accommodate not only with Sobolev corrections, but also with additional time shifts as in Subsection 4.10: for instance, in (10.11) and (10.13) we could add

$$(10.16) \quad \sup_{t \geq \tau \geq 0} \left\| \Omega^k X_{t,\tau} - \text{Id} \right\|_{\mathcal{Z}^{\widehat{\lambda}_k(1+b), \lambda_k^* - \widehat{\lambda}_k}, (\mu_k^*, \gamma)}_{\left(\tau - \frac{bt}{1+b}, 0\right)} \leq 2 \mathcal{R}_2^k(\tau, t),$$

$$(10.17) \quad \sup_{t \geq \tau \geq 0} \left\| \Omega^n X_{t,\tau} - \Omega^k X_{t,\tau} \right\|_{\mathcal{Z}^{\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n}, (\mu_n^*, \gamma)}_{\left(\tau - \frac{bt}{1+b}, 0\right)} \leq 2(\mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t)),$$

where $\lambda_k > \lambda_k^* > \widehat{\lambda}_k > \lambda_{k+1}$. The additional time-shift can be dispensed with for the proof of Theorem 2.5, but we mention it since it is not too difficult to incorporate, it played a key role in a previous version of our proof, and we suspect that it can be useful in rougher problems where the interaction does not depend only on positions.

An important property of the functions $\mathcal{R}_1^{k,n}(\tau, t)$, $\mathcal{R}_2^{k,n}(\tau, t)$ is their fast decay as $\tau \rightarrow \infty$ and as $k \rightarrow \infty$, uniformly in $n \geq k$; this is due to the fast convergence of the sequence $(\delta_k)_{k \in \mathbb{N}}$. Eventually, if $r \in \mathbb{N}$ is given, we shall have

$$(10.18) \quad \forall r \geq 1, \quad \mathcal{R}_1^{k,n}(\tau, t) \leq \omega_{k,n}^{r,1}(\tau, t), \quad \mathcal{R}_2^{k,n}(\tau, t) \leq \omega_{k,n}^{r,2}(\tau, t)$$

with

$$\omega_{k,n}^{r,1}(\tau, t) := C_\omega^r \left(\sum_{j=k+1}^n \frac{\delta_j}{(2\pi(\lambda_j - \lambda_j^*))^{1+r}} \right) \frac{\min\{(t - \tau); 1\}}{(1 + \tau)^r},$$

and

$$\omega_{k,n}^{r,2}(\tau, t) := C_\omega^r \left(\sum_{j=k+1}^n \frac{\delta_j}{(2\pi(\lambda_j - \lambda_j^*))^{2+r}} \right) \frac{\min\{(t - \tau)^2/2; 1\}}{(1 + \tau)^r}$$

for some absolute constant C_ω^r depending only on r (we denote also $\omega_{0,n}^{r,1} = \omega_n^{r,1}$ and $\omega_{0,n}^{r,2} = \omega_n^{r,2}$).

From the estimates on the characteristics and (\mathbf{E}_h^n) will follow the following “secondary controls” on the distribution function:

$$(10.19) \quad (\tilde{\mathbf{E}}_h^n) \quad \forall k \in \{1, \dots, n\}, \quad \forall p \in [1, \bar{p}],$$

$$\left\{ \begin{array}{l} \sup_{t \geq \tau \geq 0} \left\| (\nabla_x h_\tau^k) \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}}^{\tau - \frac{bt}{1+b}} \leq \delta_k \\ \sup_{t \geq \tau \geq 0} \left\| \nabla_x (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}}^{\tau - \frac{bt}{1+b}} \leq \delta_k \\ \left\| ((\nabla_v + \tau \nabla_x) h_\tau^k) \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}}^{\tau - \frac{bt}{1+b}} \leq \delta_k \\ \left\| (\nabla_v + \tau \nabla_x) (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}}^{\tau - \frac{bt}{1+b}} \leq \delta_k \\ \sup_{t \geq \tau \geq 0} \frac{1}{(1+\tau)^2} \left\| (\nabla \nabla h_\tau^k) \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; 1}}^{\tau - \frac{bt}{1+b}} \leq \delta_k \\ \sup_{t \geq \tau \geq 0} (1+\tau)^2 \left\| (\nabla h_\tau^k) \circ \Omega_{t,\tau}^{k-1} - \nabla (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; 1}}^{\tau - \frac{bt}{1+b}} \leq \delta_k. \end{array} \right.$$

The transition from stage n to stage $n+1$ can be summarized as follows:

$$(\tilde{\mathbf{E}}_\rho^n) \xrightarrow{(\mathbf{A}_n)} \left[(\mathbf{E}_\Omega^n) + (\tilde{\mathbf{E}}_\Omega^n) + (\mathbf{E}_{\nabla\Omega}^n) \right]$$

$$\left[(\mathbf{E}_\rho^n) + (\mathbf{E}_\Omega^n) + (\tilde{\mathbf{E}}_\Omega^n) + (\mathbf{E}_{\nabla\Omega}^n) + (\mathbf{E}_h^n) + (\tilde{\mathbf{E}}_h^n) \right] \xrightarrow{(\mathbf{B}_n)} \left[(\mathbf{E}_\rho^{n+1}) + (\tilde{\mathbf{E}}_\rho^{n+1}) + (\mathbf{E}_h^{n+1}) + (\tilde{\mathbf{E}}_h^{n+1}) \right].$$

The first implication (\mathbf{A}_n) is proven by an amplification of the technique used in Section 5; ultimately, it relies on repeated application of Picard’s fixed point theorem in analytic norms. The second implication (\mathbf{B}_n) is the harder part; it uses the machinery from Sections 6 and 7, together with the idea of propagating simultaneously a *shifted* \mathcal{Z} norm for the kinetic distribution and an \mathcal{F} norm for the density.

In both implications, the stratification of error estimates will prevent the blow up of constants. So we shall decompose the force field F^n generated by f^n as

$$F^n = F[f^n] = E^1 + \dots + E^n,$$

where $E^k = F[h^k] = -\nabla W * \rho[h^k]$.

The plan of the estimates is as follows. We shall construct inductively a sequence of constant coefficients

$$\lambda^\sharp > \lambda_1 > \lambda_1^* > \lambda_2 > \dots > \lambda_n > \lambda_n^* > \lambda_{n+1} > \dots$$

$$\mu^\sharp > \mu_1 > \mu_1^* > \mu_2 > \dots > \mu_n > \mu_n^* > \mu_{n+1} > \dots$$

converging respectively to λ_∞ and μ_∞ , and a sequence $(\delta_k)_{k \in \mathbb{N}}$ decreasing very fast to zero. For simplicity we shall let

$$\mathcal{R}^n(\tau, t) = \mathcal{R}_1^n(\tau, t) + \mathcal{R}_2^n(\tau, t), \quad \mathcal{R}^{k,n}(\tau, t) = \mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t),$$

and assume $2\pi(\lambda_j - \lambda_j^*) \leq 1$; so

(10.20)

$$\mathcal{R}^{k,n}(\tau, t) \leq C_\omega^r \left(\sum_{j=k+1}^n \frac{\delta_j}{(2\pi(\lambda_j - \lambda_j^*))^{2+r}} \right) \frac{\min\{t - \tau; 1\}}{(1 + \tau)^r}, \quad \mathcal{R}^{0,n} = \mathcal{R}^n.$$

It will be sufficient to work with some fixed r , large enough (as we shall see, $r = 4$ will do).

To go from stage n to stage $n + 1$, we shall do as follows:

- Implication (\mathbf{A}_n) (subsection 10.3):

Step 1. estimate $\Omega^n - \text{Id}$ (the bound should be uniform in n);

Step 2. estimate $\Omega^n - \Omega^k$ ($k \leq n - 1$; the error should be small when $k \rightarrow \infty$);

Step 3. estimate $\nabla \Omega^n - I$;

Step 4. estimate $(\Omega^k)^{-1} \circ \Omega^n$;

- Implication (\mathbf{B}_n) (subsection 10.4):

Step 5. estimate h^k and its derivatives along the composition by Ω^n ;

Step 6. estimate $\rho[h^{n+1}]$, using Sections 6 and 7;

Step 7. estimate $F[h^{n+1}]$ from $\rho[h^{n+1}]$;

Step 8. estimate $h^{n+1} \circ \Omega^n$;

Step 9. estimate derivatives of h^{n+1} composed with Ω^n ;

Step 10. show that for h^{n+1} , ∇ and composition by Ω^n asymptotically commute.

10.3. Estimates on the characteristics. In this subsection, we suppose that estimates (\mathbf{E}_ρ^n) is proven, and we establish $(\mathbf{E}_\Omega^n) + (\tilde{\mathbf{E}}_\Omega^n) + (\mathbf{E}_{\nabla\Omega}^n)$. Let $\lambda_n^* < \lambda_n$, $\mu_n^* < \mu_n$ to be fixed later on.

For the reasons alluded to above, we shall perform the first two steps in hybrid norms with two time-shifts, to show how this works; then for all the rest we shall stick to just one time-shift.

10.3.1. Step 1: Estimate of $\Omega^n - \text{Id}$. This is the first and archetypal estimate. Let $\hat{\lambda}_n < \lambda_n^*$; we shall estimate $\Omega^n X_{t,\tau} - \text{Id}$ in the two-shift hybrid norm $\mathcal{Z}_{\left(\tau - \frac{bt}{1+b}, 0\right)}^{(\hat{\lambda}_n(1+b), \lambda_n^* - \hat{\lambda}_n), (\mu_n^*, \gamma)}$.

(To convert the proofs for a single time-shift, just take $\hat{\lambda}_n \rightarrow \lambda_n^*$.)

The Sobolev correction γ will play no role here in the proofs, and for simplicity we shall forget it in the computations. (Use Proposition 4.32 whenever needed.)

Since we expect the characteristics for the force field F^n to be close to the free transport characteristics, it is natural to write

$$(10.21) \quad X_{t,\tau}^n(x, v) = x - v(t - \tau) + Z_{t,\tau}^n(x, v),$$

where $Z_{t,\tau}^n$ solves

$$(10.22) \quad \begin{cases} \frac{\partial^2}{\partial \tau^2} Z_{t,\tau}^n(x, v) = F^n\left(\tau, x - v(t - \tau) + Z_{t,\tau}^n(x, v)\right) \\ Z_{t,t}^n(x, v) = 0, \quad \partial_\tau Z_{t,\tau}^n \Big|_{\tau=t} (x, v) = 0. \end{cases}$$

(With respect to Section 5 we have dropped the parameter ε , to take advantage of the “stratified” nature of F^n ; anyway this parameter was cosmetic.) So if we fix $t > 0$, $(Z_{t,\tau}^n)$ is a fixed point of the map

$$\Psi : (W_{t,\tau})_{0 \leq \tau \leq t} \longmapsto (Z_{t,\tau})_{0 \leq \tau \leq t}$$

defined by

$$(10.23) \quad \begin{cases} \frac{\partial^2}{\partial \tau^2} Z_{t,\tau} = F^n\left(\tau, x - v(t - \tau) + W_{t,\tau}\right) \\ Z_{t,t} = 0, \quad \partial_\tau Z_{t,\tau} \Big|_{\tau=t} = 0. \end{cases}$$

We shall estimate $Z_{t,\tau}^n$ in the two-shift norm $\mathcal{Z}_{\left(t - \frac{bt}{1+b}, t - \tau\right)}^{(\hat{\lambda}_n(1+b), \lambda_n^* - \hat{\lambda}_n), \mu_n^*}$.

We first bound $(Z_0^n)_{t,\tau} = \Psi(0)$. Explicitly,

$$(Z_0^n)_{t,\tau}(x, v) = \int_\tau^t (s - \tau) F^n(s, x - v(t - s)) ds.$$

By Proposition 4.36 (i)(iii) [Propositions 4.15 (i) and 4.19 for one shift], for any $\sigma \in [0, \tau]$,

$$\begin{aligned}
(10.24) \quad & \left\| (Z_0^n)_{t,\tau} \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} \\
& \left(t - \frac{bt}{1+b}, t - \sigma \right) \\
& \leq \int_{\tau}^t (s - \tau) \left\| F^n(s, x - v(t - s)) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} ds \\
& = \int_{\tau}^t (s - \tau) \left\| F^n(s, \cdot) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} ds \\
& \quad \left(s - \frac{bt}{1+b}, s - \sigma \right) \\
& = \int_{\tau}^t (s - \tau) \left\| F^n(s, \cdot) \right\|_{\mathcal{F}^{\nu(s,t,\sigma)}} ds,
\end{aligned}$$

where

$$(10.25) \quad \nu(s, t, \sigma) = \widehat{\lambda}_n |s - b(t - s)| + (\lambda_n^* - \widehat{\lambda}_n)(s - \sigma) + \mu_n^*.$$

First case: If $s \geq bt/(1 + b)$, then

$$\begin{aligned}
(10.26) \quad & \nu(s, t, \sigma) = \widehat{\lambda}_n (s - b(t - s)) + (\lambda_n^* - \widehat{\lambda}_n)(s - \sigma) + \mu_n^* \\
& \leq \lambda_n^* s - \left[\widehat{\lambda}_n b(t - s) + (\lambda_n^* - \widehat{\lambda}_n) \sigma \right] + \mu_n^*.
\end{aligned}$$

In particular,

$$(10.27) \quad \nu(s, t, \sigma) \leq \lambda_n^* s + \mu_n^* \leq \lambda_k s + \mu_k - (\lambda_k - \lambda_n^*) s \quad (1 \leq k \leq n).$$

Second case: If $s < bt/(1 + b)$, then necessarily $s \leq B \leq T$. Taking into account (10.6), we have

$$(10.28) \quad \nu(s, t, \sigma) = \widehat{\lambda}_n bt + (\lambda_n^* - 2\widehat{\lambda}_n - \widehat{\lambda}_n b)s - (\lambda_n^* - \widehat{\lambda}_n)\sigma + \mu_n^*$$

$$(10.29) \quad \leq \widehat{\lambda}_n B + \mu_n^* - (\lambda_k - \lambda_n^*)s.$$

(Of course, the assumption $\lambda^\sharp - \lambda_\infty \leq \min\{1, \lambda_\infty/2\}$ implies $\lambda_n^* - 2\widehat{\lambda}_n \leq 0$.) In particular, by (10.7),

$$(10.30) \quad \nu(s, t, \sigma) \leq \mu^\sharp - (\lambda_k - \lambda_n^*)s \quad (1 \leq k \leq n).$$

We plug these bounds into (10.24), then use $\widehat{E}^k(s, 0) = 0$ and the bounds (10.10) and (10.27) (for large times), and (10.3) and (10.30) (for short times). This yields

(10.31)

$$\begin{aligned}
& \| (Z_0^n)_{t, \tau} \|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)} \\
& \leq \sum_{k=1}^n \left(\int_{\tau \vee \frac{bt}{1+b}}^t (s - \tau) \| E^k(s, \cdot) \|_{\mathcal{F}^{\lambda_k s + \mu_k - (\lambda_k - \lambda_n^*)s}} ds \right. \\
& \quad \left. + \int_{\tau}^{\tau \vee \frac{bt}{1+b}} (s - \tau) \| E^k(s, \cdot) \|_{\mathcal{F}^{\mu^\# - (\lambda_k - \lambda_n^*)s}} ds \right) \\
& \leq \sum_{k=1}^n \left(\int_{\tau \vee \frac{bt}{1+b}}^t (s - \tau) e^{-2\pi(\lambda_k - \lambda_n^*)s} \| E^k(s, \cdot) \|_{\mathcal{F}^{\lambda_k s + \mu_k}} ds \right. \\
& \quad \left. + \int_{\tau}^{\tau \vee \frac{bt}{1+b}} (s - \tau) e^{-2\pi(\lambda_k - \lambda_n^*)s} \| E^k(s, \cdot) \|_{\mathcal{F}^{\mu^\#}} ds \right) \\
& \leq \sum_{k=1}^n \delta_k \int_{\tau}^t (s - \tau) e^{-2\pi(\lambda_k - \lambda_n^*)s} ds \\
& \leq \sum_{k=1}^n \delta_k e^{-2\pi(\lambda_k - \lambda_n^*)\tau} \min \left\{ \frac{(t - \tau)^2}{2}; \frac{1}{(2\pi(\lambda_k - \lambda_n^*))^2} \right\} \leq \mathcal{R}_2^n(\tau, t).
\end{aligned}$$

Let us define the norm

$$\left\| \left\| (Z_{t, \tau})_{0 \leq \tau \leq t} \right\| \right\|_n := \sup_{0 \leq \sigma \leq \tau \leq t} \frac{\| Z_{t, \tau} \|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)}}{\mathcal{R}_2^n(\tau, t)}.$$

(Note the difference with Section 5: now the regularity exponents depend on time(s).) Inequality (10.31) means that $\left\| \left\| \Psi(0) \right\| \right\|_n \leq 1$. We shall check that Ψ is $(1/2)$ -Lipschitz on the ball $B(0, 2)$ in the norm $\left\| \left\| \cdot \right\| \right\|_n$. This will be subtle: the uniform bounds on the size of the force field, coming from the preceding steps, will allow to get good decaying exponentials, which in turn will imply uniform error bounds at the present stage.

So let $W, \widetilde{W} \in B(0, 2)$, and let $Z = \Psi(W)$, $\widetilde{Z} = \Psi(\widetilde{W})$. As in Section 5, we write

$$Z_{t,\tau} - \widetilde{Z}_{t,\tau} = \int_0^1 \int_\tau^t (s - \tau) \nabla_x F^n \left(s, x - v(t - s) + \left(\theta W_{t,s} + (1 - \theta) \widetilde{W}_{t,s} \right) \right) \cdot (W_{t,s} - \widetilde{W}_{t,s}) ds d\theta,$$

and deduce

$$\left\| \left\| (Z_{t,\tau} - \widetilde{Z}_{t,\tau})_{0 \leq \tau \leq t} \right\|_n \right\| \leq A(t) \left\| \left\| (W_{t,s} - \widetilde{W}_{t,s})_{0 \leq s \leq t} \right\|_n \right\|,$$

where

$$A(t) = \sup_{0 \leq \sigma \leq \tau \leq s \leq t} \frac{\mathcal{R}_2^n(s, t)}{\mathcal{R}_2^n(\tau, t)} \times \int_0^1 \int_\tau^t (s - \tau) \left\| \nabla_x F^n \left(s, x - v(t - s) + (\theta W_{t,s} + (1 - \theta) \widetilde{W}_{t,s}) \right) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} \Big|_{\left(t - \frac{bt}{1+b}, t - \sigma \right)} ds d\theta.$$

(Note: this works because the condition $\sigma \leq \tau$ automatically implies $\sigma \leq s$.) For $\tau \leq s$ we have $\mathcal{R}_2^n(s, t) \leq \mathcal{R}_2^n(\tau, t)$. Also, by Proposition 4.36(v) (applied with $V = 0$, $b = -(t - s)$ and $\sigma, \sigma' = 0$ in that statement) and (iii) [or Propositions 4.25 and 4.15 for one time-shift], we find

$$A(t) \leq \sup_{0 \leq \tau \leq t} \int_\tau^t (s - \tau) \left\| \nabla_x F^n(s, \cdot) \right\|_{\mathcal{F}^{\nu(s,t,\sigma) + e(s,t,\sigma)}} ds,$$

where ν is defined by (10.25) and the “error” $e(s, t, \tau)$ arising from composition is given by

$$\begin{aligned} e(s, t, \sigma) &= \sup_{0 \leq \theta \leq 1} \left\| \theta W_{t,s} + (1 - \theta) \widetilde{W}_{t,s} \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} \Big|_{\left(t - \frac{bt}{1+b}, t - \sigma \right)} \\ &\leq 2 \mathcal{R}_2^n(s, t). \end{aligned}$$

(Again this works because $\sigma \leq \tau \leq s$.)

Since

$$\mathcal{R}_2^n(s, t) \leq \omega_n^{1,2}(s, t) := C_\omega^1 \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^3} \right) \frac{\min \{(t - s)^2/2; 1\}}{(1 + s)}$$

we have, for all $0 \leq s \leq t$,

$$(10.32) \quad 2 \mathcal{R}_2^n(s, t) \leq \frac{\widehat{\lambda}_n}{2} b(t - s) 1_{s \geq bt/(1+b)} + \frac{\mu^\sharp - \mu_n^*}{2} 1_{s \leq bt/(1+b)},$$

as soon as
(10.33)

$$(\mathbf{C}_1) \quad \forall n \geq 1, \quad 2C_\omega^1 \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_n^*))^3} \right) \leq \min \left\{ \frac{1}{3} \frac{\widehat{\lambda}_n}{2} B; \frac{\mu^\sharp - \mu_n^*}{2} \right\}.$$

We shall check later in Subsection 10.5 the feasibility of this condition (as well as a number of other forthcoming ones).

The extra error term in the exponent is sufficiently small to be absorbed by what we throw away in (10.26)–(10.27) or (10.28)–(10.29)–(10.30). So we obtain, as in the estimate of Z_0^n , for any $k \in \{1, \dots, n\}$,

$$(\nu + e)(s, t, \sigma) \begin{cases} \leq \lambda_k s + \mu_k - (\lambda_k - \lambda_n^*) s & \text{for } s \geq bt/(1+b) \\ \leq \mu^\sharp - (\lambda_k - \lambda_n^*) s & \text{for } s \leq bt/(1+b), \end{cases}$$

and we deduce (using (10.10) and the fact that γ is greater than 1)

$$\begin{aligned} A(t) &\leq \sup_{0 \leq \tau \leq t} \sum_{k=1}^n \left(\int_{\tau \vee \frac{bt}{1+b}}^t (s - \tau) \|\nabla_x E^k(s, \cdot)\|_{\mathcal{F}^{\lambda_k s + \mu_k - (\lambda_k - \lambda_n^*) s}} ds \right. \\ &\quad \left. + \int_{\tau}^{\tau \vee \frac{bt}{1+b}} (s - \tau) \|\nabla_x E^k(s, \cdot)\|_{\mathcal{F}^{\mu^\sharp - (\lambda_k - \lambda_n^*) s}} ds \right) \\ &\leq \sup_{0 \leq \tau \leq t} \sum_{k=1}^n \delta_k \int_{\tau}^t (s - \tau) e^{-(\lambda_k - \lambda_n^*) s} ds \leq \sup_{0 \leq \tau \leq t} \mathcal{R}_2^n(\tau, t) = \mathcal{R}_2^n(0, t) \\ &\leq \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_n^*))^2}. \end{aligned}$$

If the latter quantity is bounded above by $1/2$, then Ψ is $(1/2)$ -Lipschitz and we may apply the fixed point result from Theorem A.2. Therefore, under the condition (whose feasibility will be checked later)

$$(10.34) \quad (\mathbf{C}_2) \quad \forall n \geq 1, \quad \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_n^*))^2} \leq \frac{1}{2}$$

we deduce, choosing $\sigma = \tau$,

$$\|Z_{t,\tau}^n\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*}} \leq 2 \mathcal{R}_2^n(\tau, t).$$

After that, the estimates on the scattering follow exactly as in Section 5: writing $\Omega_{t,\tau}^n = (\Omega^n X_{t,\tau}, \Omega^n V_{t,\tau})$, recalling the dependence on γ again, we end up with

$$(10.35) \quad \begin{cases} \left\| \Omega^n X_{t,\tau} - x \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), (\mu_n^*, \gamma)}_{\tau - \frac{bt}{1+b}}} \leq 2 \mathcal{R}_2^n(\tau, t) \\ \left\| \Omega^n V_{t,\tau} - v \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), (\mu_n^*, \gamma)}_{\tau - \frac{bt}{1+b}}} \leq \mathcal{R}_1^n(\tau, t); \end{cases}$$

$$(10.36) \quad \begin{cases} \left\| \Omega^n X_{t,\tau} - x \right\|_{\mathcal{Z}^{(\hat{\lambda}_n(1+b), \lambda_n^* - \hat{\lambda}_n), (\mu_n^*, \gamma)}_{\left(\tau - \frac{bt}{1+b}, 0\right)}} \leq 2 \mathcal{R}_2^n(\tau, t) \\ \left\| \Omega^n V_{t,\tau} - v \right\|_{\mathcal{Z}^{(\hat{\lambda}_n(1+b), \lambda_n^* - \hat{\lambda}_n), (\mu_n^*, \gamma)}_{\left(\tau - \frac{bt}{1+b}, 0\right)}} \leq \mathcal{R}_1^n(\tau, t). \end{cases}$$

10.3.2. *Step 2: Estimate of $\Omega^n - \Omega^k$.* In this step our goal is to estimate $\Omega^n - \Omega^k$ for $1 \leq k \leq n-1$. The point is that the error should be small as $k \rightarrow \infty$, uniformly in n , so we can't just write $\|\Omega^n - \Omega^k\| \leq \|\Omega^n - \text{Id}\| + \|\Omega^k - \text{Id}\|$. Instead, we start again from the differential equation satisfied by Z^k and Z^n :

$$\begin{aligned} & \frac{\partial^2}{\partial \tau^2} (Z_{t,\tau}^n - Z_{t,\tau}^k)(x, v) \\ &= F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n(x, v) \right) - F^k \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k(x, v) \right) \\ &= \left[F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n \right) - F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k \right) \right] \\ & \quad + (F^n - F^k) \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k \right). \end{aligned}$$

This, together with the boundary conditions $Z_{t,t}^n - Z_{t,t}^k = 0$, $\partial_\tau (Z_{t,\tau}^n - Z_{t,\tau}^k)|_{\tau=t} = 0$, implies

$$\begin{aligned} & Z_{t,\tau}^n - Z_{t,\tau}^k \\ &= \int_0^1 \int_\tau^t (s - \tau) \nabla_x F^n \left(s, x - v(t - s) + (\theta Z_{t,s}^k + (1 - \theta) Z_{t,s}^n) \right) \cdot (Z_{t,s}^n - Z_{t,s}^k) ds d\theta \\ & \quad + \int_\tau^t (s - \tau) (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k(x, v) \right) ds. \end{aligned}$$

We fix t and define the norm

$$\left\| \left\| (Z_{t,\tau})_{0 \leq \tau \leq t} \right\| \right\|_{k,n} := \sup_{0 \leq \sigma \leq \tau \leq t} \frac{\|Z_{t,\tau}\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)}}{\mathcal{R}_2^{k,n}(\tau, t)},$$

where $\mathcal{R}_2^{k,n}$ is defined in (10.14). Using the bounds on Z^n, Z^k in $\left\| \left\| \cdot \right\| \right\|_n$ (since $\left\| \left\| \cdot \right\| \right\|_n \leq \left\| \left\| \cdot \right\| \right\|_k$ by using the fact that $\mathcal{R}_2^k \leq \mathcal{R}_2^n$) and proceeding as before, we get

$$(10.37) \quad \left\| \left\| (Z_{t,\tau}^n - Z_{t,\tau}^k)_{0 \leq \tau \leq t} \right\| \right\|_{k,n} \leq \frac{1}{2} \left\| \left\| (Z_{t,\tau}^n - Z_{t,\tau}^k)_{0 \leq \tau \leq t} \right\| \right\|_{k,n} \\ + \left\| \left\| \left(\int_{\tau}^t (s - \tau) (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k \right) ds \right)_{0 \leq \tau \leq t} \right\| \right\|_{k,n}.$$

We estimate, for $0 \leq \sigma \leq \tau$

$$\begin{aligned} & \left\| (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k \right) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)}} \\ &= \left\| (F^n - F^k) \left(s, X_{t,s}^k \right) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)}} \\ &= \left\| (F^n - F^k) \left(s, \Omega_{t,s}^k \right) \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(s - \frac{bt}{1+b}, s - \sigma \right)}} \leq \left\| (F^n - F^k) \left(s, \cdot \right) \right\|_{\mathcal{F}^{\nu(s,t,\tau) + e(s,t,\tau)}}, \end{aligned}$$

where the last inequality follows from Proposition 4.36, ν is again given by (10.25), and

$$\begin{aligned} e(s, t, \sigma) &= \left\| \Omega^k X_{t,s} - \text{Id} \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(s - \frac{bt}{1+b}, s - \sigma \right)}} \\ &\leq 2 \mathcal{R}_2^k(s, t) \leq 2 \mathcal{R}_2^n(s, t). \end{aligned}$$

The same reasoning as in Step 1 yields, under assumptions (\mathbf{C}_1) - (\mathbf{C}_2) , for $k+1 \leq j \leq n$:

$$(\nu + e)(s, t, \sigma) \begin{cases} \leq \lambda_j s + \mu_j - (\lambda_j - \lambda_n^*) s & \text{for } s \geq bt/(1+b) \\ \leq \mu^\sharp - (\lambda_j - \lambda_n^*) s & \text{for } s \leq bt/(1+b), \end{cases}$$

and so

$$\|F_s^n - F_s^k\|_{\mathcal{F}^{\nu+e}} \leq \sum_{j=k+1}^n \delta_j e^{-2\pi(\lambda_j - \lambda_n^*)s}.$$

For any $0 \leq \sigma \leq \tau$, by integrating in time we find

$$\begin{aligned} & \left\| \int_{\tau}^t (s - \tau) (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k \right) ds \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), \mu_n^*} \left(t - \frac{bt}{1+b}, t - \sigma \right)}} \\ & \leq \int_{\tau}^t (s - \tau) \sum_{j=k+1}^n \delta_j e^{-2\pi(\lambda_j - \lambda_n^*)s} ds \leq \mathcal{R}_2^{k,n}(\tau, t). \end{aligned}$$

Therefore

$$\left\| \left(\int_{\tau}^t (s - \tau) (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k \right) ds \right) \right\|_{0 \leq \tau \leq t} \left\| \right\|_{k,n} \leq 1$$

and by (10.37)

$$\left\| (Z_{t,\tau}^n - Z_{t,\tau}^k) \right\|_{0 \leq \tau \leq t} \left\| \right\|_{k,n} \leq 2.$$

Picking $\sigma = \tau$ and recalling the Sobolev correction, we deduce

$$(10.38) \quad \left\| \Omega^n X_{t,\tau} - \Omega^k X_{t,\tau} \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), (\mu_n^*, \gamma)} \left(\tau - \frac{bt}{1+b}, 0 \right)}} \leq 2 \mathcal{R}_2^{k,n}(\tau, t).$$

For the velocity component, say U , we write

$$\begin{aligned} & \frac{\partial}{\partial \tau} (U_{t,\tau}^n - U_{t,\tau}^k)(x, v) \\ & = F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n(x, v) \right) - F^k \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k(x, v) \right) \\ & = \left[F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n \right) - F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k \right) \right] \\ & \quad + (F^n - F^k) \left(\tau, x - v(t - \tau) + Z_{t,\tau}^k \right). \end{aligned}$$

where Z^n, Z^k were estimated above, and the boundary conditions are $U_{t,t}^n - U_{t,t}^k = 0$. Thus

$$\begin{aligned} & U_{t,\tau}^n - U_{t,\tau}^k \\ & = \int_0^1 \int_{\tau}^t \nabla_x F^n \left(s, x - v(t - s) + (\theta Z_{t,s}^k + (1 - \theta) Z_{t,s}^n) \right) \cdot (Z_{t,s}^n - Z_{t,s}^k) ds d\theta \\ & \quad + \int_{\tau}^t (F^n - F^k) \left(s, x - v(t - s) + Z_{t,s}^k(x, v) \right) ds, \end{aligned}$$

and from this one easily derives the similar estimate

$$(10.39) \quad \left\| \Omega^n V_{t,\tau} - \Omega^k V_{t,\tau} \right\|_{\mathcal{Z}^{(\widehat{\lambda}_n(1+b), \lambda_n^* - \widehat{\lambda}_n), (\mu_n^*, \gamma)}_{\left(\tau - \frac{bt}{1+b}, 0\right)}} \leq \mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t),$$

as well as

$$\begin{cases} \left\| \Omega^n X_{t,\tau} - \Omega^k X_{t,\tau} \right\|_{\mathcal{Z}^{(\lambda_n^*(1+b), \mu_n^*)}_{\tau - \frac{bt}{1+b}}} \leq 2 \mathcal{R}_2^{k,n}(t, \tau) \\ \left\| \Omega^n V_{t,\tau} - \Omega^k V_{t,\tau} \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*}_{\tau - \frac{bt}{1+b}}} \leq \mathcal{R}_1^{k,n}(t, \tau) + \mathcal{R}_2^{k,n}(t, \tau). \end{cases}$$

10.3.3. *Step 3: Estimate of $\nabla \Omega^n$.* Now we establish a control on the derivative of the scattering. Of course, we could deduce such a control from the bound on $\Omega^n - \text{Id}$ and Proposition 4.32(vi): for instance, if $\lambda_n^{**} < \lambda_n^*$, $\mu_n^{**} < \mu_n^*$, then

$$(10.40) \quad \left\| \nabla \Omega_{t,\tau}^n - I \right\|_{\mathcal{Z}^{\lambda_n^{**}(1+b), (\mu_n^{**}, \gamma)}_{\tau - \frac{bt}{1+b}}} \leq \frac{C \mathcal{R}_2^n(\tau, t)}{\min \{ \lambda^* - \lambda^{**}; \mu^* - \mu^{**} \}}.$$

But this bound involves very large constants, and is useless in practice. Better estimates can be obtained by using again the equation (10.22). Writing

$$(\Omega_{t,\tau}^n - \text{Id})(x, v) = \left(Z_{t,\tau}^n(x + v(t - \tau), v), \dot{Z}_{t,\tau}^n(x + v(t - \tau), v) \right),$$

where the dot stands for $\partial/\partial\tau$, we get by differentiation

$$\nabla_x \Omega_{t,\tau}^n - (\text{Id}, 0) = \left(\nabla_x Z_{t,\tau}^n(x + v(t - \tau), v), \nabla_x \dot{Z}_{t,\tau}^n(x + v(t - \tau), v) \right),$$

$$\nabla_v \Omega_{t,\tau}^n - (0, \text{Id}) = \left((\nabla_v + (t - \tau) \nabla_x) Z_{t,\tau}^n(x + v(t - \tau), v), (\nabla_v + (t - \tau) \nabla_x) \dot{Z}_{t,\tau}^n(x + v(t - \tau), v) \right).$$

Let us estimate for instance $\nabla_x \Omega - (\text{Id}, 0)$, or equivalently $\nabla_x Z_{t,\tau}^n$. By differentiating (10.22), we obtain

$$\frac{\partial^2}{\partial \tau^2} \nabla_x Z_{t,\tau}^n(x, v) = \nabla_x F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n(x, v) \right) \cdot (\text{Id} + \nabla_x Z_{t,\tau}^n).$$

So $\nabla_x Z_{t,\tau}^n$ is a fixed point of $\Psi : W \mapsto Q$, where W and Q are functions of $\tau \in [0, t]$ satisfying

$$\begin{cases} \frac{\partial^2 Q}{\partial \tau^2} = \nabla_x F^n(\tau, x - v(t - \tau) + Z_{t,\tau}^n)(I + W), \\ Q(t) = 0, \quad \partial_\tau Q(t) = 0. \end{cases}$$

We treat this in the same way as in Steps 1 and 2, and find on Q_x (the x component of Q) the same estimates as we had previously on the x component of Ω . For the velocity component, a direct estimate from the integral equation expressing the

velocity in terms of F yields a control by $\mathcal{R}_1^n + \mathcal{R}_2^n$. Finally for $\nabla_v \Omega$ this is similar, noting that $(\nabla_v + (t - \tau)\nabla_x)(x - v(t - \tau)) = 0$, the differential equation being for instance:

$$\frac{\partial^2}{\partial \tau^2} (\nabla_v + (t - \tau)\nabla_x) Z_{t,\tau}^n(x, v) = \nabla_x F^n \left(\tau, x - v(t - \tau) + Z_{t,\tau}^n(x, v) \right) \cdot ((\nabla_v + (t - \tau)\nabla_x) Z_{t,\tau}^n).$$

In the end we obtain

$$(10.41) \quad \begin{cases} \sup_{t \geq \tau \geq 0} \left\| \nabla \Omega^n X_{t,\tau} - I \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \leq 2 \mathcal{R}_2^n(\tau, t), \\ \sup_{t \geq \tau \geq 0} \left\| \nabla \Omega^n V_{t,\tau} - I \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \leq \mathcal{R}_1^n(\tau, t) + \mathcal{R}_2^n(\tau, t). \end{cases}$$

10.3.4. *Step 4: Estimate of $(\Omega^k)^{-1} \circ \Omega^n$.* We do this by applying Proposition 4.28 with $F = \Omega^k$, $G = \Omega^n$. (Note: we cannot exchange the roles of Ω^k and Ω^n in this step, because we have a better information on the regularity of Ω^k .) Let $\varepsilon = \varepsilon(d)$ be the small constant appearing in Proposition 4.28. If

$$(10.42) \quad (\mathbf{C}_3) \quad \forall k \geq 1, \quad 3 \mathcal{R}_2^k(\tau, t) + \mathcal{R}_1^k(\tau, t) \leq \varepsilon,$$

then $\|\nabla \Omega_{t,\tau}^k - I\|_{\mathcal{Z}_{\tau - bt/(1+b)}^{\lambda_k^*(1+b), \mu_k^*}} \leq \varepsilon$; if in addition

$$(10.43) \quad (\mathbf{C}_4) \quad \forall k \in \{1, \dots, n-1\}, \quad \forall t \geq \tau, \\ 2(1 + \tau)(1 + B)(3 \mathcal{R}_2^{k,n} + \mathcal{R}_1^{k,n})(\tau, t) \leq \max\{\lambda_k^* - \lambda_n^*; \mu_k^* - \mu_n^*\},$$

then

$$\begin{cases} \lambda_n^*(1 + b) + 2 \|\Omega^n - \Omega^k\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \leq \lambda_k^*(1 + b) \\ \mu_n^* + 2 \left(1 + \left| \tau - \frac{bt}{1+b} \right| \right) \|\Omega^n - \Omega^k\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \leq \mu_k^*. \end{cases}$$

(Once again, short times should be treated separately. Further note that the need for the factor $(1 + \tau)$ in (\mathbf{C}_4) ultimately comes from the fact that we are composing also in the v variable, see the coefficient σ in the last norm of (4.29).) Then Proposition 4.28 (ii) yields

$$\begin{aligned} \left\| (\Omega_{t,\tau}^k)^{-1} \circ \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} &\leq 2 \left\| \Omega_{t,\tau}^k - \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \\ &\leq 4 (\mathcal{R}_1^{k,n} + \mathcal{R}_2^{k,n})(\tau, t). \end{aligned}$$

10.3.5. *Partial conclusion.* At this point we have established $(\mathbf{E}_\Omega^n) + (\widetilde{\mathbf{E}}_\Omega^n) + (\mathbf{E}_{\nabla \Omega}^n)$.

10.4. **Estimates on the density and distribution along characteristics.** In this subsection we establish $(\mathbf{E}_\rho^{n+1}) + (\tilde{\mathbf{E}}_\rho^{n+1}) + (\mathbf{E}_h^{n+1}) + (\tilde{\mathbf{E}}_h^{n+1})$.

10.4.1. *Step 5: Estimate of $h^k \circ \Omega^n$ and $(\nabla h^k) \circ \Omega^n$ ($k \leq n$).* Let $k \in \{1, \dots, n\}$. Since

$$h_\tau^k \circ \Omega_{t,\tau}^n = (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \circ ((\Omega_{t,\tau}^{k-1})^{-1} \circ \Omega_{t,\tau}^n),$$

the control on $h^k \circ \Omega^n$ will follow from the control on $h^k \circ \Omega^{k-1}$ in (\mathbf{E}_h^n) , together with the control on $(\Omega^{k-1})^{-1} \circ \Omega^n$ in $(\tilde{\mathbf{E}}_\Omega^n)$. If

$$(10.44) \quad (1 + \tau) \left\| (\Omega_{t,\tau}^{k-1})^{-1} \circ \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*}} \leq \min\{(\lambda_k - \lambda_n^*); (\mu_k - \mu_n^*)\},$$

then we can apply Proposition 4.25 and get, for any $p \in [1, \bar{p}]$, and $t \geq \tau \geq 0$,

$$(10.45) \quad \left\| h_\tau^k \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*; p}} \leq \left\| h_\tau^k \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; p}} \leq \delta_k.$$

In turn, (10.44) is satisfied if

$$(10.46) \quad (\mathbf{C}_5) \quad \forall k \in \{1, \dots, n\}, \quad \forall \tau \in [0, t], \\ 4(1 + \tau) (\mathcal{R}_1^{k,n}(\tau, t) + \mathcal{R}_2^{k,n}(\tau, t)) \leq \min\{\lambda_k - \lambda_n^*; \mu_k - \mu_n^*\};$$

we shall check later the feasibility of this condition.

Then, by the same argument, we also have

$$\forall k \in \{1, \dots, n\}, \quad \forall p \in [1, \bar{p}], \\ \sup_{t \geq \tau \geq 0} \left\| (\nabla_x h_\tau^k) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*; p}} + \left\| ((\nabla_v + \tau \nabla_x) h_\tau^k) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^*(1+b), \mu_n^*; p}} \leq \delta_k.$$

10.4.2. *Step 6: estimate on $\rho[h^{n+1}]$.* This step is the first where we shall use the Vlasov equation. Starting from (8.2), we apply the method of characteristics to get, as in Section 8,

$$(10.47) \quad h^{n+1}(t, X_{0,t}^n(x, v), V_{0,t}^n(x, v)) = \int_0^t \Sigma^{n+1}(\tau, X_{0,\tau}^n(x, v), V_{0,\tau}^n(x, v)) d\tau,$$

where

$$\Sigma^{n+1} = -\left(F[h^{n+1}] \cdot \nabla_v f^n + F[h^n] \cdot \nabla_v h^n \right).$$

We compose this with $(X_{t,0}^n, V_{t,0}^n)$ and apply (5.2) to get

$$h^{n+1}(t, x, v) = \int_0^t \Sigma^{n+1}(\tau, X_{t,\tau}^n(x, v), V_{t,\tau}^n(x, v)) d\tau,$$

and so, by integration in the v variable,

$$\begin{aligned}
(10.48) \quad \rho[h^{n+1}](t, x) &= \int_0^t \int_{\mathbb{R}^d} \Sigma^{n+1}(\tau, X_{t,\tau}^n(x, v), V_{t,\tau}^n(x, v)) \, dv \, d\tau \\
&= - \int_0^t \int_{\mathbb{R}^d} (R_{\tau,t}^{n+1} \cdot G_{\tau,t}^n)(x - v(t - \tau), v) \, dv \, d\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^d} (R_{\tau,t}^n \cdot H_{\tau,t}^n)(x - v(t - \tau), v) \, dv \, d\tau,
\end{aligned}$$

where (with a slight inconsistency in the notation)

$$(10.49) \quad \begin{cases} R_{\tau,t}^{n+1} = F[h^{n+1}] \circ \Omega_{t,\tau}^n, & R_{\tau,t}^n = F[h^n] \circ \Omega_{t,\tau}^n, \\ G_{\tau,t}^n = (\nabla_v f^n) \circ \Omega_{t,\tau}^n, & H_{\tau,t}^n = (\nabla_v h^n) \circ \Omega_{t,\tau}^n. \end{cases}$$

Since the free transport semigroup and $\Omega_{t,\tau}^n$ are measure-preserving,

$$\begin{aligned}
\forall 0 \leq \tau \leq t, \quad & \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (R_{\tau,t}^{n+1} \cdot G_{\tau,t}^n)(x - v(t - \tau), v) \, dv \, dx \\
&= \int \int R_{\tau,t}^{n+1} \cdot G_{\tau,t}^n \, dv \, dx \\
&= \int \int F[h^{n+1}] \cdot \nabla_v f^n \, dv \, dx \\
&= \int \int \nabla_v \cdot (F[h^{n+1}] f^n) \, dv \, dx = 0,
\end{aligned}$$

and similarly

$$\forall 0 \leq \tau \leq t, \quad \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (R_{\tau,t}^n \cdot H_{\tau,t}^n)(x - v(t - \tau), v) \, dv \, dx = 0.$$

This will allow us to apply the inequalities from Section 6.

Substep a. Let us first deal with the source term

$$(10.50) \quad \sigma^{n,n}(t, x) := \int_0^t \int_{\mathbb{R}^d} (R_{\tau,t}^n \cdot H_{\tau,t}^n)(x - v(t - \tau), v) \, dv \, d\tau.$$

By Proposition 6.2,

$$(10.51) \quad \|\sigma^{n,n}(t, \cdot)\|_{\mathcal{F}\lambda_n^* t + \mu_n^*} \leq \int_0^t \|R_{\tau,t}^n\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*} \left|_{\tau - \frac{b\tau}{1+b}}\right| \|H_{\tau,t}^n\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*; 1} \left|_{\tau - \frac{b\tau}{1+b}}\right| d\tau.$$

On the one hand, we have from Step 5

$$\|H_{\tau,t}^n\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*};1} \leq 2(1+\tau)\delta_n.$$

On the other hand, under condition (\mathbf{C}_1) , we may apply Proposition 4.25 (choosing $\sigma = 0$ in that proposition) to get

$$\|R_{\tau,t}^n\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*}} \leq \|F[h_\tau^n]\|_{\mathcal{F}^{\nu_n}},$$

where

$$\begin{aligned} \nu_n(t, \tau) &= \mu_n^* + \lambda_n^*(1+b) \left| \tau - \frac{bt}{1+b} \right| + \|\Omega^n X_{t,\tau} - \text{Id}\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*}; \tau - \frac{bt}{1+b}} \\ &\leq \mu_n^* + \lambda_n^*(1+b) \left| \tau - \frac{bt}{1+b} \right| + 2\mathcal{R}_2^n(\tau, t). \end{aligned}$$

Proceeding as in Step 1 (treating small times separately), we deduce

$$\begin{aligned} \|R_{\tau,t}^n\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*}; \tau - \frac{bt}{1+b}} &\leq \|F[h_\tau^n]\|_{\mathcal{F}^{\nu_n}} \leq e^{-2\pi(\lambda_n - \lambda_n^*)\tau} \|F[h_\tau^n]\|_{\mathcal{F}^{\bar{\nu}_n}} \\ &\leq C_F e^{-2\pi(\lambda_n - \lambda_n^*)\tau} \|\rho[h_\tau^n]\|_{\mathcal{F}^{\bar{\nu}_n}} \leq C_F e^{-2\pi(\lambda_n - \lambda_n^*)\tau} \delta_n, \end{aligned}$$

with

$$(10.52) \quad \begin{cases} \bar{\nu}_n(\tau, t) := \mu_n^\sharp & \text{when } 0 \leq \tau \leq bt/(1+b) \\ \bar{\nu}_n(\tau, t) := \lambda_n \tau + \mu_n & \text{when } \tau \geq bt/(1+b). \end{cases}$$

(We have used the gradient structure of the force to convert (gliding) regularity into decay.) Thus

$$(10.53) \quad \begin{aligned} \int_0^t \|R_{\tau,t}^n\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*}; \tau - \frac{bt}{1+b}} \|H_{\tau,t}^n\|_{\mathcal{Z}^{\lambda_n^*(1+b),\mu_n^*};1} d\tau \\ \leq 2C_F \delta_n^2 \int_0^t e^{-2\pi(\lambda_n - \lambda_n^*)\tau} (1+\tau) d\tau \\ \leq \frac{2C_F \delta_n^2}{(\pi(\lambda_n - \lambda_n^*))^2}. \end{aligned}$$

(Note: This is the power 2 which is responsible for the very fast convergence of the Newton scheme.)

Substep b. Now let us handle the term

$$(10.54) \quad \sigma^{n,n+1}(t, x) := \int_0^t \int (R_{v,\tau,t}^{n+1} \cdot G_{\tau,t}^n)(x - v(t - \tau), v) dv d\tau.$$

This is the focal point of all our analysis, because it is in this term that the self-consistent nature of the Vlasov equation appears. In particular, we will make crucial use of the time-cheating trick to overcome the loss of regularity implied by composition; and also the other bilinear estimates (regularity extortion) from Section 6, as well as the time-response study from Section 7. Particular care should be given to the zero spatial mode of G^n , which is associated with instantaneous response (no echo). In the linearized equation we did not see this problem because the contribution of the zero mode was vanishing!

We start by introducing

$$(10.55) \quad \overline{G}_{\tau,t}^n = \nabla_v f^0 + \sum_{k=1}^n \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}),$$

and we decompose $\sigma^{n,n+1}$ as

$$(10.56) \quad \sigma^{n,n+1} = \overline{\sigma}^{n,n+1} + \mathcal{E} + \overline{\mathcal{E}},$$

where

$$(10.57) \quad \overline{\sigma}^{n,n+1}(t, x) = \int_0^t \int F[h_\tau^{n+1}] \cdot \overline{G}_{\tau,t}^n(x - v(t - \tau), v) dv d\tau$$

and the error terms \mathcal{E} and $\overline{\mathcal{E}}$ are defined by

$$(10.58) \quad \mathcal{E}(t, x) = \int_0^t \int \left((F[h_\tau^{n+1}] \circ \Omega_{t,\tau}^n - F[h_\tau^{n+1}]) \cdot G^n \right) (\tau, x - v(t - \tau), v) dv d\tau,$$

$$(10.59) \quad \overline{\mathcal{E}}(t, x) = \int_0^t \int \left(F[h_\tau^{n+1}] \cdot (G^n - \overline{G}^n) \right) (\tau, x - v(t - \tau), v) dv d\tau.$$

We shall first estimate \mathcal{E} and $\overline{\mathcal{E}}$.

Control of \mathcal{E} : This is based on the time-cheating trick from Section 6, and the regularity of the force. By Proposition 6.2,

$$(10.60) \quad \|\mathcal{E}(t, \cdot)\|_{\mathcal{F}\lambda_n^* t + \mu_n^*} \leq \int_0^t \left\| F[h_\tau^{n+1}] \circ \Omega_{t,\tau}^n - F[h_\tau^{n+1}] \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*} \times \\ \left\| G^n \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*; 1} d\tau.$$

From (10.1) and Step 5,

$$(10.61) \quad \left\| G^n \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*; 1} \leq \left\| \nabla_v f^0 \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*; 1} + \sum_{k=1}^n \left\| \nabla_v h_\tau^k \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*; 1} \\ \leq C'_0 + \left(\sum_{k=1}^n \delta_k \right) (1 + \tau),$$

where C'_0 comes from the contribution of f^0 .

Next, by Propositions 4.24 and 4.25 (with $V = 0$, $\tau = \sigma$, $b = 0$),

$$(10.62) \quad \left\| F[h_\tau^{n+1}] \circ \Omega_{t,\tau}^n - F[h_\tau^{n+1}] \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*} \\ \leq \left(\int_0^1 \left\| \nabla F[h_\tau^{n+1}] \circ (\text{Id} + \theta(\Omega_{t,\tau}^n - \text{Id})) \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*} d\theta \right) \left\| \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*} \\ \leq \left\| \nabla F[h_\tau^{n+1}] \right\|_{\mathcal{F}\nu_n} \left\| \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*},$$

where

$$\nu_n = \mu_n^* + \lambda_n^*(1+b) \left| \tau - \frac{bt}{1+b} \right| + \left\| \Omega^n X_{t,\tau} - \text{Id} \right\|_{\mathcal{Z}\lambda_n^*(1+b), \mu_n^*}.$$

Small times are taken care of, as usual, by the initial regularity layer, so we only focus on the case $\tau \geq bt/(1+b)$; then

$$\nu_n \leq (\lambda_n^* \tau + \mu_n^*) - \lambda_n^* b(t - \tau) + 2\mathcal{R}^n(\tau, t) \\ \leq (\lambda_n^* \tau + \mu_n^*) - \lambda_n^* \frac{B(t - \tau)}{1 + t} + 4C_\omega^1 \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^3} \right) \frac{\min\{t - \tau; 1\}}{1 + \tau}.$$

To make sure that $\nu_n \leq \lambda_n^* \tau + \mu_n^*$, we assume that

$$(10.63) \quad (\mathbf{C}_6) \quad 4C_\omega^1 \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^3} \leq \frac{\lambda_\infty^* B}{3},$$

and we note that

$$\frac{\min\{t - \tau; 1\}}{1 + \tau} \leq 3 \left(\frac{t - \tau}{1 + t} \right).$$

(This is easily seen by separating four cases: (a) $t \leq 2$, (b) $t \geq 2$ and $t - \tau \leq 1$, (c) $t \geq 2$ and $t - \tau \geq 1$ and $\tau \leq t/2$, (d) $t \geq 2$ and $t - \tau \geq 1$ and $\tau \geq t/2$.)

Then, since $\gamma \geq 1$, we have

$$\begin{aligned} \|\nabla F[h_\tau^{n+1}]\|_{\mathcal{F}\nu_n} &\leq \|\nabla F[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*} \\ &\leq \|F[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*, \gamma} \\ &\leq C_F \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*}. \end{aligned}$$

(Note: Applying Proposition 4.10 instead of the regularity coming from the interaction would consume more regularity than we can afford to.)

Plugging this back into (10.62), we get

$$\begin{aligned} &\left\| F[h_\tau^{n+1}] \circ \Omega_{t,\tau}^n - F[h_\tau^{n+1}] \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \\ &\leq 2\mathcal{R}^n(\tau, t) C_F \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*} \\ &\leq 2C_\omega^3 C_F \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^5} \right) \frac{1}{(1 + \tau)^3} \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*}. \end{aligned}$$

Recalling (10.58) and (10.61), applying Proposition 4.24, we conclude that

$$(10.64) \quad \|\mathcal{E}(t, \cdot)\|_{\mathcal{F}\lambda_n^* t + \mu_n^*} \leq 2C_\omega^3 C_F \left(C'_0 + \sum_{k=1}^n \delta_k \right) \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^5} \right) \int_0^t \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*} \frac{d\tau}{(1 + \tau)^2}.$$

(We could be a bit more precise; anyway we cannot go further since we do not yet have an estimate on $\rho[h^{n+1}]$.)

Control of $\bar{\mathcal{E}}$: This will use the control on the derivatives of h^k . We start again from Proposition 6.2:

$$(10.65) \quad \|\bar{\mathcal{E}}(t, \cdot)\|_{\mathcal{F}\lambda_n^* t + \mu_n^*} \leq \int_0^t \|G^n - \bar{G}^n\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \|F[h_\tau^{n+1}]\|_{\mathcal{F}\nu_n} d\tau,$$

where $\nu_n = \lambda_n^*(1+b)|\tau - bt/(1+b)| + \mu_n^*$. We focus again on the case $\tau \geq bt/(1+b)$, so that (with crude estimates)

$$\|F[h_\tau^{n+1}]\|_{\mathcal{F}\nu_n} \leq \|F[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*} \leq C_F \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}\lambda_n^* \tau + \mu_n^*},$$

and the problem is to control $G^n - \bar{G}^n$:

$$(10.66) \quad \|G^n - \bar{G}^n\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \leq \left\| (\nabla_v f^0) \circ \Omega_{t,\tau}^n - \nabla_v f^0 \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \\ + \sum_{k=1}^n \left\| (\nabla_v h_\tau^k) \circ \Omega_{t,\tau}^n - (\nabla_v h_\tau^k) \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} + \sum_{k=1}^n \left\| (\nabla_v h_\tau^k) \circ \Omega_{t,\tau}^{k-1} - \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}}.$$

By induction hypothesis ($\tilde{\mathbf{E}}_h^n$), and since the $\mathcal{Z}_\tau^{\lambda, \mu}$ norms are increasing as a function of λ, μ ,

$$\sum_{k=1}^n \left\| (\nabla_v h_\tau^k) \circ \Omega_{t,\tau}^{k-1} - \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \leq \left(\sum_{k=1}^n \delta_k \right) \frac{1}{(1+\tau)^2}.$$

It remains to treat the first and second terms in the right-hand side of (10.66). This is done by inversion/composition as in Step 5; let us consider for instance the contribution of h^k , $k \geq 1$:

$$\left\| \nabla_v h_\tau^k \circ \Omega_{t,\tau}^n - \nabla_v h_\tau^k \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \\ \leq \int_0^1 \left\| \nabla \nabla_v h_\tau^k \circ \left((1-\theta) \Omega_{t,\tau}^n + \theta \Omega_{t,\tau}^{k-1} \right) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \left\| \Omega_{t,\tau}^n - \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} d\theta \\ \leq 2 \left\| \nabla \nabla_v h_\tau^k \circ \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_k^*(1+b), \mu_k^*; 1}} \left\| \Omega_{t,\tau}^n - \Omega_{t,\tau}^{k-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*; 1}} \\ \leq 4 \delta_k (1+\tau)^2 \mathcal{R}^{k-1, n}(\tau, t) \\ \leq 4 C_\omega^4 \delta_k \left(\sum_{j=k}^n \frac{\delta_j}{(2\pi(\lambda_j - \lambda_j^*))^6} \right) \frac{1}{(1+\tau)^2},$$

where in the but-to-last step we used $(\widetilde{\mathbf{E}}_{\Omega}^n)$, $(\widetilde{\mathbf{E}}_{\rho}^n)$, Propositions 4.24 and 4.28, Condition (\mathbf{C}_5) and the same reasoning as in Step 5.

Summing up all contributions and inserting in (10.65) yields

$$(10.67) \quad \|\overline{\mathcal{E}}(t, \cdot)\|_{\mathcal{F}^{\lambda_n^* t + \mu_n^*}} \\ \leq 4 C_F \left[C_{\omega}^4 \left(C_0' + \sum_{k=1}^n \delta_k \right) \left(\sum_{j=1}^n \frac{\delta_j}{(2\pi(\lambda_j - \lambda_n^*))^6} \right) + \sum_{k=1}^n \delta_k \right] \int_0^t \|\rho[h_{\tau}^{n+1}]\|_{\mathcal{F}^{\lambda_n^* \tau + \mu_n^*}} \frac{d\tau}{(1+\tau)^2}.$$

Main contribution: Now we consider $\overline{\sigma}^{n,n+1}$, which we decompose as

$$\overline{\sigma}_t^{n,n+1} = \overline{\sigma}_{t,0}^{n,n+1} + \sum_{k=1}^n \overline{\sigma}_{t,k}^{n,n+1},$$

where

$$\overline{\sigma}_{t,0}^{n,n+1}(x) = \int_0^t \int F[h_{\tau}^{n+1}](\tau, x - v(t-\tau), v) \cdot \nabla_v f^0(v) dv d\tau,$$

$$\overline{\sigma}_{t,k}^{n,n+1}(x) = \int_0^t \int \left(F[h_{\tau}^{n+1}] \cdot \nabla_v (h_{\tau}^k \circ \Omega_{t,\tau}^{k-1}) \right) (\tau, x - v(t-\tau), v) dv d\tau.$$

Note that their zero mode vanishes. For any $k \geq 1$, we apply Theorem 6.9 to get

$$\|\overline{\sigma}_{t,k}^{n,n+1}\|_{\mathcal{F}^{\lambda_n^* t + \mu_n^*}} \leq \int_0^t K_1^{n,h^k}(t, \tau) \|F[h_{\tau}^{n+1}]\|_{\mathcal{F}^{\nu'_{n,\gamma}}} d\tau \\ + \int_0^t K_0^{n,h^k}(t, \tau) \|F[h_{\tau}^{n+1}]\|_{\mathcal{F}^{\nu'_{n,\gamma}}} d\tau,$$

where

- $\nu'_n = \lambda_n^*(1+b) \left| \tau - \frac{bt}{1+b} \right| + \mu'_n$
- $K_1^{n,h^k}(t, \tau) = \sup_{0 \leq \tau \leq t} \left(\frac{\left\| \nabla_v (h_{\tau}^k \circ \Omega_{t,\tau}^{k-1}) - \langle \nabla_v (h_{\tau}^k \circ \Omega_{t,\tau}^{k-1}) \rangle \right\|_{\mathcal{Z}^{\lambda_k + b\lambda_n^*, \mu_k}}}{1+\tau} \right) K_1^{n,k},$
- $K_1^{n,k}(t, \tau) = (1+\tau) \sup_{k \neq 0, \ell \neq 0} e^{-2\pi \left(\frac{\mu_k - \mu_n^*}{2} \right) |\ell|} \left(\frac{e^{-2\pi(\mu'_n - \mu_n^*)|k-\ell|}}{1+|k-\ell|^{\gamma}} \right) e^{-2\pi(\lambda_k - \lambda_n^*)|k(t-\tau) + \ell\tau|},$
- $K_0^{n,h^k}(t, \tau) = \left(\sup_{0 \leq \tau \leq t} \left\| \nabla_v \langle h_{\tau}^k \circ \Omega_{t,\tau}^{k-1} \rangle \right\|_{\mathcal{C}^{\lambda_k(1+b); 1}} \right) K_0^{n,k},$

- $K_0^{n,k}(t, \tau) = e^{-2\pi\left(\frac{\lambda_k - \lambda_n^*}{2}\right)(t-\tau)}$.

We assume

$$(10.68) \quad \mu'_n = \mu_n^* + \eta \left(\frac{t - \tau}{1 + t} \right), \quad \eta > 0 \quad \text{small},$$

and check that $\nu'_n \leq \lambda_n^* \tau + \mu_n^*$. Leaving apart the small-time case, we assume $\tau \geq bt/(1+b)$, so that

$$\nu'_n = (\lambda_n^* \tau + \mu_n^*) - \frac{B \lambda_n^* (t - \tau)}{1 + t} + \eta \left(\frac{t - \tau}{1 + t} \right),$$

which is indeed bounded above by $\lambda_n^* \tau + \mu_n^*$ as soon as

$$(10.69) \quad \eta \leq B \lambda_\infty^*.$$

Then, with the notation (7.9),

$$(10.70) \quad K_1^{n,k}(t, \tau) \leq K_1^{(\alpha_{n,k}), \gamma}(t, \tau),$$

with

$$(10.71) \quad \alpha_{n,k} = \min \left\{ \frac{\mu_k - \mu_n^*}{2}; \lambda_k - \lambda_n^*; \eta \right\}.$$

From the controls on h^k (assumption $(\tilde{\mathbf{E}}_h^n)$) we have

$$\begin{aligned} \left\| \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) - \langle \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \rangle \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; 1}}{}_{\tau - \frac{bt}{1+b}} &\leq \left\| \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}^{\lambda_k(1+b), \mu_k; 1}}{}_{\tau - \frac{bt}{1+b}} \\ &\leq \delta_k (1 + \tau); \end{aligned}$$

and

$$\begin{aligned} \left\| \langle \nabla_v (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \rangle \right\|_{\mathcal{C}^{\lambda_k(1+b); 1}} &= \left\| \langle (\nabla_v + \tau \nabla_x) (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \rangle \right\|_{\mathcal{C}^{\lambda_k(1+b); 1}} \\ &\leq \left\| (\nabla_v + \tau \nabla_x) (h_\tau^k \circ \Omega_{t,\tau}^{k-1}) \right\|_{\mathcal{Z}^{\lambda_k(1+b); 1}}{}_{\tau - \frac{bt}{1+b}} \\ &\leq \delta_k. \end{aligned}$$

After controlling $F[h^{n+1}]$ by $\rho[h^{n+1}]$, we end up with

$$(10.72) \quad \begin{aligned} \|\bar{\sigma}_{t,k}^{n,n+1}\|_{\mathcal{F}^{\lambda_n^* t + \mu_n^*}} &\leq C_F \int_0^t \left(\sum_{k=1}^n \delta_k K_1^{(\alpha_{n,k}), \gamma}(t, \tau) \right) \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}^{\lambda_n^* \tau + \mu_n^*}} d\tau \\ &\quad + C_F \int_0^t \left(\sum_{k=1}^n \delta_k e^{-2\pi \left(\frac{\lambda_k - \lambda_n^*}{2}\right)(t-\tau)} \right) \|\rho[h_\tau^{n+1}]\|_{\mathcal{F}^{\lambda_n^* \tau + \mu_n^*}} d\tau, \end{aligned}$$

with $\alpha_{n,k}$ defined by (10.71).

Substep c. Gathering all previous controls, we obtain the following integral inequality for $\varphi = \rho[h^{n+1}]$:

$$(10.73) \quad \begin{aligned} \left\| \varphi(t, x) - \int_0^t \int (\nabla W * \varphi)(\tau, x - v(t-\tau)) \cdot \nabla_v f^0(v) dv d\tau \right\|_{\mathcal{F}^{\lambda_n^* t + \mu_n^*}} \\ \leq A_n + \int_0^t \left[K_1^n(t, \tau) + K_0^n(t, \tau) + \frac{c_0^n}{(1+\tau)^2} \right] \|\varphi(\tau, \cdot)\|_{\mathcal{F}^{\lambda_n^* \tau + \mu_n^*}}, \end{aligned}$$

where, by (10.53), (10.64) and (10.67),

$$(10.74) \quad \begin{aligned} A_n &= \sup_{t \geq 0} \|\sigma^{n,n}(t, \cdot)\|_{\mathcal{F}^{\lambda_n^* t + \mu_n^*}} \leq \frac{2 C_F \delta_n^2}{(\pi(\lambda_n - \lambda_n^*))^2}, \\ K_1^n(t, \tau) &= \left(C_F \sum_{k=1}^n \delta_k \right) K_1^{(\alpha_n), \gamma}, \quad \alpha_n = \alpha_{n,n} = \min \left\{ \frac{\mu_n - \mu_n^*}{2}; \lambda_n - \lambda_n^*; \eta \right\}, \\ K_0^n(t, \tau) &= C_F \sum_{k=1}^n \delta_k e^{-2\pi \left(\frac{\lambda_k - \lambda_n^*}{2}\right)(t-\tau)}, \\ c_0^n &= 3 C_F C_\omega^4 \left(C_0' + \sum_{k=1}^n \delta_k \right) \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^6} \right) + \sum_{k=1}^n \delta_k. \end{aligned}$$

(We are cheating a bit when writing (10.73), because in fact one should take into account small times separately; but this does not cause any real difficulty.)

We easily estimate K_0^n :

$$\int_0^t K_0^n(t, \tau) d\tau \leq C_F \sum_{k=1}^n \frac{\delta_k}{\pi(\lambda_k - \lambda_n^*)},$$

$$\int_{\tau}^{\infty} K_0^n(t, \tau) dt \leq C_F \sum_{k=1}^n \frac{\delta_k}{\pi(\lambda_k - \lambda_n^*)},$$

$$\left(\int_0^t K_0^n(t, \tau)^2 d\tau \right)^{1/2} \leq C_F \sum_{k=1}^n \frac{\delta_k}{\sqrt{2\pi(\lambda_k - \lambda_n^*)}}.$$

Let us assume that α_n is smaller than $\bar{\alpha}(\gamma)$ appearing in Theorem 7.8, and that

$$(10.75) \quad (\mathbf{C}_7) \quad 3 C_F C_{\omega}^4 \left(C'_0 + \sum_{k=1}^n \delta_k + 1 \right) \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^6} \right) \leq \frac{1}{4},$$

$$(10.76) \quad (\mathbf{C}_8) \quad C_F \sum_{k=1}^n \frac{\delta_k}{\sqrt{2\pi(\lambda_k - \lambda_k^*)}} \leq \frac{1}{2},$$

$$(10.77) \quad (\mathbf{C}_9) \quad C_F \sum_{k=1}^n \frac{\delta_k}{\pi(\lambda_k - \lambda_k^*)} \leq \max \left\{ \frac{1}{4}; \chi \right\},$$

(note that in these conditions we have strenghtened the inequalities by replacing $\lambda_k - \lambda_n^*$ by $\lambda_k - \lambda_k^*$ where $\chi > 0$ is also defined by Theorem 7.8. Applying that theorem, we deduce that for any $\varepsilon \in (0, \alpha_n)$ and $t \geq 0$,

$$(10.78) \quad \|\rho_t^{n+1}\|_{\mathcal{F}\lambda_n^*t+\mu_n^*} \leq C A_n e^{C T_{\varepsilon,n}} \left(1 + \frac{c_n}{\alpha_n \varepsilon^{3/2}} \right) e^{C c_n (1+T_{\varepsilon,n}^2)} e^{\varepsilon t},$$

$$\text{where } c_n = 2 C_F \left(\sum_{k=1}^n \delta_k \right) \quad \text{and} \quad T_{\varepsilon,n} = C_{\gamma} \max \left\{ \left(\frac{c_n + c_n^2}{\alpha_n^5 \varepsilon^{2+\gamma}} \right)^{1/(\gamma-1)}; \frac{1}{\varepsilon^{1/3}} \right\}.$$

Pick up $\lambda_n^{\dagger} < \lambda_n^*$ and choose $\varepsilon = 2\pi(\lambda_n^* - \lambda_n^{\dagger})$; recalling that $\widehat{\rho}^{n+1}(t, 0) = 0$, and that our conditions imply an upper bound on c_n , we deduce the *uniform control*

$$(10.79) \quad \|\rho_t^{n+1}\|_{\mathcal{F}\lambda_n^{\dagger}t+\mu_n^*} \leq e^{-2\pi(\lambda_n^* - \lambda_n^{\dagger})t} \|\rho_t^{n+1}\|_{\mathcal{F}\lambda_n^*t+\mu_n^*}$$

$$\leq C A_n \left(1 + \frac{c_n}{\alpha_n (2\pi(\lambda_n^* - \lambda_n^{\dagger}))^{3/2}} \right) e^{C(1+T_n^2)},$$

where

$$(10.80) \quad T_n = C \left(\frac{c_n}{\alpha_n^5 (2\pi(\lambda_n^* - \lambda_n^{\dagger}))^{2+\gamma}} \right)^{\frac{1}{\gamma-1}}.$$

10.4.3. *Step 7: estimate on $F[h^{n+1}]$.* As an immediate consequence of (10.2) and (10.79), we have

$$(10.81) \quad \sup_{t \geq 0} \|F[\rho_t^{n+1}]\|_{\mathcal{F}\lambda_n^\dagger t + \mu_n^*, \gamma} \leq CA_n \left(1 + \frac{c_n}{\alpha_n (2\pi(\lambda_n^* - \lambda_n^\dagger))^{3/2}}\right) e^{C(1+T_n^2)}.$$

10.4.4. *Step 8: estimate of $h^{n+1} \circ \Omega^n$.* In this step we shall use again the Vlasov equation. We rewrite (10.47) as

$$h^{n+1}(\tau, X_{0,\tau}^n(x, v), V_{0,\tau}^n(x, v)) = \int_0^\tau \Sigma^{n+1}(s, X_{0,s}^n(x, v), V_{0,s}^n(x, v)) ds;$$

but now we compose with $(X_{t,0}^n, V_{t,0}^n)$, where $t \geq \tau$ is arbitrary. This gives

$$h^{n+1}(\tau, X_{t,\tau}^n(x, v), V_{t,\tau}^n(x, v)) = \int_0^\tau \Sigma^{n+1}(s, X_{t,s}^n(x, v), V_{t,s}^n(x, v)) ds.$$

Then for any $p \in [1, \bar{p}]$ and $\lambda_n^b < \lambda_n^\dagger$, using Propositions 4.19 and 4.24, and the notation (10.49), we get

$$\begin{aligned} \|h_\tau^{n+1} \circ \Omega_{t,\tau}^n\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} &= \|h_\tau^{n+1} \circ (X_{t,\tau}^n, V_{t,\tau}^n)\|_{\mathcal{Z}_{t - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} \\ &\leq \int_0^\tau \|\Sigma^{n+1}(s, X_{t,s}^n, V_{t,s}^n)\|_{\mathcal{Z}_{t - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} ds = \int_0^\tau \|\Sigma^{n+1}(s, \Omega_{t,s}^n)\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} ds \\ &\leq \int_0^\tau \|R_{s,t}^{n+1}\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*}} \|G_{s,t}^n\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} ds \\ &\quad + \int_0^\tau \|R_{s,t}^n\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*}} \|H_{s,t}^n\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} ds. \end{aligned}$$

Then (proceeding as in Step 6 to check that the exponents lie in the appropriate range)

$$\|R_{s,t}^{n+1}\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*}} \leq C_F e^{-2\pi(\lambda_n^\dagger - \lambda_n^b)s} \|\rho_s^{n+1}\|_{\mathcal{F}\bar{\nu}_n(s)}$$

and

$$\|R_{s,t}^n\|_{\mathcal{Z}_{s - \frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*}} \leq C_F e^{-2\pi(\lambda_n^\dagger - \lambda_n^b)s} \|\rho_s^n\|_{\mathcal{F}\bar{\nu}_n(s)} \leq C_F e^{-2\pi(\lambda_n^\dagger - \lambda_n^b)s} \delta_n$$

with

$$\begin{cases} \bar{\nu}_n(s, t) := \mu^\sharp & \text{when } s \leq bt/(1+b) \\ \bar{\nu}_n(s, t) := \lambda_n^\dagger s + \mu_n^* & \text{when } s \geq bt/(1+b). \end{cases}$$

On the other hand, from the induction assumption $(\mathbf{E}_n^{\mathbf{n}})$ - $(\tilde{\mathbf{E}}_n^{\mathbf{n}})$ (and again control of composition *via* Proposition 4.25...),

$$\|H_{s,t}^n\|_{\mathcal{Z}_{s-\frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} \leq 2(1+s)\delta_n$$

and

$$\|G_{s,t}^n\|_{\mathcal{Z}_{s-\frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} \leq 2(1+s) \left(\sum_{k=1}^n \delta_k \right).$$

We deduce that

$$y(t, \tau) := \|h_\tau^{n+1} \circ \Omega_{t,\tau}^n\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}}$$

satisfies

$$\begin{aligned} y(t, \tau) \leq 2C_F \left(\sum_{k=1}^n \delta_k \right) \int_0^\tau e^{-2\pi(\lambda_n^\dagger - \lambda_n^b)s} \|\rho_s^{n+1}\|_{\mathcal{F}^{\bar{\nu}_n(s)}} (1+s) ds \\ + 2C_F \delta_n^2 \int_0^\tau e^{-2\pi(\lambda_n^\dagger - \lambda_n^b)s} (1+s) ds; \end{aligned}$$

so

$$(10.82) \quad \forall t \geq \tau \geq 0,$$

$$\|h_\tau^{n+1} \circ \Omega_{t,\tau}^n\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{(1+b)\lambda_n^b, \mu_n^*; p}} \leq \frac{4C_F \max\{(\sum_{k=1}^n \delta_k); 1\}}{(2\pi(\lambda_n^\dagger - \lambda_n^b))^2} \left(\delta_n^2 + \sup_{s \geq 0} \|\rho_s^{n+1}\|_{\mathcal{F}^{\bar{\nu}_n(s)}} \right).$$

10.4.5. *Step 9: Crude estimates on the derivatives of h^{n+1} .* Again we choose $p \in [1, \bar{p}]$. From the previous step and Proposition 4.27 we deduce, for any λ_n^\dagger such that $\lambda_n^\dagger < \lambda_n^b < \lambda_n^\ddagger$, and any $\mu_n^\dagger < \mu_n^*$,

$$\begin{aligned} (10.83) \quad & \left\| \nabla_x (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; p}} + \left\| (\nabla_v + \tau \nabla_x) (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; p}} \\ & \leq \frac{C(d)}{\min\{\lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}} \|h_\tau^{n+1} \circ \Omega_{t,\tau}^n\|_{\mathcal{Z}_{\tau-\frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}}; \end{aligned}$$

and

$$(10.84) \quad \left\| \nabla (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; p}} \leq \frac{C(d)(1+\tau)}{\min\{\lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}}.$$

Similarly,

$$(10.85) \quad \left\| \nabla \nabla (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; p}} \leq \frac{C(d)(1+\tau)^2}{\min\{\lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}^2} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}}.$$

10.4.6. *Step 10: Chain-rule and refined estimates on derivatives of h^{n+1} .* From Step 3 we have

$$(10.86) \quad \left\| \nabla \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger}} + \left\| (\nabla \Omega_{t,\tau}^n)^{-1} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger}} \leq C(d)$$

and (via Proposition 4.27)

$$(10.87) \quad \begin{aligned} \left\| \nabla \nabla \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger}} &\leq \frac{C(d)(1+\tau)}{\min\{\lambda_n^* - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}} \left\| \nabla \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^*(1+b), \mu_n^*}} \\ &\leq \frac{C(d)(1+\tau)}{\min\{\lambda_n^* - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}}. \end{aligned}$$

Combining these bounds with Step 9, Proposition 4.24 and the identities

$$(10.88) \quad \begin{cases} (\nabla h) \circ \Omega = (\nabla \Omega)^{-1} \nabla (h \circ \Omega) \\ (\nabla \nabla h) \circ \Omega = (\nabla \Omega)^{-2} \nabla \nabla (h \circ \Omega) - (\nabla \Omega)^{-1} \nabla^2 \Omega (\nabla \Omega)^{-1} (\nabla h \circ \Omega), \end{cases}$$

we get

$$(10.89) \quad \begin{aligned} \left\| (\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; 1}} &\leq C(d) \left\| \nabla (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\dagger(1+b), \mu_n^\dagger; 1}} \\ &\leq \frac{C(d)(1+\tau)}{\min\{\lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger\}} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; 1}} \end{aligned}$$

and

(10.90)

$$\begin{aligned}
& \left\| (\nabla^2 h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; 1}} \\
& \leq C(d) \left[\left\| \nabla^2 (h_\tau^{n+1} \circ \Omega_{t,\tau}^n) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; 1}} + \left\| \nabla^2 \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; 1}} \left\| (\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; 1}} \right] \\
& \leq \frac{C(d) (1 + \tau)^2}{\min \{ \lambda_n^b - \lambda_n^\ddagger; \mu_n^* - \mu_n^\ddagger \}^2} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}}.
\end{aligned}$$

This gives us the bounds $\|(\nabla h_\tau^{n+1}) \circ \Omega_\tau^n\| = O(1 + \tau)$, $\|(\nabla^2 h_\tau^{n+1}) \circ \Omega_\tau^n\| = O((1 + \tau)^2)$, which are optimal if one does not distinguish between the x and v variables. We shall now refine these estimates. First we write

$$\nabla(h_\tau^{n+1} \circ \Omega_{t,\tau}^n) - (\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n = \nabla(\Omega_{t,\tau}^n - \text{Id}) \cdot [(\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n],$$

and we deduce (*via* Propositions 4.24 and 4.27)

(10.91)

$$\begin{aligned}
& \left\| \nabla(h_\tau^{n+1} \circ \Omega_{t,\tau}^n) - (\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; p}} \\
& \leq \left\| \nabla(\Omega_{t,\tau}^n - \text{Id}) \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger}} \left\| (\nabla h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^\ddagger(1+b), \mu_n^\ddagger; p}} \\
& \leq C(d) \left(\frac{1 + \tau}{\min \{ \lambda_n^b - \lambda_n^\ddagger; \mu_n^* - \mu_n^\ddagger \}} \right)^2 \left\| \Omega_{t,\tau}^n - \text{Id} \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*}} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}} \\
& \leq \frac{C(d) C_\omega^4}{\min \{ \lambda_n^b - \lambda_n^\ddagger; \mu_n^* - \mu_n^\ddagger \}^2} \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^6} \right) (1 + \tau)^{-2} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}_{\tau - \frac{bt}{1+b}}^{\lambda_n^b(1+b), \mu_n^*; p}}.
\end{aligned}$$

(Note: $\Omega^n - \text{Id}$ brings the time-decay, while h^{n+1} brings the smallness.)

This shows that $(\nabla h^{n+1}) \circ \Omega^n \simeq \nabla(h^{n+1} \circ \Omega^n)$ as $\tau \rightarrow \infty$. In view of Step 9, this also implies the refined gradient estimates

$$(10.92) \quad \left\| (\nabla_x h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^\dagger, \mu_n^\dagger; p}}{}_{\tau - \frac{bt}{1+b}} + \left\| ((\nabla_v + \tau \nabla_x) h_\tau^{n+1}) \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^\dagger, \mu_n^\dagger; p}}{}_{\tau - \frac{bt}{1+b}} \leq \bar{C} \left\| h_\tau^{n+1} \circ \Omega_{t,\tau}^n \right\|_{\mathcal{Z}^{\lambda_n^b(1+b), \mu_n^*; p}}{}_{\tau - \frac{bt}{1+b}},$$

with

$$\bar{C} = C(d) \left[\frac{C_\omega^4}{\min \{ \lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger \}^2} \left(\sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^6} \right) + \frac{1}{\min \{ \lambda_n^b - \lambda_n^\dagger; \mu_n^* - \mu_n^\dagger \}} \right].$$

10.4.7. *Conclusion.* Given $\lambda_{n+1} < \lambda_n^*$, $\mu_{n+1} < \mu_n^*$, we define

$$\lambda_{n+1} = \lambda_n^\dagger, \quad \mu_{n+1} = \mu_n^\dagger,$$

and we impose

$$\lambda_n^* - \lambda_n^\dagger = \lambda_n^\dagger - \lambda_n^b = \lambda_n^b - \lambda_n^\dagger = \frac{\lambda_n^* - \lambda_{n+1}}{3},$$

$$\mu_n^* - \mu_n^\dagger = \frac{\mu_n^* - \mu_{n+1}}{2}.$$

Then from (10.79), (10.81), (10.82), (10.83), (10.90), (10.91) and (10.92) we see that (\mathbf{E}_ρ^{n+1}) , $(\tilde{\mathbf{E}}_\rho^{n+1})$, (\mathbf{E}_h^{n+1}) , $(\tilde{\mathbf{E}}_h^{n+1})$ have all been established in the present subsection, with

$$(10.93) \quad \delta_{n+1} = \frac{C(d) C_F (1 + C_F) (1 + C_\omega^4) e^{C(1+T_n^2)}}{\min \{ \lambda_n^* - \lambda_{n+1}; \mu_n^* - \mu_{n+1} \}^4} \left(1 + \sum_{k=1}^n \frac{\delta_k}{(2\pi(\lambda_k - \lambda_k^*))^6} \right) \frac{\delta_n^2}{\lambda_n - \lambda_n^*}.$$

10.5. **Convergence of the scheme.** For any $n \geq 1$, we set

$$(10.94) \quad \lambda_n - \lambda_n^* = \lambda_n^* - \lambda_{n+1} = \mu_n - \mu_n^* = \mu_n^* - \mu_{n+1} = \frac{\Lambda}{n^2},$$

for some $\Lambda > 0$. By choosing Λ small enough, we can make sure that the conditions $2\pi(\lambda_k - \lambda_k^*) < 1$, $2\pi(\mu_k - \mu_k^*) < 1$ are satisfied for all k , as well as the other smallness assumptions made throughout this section. Moreover, we have $\lambda_k - \lambda_k^* \geq \Lambda/k^2$, so conditions (\mathbf{C}_1) to (\mathbf{C}_9) will be satisfied if

$$\sum_{k=1}^n k^{12} \delta_k \leq \Lambda^6 \omega, \quad \sum_{j=k+1}^n j^6 \delta_j \leq \Lambda^3 \omega \left(\frac{1}{k^2} - \frac{1}{n^2} \right),$$

for some small explicit constant $\omega > 0$, depending on the other constants appearing in the problem. Both conditions are satisfied if

$$(10.95) \quad \sum_{k=1}^{\infty} k^{12} \delta_k \leq \Lambda^6 \omega.$$

Then from (10.80) we have $T_n \leq C_\gamma n^{\frac{14+2\gamma}{\gamma-1}} \Lambda^{-\frac{7+\gamma}{\gamma-1}}$, so the induction relation on δ_n allows

$$(10.96) \quad \delta_1 \leq C \delta, \quad \delta_{n+1} = C \frac{n^{10}}{\Lambda^5} e^{C n^{\frac{14+2\gamma}{\gamma-1}} / \Lambda^{\frac{7+\gamma}{\gamma-1}}} \delta_n^2.$$

To establish this relation we also assumed that δ_n is bounded below by $C_F \zeta_n$, the error coming from the short-time iteration; but this follows easily by construction, since the constraints imposed on δ_n are much worse than those on ζ_n .

Having fixed Λ , we will check that for δ small enough, (10.96) implies both the fast convergence of $(\delta_k)_{k \in \mathbb{N}}$, and the condition (10.95), which will justify *a posteriori* the derivation of (10.96). (An easy induction is enough to turn this into a rigorous reasoning.)

For this we fix $a \in (1, a_I)$, $0 < z < z_I < 1$, and we check by induction

$$(10.97) \quad \forall n \geq 1, \quad \delta_n \leq \Delta z^{a^n}.$$

If Δ is given, (10.97) holds for $n = 1$ as soon as $\delta \leq (\Delta/C) z^a$. Then, to go from stage n to stage $n + 1$, we should check that

$$\frac{C n^{10}}{\Lambda^5} e^{C n^{\frac{14+2\gamma}{\gamma-1}} / \Lambda^{\frac{7+\gamma}{\gamma-1}}} \Delta^2 z^{2a^n} \leq \Delta z^{a^{n+1}};$$

this is true if

$$\frac{1}{\Delta} \geq \frac{C}{\Lambda^5} \sup_{n \in \mathbb{N}} \left(n^{10} e^{C n^{\frac{14+2\gamma}{\gamma-1}} / \Lambda^{\frac{7+\gamma}{\gamma-1}}} z^{(2-a) a^n} \right).$$

Since $a < 2$, the supremum on the right-hand side is finite, and we just have to choose Δ small enough. Then, reducing Δ further if necessary, we can ensure (10.95). This concludes the proof.

Remark 10.1. This argument almost fully exploits the bi-exponential convergence of the Netwon scheme: a convergence like, say, $O(e^{-n^{1000}})$, would not be enough to treat values of γ which are close to 1.

11. CONVERGENCE IN LARGE TIME

In this section we prove Theorem 2.5 as a simple consequence of the uniform bounds established in Section 10.

So let f^0, L, W satisfy the assumptions of Theorem 2.5. To simplify notation we assume $L = 1$.

The second part of Assumption (2.12) precisely means that $f^0 \in \mathcal{C}^{\lambda;1}$. We shall actually assume a slightly more precise condition, namely that for some $\bar{p} \in [1, \infty]$,

$$(11.1) \quad \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|\nabla_v^n f^0\|_{L^p(\mathbb{R}^d)} \leq C_0 < +\infty, \quad \forall p \in [1, \bar{p}].$$

(It is sufficient to take $\bar{p} = 1$ to get Theorem 2.5; but if this bound is available for some $\bar{p} > 1$ then it will be propagated by the iteration scheme, and result in more precise bounds.) Then we pick up $\underline{\lambda} \in (0, \lambda)$, $\underline{\mu} \in (0, \mu)$, $\beta > 0$, $\beta' \in (0, \beta)$. By symmetry, we only consider nonnegative times.

If f_i is an initial datum satisfying the smallness condition (2.13), then by Theorem 4.20, we have a smallness estimate on $\|f_i - f^0\|_{\mathcal{Z}^{\lambda', \mu'; p}}$ for all $p \in [1, \bar{p}]$, $\lambda' < \lambda$, $\mu' < \mu$. Then, as in Subsection 4.12 we can estimate the solution h^1 to the linearized equation

$$(11.2) \quad \begin{cases} \partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0 \\ h^1(0, \cdot) = f_i - f^0, \end{cases}$$

and we recover uniform bounds in $\mathcal{Z}^{\lambda^\sharp, \mu^\sharp; p}$ spaces, for any $\lambda^\sharp \in (\underline{\lambda}, \lambda)$, $\mu^\sharp \in (\underline{\mu}, \mu)$. More precisely,

$$(11.3) \quad \sup_{t \geq 0} \|\rho[h_t^1]\|_{\mathcal{F}^{\lambda^\sharp t + \mu^\sharp}} + \sup_{t \geq 0} \|h^1(t, \cdot)\|_{\mathcal{Z}_t^{\lambda^\sharp, \mu^\sharp; p}} \leq C \delta,$$

with $C = C(d, \lambda', \lambda^\sharp, \mu', \mu^\sharp, W, f^0)$ (this is of course assuming ε in Theorem 2.5 to be small enough).

We now set $\lambda_1 = \lambda'$, $\bar{\lambda}_1 = \lambda^\sharp$, and we run the iterative scheme of Sections 9–10 for all $n \geq 2$. If ε is small enough, up to slightly lowering λ_1 , we may choose all parameters in such a way that

$$\lambda_k, \bar{\lambda}_k, \lambda_k^* \xrightarrow[k \rightarrow \infty]{} \lambda_\infty > \underline{\lambda}, \quad \mu_k, \bar{\mu}_k, \mu_k^* \xrightarrow[k \rightarrow \infty]{} \mu_\infty > \underline{\mu};$$

then we pick up $B > 0$ such that

$$\mu_\infty - \lambda_\infty(1 + B)B \geq \mu'_\infty > \underline{\mu},$$

and we let $b(t) = B/(1 + t)$.

As a result of the scheme, we have, for all $k \geq 2$,

$$(11.4) \quad \sup_{t \geq \tau \geq 0} \|h_\tau^k \circ \Omega_{t,\tau}^{k-1}\|_{\mathcal{Z}^{\lambda_\infty(1+b), \mu_\infty; 1}} \leq \delta_k,$$

where $\sum_{k=2}^\infty \delta_k \leq C \delta$ and Ω^k is the scattering associated to the force field generated by $h^1 + \dots + h^k$. Choosing $t = \tau$ in (11.4) yields

$$\sup_{t \geq 0} \|h_t^k\|_{\mathcal{Z}^{\lambda_\infty(1+B), \mu_\infty; 1}} \leq \delta_k.$$

By Proposition 4.17, this implies

$$\sup_{t \geq 0} \|h_t^k\|_{\mathcal{Z}_t^{\lambda_\infty(1+B), \mu_\infty - \lambda_\infty(1+B)B; 1}} \leq \delta_k.$$

In particular, we have a uniform estimate on h_t^k in $\mathcal{Z}_t^{\lambda_\infty, \mu'_\infty; 1}$. Summing up over k yields for $f = f^0 + \sum_{k \geq 1} h^k$ the estimate

$$(11.5) \quad \sup_{t \geq 0} \|f(t, \cdot)\|_{\mathcal{Z}_t^{\lambda_\infty, \mu'_\infty; 1}} \leq C \delta.$$

Passing to the limit in the Newton scheme, one shows that f solves the nonlinear Vlasov equation with initial datum f_i . (Once again we do not check details; to be rigorous one would need to establish moment estimates, locally in time, before passing to the limit.) This implies in particular that f stays nonnegative at all times.

Applying Theorem 4.20 again, we deduce from (11.5)

$$\sup_{t \geq 0} \|f(t, \cdot) - f^0\|_{\mathcal{Y}_t^{\lambda, \mu}} \leq C \delta;$$

or equivalently, with the notation used in Theorem 2.5,

$$(11.6) \quad \sup_{t \geq 0} \|f(t, x - vt, v) - f^0(v)\|_{\lambda, \mu} \leq C \delta.$$

Moreover, $\rho = \int f dv$ satisfies similarly

$$\sup_{t \geq 0} \|\rho(t, \cdot)\|_{\mathcal{F}^{\lambda_\infty t + \mu_\infty}} \leq C \delta.$$

It follows that $|\widehat{\rho}(t, k)| \leq C \delta e^{-2\pi\lambda_\infty|k|t} e^{-2\pi\mu_\infty|k|}$, for any $k \neq 0$. On the one hand, by Sobolev embedding, we deduce that for any $r \in \mathbb{N}$,

$$\|\rho(t, \cdot) - \langle \rho \rangle\|_{C^r(\mathbb{T}^d)} \leq C_r \delta e^{-2\pi\lambda' t};$$

on the other hand, multiplying $\widehat{\rho}$ by the Fourier transform of ∇W , we see that the force $F = F[f]$ satisfies

$$(11.7) \quad \forall t \geq 0, \quad \forall k \in \mathbb{Z}^d, \quad |\widehat{F}(t, k)| \leq C \delta e^{-2\pi\lambda'|k|t} e^{-2\pi\mu'|k|},$$

for some $\lambda' > \underline{\lambda}$, $\mu' > \underline{\mu}$.

Now, from (11.6) we have, for any $(k, \eta) \in \mathbb{Z}^d \times \mathbb{R}^d$, and any $t \geq 0$,

$$(11.8) \quad \left| \tilde{f}(t, k, \eta + kt) - \tilde{f}^0(\eta) \right| \leq C \delta e^{-2\pi\mu'|k|} e^{-2\pi\lambda'|\eta|},$$

so

$$(11.9) \quad |\tilde{f}(t, k, \eta)| \leq |\tilde{f}^0(\eta + kt)| + C \delta e^{-2\pi\mu'|k|} e^{-2\pi\lambda'|\eta+kt|}.$$

In particular, for any $k \neq 0$, and any $\eta \in \mathbb{R}^d$,

$$(11.10) \quad \tilde{f}(t, k, \eta) = O(e^{-2\pi\lambda't}).$$

Thus f is asymptotically close (in the weak topology) to its spatial average $g = \langle f \rangle = \int f dx$. Taking $k = 0$ in (11.8) shows that, for any $\eta \in \mathbb{R}^d$,

$$(11.11) \quad |\tilde{g}(t, \eta) - \tilde{f}^0(\eta)| \leq C \delta e^{-2\pi\lambda'|\eta|}.$$

Also, from the nonlinear Vlasov equation, for any $\eta \in \mathbb{R}^d$ we have

$$\begin{aligned} \tilde{g}(t, \eta) &= \tilde{f}_i(0, \eta) - \int_0^t \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} F(\tau, x) \cdot \nabla_v f(\tau, x, v) e^{-2i\pi\eta \cdot v} dv dx d\tau \\ &= \tilde{f}_i(0, \eta) - 2i\pi \sum_{\ell \in \mathbb{Z}^d} \int_0^t \widehat{F}(\tau, \ell) \cdot \eta \tilde{f}(\tau, -\ell, \eta) d\tau. \end{aligned}$$

Using the bounds (11.7) and (11.10), it is easily shown that the above time-integral converges exponentially fast as $t \rightarrow \infty$, with rate $O(e^{-\lambda''t})$ for any $\lambda'' < \lambda'$, to its limit

$$(11.12) \quad \tilde{g}_\infty(\eta) = \tilde{f}_i(0, \eta) - 2i\pi \sum_{\ell \in \mathbb{Z}^d} \int_0^\infty \widehat{F}(\tau, \ell) \cdot \eta \tilde{f}(\tau, -\ell, \eta) d\tau.$$

By passing to the limit in (11.11) we see that

$$|\tilde{g}_\infty(\eta) - \tilde{f}^0(\eta)| \leq C \delta e^{-2\pi\lambda'|\eta|},$$

and this concludes the proof of Theorem 2.5.

12. COULOMB/NEWTON INTERACTION

In this section we establish Landau damping on exponentially long times for Coulomb/Newton interactions ($\gamma = 1$), as stated in Theorem 2.6. This is obtained by a slight modification of the proof of Theorem 2.5; rather than going through the details again, we shall just give the reader the new ingredients.

Sketch of Proof of Theorem 2.6. In the iteration scheme of Section 10, the only place where we used $\gamma > 1$ (and not just $\gamma \geq 1$) is in Step 6, when it comes to the echo response *via* Theorem 7.8. Now, in the case $\gamma = 1$, the formula for K_1^n should be

$$K_1^n(t, \tau) = \sum_{k=1}^n \delta_k K_1^{(\alpha_{n,k}),1}(t, \tau),$$

with $\alpha_{n,k} = \min \{(\mu_k - \mu_n^*)/2; \lambda_k - \lambda_n^*; \eta\}$. By Theorem 7.8 (iii) this induces, in addition to other well-behaved factors, an uncontrolled exponential growth $O(e^{\epsilon_n t})$, with

$$\epsilon_n = \Gamma \sum_{k=1}^n \frac{\delta_k}{\alpha_{n,k}^3};$$

in particular ϵ_n will remain bounded and $O(\delta)$ throughout the scheme.

Let us replace (10.94) by

$$\lambda_n - \lambda_n^* = \lambda_n^* - \lambda_{n+1} = \mu_n - \mu_n^* = \mu_n^* - \mu_{n+1} = \frac{\Lambda}{n(\log(e+n))^2},$$

where $\Lambda > 0$ is very small. (This is allowed since the series $\sum 1/(n(\log(e+n))^2)$ converges — the power 2 could of course be replaced by any $r > 1$.) Then during the first stages of the iteration we can absorb the $O(e^{\epsilon_n t})$ factor by the loss of regularity if, say,

$$\epsilon_n \leq \frac{\Lambda}{2n(\log(e+n))^2}.$$

Recalling that $\epsilon_n = O(\delta)$, this is satisfied as soon as

$$(12.1) \quad n \leq N := \frac{K}{\delta(\log(1/\delta))^2},$$

where $K > 0$ is a positive constant depending on the other parameters of the problem but of course not on δ . So during these first stages we get the same long-time estimates as in Section 10.

For $n > N$ we cannot rely on the loss of regularity any longer; at this stage the error is about

$$\delta_N \leq C \delta^{a^N},$$

where $1 < a < 2$. To get the bounds for larger values of n , we use impose a restriction on the time-interval, say $0 \leq t \leq T_{\max}$. Allowing a degradation of the rate δ^{a^n} into $\delta^{\underline{a}^n}$ with $\underline{a} < a$, we see that the new factor $e^{\epsilon_n T_{\max}}$ can be eaten up by the scheme if

$$e^{\epsilon_n T_{\max}} \delta^{(\underline{a}-a)\underline{a}^n} \leq 1, \quad \forall n \geq N.$$

This is satisfied if

$$T_{\max} = O\left(\underline{a}^N \frac{\log \frac{1}{\delta}}{\delta}\right).$$

Recalling (12.1), we see that the latter condition holds true if

$$T_{\max} = O\left(A \frac{1}{\delta^{(\log \delta)^2}} \frac{\log \frac{1}{\delta}}{\delta}\right)$$

for some well-chosen constant $A > 1$. Then we can complete the iteration, and end up with a bound like

$$\|f_t - f_i\|_{\mathcal{Z}_t^{\lambda', \mu'}} \leq C \delta \quad \forall t \in [0, T_{\max}],$$

where C is another constant independent of δ . The conclusion follows easily. \square

Remark 12.1. The constants involved in the preceding argument are somewhat dreadful. If one is ready to relax the exponential bound (2.20) into a weaker bound $T \leq e^{K/\sqrt{\delta}}$, then much better constants can be obtained by using Theorem 7.8 (iii') instead of Theorem 7.8 (iii). To summarize, there are three time scales in our argument (discarding logarithmic corrections in the exponentials):

- $t = O(\delta^{-1})$: linearization provides a close approximation of the full solution;
- $t = O(A^{1/\sqrt{\delta}})$: nonlinear damping with very large but decent constants;
- $t = O(A^{1/\delta})$: nonlinear damping with crazy constants.

13. EXPANSIONS AND COUNTEREXAMPLES

A most important consequence of the proof of Theorem 2.5 is that the asymptotic behavior of the solution of the nonlinear Vlasov equation can in principle be determined at arbitrary precision as the size of the perturbation goes to 0. Indeed, if we define $g_\infty^k(v)$ as the large-time limit of h^k (say in positive time), then $\|g^k\| = O(\delta_k)$, so $f^0 + g_\infty^1 + \dots + g_\infty^n$ converges very fast to f_∞ . In other words, to investigate the properties of the time-asymptotics of the system, we may freely exchange the limits $t \rightarrow \infty$ and $\delta \rightarrow 0$, perform expansions, etc. This at once puts on rigorous grounds many asymptotic expansions used by various authors — who so far implicitly postulated the possibility of this exchange.

With this in mind, let us estimate the first corrections to the linearized theory, in the regime of a very small perturbation and small interaction strength (which can be achieved by a proper scaling of physical quantities). We shall work in dimension $d = 1$ and in a periodic box of length $L = 1$.

13.1. Simple excitation. For a start, let us consider the case where the perturbation affects only the first spatial frequency. We let

- $f^0(v) = e^{-\pi v^2}$: the homogeneous (Maxwellian) distribution;
- $\varepsilon \rho_i(x) = \varepsilon \cos(2\pi x)$: the initial space density perturbation;
- $\varepsilon \rho_i(x) \theta(v)$: the initial perturbation of the distribution function; we denote by φ the Fourier transform of θ ;
- αW : the interaction potential, with $W(-x) = W(x)$. We do not specify its form, but it should satisfy the assumptions in Theorem 2.5.

We work in the asymptotic regime $\varepsilon \rightarrow 0, \alpha \rightarrow 0$. We will not write norms explicitly, but all our computations can be made in the norms introduced in Section 4, with small losses in the regularity indices — as we have done in all this paper.

The first-order correction $h^1 = O(\varepsilon)$ to f^0 is provided by the solution of the linearized equation (3.3), here taking the form

$$\partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0,$$

with initial datum $h^1(0, \cdot) = h_i := f_i - f^0$. As in Section 3 we get a closed equation for the associated density $\rho[h^1]$:

$$\widehat{\rho}[h^1](t, k) = \widetilde{h}_i(k, kt) - 4\pi^2 \alpha \widehat{W}(k) \int_0^t \widehat{\rho}[h^1](\tau, k) e^{-\pi(k(t-\tau))^2} (t - \tau) k^2 d\tau.$$

It follows that $\widehat{\rho}[h^1](t, k) = 0$ for $k \neq \pm 1$, so the behavior of $\widehat{\rho}[h^1]$ is entirely determined by $u_1(t) = \widehat{\rho}[h^1](t, 1)$ and $u_{-1}(t) = \widehat{\rho}[h^1](t, -1)$, which satisfy

$$\begin{aligned} (13.1) \quad u_1(t) &= \frac{\varepsilon}{2} \varphi(t) - 4\pi^2 \alpha \widehat{W}(1) \int_0^t u_1(\tau) e^{-\pi(t-\tau)^2} (t - \tau) d\tau \\ &= \frac{\varepsilon}{2} \left[\varphi(t) + O(\alpha) \right]. \end{aligned}$$

$$\begin{aligned} (13.2) \quad u_{-1}(t) &= \frac{\varepsilon}{2} \varphi(-t) - 4\pi^2 \alpha \widehat{W}(1) \int_0^t u_{-1}(\tau) e^{-\pi(t-\tau)^2} (t - \tau) d\tau \\ &= \frac{\varepsilon}{2} \left[\varphi(-t) + O(\alpha) \right]. \end{aligned}$$

The corresponding force, in Fourier transform, is given by $\widehat{F}^1(t, 1) = -2i\pi\alpha \widehat{W}(1) u_1(t)$ and $\widehat{F}^1(t, -1) = 2i\pi\alpha \widehat{W}(1) u_{-1}(t)$.

From this we also deduce the Fourier transform of h^1 itself:

$$(13.3) \quad \tilde{h}^1(t, k, \eta) = \tilde{h}_i(k, \eta + kt) - 4\pi^2\alpha \widehat{W}(k) \int_0^t \widehat{\rho}[h^1](\tau, k) e^{-\pi(\eta+k(t-\tau))^2} (\eta + k(t-\tau)) \cdot k \, d\tau;$$

this is 0 if $k \neq \pm 1$, while

$$(13.4) \quad \begin{aligned} \tilde{h}^1(t, 1, \eta) &= \frac{\varepsilon}{2} \varphi(\eta + t) - 4\pi^2\alpha \widehat{W}(1) \int_0^t u_1(\tau) e^{-\pi(\eta+(t-\tau))^2} (\eta + (t-\tau)) \, d\tau \\ &= \frac{\varepsilon}{2} [\varphi(\eta + t) + O(\alpha)], \end{aligned}$$

$$(13.5) \quad \begin{aligned} \tilde{h}^1(t, -1, \eta) &= \frac{\varepsilon}{2} \varphi(\eta - t) + 4\pi^2\alpha \widehat{W}(1) \int_0^t u_{-1}(\tau) e^{-\pi(\eta-(t-\tau))^2} (\eta - (t-\tau)) \, d\tau \\ &= \frac{\varepsilon}{2} [\varphi(\eta - t) + O(\alpha)]. \end{aligned}$$

To get the next order correction, we solve, as in Section 10,

$$\partial_t h^2 + v \cdot \nabla_x h^2 + F[h^1] \cdot \nabla_v h^2 + F[h^2] \cdot (\nabla_v f^0 + \nabla_v h^1) = -F[h^1] \cdot \nabla_v h^1,$$

with zero initial datum. Since $h^2 = O(\varepsilon^2)$, we may neglect the terms $F[h^1] \cdot \nabla_v h^2$ and $F[h^2] \cdot \nabla_v h^1$ which are both $O(\alpha \varepsilon^3)$. So it is sufficient to solve

$$(13.6) \quad \partial_t h'_2 + v \cdot \nabla_x h'_2 + F[h'_2] \cdot \nabla_v f^0 = -F[h^1] \cdot \nabla_v h^1$$

with vanishing initial datum. As $t \rightarrow \infty$, we know that the solution $h'_2(t, x, v)$ is asymptotically close to its spatial average $\langle h'_2 \rangle = \int h'_2 \, dx$. Taking the integral over \mathbb{T}^d in (13.6) yields

$$\partial_t \langle h'_2 \rangle = -\langle F[h^1] \cdot \nabla_v h^1 \rangle.$$

Since h^1 converges to $\langle h_i \rangle$, the deviation of f to $\langle f_i \rangle$ is given, at order ε^2 , by

$$\begin{aligned} g(v) &= - \int_0^{+\infty} \langle F[h_1] \cdot \nabla_v h^1 \rangle(t, v) \, dt \\ &= - \int_0^{+\infty} \sum_{k \in \mathbb{Z}} \widehat{F}[h^1](t, -k) \cdot \nabla_v \widehat{h}^1(t, k, v) \, dt. \end{aligned}$$

Applying the Fourier transform and using (13.1)-(13.2)-(13.4)-(13.5), we deduce

$$\begin{aligned}
\tilde{g}(\eta) &= - \int_0^{+\infty} \sum_{k \in \mathbb{Z}} \widehat{F}[h^1](t, -k) \cdot \widetilde{\nabla_v h^1}(t, k, \eta) dt \\
&= - \int_0^{+\infty} \widehat{F}[h^1](t, -1) (2i\pi\eta) \tilde{h}^1(t, 1, \eta) dt \\
&\quad - \int_0^{+\infty} \widehat{F}[h^1](t, 1) (2i\pi\eta) \tilde{h}^1(t, -1, \eta) dt \\
&= \pi^2 \varepsilon^2 \alpha \widehat{W}(1) \eta \left(\int_0^{+\infty} \varphi(-t) \varphi(\eta + t) dt - \int_0^{+\infty} \varphi(t) \varphi(\eta - t) dt + O(\alpha) \right) \\
&= -\pi^2 \varepsilon^2 \alpha \widehat{W}(1) \eta \left(\int_{-\infty}^{+\infty} \varphi(t) \varphi(\eta - t) \text{sign}(t) dt + O(\alpha) \right).
\end{aligned}$$

Summarizing:

$$(13.7) \quad \begin{cases} \lim_{t \rightarrow \infty} \tilde{f}(t, k, \eta) = 0 & \text{if } k \neq 0 \\ \lim_{t \rightarrow \infty} \tilde{f}(t, 0, \eta) = \tilde{f}_i(t, 0, \eta) - \varepsilon^2 \alpha (\pi^2 \widehat{W}(1)) \eta \left(\int_{-\infty}^{+\infty} \varphi(t) \varphi(\eta - t) \text{sign}(t) dt + O(\alpha) \right). \end{cases}$$

Since φ is an arbitrary analytic profile, this simple calculation already shows that

- the asymptotic profile is not necessarily the spatial mean of the initial datum;
- the asymptotic profile in general depends on the interaction, and cannot be predicted by a “statistical” recipe based only on the initial datum (e.g. maximum of entropy given the constraints).

Assuming $\varepsilon \ll \alpha$, higher order expansions in α can be obtained by bootstrap on the equations (13.1)-(13.2)-(13.4)-(13.5): for instance,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \tilde{f}(t, 0, \eta) &= \tilde{f}_i(0, \eta) - \varepsilon^2 \alpha (\pi^2 \widehat{W}(1)) \eta \int_{-\infty}^{+\infty} \varphi(t) \varphi(\eta - t) \text{sign}(t) dt \\
&\quad - \varepsilon^2 \alpha^2 (2\pi^2 \widehat{W}(1))^2 \eta \left\{ \int_0^\infty \int_0^t \left(\varphi(\eta + t) \varphi(-\tau) - \varphi(\eta - t) \varphi(\tau) \right) e^{-\pi(t-\tau)^2} (t - \tau) \right. \\
&\quad \quad \quad \left. + \varphi(\tau) \varphi(-t) e^{-\pi(\eta+(t-\tau))^2} (\eta + (t - \tau)) \right. \\
&\quad \quad \quad \left. + \varphi(-\tau) \varphi(t) e^{-\pi(\eta-(t-\tau))^2} (\eta - (t - \tau)) \right\} d\tau + O(\varepsilon^2 \alpha^3).
\end{aligned}$$

What about the limit in negative time? Reversing time is equivalent to changing $f(t, x, v)$ into $f(t, x, -v)$ and letting time go forward. So we define $S(v) := -v$, $T(\varphi)(\eta) := \varepsilon^2 \alpha \pi^2 \widehat{W}(1) \eta \int_{-\infty}^{+\infty} \varphi(t) \varphi(\eta - t) \text{sign}(t) dt$; then $T(\varphi \circ S) = T(\varphi) \circ S$, which means that the solutions constructed above are always *homoclinic* at order $O(\varepsilon^2 \alpha)$. The same is true for the more precise expansions at order $O(\varepsilon^2 \alpha^2)$, and in fact it can be checked that the whole distribution f^2 is homoclinic; in other words, f is homoclinic up to possible corrections of order $O(\varepsilon^4)$. To exhibit heteroclinic deviations, we shall consider more general perturbations.

13.2. General perturbation. Let us now consider a “general” initial datum $f_i(x, v)$ close to $f^0(v)$, and expand the solution f . We write $\varepsilon \varphi_k(\eta) = (f_i - f^0) \widetilde{}(k, \eta)$ and $\rho^m = \rho[h^m]$. The interaction potential is assumed to be of the form αW with $\alpha \ll 1$ and $W(x) = W(-x)$. The first equations of the Newton scheme are

$$(13.8) \quad \widehat{\rho}^1(t, k) = \varepsilon \varphi_k(kt) - 4\pi^2 \alpha \widehat{W}(k) \int_0^t \widehat{\rho}^1(\tau, k) \widetilde{f}^0(k(t - \tau)) |k|^2 (t - \tau) d\tau,$$

$$(13.9) \quad \widetilde{h}^1(t, k, \eta) = \varepsilon \varphi_k(\eta + kt) - 4\pi^2 \alpha \widehat{W}(k) \int_0^t \widehat{\rho}^1(\tau, k) \widetilde{f}^0(\eta + k(t - \tau)) k \cdot (\eta + k(t - \tau)) d\tau,$$

$$(13.10) \quad \begin{aligned} \widetilde{h}^2(t, k, \eta) = & -4\pi^2 \alpha \widehat{W}(k) \int_0^t \widehat{\rho}^2(\tau, k) \widetilde{f}^0(\eta + k(t - \tau)) k \cdot (\eta + k(t - \tau)) d\tau \\ & - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \widehat{\rho}^1(\tau, \ell) \widetilde{h}^1(\tau, k - \ell, \eta + k(t - \tau)) \ell \cdot (\eta + k(t - \tau)) d\tau \\ & - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \widehat{\rho}^2(\tau, \ell) \widetilde{h}^1(\tau, k - \ell, \eta + k(t - \tau)) \ell \cdot (\eta + k(t - \tau)) d\tau \\ & - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \widehat{\rho}^1(\tau, \ell) \widetilde{h}^2(\tau, k - \ell, \eta + k(t - \tau)) \ell \cdot (\eta + k(t - \tau)) d\tau, \end{aligned}$$

$$\begin{aligned}
(13.11) \quad \tilde{\rho}^2(t, k) &= -4\pi^2 \alpha \widehat{W}(k) \int_0^t \tilde{\rho}^2(\tau, k) \tilde{f}^0(k(t-\tau)) |k|^2 (t-\tau) d\tau \\
&\quad - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \tilde{\rho}^1(\tau, \ell) \tilde{h}^1(\tau, k-\ell, k(t-\tau)) \ell \cdot k (t-\tau) d\tau \\
&\quad - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \tilde{\rho}^2(\tau, \ell) \tilde{h}^1(\tau, k-\ell, k(t-\tau)) \ell \cdot k (t-\tau) d\tau \\
&\quad - 4\pi^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \tilde{\rho}^1(\tau, \ell) \tilde{h}^2(\tau, k-\ell, k(t-\tau)) \ell \cdot k (t-\tau) d\tau.
\end{aligned}$$

Here k and ℓ run over \mathbb{Z}^d .

From (13.8)–(13.9) we see that ρ^1 and h^1 depend *linearly* on ε , and

$$(13.12) \quad \tilde{\rho}^1(t, k) = \varepsilon [\varphi_k(kt) + O(\alpha)], \quad \tilde{h}^1(t, k, \eta) = \varepsilon [\varphi_k(\eta + kt) + O(\alpha)].$$

Then from (13.10)–(13.11), ρ^2 and h^2 are $O(\varepsilon^2 \alpha)$; so by plugging (13.12) in these equations we obtain

$$(13.13) \quad \tilde{\rho}^2(t, k) = -4\pi^2 \varepsilon^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \varphi_{\ell}(\ell\tau) \varphi_{k-\ell}(kt-\ell\tau) \ell \cdot k (t-\tau) d\tau + O(\varepsilon^2 \alpha^2) + O(\varepsilon^3 \alpha),$$

$$(13.14) \quad \tilde{h}^2(t, k, \eta) = -4\pi^2 \varepsilon^2 \alpha \int_0^t \sum_{\ell} \widehat{W}(\ell) \varphi_{\ell}(\ell\tau) \varphi_{k-\ell}(\eta+kt-\ell\tau) \ell \cdot (\eta+k(t-\tau)) d\tau + O(\varepsilon^2 \alpha^2) + O(\varepsilon^3 \alpha).$$

We plug these bounds again in the right-hand side of (13.10) to find

$$(13.15) \quad \tilde{h}^2(t, 0, \eta) = (\mathbb{II})_{\varepsilon}(t, \eta) + (\mathbb{III})_{\varepsilon}(t, \eta) + O(\varepsilon^3 \alpha^3),$$

where

$$(\mathbb{II})_{\varepsilon}(t, \eta) = -4\pi^2 \alpha \int_0^t \sum_{\ell} (\ell \cdot \eta) \widehat{W}(\ell) \tilde{\rho}^1(\tau, \ell) \tilde{h}^1(\tau, -\ell, \eta) d\tau$$

is quadratic in ε , and $(\mathbb{III})_\varepsilon(t, \eta)$ is a third-order correction:

(13.16)

$$\begin{aligned}
 (\mathbb{III})_\varepsilon(t, \eta) = & 16\pi^4 \varepsilon^3 \alpha^2 \sum_{m, \ell \in \mathbb{Z}^d} \widehat{W}(\ell) \widehat{W}(m) \\
 & \int_0^t \int_0^\tau \varphi_m(ms) \left\{ \varphi_{\ell-m}(\ell\tau - ms) \varphi_{-\ell}(\eta - \ell\tau) (\ell \cdot m) (\tau - s) \right. \\
 & \left. + \varphi_\ell(\ell\tau) \varphi_{-\ell-m}(\eta - \ell\tau - ms) m \cdot (\eta - \ell(\tau - s)) \right\} (\ell \cdot \eta) ds d\tau.
 \end{aligned}$$

If \tilde{f}^0 is even, changing φ_k into $\varphi_k(-\cdot)$ and η into $-\eta$ amounts to change k into $-k$ at the level of (13.8)–(13.9); but then $(\mathbb{II})_\varepsilon$ is invariant under this operation. We conclude that f is always homoclinic at second order in ε , and we consider the influence of the third-order term (13.16). Let

$$C[\varphi](\eta) := \lim_{t \rightarrow \infty} (\mathbb{III})_\varepsilon(t, \eta).$$

After some relabelling, we find

$$\begin{aligned}
 (13.17) \quad C[\varphi](\eta) = & 16\pi^4 \varepsilon^3 \alpha^2 \sum_{k, \ell \in \mathbb{Z}^d} \widehat{W}(k) \widehat{W}(\ell) \\
 & \int_0^\infty \int_0^t \varphi_\ell(\ell\tau) \left\{ \varphi_{k-\ell}(kt - \ell\tau) \varphi_{-k}(\eta - kt) (k \cdot \ell) (t - \tau) \right. \\
 & \left. + \varphi_k(kt) \varphi_{-k-\ell}(\eta - kt - \ell\tau) \ell \cdot (\eta - k(t - \tau)) \right\} (k \cdot \eta) d\tau dt.
 \end{aligned}$$

Now assume that $\varphi_{-k} = \sigma \varphi_k$ with $\sigma = \pm 1$. ($\sigma = 1$ means that the perturbation is even in x ; $\sigma = -1$ that it is odd.) Using the symmetry $(k, \ell) \leftrightarrow (-k, -\ell)$ one can check that

$$C[\varphi \circ S] \circ S = \sigma C[\varphi],$$

where $S(z) = -z$. In particular, if the perturbation is odd in x , then the third-order correction imposes a heteroclinic behavior for the solution, as soon as $C[\varphi] \neq 0$.

To construct an example where $C[\varphi] \neq 0$, we set $d = 1$, $f^0 = \text{Gaussian}$, $f_i - f^0 = \sin(2\pi x) \theta_1(v) + \sin(4\pi x) \theta_2(v)$, $\varphi_1 = -\varphi_{-1} = \tilde{\theta}_1/2$, $\varphi_2 = -\varphi_{-2} = \tilde{\theta}_2/2$. The six pairs (k, ℓ) contributing to (13.17) are $(-1, 1)$, $(1, -1)$, $(1, 2)$, $(2, 1)$, $(-1, -2)$, $(-2, -1)$. By playing on the respective sizes of $\widehat{W}(1)$ and $\widehat{W}(2)$ (which amounts in fact to changing the size of the box), it is sufficient to consider the terms with coefficient

$\widehat{W}(1)^2$, *i.e.*, the pairs $(-1, 1)$ and $(1, -1)$. Then the corresponding bit of $C[\varphi](\eta)$ is

$$\begin{aligned} -16\pi^4 \varepsilon^3 \alpha^2 \widehat{W}(1)^2 \eta \int_0^\infty \int_0^t & \left[\varphi_1(\tau) \varphi_1(\eta + t) \varphi_2(-t + \tau) (t - \tau) \right. \\ & + \varphi_1(\tau) \varphi_1(t) \varphi_2(\eta + t - \tau) (\eta + t - \tau) \\ & + \varphi_1(-\tau) \varphi_1(\eta - t) \varphi_2(t + \tau) (t - \tau) \\ & \left. + \varphi_1(-\tau) \varphi_1(t) \varphi_2(\eta - t + \tau) (t - \tau - \eta) \right] d\tau dt. \end{aligned}$$

If we let φ_1 and φ_2 vary in such a way that they become positive and almost concentrated on \mathbb{R}_+ , the only remaining term is the one in $\varphi_1(\tau) \varphi_2(\eta + t - \tau) \varphi_1(t)$, and its contribution is negative for $\eta > 0$. So, at least for certain values of $W(1)$ and $W(2)$ there is a choice of analytic functions φ_1 and φ_2 , such that $C[\varphi] \neq 0$. This demonstrates the existence of heteroclinic trajectories.

To summarize: At first order in ε , the convergence is to the spatial average; at second order there is a homoclinic correction; at third order, if at least three modes with zero sum are excited, there is possibility of heteroclinic behavior.

14. BEYOND LANDAU DAMPING

We conclude this paper with some general comments about the physical implications of Landau damping.

The counterexamples from Section 13 show in particular that there is no “universal” large-time behavior of the solution of the nonlinear Vlasov equation in terms of just, say, conservation laws and the initial datum; the *dynamics* also have to enter explicitly. One can also interpret this as a lack of ergodicity: the nonlinearity is not sufficient to make the system explore the space of all “possible” distributions and to choose the most favorable one, whatever this means. Failure of ergodicity for a system of finitely many particles was already known to occur, in relation to the KAM theorem; this is mentioned *e.g.* in [55, p. 257] for the vortex system. There it is hoped that such behavior disappears as the dimension goes to infinity; but now we see that it also exists even in the infinite-dimensional setting of the Vlasov equation — at least for any interaction less singular than Coulomb/Newton.

At first, this seems to be bad news for the statistical theory of the Vlasov equation, pioneered by Lynden-Bell [51] and explored by various authors [16, 58, 70, 78, 84], since even the sophisticated variants of this theory try to predict the likely final states in terms of just the characteristics of the initial data. In this sense, our

results provide support for an objection raised by Isichenko [38, p. 2372] against the statistical theory.

However, looking more closely at our proofs and results, proponents of the statistical theory will have a lot to rejoice about.

To start with, our results are the first to rigorously establish that the nonlinear Vlasov equation does enjoy some asymptotic “stabilization” property in large time, without the help of any extra diffusion or ensemble averaging.

Next, the whole analysis is perturbative: each stable spatially homogeneous distribution will have its small “basin of damping”, and it may be that some distributions are “much more stable” than others, say in the sense of having a larger basin.

Even more importantly, in Section 7 we have crucially used the smoothness to overcome the potentially destabilizing nonlinear effects; we have even explained why the critical regularity for Coulomb or Newton interaction is analyticity. So any theory based on nonsmooth, or even non-analytic, functions, might not be constrained by Landau damping. This certainly applies to a statistical theory, for which analytic functions should be a zero-probability set.

Finally, to overcome the nonlinearity, not only were we led to use the analytic regularity, but we also had to cope with *huge* constants — even qualitatively larger than those appearing in classical KAM theory — at least on very large time scales. If one believes in the explanatory virtues of proofs, these large constants might be the indication that Landau damping is a thin effect, which might be neglected when it comes to predict the “final” state in a “turbulent” situation.

Further work needs to be done to understand whether these considerations apply equally to the electrostatic and gravitational cases, or whether the electrostatic case is favored in these respects.

Although the underlying mathematical and physical mechanisms differ, nonlinear Landau damping (as defined by Theorem 2.5) may arguably be to the theory of Vlasov equation what the KAM theorem is to the theory of Hamiltonian systems. Like the KAM theorem, it might be conceptually important in theory and practice, and still be severely limited.¹⁷

Beyond the range of application of KAM theory lies the softer, more robust *weak KAM theory* developed by Fathi [24] in relation to Aubry–Mather theory. By a nice coincidence, a Vlasov version of the weak KAM theory has just been developed by Gangbo and Tudorascu [27], although with no relation to Landau damping. Making

¹⁷It is a well-known scientific paradox that the KAM theorem was at the same time tremendously influential in the science of the twentieth century, and so restrictive that its assumptions are essentially never satisfied in practice.

the connection is just one of the many developments which may be explored in the future.

APPENDIX

In this appendix we gather some elementary tools, our conventions, and some reminders about calculus.

A.1. Calculus in dimension d . If $n \in \mathbb{N}_0^d$ we define

$$n! = n_1! \dots n_d!$$

and

$$\binom{n}{m} = \binom{n_1}{m_1} \dots \binom{n_d}{m_d}.$$

If $z \in \mathbb{C}^d$ and $n \in \mathbb{Z}^d$, we let

$$\|z\| = |z_1| + \dots + |z_d|; \quad z^n = z_1^{n_1} \dots z_d^{n_d} \in \mathbb{C}; \quad |z|^n = |z^n|.$$

In particular, if $z \in \mathbb{C}^d$ we have

$$e^{\|z\|} = e^{|z_1| + \dots + |z_d|} = \sum_{n \in \mathbb{N}_0^d} \frac{\|z\|^n}{n!}.$$

We may occasionally write $e^{|z|}$ instead of $e^{\|z\|}$.

A.2. Multi-dimensional differential calculus. The Leibniz formula for functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ is

$$(fg)^{(n)} = \sum_{m \leq n} \binom{n}{m} f^{(m)} g^{(n-m)},$$

where of course $f^{(n)} = d^n f / dx^n$. The expression of derivatives of composed functions is given by the Faà di Bruno formula:

$$(f \circ G)^{(n)} = \sum_{\sum j m_j = n} \frac{n!}{m_1! \dots m_n!} \left(f^{(m_1 + \dots + m_n)} \circ G \right) \prod_{j=1}^n \left(\frac{G^{(j)}}{j!} \right)^{m_j}.$$

These formulas remain valid in several dimensions, provided that one defines, for a multi-index $n = (n_1, \dots, n_d)$,

$$f^{(n)} = \frac{\partial^{n_1}}{\partial x_1^{n_1}} \dots \frac{\partial^{n_d}}{\partial x_d^{n_d}} f.$$

They also remain true if $(\partial_1, \dots, \partial_d)$ is replaced by a d -tuple of *commuting* derivation operators.

As a consequence, we shall establish the following Leibniz-type formula for operators that are combinations of gradients and multiplications.

Lemma A.1. *Let f and g be functions of $v \in \mathbb{R}^d$, and $a, b \in \mathbb{C}^d$. Then for any $n \in \mathbb{N}^d$,*

$$(\nabla_v + (a + b))^n (fg) = \sum_{m \leq n} \binom{n}{m} (\nabla_v + a)^m f (\nabla_v + b)^{n-m} g.$$

Proof. The right-hand side is equal to

$$\sum_{m,q,r} \binom{n}{m} \binom{m}{q} \binom{n-m}{r} \nabla_v^q f \nabla_v^r g a^{m-q} b^{n-m-r}.$$

After changing indices $p = q + r$, $s = m - q$, this becomes

$$\begin{aligned} \sum_{s,p,r} \binom{n}{p} \binom{p}{r} \binom{n-p}{s} \nabla_v^r g \nabla_v^{p-r} f a^s b^{n-p-s} &= \sum_p \binom{n}{p} \nabla_v^p (fg) (a + b)^{n-p} \\ &= (\nabla_v + (a + b))^n (fg). \end{aligned}$$

□

A.3. Fourier transform. If f is a function $\mathbb{R}^d \rightarrow \mathbb{R}$, we define

$$(A.1) \quad \tilde{f}(\eta) = \int_{\mathbb{R}^d} e^{-2i\pi\eta \cdot v} f(v) dv;$$

then we have the usual formulas

$$f(v) = \int_{\mathbb{R}^d} \tilde{f}(\eta) e^{2i\pi\eta \cdot v} d\eta; \quad \widehat{\nabla} f(\eta) = 2i\pi\eta \tilde{f}(\eta).$$

Let $\mathbb{T}^d = \mathbb{R}^d / (L\mathbb{Z}^d)$. If f is a function $\mathbb{T}_L^d \rightarrow \mathbb{R}$, we define

$$(A.2) \quad \widehat{f}^{(L)}(k) = \int_{\mathbb{T}^d} e^{-2i\pi \frac{k}{L} \cdot x} f(x) dx;$$

then we have

$$f(x) = \frac{1}{L^d} \sum_{k \in \mathbb{Z}^d} \widehat{f}^{(L)}(k) e^{2i\pi \frac{k}{L} \cdot x}; \quad \widehat{\nabla} f^{(L)}(k) = 2i\pi \frac{k}{L} \widehat{f}^{(L)}(k).$$

If f is a function $\mathbb{T}_L^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, we define

$$(A.3) \quad \widetilde{f}^{(L)}(k, \eta) = \int_{\mathbb{T}_L^d} \int_{\mathbb{R}^d} e^{-2i\pi \frac{k}{L} \cdot x} e^{-2i\pi\eta \cdot v} f(x, v) dx dv;$$

so that the reconstruction formula reads

$$f(x, v) = \frac{1}{L^d} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \tilde{f}^{(L)}(k, \eta) e^{2i\pi \frac{k}{L} \cdot x} e^{2i\pi \eta \cdot v} dv.$$

When $L = 1$ we do not specify it: so we just write

$$\hat{f} = \hat{f}^{(1)}; \quad \tilde{f} = \tilde{f}^{(1)}.$$

(There is no risk of confusion since in that case formulae (A.3) and (A.1) coincide.)

A.4. Fixed point theorem. The following theorem is one of the many variants of the Picard fixed point theorem:

Theorem A.2 (Fixed point theorem). *Let E be a Banach space, $F : E \rightarrow E$, and $R = 2\|F(0)\|$. If F is $(1/2)$ -Lipschitz $B(0, R) \rightarrow E$, then it has a unique fixed point in $B(0, R)$.*

Proof. Uniqueness is obvious. To prove existence, run the classical Picard iterative scheme initialized at 0: $x_0 = 0$, $x_1 = F(0)$, $x_2 = F(F(0))$, etc. It is clear that (x_n) is a Cauchy sequence and $\|x_n\| \leq \|F(0)\|(1 + \dots + 1/2^n) \leq 2\|F(0)\|$, so x_n converges in $B(0, R)$ to a fixed point of F . \square

REFERENCES

- [1] AKHIEZER, A., AKHIEZER, I., POLOVIN, R., SITENKO, A., AND STEPANOV, K. *Plasma electrodynamics. Vol. I: Linear theory, Vol. II: Non-linear theory and fluctuations*. Pergamon Press, 1975 (English Edition). Translated by D. ter Haar.
- [2] ALINHAC, S., AND GÉRARD, P. *Opérateurs pseudo-différentiels et théorème de Nash–Moser*. Savoirs Actuels. InterEditions, Paris, 1991. English translation in *Graduate Studies in Mathematics, Vol. 82*, American Mathematical Society, Providence, 2007.
- [3] ALINHAC, S., AND GÉRARD, P. *Pseudo-differential operators and the Nash–Moser theorem*, vol. 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson.
- [4] BACH, V., FRÖHLICH, J., AND SIGAL, I. M. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Comm. Math. Phys.* 207, 2 (1999), 249–290.
- [5] BACKUS, G. Linearized plasma oscillations in arbitrary electron distributions. *J. Math. Phys.* 1 (1960), 178–191, 559.
- [6] BALESCU, R. *Statistical Mechanics of Charged Particles*. Wiley-Interscience, New York, 1963.
- [7] BATT, J., AND REIN, G. Global classical solutions of the periodic Vlasov–Poisson system in three dimensions. *C. R. Acad. Sci. Paris Sér. I Math.* 313, 6 (1991), 411–416.
- [8] BENACHOUR, S. Analyticité des solutions des équations de Vlasov–Poisson. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 16, 1 (1989), 83–104.

- [9] BERNSTEIN, I. B., GREENE, J. M., AND KRUSKAL, M. D. Exact non-linear plasma oscillations. *Phys. Rev. (2)* **108** (1957), 546–550.
- [10] BINNEY, J., AND TREMAINE, S. *Galactic Dynamics*, first ed. Princeton Series in Astrophysics. Princeton University Press, 1987.
- [11] BINNEY, J., AND TREMAINE, S. *Galactic Dynamics*, second ed. Princeton Series in Astrophysics. Princeton University Press, 2008.
- [12] BOURGAIN, J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* **3**, 2 (1993), 107–156.
- [13] CAGLIOTI, E., AND MAFFEI, C. Time asymptotics for solutions of Vlasov–Poisson equation in a circle. *J. Statist. Phys.* **92**, 1-2 (1998), 301–323.
- [14] CASE, K. Plasma oscillations. *Ann. Phys.* **7** (1959), 349–364.
- [15] CHAVANIS, P. H. Quasilinear theory of the 2D Euler equation. *Phys. Rev. Lett.* **84**, 24 (2000), 5512–5515.
- [16] CHAVANIS, P. H., SOMMERIA, J., AND ROBERT, R. Statistical mechanics of two-dimensional vortices and collisionless stellar systems. *Astrophys. J.* **471** (1996), 385–399.
- [17] CHEMIN, J.-Y. Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray. In *Actes des Journées Mathématiques à la Mémoire de Jean Leray*, vol. 9 of *Sémin. Congr. Soc. Math. France*, Paris, 2004, pp. 99–123.
- [18] CHERCHIA, L. A. N. Kolmogorov’s 1954 paper on nearly-integrable Hamiltonian systems. A comment on: “On conservation of conditionally periodic motions for a small change in Hamilton’s function” [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 527–530; mr0068687]. *Regul. Chaotic Dyn.* **13**, 2 (2008), 130–139.
- [19] DEGOND, P. Spectral theory of the linearized Vlasov–Poisson equation. *Trans. Amer. Math. Soc.* **294**, 2 (1986), 435–453.
- [20] DESVILLETES, L., AND VILLANI, C. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker–Planck equation. *Comm. Pure Appl. Math.* **54**, 1 (2001), 1–42.
- [21] ELSKENS, Y. Irreversible behaviours in Vlasov equation and many-body Hamiltonian dynamics: Landau damping, chaos and granularity in the kinetic limit. In *Topics in Kinetic Theory* (2005), T. Passot, C. Sulem, and P. L. Sulem, Eds., vol. 46 of *Fields Institute Communications*, Amer. Math. Soc., Providence, pp. 89–108.
- [22] ELSKENS, Y., AND ESCANDE, D. F. *Microscopic dynamics of plasmas and chaos*. Institute of Physics, Bristol, 2003.
- [23] ESCANDE, D. F. Wave–particle interaction in plasmas: a qualitative approach. Lecture Notes for the 2008 Edition of the Houches Summer School.
- [24] FATHI, A. Weak KAM theory in Lagrangian dynamics. Cambridge University Press, book to appear.
- [25] FILBET, F. Numerical simulations available online at <http://math.univ-lyon1.fr/~filbet/publication.html>.
- [26] FRIDMAN, A. M., AND POLYACHENKO, V. L. *Physics of gravitating systems. Vol. I. Equilibrium and stability; Vol. II. Nonlinear collective processes: nonlinear waves, solitons, collisionless shocks, turbulence. Astrophysical applications*. Springer-Verlag, New York, 1984.

- [27] GANGBO, W., AND TUDORASCU, A. Lagrangian dynamics on an infinite-dimensional torus; a weak KAM theorem. Preprint, 2009.
- [28] GLASSEY, R., AND SCHAEFFER, J. Time decay for solutions to the linearized Vlasov equation. *Transport Theory Statist. Phys.* 23, 4 (1994), 411–453.
- [29] GLASSEY, R., AND SCHAEFFER, J. On time decay rates in Landau damping. *Comm. Partial Differential Equations* 20, 3-4 (1995), 647–676.
- [30] GOULD, R., O’NEIL, T., AND MALMBERG, J. Plasma wave echo. *Phys. Rev. Letters* 19, 5 (1967), 219–222.
- [31] GROSS, L. Logarithmic Sobolev inequalities. *Amer. J. Math.* 97 (1975), 1061–1083.
- [32] GUO, Y., AND REIN, G. A non-variational approach to nonlinear stability in stellar dynamics applied to the King model. *Comm. Math. Phys.* 271, 2 (2007), 489–509.
- [33] GUO, Y., AND STRAUSS, W. A. Nonlinear instability of double-humped equilibria. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 12, 3 (1995), 339–352.
- [34] HAURAY, M., AND JABIN, P.-E. N -particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.* 183, 3 (2007), 489–524.
- [35] HAYES, J. N. On non-Landau damped solutions to the linearized Vlasov equation. *Nuovo Cimento (10)* 30 (1963), 1048–1063.
- [36] HEATH, R., GAMBA, I., MORRISON, P., AND MICHLER, C. A discontinuous Galerkin method for the Vlasov–Poisson system. Work in progress, 2009.
- [37] HWANG, J.-H., AND VELÁZQUEZ, J. On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem. Preprint, 2008.
- [38] ISICHENKO, M. Nonlinear Landau damping in collisionless plasma and inviscid fluid. *Phys. Rev. Lett.* 78, 12 (1997), 2369–2372.
- [39] JABIN, P.-E. Averaging lemmas and dispersion estimates for kinetic equations. To appear in *Rivista di Matematica della Università di Parma*, contribution to the special issue devoted to the Summer School 2008 “Methods and models of kinetic theory”.
- [40] KAGANOVICH, I. D. Effects of collisions and particle trapping on collisionless heating. *Phys. Rev. Lett.* 82, 2 (1999), 327–330.
- [41] KANDRUP, H. Violent relaxation, phase mixing, and gravitational Landau damping. *Astrophysical Journal* 500 (1998), 120–128.
- [42] KIESSLING, M. Personal communication.
- [43] KIESSLING, M. K.-H. The “Jeans swindle”: a true story—mathematically speaking. *Adv. in Appl. Math.* 31, 1 (2003), 132–149.
- [44] KRALL, N., AND TRIVELPIECE, A. *Principles of plasma physics*. San Francisco Press, 1986.
- [45] KUKSIN, S. B. *Nearly integrable infinite-dimensional Hamiltonian systems*, vol. 1556 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1993.
- [46] KUKSIN, S. B. *Analysis of Hamiltonian PDEs*, vol. 19 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.
- [47] LANDAU, L. On the vibration of the electronic plasma. *J. Phys. USSR* 10 (1946), 25. English translation in *JETP* 16, 574. Reproduced in *Collected papers of L.D. Landau*, edited and with an introduction by D. ter Haar, Pergamon Press, 1965, pp. 445–460; and in *Men of Physics: L.D. Landau*, Vol. 2, Pergamon Press, D. ter Haar, ed. (1965).¹⁸

¹⁸There is a misprint in formula (17) of this reference (p. 104): replace $e^{-(ka)^2/2}$ by $e^{-1/(2(ka)^2)}$.

- [48] LIFSHITZ, E. M., AND PITAEVSKIĬ, L. P. *Course of theoretical physics [“Landau–Lifshits”]. Vol. 10.* Pergamon Press, Oxford, 1981. Translated from the Russian by J. B. Sykes and R. N. Franklin.
- [49] LIONS, P.-L., AND PERTHAME, B. Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system. *Invent. Math.* 105, 2 (1991), 415–430.
- [50] LYNDEN-BELL, D. The stability and vibrations of a gas of stars. *Mon. Not. R. astr. Soc.* 124, 4 (1962), 279–296.
- [51] LYNDEN-BELL, D. Statistical mechanics of violent relaxation in stellar systems. *Mon. Not. R. astr. Soc.* 136 (1967), 101–121.
- [52] MALMBERG, J., AND WHARTON, C. Collisionless damping of electrostatic plasma waves. *Phys. Rev. Lett.* 13, 6 (1964), 184–186.
- [53] MALMBERG, J., WHARTON, C., GOULD, R., AND O’NEIL, T. Plasma wave echo experiment. *Phys. Rev. Letters* 20, 3 (1968), 95–97.
- [54] MANFREDI, G. Long-time behavior of nonlinear Landau damping. *Phys. Rev. Lett.* 79, 15 (1997), 2815–2818.
- [55] MARCHIORO, C., AND PULVIRENTI, M. *Mathematical theory of incompressible nonviscous fluids.* Springer-Verlag, New York, 1994.
- [56] MASLOV, V. P., AND FEDORYUK, M. V. The linear theory of Landau damping. *Mat. Sb. (N.S.)* 127(169), 4 (1985), 445–475, 559.
- [57] MEDVEDEV, M. V., DIAMOND, P. H., ROSENBLUTH, M. N., AND SHEVCHENKO, V. I. Asymptotic theory of nonlinear Landau damping and particle trapping in waves of finite amplitude. *Phys. Rev. Lett.* 81, 26 (1998), 5824–5827.
- [58] MILLER, J. Statistical mechanics of Euler equations in two dimensions. *Phys. Rev. Lett.* 65, 17 (1990), 2137–2140.
- [59] MOSER, J. A rapidly convergent iteration method and non-linear differential equations. II. *Ann. Scuola Norm. Sup. Pisa (3)* 20 (1966), 499–535.
- [60] MOSER, J. Recollections. In *The Arnoldfest (Toronto, ON, 1997)*, vol. 24 of *Fields Inst. Commun.* Amer. Math. Soc., Providence, RI, 1999, pp. 19–21. Concerning the early development of KAM theory.
- [61] MOUHOT, C., AND VILLANI, C. On the inviscid damping around vortices and shear-flows for two-dimensional fluids. Work in progress.
- [62] MOUHOT, C., AND VILLANI, C. Landau damping. Preprint, 2009.
- [63] NASH, J. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)* 63 (1956), 20–63.
- [64] NEHOROSĚV, N. N. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. *Uspehi Mat. Nauk* 32, 6(198) (1977), 5–66, 287.
- [65] NEHOROSĚV, N. N. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II. *Trudy Sem. Petrousk.*, 5 (1979), 5–50.
- [66] O’NEIL, T. Collisionless damping of nonlinear plasma oscillations. *Phys. Fluids* 8, 12 (1965), 2255–2262.
- [67] O’NEIL, T. M., AND CORONITI, F. V. The collisionless nature of high-temperature plasmas. *Rev. Mod. Phys.* 71 (Centenary), 2 (1999), S404–S410.

- [68] PENROSE, O. Electrostatic instability of a non-Maxwellian plasma. *Phys. Fluids* 3 (1960), 258–265.
- [69] REIN, G. Personal communication.
- [70] ROBERT, R. Statistical mechanics and hydrodynamical turbulence. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)* (Basel, 1995), Birkhäuser, pp. 1523–1531.
- [71] RYUTOV, D. D. Landau damping: half a century with the great discovery. *Plasma Phys. Control. Fusion* 41 (1999), A1–A12.
- [72] SÁENZ, A. W. Long-time behavior of the electric potential and stability in the linearized Vlasov theory. *J. Mathematical Phys.* 6 (1965), 859–875.
- [73] SOFFER, A., AND WEINSTEIN, M. I. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.* 133, 1 (1990), 119–146.
- [74] SPENTZOURIS, L., OSTIGUY, J., AND COLESTOCK, P. Direct measurement of diffusion rates in high energy synchrotrons using longitudinal beam echoes. *Phys. Rev. Lett.* 76, 4 (1996), 620–623.
- [75] STAHL, B., KIESSLING, M. K.-H., AND SCHINDLER, K. Phase transitions in gravitating systems and the formation of condensed objects. *Planet. Space Sci.* 43, 3/4 (1995), 271–282.
- [76] STIX, T. H. *The theory of plasma waves*. McGraw-Hill Book Co., Inc., New York, 1962.
- [77] TER HAAR, D. *Men of Physics: L.D. Landau*, vol. II of *Selected Reading of physics*. Pergamon Press, 1969.
- [78] TREMAINE, S., HÉNON, M., AND LYNDEN-BELL, D. *H*-functions and mixing in violent relaxation. *Mon. Not. R. astr. Soc.* 219 (1986), 285–297.
- [79] TURKINGTON, B. Statistical equilibrium measures and coherent states in two-dimensional turbulence. *Comm. Pure Appl. Math.* 52, 7 (1999), 781–809.
- [80] VAN KAMPEN, N. On the theory of stationary waves in plasma. *Physica* 21 (1955), 949–963.
- [81] VEKSTEIN, G. E. Landau resonance mechanism for plasma and wind-generated water waves. *Am. J. Phys.* 66, 10 (1998), 886–892.
- [82] VILLANI, C. Hypocoercivity. To appear in *Mem. Amer. Math. Soc.*
- [83] VILLANI, C. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*. North-Holland, Amsterdam, 2002, pp. 71–305.
- [84] WIECHEN, H., ZIEGLER, H. J., AND SCHINDLER, K. Relaxation of collisionless self-gravitating matter – the lowest energy state. *Mon. Not. R. astr. Soc.* 232 (1988), 623–646.
- [85] ZHOU, T., GUO, Y., AND SHU, C.-W. Numerical study on Landau damping. *Physica D* 157 (2001), 322–333.

CLÉMENT MOUHOT

UNIVERSITÉ PARIS-DAUPHINE & CNRS
 CEREMADE, UMR CNRS 7534
 PLACE DU MARCHAL DE LATTRE DE TASSIGNY
 75775 PARIS CEDEX 16
 FRANCE

E-MAIL: cmouhot@ceremade.dauphine.fr

CÉDRIC VILLANI

ENS LYON & INSTITUT UNIVERSITAIRE DE FRANCE
UMPA, UMR CNRS 5669
46 ALLÉE D'ITALIE
69364 LYON CEDEX 07
FRANCE

E-MAIL: cvillani@umpa.ens-lyon.fr