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A criterion of weak compactness for operators on subspaces of Orlicz spaces

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Abstract. *We give a criterion of weak compactness for the operators on the Morse-Transue space M^Ψ , the subspace of the Orlicz space L^Ψ generated by L^∞ .*

Mathematics Subject Classification. Primary: 46 E 30; Secondary: 46 B 20

Key-words. Morse-Transue space; Orlicz space; weakly compact operators

1 Introduction and Notation.

In 1975, C. Niculescu established a characterization of weakly compact operators T from $\mathcal{C}(S)$, where S is a compact space, into a Banach space Z ([14, 15], see [3] Theorem 15.2 too): $T: \mathcal{C}(S) \rightarrow Z$ is weakly compact if and only if there exists a Borel probability measure μ on S such that, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that:

$$\|Tf\| \leq C(\epsilon) \|f\|_{L^1(\mu)} + \epsilon \|f\|_\infty, \quad \forall f \in \mathcal{C}(S).$$

The same kind of result was proved by H. Jarchow for \mathbb{C}^* -algebras in [7], and by the first author for $A(\mathbb{D})$ and H^∞ (see [11]). The criterion for H^∞ played a key role to give an elementary proof of the equivalence between weak compactness and compactness for composition operators on H^∞ .

Beside these spaces, one natural class of Banach spaces is the class of Orlicz spaces L^Ψ . Unfortunately, we shall see that the above criterion is in

general not true for Orlicz spaces. However, it remains true when we restrict ourselves to subspaces of the Morse-Transue space M^Ψ . This space is the closure of L^∞ in the Orlicz space L^Ψ .

In this paper, we first give a characterization of the operators from a subspace of M^Ψ which fix no copy of c_0 . When the complementary function of Ψ satisfies Δ_2 , that gives a criterion of weak compactness. If moreover Ψ satisfies a growth condition, that we call Δ^0 , the criterion has a more usable formulation, analogous to those described above.

As in the case of H^∞ (but this is far less elementary), this new version obtained for subspaces of Morse-Transue spaces (Theorem 4), combined with a study of generalized Carleson measures, may be used to prove the equivalence between weak compactness and compactness for composition operators on Hardy-Orlicz spaces (see [13]), when Ψ satisfies Δ^0 .

However, we think that this characterization has an intrinsic interest for Orlicz spaces, and will be useful not only for composition operators (see Remark 5 at the end of the paper).

In this note, we shall consider Orlicz spaces defined on a probability space (Ω, \mathbb{P}) , that we shall assume non purely atomic.

By an Orlicz function, we shall understand that $\Psi: [0, \infty] \rightarrow [0, \infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function Ψ having the following additional properties: Ψ is continuous at 0, strictly convex (hence *strictly* increasing), and such that

$$\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty.$$

This is essentially to exclude the case of $\Psi(x) = ax$. The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f: \Omega \rightarrow \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_{\Omega} \Psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants C such that this integral is ≤ 1 .

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*: [0, \infty] \rightarrow [0, \infty]$ defined by:

$$\Phi(x) = \sup_{y \geq 0} (xy - \Psi(y)).$$

The extra assumptions on Ψ ensure that Φ is itself strictly convex.

Throughout this paper, we shall assume, except explicit mention of the contrary, that the *complementary* Orlicz function satisfies the Δ_2 condition ($\Phi \in \Delta_2$), *i.e.*, for some constant $K > 0$, and some $x_0 > 0$, we have:

$$\Phi(2x) \leq K \Phi(x), \quad \forall x \geq x_0.$$

This is usually expressed by saying that Ψ satisfies the ∇_2 condition ($\Psi \in \nabla_2$). This is equivalent to say that for some $\beta > 1$ and $x_0 > 0$, one has $\Psi(x) \leq \Psi(\beta x)/(2\beta)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty$. In particular, this excludes the case $L^\Psi = L^1$.

When Φ satisfies the Δ_2 condition, L^Ψ is the dual space of L^Φ .

We shall denote by M^Ψ the closure of L^∞ in L^Ψ . Equivalently (see [16], page 75), M^Ψ is the space of (classes of) functions such that:

$$\int_{\Omega} \Psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) < +\infty, \quad \forall C > 0.$$

This space is the *Morse-Transue space* associated to Ψ , and $(M^\Psi)^* = L^\Phi$, isometrically if L^Φ is provided with the Orlicz norm, and isomorphically if it is equipped with the Luxemburg norm (see [16], Chapter IV, Theorem 1.7, page 110).

We have $M^\Psi = L^\Psi$ if and only if Ψ satisfies the Δ_2 condition, and L^Ψ is reflexive if and only if both Ψ and Φ satisfy the Δ_2 condition. When the complementary function $\Phi = \Psi^*$ of Ψ satisfies it (but Ψ does not satisfy this Δ_2 condition, to exclude the reflexive case), we have (see [16], Chapter IV, Proposition 2.8, page 122, and Theorem 2.11, page 123):

$$(*) \quad (L^\Psi)^* = (M^\Psi)^* \oplus_1 (M^\Psi)^\perp,$$

or, equivalently, $(L^\Psi)^* = L^\Phi \oplus_1 (M^\Psi)^\perp$, isometrically, with the Orlicz norm on L^Φ .

For all the matter about Orlicz functions and Orlicz spaces, we refer to [16], or to [9].

2 Main result.

Our goal in this section is the following criterion of weak compactness for operators. We begin with:

Theorem 1 *Let Ψ be an arbitrary Orlicz function, and let X be a subspace of the Morse-Transue space M^Ψ . Then an operator $T: X \rightarrow Y$ from X into a Banach space Y fixes no copy of c_0 if and only if for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$(1) \quad \|Tf\| \leq \left[C_\varepsilon \int_{\Omega} \Psi\left(\varepsilon \frac{|f|}{\|f\|_\Psi}\right) d\mathbb{P} + \varepsilon \right] \|f\|_\Psi, \quad \forall f \in X.$$

Recall that saying that T fixes a copy of c_0 means that there exists a subspace X_0 of X isomorphic to c_0 such that T realizes an isomorphism between X_0 and $T(X_0)$.

Before proving that, we shall give some consequences. First, we have:

Corollary 2 *Assume that the complementary function of Ψ has Δ_2 ($\Psi \in \nabla_2$). Then for every subspace X of M^Ψ , and every operator $T: X \rightarrow Y$, T is weakly compact if and only if it satisfies (1).*

Proof. When the complementary function of Ψ has Δ_2 , one has the decomposition (*), which means that M^Ψ is M -ideal in its bidual (see [6], Chapter III); this result was first shown by D. Werner ([17]; see also [6], Chapter III, Example 1.4 (d), page 105) by a different way, using the ball intersection property; note that in these references, it is moreover assumed that Ψ does not satisfy the Δ_2 condition, but if it satisfies it, the space L^Ψ is reflexive, and so the result is obvious. But every subspace X of a Banach space which is M -ideal of its bidual has Pełczyński's property (V) ([4, 5]; see also [6], Chapter III, Theorem 3.4), which means that operators from X are weakly compact if and only if they fix no copy of c_0 . \square

With Ψ satisfying the following growth condition, the characterization (1) takes on a more usable form.

Definition 3 *We say that the Orlicz function Ψ satisfies the Δ^0 condition if for some $\beta > 1$, one has:*

$$\lim_{x \rightarrow +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$

This growth condition is a strong negation of the Δ_2 condition and it implies that the complementary function $\Phi = \Psi^*$ of Ψ satisfies the Δ_2 condition.

Note that in the following theorem, we cannot content ourselves with $\Psi \notin \Delta_2$ (i.e. $\limsup_{x \rightarrow +\infty} \Psi(\beta x)/\Psi(x) = +\infty$), instead of $\Psi \in \Delta^0$ (see Remark 3 in Section 3). An interesting question is whether the condition $\Psi \in \Delta^0$ is actually necessary for this characterization.

Theorem 4 *Assume that Ψ satisfies the Δ^0 condition, and let X be a subspace of M^Ψ . Then every linear operator T mapping X into some Banach space Y is weakly compact if and only if for some (and then for all) $1 \leq p < \infty$ and all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that:*

$$(W) \quad \|T(f)\| \leq C_\varepsilon \|f\|_p + \varepsilon \|f\|_\Psi, \quad \forall f \in X.$$

Remark 5 This theorem extends [12] Theorem II.1. As in the case of \mathbb{C}^* -algebras (see [3], Notes and Remarks, Chap. 15), there are miscellaneous applications of such a characterization.

Remark 6 Contrary to the Δ_2 condition where the constant 2 may be replaced by any constant $\beta > 1$, in this Δ^0 condition, the constant β cannot be replaced by another, as the following example shows.

Example 7 There exists an Orlicz function Ψ such that:

$$(2) \quad \lim_{x \rightarrow +\infty} \frac{\Psi(5x)}{\Psi(x)} = +\infty,$$

but

$$(3) \quad \liminf_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty.$$

Indeed, let $(c_n)_n$ be an increasing sequence of positive numbers such that $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = +\infty$, take $\psi(t) = c_n$ for $t \in (4^n, 4^{n+1}]$ and $\Psi(x) = \int_0^x \psi(t) dt$. Then (2) is verified. On the other hand, if $x_n = 2 \cdot 4^n$, one has $\Psi(x_n) \geq c_n 4^n$, and $\Psi(2x_n) \leq c_n 4^{n+1}$, so we get (3).

Before proving Theorem 4, let us note that it has the following straightforward corollary.

Corollary 8 *Let X be like in Theorem 4, and assume that \mathcal{F} is a family of operators from X into a Banach space Y with the following property: there exists a bounded sequence $(g_n)_n$ in X such that $\lim_{n \rightarrow \infty} \|g_n\|_1 = 0$ and such that an operator $T \in \mathcal{F}$ is compact whenever*

$$\lim_{n \rightarrow \infty} \|Tg_n\| = 0.$$

Then every weakly compact operator in $T \in \mathcal{F}$ is actually compact.

In the forthcoming paper [13], we prove, using a generalization of the notion of Carleson measure, that a composition operator $C_\phi: H^\Psi \rightarrow H^\Psi$ (H^Ψ is the space of analytic functions on the unit disk \mathbb{D} of the complex plane whose boundary values are in $L^\Psi(\partial\mathbb{D})$, and $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic self-map) is compact whenever

$$\lim_{r \rightarrow 1^-} \sup_{|\xi|=1} \Psi^{-1}(1/(1-r)) \|C_\phi(u_{\xi,r})\|_\Psi = 0,$$

where

$$u_{\xi,r}(z) = \left(\frac{1-r}{1-\bar{\xi}rz} \right)^2, \quad |z| < 1,$$

and we have

$$\lim_{r \rightarrow 1^-} \sup_{|\xi|=1} \Psi^{-1}(1/(1-r)) \|C_\phi(u_{\xi,r})\|_\Psi = 0$$

when C_ϕ is weakly compact and $\Psi \in \Delta^0$.

Though the situation does not fit exactly as in Corollary 8 (not because of the space H^Ψ , which is not a subspace of M^Ψ : we actually work in $HM^\Psi = H^\Psi \cap M^\Psi$ since $u_{\xi,r} \in HM^\Psi$, but because of the fact that we ask a uniform limit for $|\xi| = 1$), the same ideas allow us to get, when Ψ satisfies the condition Δ^0 , that C_ϕ is compact if and only if it is weakly compact.

Proof of Theorem 4. Assume that we have (W). We may assume that $p > 1$, since if (W) is satisfied for some $p \geq 1$, it is satisfied for all $q \geq p$. Moreover, we may assume that $L^\Psi \xrightarrow{j} L^p$ since Ψ satisfies condition Δ^0 (since we have: $\lim_{x \rightarrow +\infty} \frac{\Psi(x)}{x^r} = +\infty$, for every $r > 0$). Then $T[(1/C_\varepsilon)B_{L^p} \cap (1/\varepsilon)B_X] \subseteq 2B_Y$. Taking the polar of these sets, we get $T^*(B_{Y^*}) \subseteq (2C_\varepsilon)B_{j^*[(L^p)^*]} + (2\varepsilon)B_{X^*}$, for every $\varepsilon > 0$. By a well-known lemma of Grothendieck, we get, since $B_{j^*[(L^p)^*]}$ is weakly compact, that $T^*(B_{Y^*})$ is relatively weakly compact, *i.e.* T^* , and hence also T , is weakly compact.

Conversely, assume in Theorem 4 that T is weakly compact. We are going to show that (W) is satisfied with $p = 1$ (hence for all finite $p \geq 1$). Let $\varepsilon > 0$. Since the Δ^0 condition implies that the complementary function of Ψ satisfies the Δ_2 condition, Corollary 2 implies that, when $\|f\|_\Psi = 1$:

$$\|Tf\| \leq C_{\varepsilon/2} \int_{\Omega} \Psi((\varepsilon/2)|f|) d\mathbb{P} + \varepsilon/2.$$

As Ψ satisfies the Δ^0 condition, there is some $\beta > 1$ such that $\frac{\Psi(x)}{\Psi(\beta x)} \rightarrow 0$ as $x \rightarrow \infty$; hence, with $\kappa = \varepsilon/2C_{\varepsilon/2}$, there exists some $x_\kappa > 0$ such that $\Psi(x) \leq \kappa\Psi(\beta x)$ for $x \geq x_\kappa$. By the convexity of Ψ , one has $\Psi(x) \leq \frac{\Psi(x_\kappa)}{x_\kappa}x =: K_\kappa x$ for $0 \leq x \leq x_\kappa$. Hence, for every $x \geq 0$ $\Psi(x) \leq \kappa\Psi(\beta x) + K_\kappa x$. It follows that, for $f \in X$, with $\|f\|_\Psi = 1$

$$\int_{\Omega} \Psi((\varepsilon/2)|f|) d\mathbb{P} \leq \kappa \int_{\Omega} \Psi(\beta(\varepsilon/2)|f|) d\mathbb{P} + K_\kappa \frac{\varepsilon}{2} \|f\|_1 \leq \kappa + K_\kappa \frac{\varepsilon}{2} \|f\|_1$$

if we have chosen $\varepsilon \leq 2/\beta$. Hence:

$$\|Tf\| \leq C_{\varepsilon/2} \left(\kappa + K_\kappa \frac{\varepsilon}{2} \|f\|_1 \right) + \frac{\varepsilon}{2} = C_{\varepsilon/2} \frac{\varepsilon}{2} K_\kappa \|f\|_1 + \left(C_{\varepsilon/2} \kappa + \frac{\varepsilon}{2} \right) = C'_\varepsilon \|f\|_1 + \varepsilon,$$

which is (W). \square

Remark 9 The sufficient condition is actually a general fact, which is surely well known (see [11], Theorem 1.1, for a similar result, and [3], Theorem 15.2 for $\mathcal{C}(K)$; see also [8], page 81), and has close connection with interpolation (see [2], Proposition 1), but we have found no reference, and so we shall state it separately without proof (the proof follows that given in [3], page 310).

Proposition 10 *Let $T: X \rightarrow Y$ be an operator between two Banach spaces. Assume that there is a Banach space Z and a weakly compact map $j: X \rightarrow Z$ such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|Tx\| \leq C_\varepsilon \|jx\|_Z + \varepsilon \|x\|_X, \quad \forall x \in X.$$

Then T is weakly compact.

Note that, by the Davis-Figiel-Johnson-Pelczyński factorization theorem, we may assume that Z is reflexive. We may also assume that j is injective, because $\ker j \subseteq \ker T$, so T induces a map $\tilde{T}: X/\ker j \rightarrow Y$ with the same property as T . Indeed, if $jx = 0$, then $\|Tx\| \leq \varepsilon\|x\|$ for every $\varepsilon > 0$, and hence $Tx = 0$.

Proof of Theorem 1. Assume first that T fixes a copy of c_0 . There are hence some $\delta > 0$ and a sequence $(f_n)_n$ in X equivalent to the canonical basis of c_0 such that $\|f_n\|_\Psi = 1$ and $\|Tf_n\| \geq \delta$. In particular, there is some $M > 0$ such that, for every choice of $\varepsilon_n = \pm 1$

$$\left\| \sum_{n=1}^N \varepsilon_n f_n \right\|_\Psi \leq M, \quad \forall N \geq 1.$$

Let $(r_n)_n$ be a Rademacher sequence. We have, first by Khintchine's inequality, then by Jensen's inequality and Fubini's Theorem

$$\begin{aligned} \int_{\Omega} \Psi \left(\frac{1}{M\sqrt{2}} \left(\sum_{n=1}^N |f_n|^2 \right)^{1/2} \right) d\mathbb{P} &\leq \int_{\Omega} \Psi \left[\frac{1}{M} \int_0^1 \left| \sum_{n=1}^N r_n(t) f_n \right| dt \right] d\mathbb{P} \\ &\leq \int_{\Omega} \int_0^1 \Psi \left[\frac{1}{M} \left| \sum_{n=1}^N r_n(t) f_n \right| \right] dt d\mathbb{P} \\ &= \int_0^1 \int_{\Omega} \Psi \left[\frac{1}{M} \left| \sum_{n=1}^N r_n(t) f_n \right| \right] d\mathbb{P} dt \leq 1. \end{aligned}$$

The monotone convergence Theorem gives then

$$\int_{\Omega} \Psi \left(\frac{1}{M\sqrt{2}} \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \right) d\mathbb{P} \leq 1.$$

In particular, $\sum_{n=1}^{\infty} |f_n|^2$ is finite almost everywhere, and hence $f_n \rightarrow 0$ almost everywhere. Since $\Psi \left(\frac{1}{M\sqrt{2}} \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \right) \in L^1$, by the above inequalities, Lebesgue's dominated convergence Theorem gives:

$$\int_{\Omega} \Psi \left(\frac{|f_n|}{M\sqrt{2}} \right) d\mathbb{P} \xrightarrow{n \rightarrow \infty} 0.$$

But that contradicts (1) with $\varepsilon \leq 1/M\sqrt{2}$ and $\varepsilon < \delta$, since $\|Tf_n\| \geq \delta$.

The converse follows from the following lemma.

Lemma 11 *Let X be a subspace of M^Ψ , and let $(h_n)_n$ be a sequence in X , with $\|h_n\|_\Psi = 1$ for all $n \geq 1$, and such that, for some $M > 0$*

$$\int_{\Omega} \Psi(|h_n|/M) d\mathbb{P} \xrightarrow{n \rightarrow \infty} 0.$$

Then $(h_n)_n$ has a subsequence equivalent to the canonical basis of c_0 .

Indeed, if condition (1) is not satisfied, there exist some $\varepsilon_0 > 0$ and functions $h_n \in X$ with $\|h_n\|_\Psi = 1$ such that $\|Th_n\| \geq 2^n \int_{\Omega} \Psi(\varepsilon_0|h_n|) d\mathbb{P} + \varepsilon_0$. That implies that $\int_{\Omega} \Psi(\varepsilon_0|h_n|) d\mathbb{P}$ tends to 0, so Lemma 11 ensures that $(h_n)_n$ has a subsequence, which we shall continue to denote by $(h_n)_n$, equivalent to the canonical basis of c_0 . Then $(Th_n)_n$ is weakly unconditionally Cauchy. Since $\|Th_n\| \geq \varepsilon_0$, $(Th_n)_n$ has, by Bessaga-Pełczyński's Theorem, a further subsequence equivalent to the canonical basis of c_0 . It is then obvious that T realizes an isomorphism between the spaces generated by these subsequences. \square

Proof of Lemma 11. The proof uses the idea of the construction made in the proof of Theorem II.1 in [12], which it generalizes, but with some additional details.

By the continuity of Ψ , there exists $a > 0$ such that $\Psi(a) = 1$. Then, since Ψ is increasing, we have, for every $g \in L^\infty$, $\int_{\Omega} \Psi\left(a \frac{|g|}{\|g\|_\infty}\right) d\mathbb{P} \leq 1$, and so $\|g\|_\Psi \leq (1/a) \|g\|_\infty$.

Now, choose, for every $n \geq 1$, positive numbers $\alpha_n < a/2^{n+2}$ such that $\Psi(\alpha_n/2M) \leq 1$.

We are going to construct inductively a subsequence $(f_n)_n$ of $(h_n)_n$, a sequence of functions $g_n \in L^\infty$ and two sequences of positive numbers β_n and $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$, such that, for every $n \geq 1$

(i) if we set $M_1 = 1$ and, for $n \geq 2$

$$M_n = \max \left\{ 1, \Psi \left(\frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M} \right) \right\},$$

then $M_n \beta_n \leq 1/2^{n+1}$;

(ii) $\|f_n\|_\Psi = 1$;

- (iii) $\|f_n - g_n\|_\Psi \leq \varepsilon_n$, with ε_n such that $\beta_n \Psi(\alpha_n/2\varepsilon_n) \geq 2$;
- (iv) $\mathbb{P}(\{|g_n| > \alpha_n\}) \leq \beta_n$;
- (v) $\|\check{g}_n\|_\Psi \geq 1/2$, with $\check{g}_n = g_n \mathbb{1}_{\{|g_n| > \alpha_n\}}$.

We shall give only the inductive step, since the starting one unfolds identically. Suppose hence that the functions f_1, \dots, f_{n-1} , g_1, \dots, g_{n-1} and the numbers $\beta_1, \dots, \beta_{n-1}$ and $\varepsilon_1, \dots, \varepsilon_{n-1}$ have been constructed. Choose then $\beta_n > 0$ such that $M_n \beta_n \leq 1/2^{n+1}$. Note that $M_n \geq 1$ implies that $\beta_n \leq 1/2^{n+1}$. Since $\int_\Omega \Psi(|h_k|/M) d\mathbb{P} \rightarrow 0$ as $n \rightarrow \infty$, we can find $f_n = h_{k_n}$ such that $\|f_n\|_\Psi = 1$, and moreover

$$\mathbb{P}(\{|f_n| > \alpha_n/2\}) \leq \frac{1}{\Psi(\alpha_n/2M)} \int_\Omega \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P} \leq \frac{\beta_n}{2}.$$

Take now $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$ such that $0 < \varepsilon_n \leq \alpha_n/2\Psi^{-1}(2/\beta_n)$ and $g_n \in L^\infty$ such that $\|f_n - g_n\|_\Psi \leq \varepsilon_n$. Then, since

$$\mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \Psi\left(\frac{\alpha_n}{2\varepsilon_n}\right) \leq \int_\Omega \Psi\left(\frac{|f_n - g_n|}{\varepsilon_n}\right) d\mathbb{P} \leq 1,$$

we have

$$\begin{aligned} \mathbb{P}(\{|g_n| > \alpha_n\}) &\leq \mathbb{P}(\{|f_n| > \alpha_n/2\}) + \mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \\ &\leq \frac{\beta_n}{2} + \frac{1}{\Psi(\alpha_n/2\varepsilon_n)} \leq \beta_n. \end{aligned}$$

To end the construction, it remains to note that

$$\begin{aligned} \|f_n - \check{g}_n\|_\Psi &\leq \|f_n - g_n\|_\Psi + \|\check{g}_n - g_n\|_\Psi \leq \varepsilon_n + \frac{1}{a} \|\check{g}_n - g_n\|_\infty \\ &\leq \frac{1}{2^{n+1}} + \frac{\alpha_n}{a} \leq \frac{1}{2^n} \leq \frac{1}{2} \end{aligned}$$

and so

$$\|\check{g}_n\|_\Psi \geq \|f_n\|_\Psi - \|f_n - \check{g}_n\|_\Psi \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

This ends the inductive construction.

Consider now

$$\check{g} = \sum_{n=1}^{+\infty} |\check{g}_n|.$$

Set $A_n = \{|g_n| > \alpha_n\}$ and, for $n \geq 1$

$$B_n = A_n \setminus \bigcup_{j>n} A_j.$$

We have $\mathbb{P}(\limsup A_n) = 0$, because

$$\sum_{n \geq 1} \mathbb{P}(A_n) \leq \sum_{n \geq 1} \beta_n \leq \sum_{n \geq 1} \frac{1}{2^n} < +\infty.$$

Now \check{g} vanishes out of $\bigcup_{n \geq 1} B_n \cup (\limsup A_n)$ and we have

$$\begin{aligned} \int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} &\leq \int_{\Omega} \Psi\left(\frac{|g_n|}{2M}\right) d\mathbb{P} \\ &\leq \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{2M} + \frac{|f_n|}{2M}\right) d\mathbb{P} \\ &\leq \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\right) d\mathbb{P} + \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P}. \end{aligned}$$

The first integral is less than ε_n/M , because $\Psi(at) \leq a\Psi(t)$ for $0 \leq a \leq 1$ and $\varepsilon_n/M \leq 1$, so that

$$\int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\right) d\mathbb{P} \leq \frac{\varepsilon_n}{M} \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{\varepsilon_n}\right) d\mathbb{P} \leq \frac{\varepsilon_n}{M} \leq \frac{1}{2^{n+1}}$$

(since $\|f_n - g_n\|_{\Psi} \leq \varepsilon_n$). Since

$$\int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P} \leq \frac{\beta_n}{2} \Psi(\alpha_n/2M) \leq \beta_n/2,$$

we obtain

$$\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} \leq \frac{1}{2^{n+2}} + \frac{\beta_n}{4}.$$

Therefore, since $\mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \beta_n$, we have

$$\begin{aligned}
\int_{\Omega} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P} &= \sum_{n=1}^{+\infty} \int_{B_n} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P} \\
&\leq \sum_{n=1}^{+\infty} \int_{B_n} \frac{1}{2} \left[\Psi\left(\frac{\|g_1\|_{\infty} + \cdots + \|g_{n-1}\|_{\infty}}{2M}\right) + \Psi\left(\frac{|\check{g}_n|}{2M}\right) \right] d\mathbb{P} \\
&\quad \text{by convexity of } \Psi \text{ and because } \check{g}_j = 0 \text{ on } B_n \text{ for } j > n \\
&\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(M_n \beta_n + \frac{1}{2^{n+2}} + \frac{\beta_n}{4} \right) \\
&\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} \right) \leq 1,
\end{aligned}$$

which proves that $\check{g} \in L^{\Psi}$, and consequently that the series $\sum_{n \geq 1} \check{g}_n$ is weakly unconditionally Cauchy in L^{Ψ} :

$$\sup_{n \geq 1} \sup_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k \check{g}_k \right\|_{\Psi} \leq \sup_{n \geq 1} \left\| \sum_{k=1}^n |\check{g}_k| \right\|_{\Psi} \leq \|\check{g}\|_{\Psi} \leq 4M.$$

Since $\|\check{g}_n\|_{\Psi} \geq 1/2$, $(\check{g}_n)_{n \geq 1}$ has, by Bessaga-Pelczyński's theorem, a subsequence $(\check{g}_{n_k})_{k \geq 1}$ which is equivalent to the canonical basis of c_0 . The corresponding subsequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$ remains equivalent to the canonical basis of c_0 , since

$$\sum_{n=1}^{+\infty} \|f_n - \check{g}_n\|_{\Psi} \leq \sum_{n=1}^{+\infty} \varepsilon_n + \frac{\alpha_n}{a} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} < 1$$

and the assertion follows. \square

3 Comments

Remark 12 Let us note that the assumption $X \subseteq M^{\Psi}$ in Theorem 4 cannot be relaxed in general. In fact, suppose that X is a subspace of L^{Ψ} containing L^{∞} , and let $\xi \in (M^{\psi})^{\perp} \subseteq (L^{\Psi})^*$, $\xi \neq 0$. Being of rank one, ξ is trivially weakly compact. Suppose that it satisfies (W). Let $f \in X$ with

norm 1, and let $\varepsilon > 0$. For t large enough and $f_t = f \mathbb{1}_{\{|f| \leq t\}}$, we have $\|f - f_t\|_2 \leq \varepsilon/C_\varepsilon$. Moreover, $f_t \in L^\infty \subseteq X$ and $\|f_t\|_\Psi \leq \|f\|_\Psi = 1$. Since ξ vanishes on L^∞ and $f - f_t \in X$, we get

$$|\xi(f)| = |\xi(f - f_t)| \leq C_\varepsilon \|f - f_t\|_2 + \varepsilon \|f - f_t\|_\Psi \leq 3\varepsilon.$$

This implies that $\xi(f) = 0$. Since this occurs for every $\xi \in (M^\Psi)^\perp$, we get that $X \subseteq M^\Psi$ (and actually $X = M^\Psi$ since X contains L^∞).

In particular Theorem 4 does not hold for $X = L^\Psi$.

Remark 13 However, condition (W) remains true for bi-adjoint operators coming from subspaces of M^Ψ : if $T: X \subseteq M^\Psi \rightarrow Y$ satisfies the condition (W), then $T^{**}: X^{**} \rightarrow Y^{**}$ also satisfies it. Indeed, for every $\varepsilon > 0$, we get an equivalent norm $\|\cdot\|_\varepsilon$ on X by putting

$$\|f\|_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.$$

Hence if $f \in X^{**}$, there exists a net $(f_\alpha)_\alpha$ of elements in X , with $\|f_\alpha\|_\varepsilon \leq \|f\|_\varepsilon$ which converges weak-star to f . Then $(Tf_\alpha)_\alpha$ converges weak-star to $T^{**}f$, and

$$\begin{aligned} \|T^{**}f\| &\leq \liminf_\alpha \|Tf_\alpha\| \leq \liminf_\alpha (C_\varepsilon \|f_\alpha\|_2 + \varepsilon \|f_\alpha\|_\Psi) \\ &= \liminf_\alpha \|f_\alpha\|_\varepsilon \leq \|f\|_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi. \end{aligned}$$

Hence, from Proposition 10 above, for such a T , T^{**} is weakly compact if and only if it satisfies (W). We shall use this fact in the forthcoming paper [13].

Remark 14 In Theorem 4, we cannot only assume that $\Psi \notin \Delta_2$, instead of $\Psi \in \Delta^0$, as the following example shows. It also shows that in Corollary 2, we cannot obtain condition (W) instead of condition (1).

Example 15 Let us define:

$$\psi(t) = \begin{cases} t & \text{for } 0 \leq t < 1, \\ (k!)(k+2)t - k!(k+1)! & \text{for } k! \leq t \leq (k+1)!, \quad k \geq 1, \end{cases}$$

$(\psi(k!)) = (k!)^2$ for every integer $k \geq 1$ and ψ is linear between $k!$ and $(k+1)!$, and

$$\Psi(x) = \int_0^x \psi(t) dt.$$

Since $t^2 \leq \psi(t)$ for all $t \geq 0$, one has $x^3/3 \leq \Psi(x)$ for all $x \geq 0$. Then

$$\Psi(2.n!) \geq \int_{n!}^{2.n!} \psi(t) dt = n!(n+2)\frac{3}{2}(n!)^2 - (n!)^2(n+1)! = (n!)^3\left(\frac{n}{2} + 2\right),$$

whereas

$$\Psi(n!) = \int_0^{n!} \psi(t) dt \leq (n!)^2 n! = (n!)^3;$$

hence

$$\frac{\Psi(2.n!)}{\Psi(n!)} \geq \frac{n}{2} + 2,$$

and so

$$\limsup_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} = +\infty,$$

which means that $\Psi \notin \Delta_2$. On the other hand, for every $\beta > 1$

$$\Psi(n!/\beta) \geq \frac{1}{3}\left(\frac{n!}{\beta}\right)^3 = \frac{(n!)^3}{3\beta^3},$$

so

$$\frac{\Psi(n!)}{\Psi(n!/\beta)} \leq \frac{(n!)^3}{(n!)^3/3\beta^3} = 3\beta^3;$$

hence

$$\liminf_{x \rightarrow +\infty} \frac{\Psi(2x)}{\Psi(x)} \leq 3\beta^3,$$

and $\Psi \notin \Delta^0$ (actually, this will follow too from the fact that Theorem 4 is not valid for this Ψ).

Moreover, the conjugate function of Ψ satisfies the condition Δ_2 . Indeed, since ψ is convex, one has $\psi(2u) \geq 2\psi(u)$ for all $u \geq 0$, and hence:

$$\Psi(2x) = \int_0^{2x} \psi(t) dt = 2 \int_0^x \psi(2u) du \geq 2 \int_0^x 2\psi(u) du = 4\Psi(x),$$

and as it was seen in the Introduction that means that $\Psi \in \nabla_2$.

Now, we have $x^3/3 \leq \Psi(x)$ for all $x \geq 0$; therefore $\|\cdot\|_3 \leq 3^{1/3}\|\cdot\|_\Psi$. In particular, we have an inclusion map $j: M^\Psi \hookrightarrow L^3$, which is, of course, weakly compact. Nevertheless, assuming that \mathbb{P} is diffuse, condition (W) is not verified by j , when $\varepsilon < 1$. Indeed, as we have seen before, one has

$\Psi(n!) \leq (n!)^3$. Hence, if we choose a measurable set A_n such that $\mathbb{P}(A_n) = 1/\Psi(n!)$, we have

$$\|\mathbb{1}_{A_n}\|_{\Psi} = \frac{1}{\Psi^{-1}(1/\mathbb{P}(A_n))} = \frac{1}{n!};$$

whereas:

$$\|\mathbb{1}_{A_n}\|_3 = \mathbb{P}(A_n)^{1/3} = \frac{1}{\Psi(n!)^{1/3}} \geq \frac{1}{n!}$$

and

$$\|\mathbb{1}_{A_n}\|_2 = \mathbb{P}(A_n)^{1/2} \leq \left[\frac{3}{(n!)^3} \right]^{1/2} = \frac{\sqrt{3}}{(n!)^{3/2}}.$$

If condition (W) were true, we should have, for every $n \geq 1$

$$\frac{1}{n!} \leq C_{\varepsilon} \frac{\sqrt{3}}{(n!)^{3/2}} + \varepsilon \frac{1}{n!},$$

that is $\sqrt{n!} \leq \sqrt{3} \frac{C_{\varepsilon}}{1-\varepsilon}$, which is of course impossible for n large enough.

Remark 16 In the case of the whole space M^{Ψ} , we can give a direct proof of the necessity in Theorem 4. Indeed, suppose that $T: M^{\Psi} \rightarrow X$ is weakly compact. Then $T^*: X^* \rightarrow L^{\Phi} = (M^{\Psi})^*$ is weakly compact, and so the set $K = T^*(B_{X^*})$ is relatively weakly compact.

Since Φ satisfies the Δ^0 condition, it follows from [1] (Corollary 2.9) that K has equi-absolutely continuous norms. Hence, for every $\varepsilon > 0$, we can find $\delta_{\varepsilon} > 0$ such that:

$$m(A) \leq \delta_{\varepsilon} \quad \Rightarrow \quad \|g\mathbb{1}_A\|_{\Phi} \leq \varepsilon/2, \quad \forall g \in T^*(B_{X^*}).$$

But (the factor 1/2 appears because we use the Luxemburg norm on the dual, and not the Orlicz norm: see [16], Proposition III.3.4)

$$\begin{aligned} \sup_{g \in T^*(B_{X^*})} \|g\mathbb{1}_A\|_{\Phi} &\geq \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_{\Psi} \leq 1} | \langle f, (T^*u)\mathbb{1}_A \rangle | \\ &= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_{\Psi} \leq 1} \left| \int f(T^*u)\mathbb{1}_A \, dm \right| \\ &= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_{\Psi} \leq 1} | \langle T(f\mathbb{1}_A), u \rangle | = \frac{1}{2} \sup_{\|f\|_{\Psi} \leq 1} \|T(f\mathbb{1}_A)\|; \end{aligned}$$

so

$$m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \sup_{\|f\|_\Psi \leq 1} \|T(f\mathbf{1}_A)\| \leq \varepsilon.$$

Now, we have

$$m(|f| \geq \|f\|_2/\delta_\varepsilon) \leq \frac{\delta_\varepsilon}{\|f\|_2} \int |f| dm = \frac{\delta_\varepsilon}{\|f\|_2} \|f\|_1 \leq \delta_\varepsilon;$$

hence, with $A = \{|f| \geq \|f\|_2/\delta_\varepsilon\}$, we get, for $\|f\|_\Psi \leq 1$:

$$\|Tf\| \leq \|T(f\mathbf{1}_A)\| + \|T(f\mathbf{1}_{A^c})\| \leq \varepsilon + \|T\| \frac{\|f\|_2}{\delta_\varepsilon}$$

since $|f\mathbf{1}_{A^c}| \leq \|f\|_2/\delta_\varepsilon$ implies $\|f\mathbf{1}_{A^c}\|_\Psi \leq \|f\mathbf{1}_{A^c}\|_\infty \leq \|f\|_2/\delta_\varepsilon$.

Remark 17 Conversely, E. Lavergne ([10]) recently uses our Theorem 4 to give a proof of the above quoted result of J. Alexopoulos ([1], Corollary 2.9), and uses it to show that, when $\Psi \in \Delta^0$, then the reflexive subspaces of L^Φ (where Φ is the conjugate of Ψ) are closed for the L^1 -norm.

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