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# On some random thin sets of integers

Daniel LI – Hervé QUEFFÉLEC – Luis RODRÍGUEZ-PIAZZA

## Abstract

*We show how different random thin sets of integers may have different behaviour. First, using a recent deviation inequality of Boucheron, Lugosi and Massart, we give a simpler proof of one of our results in Some new thin sets of integers in Harmonic Analysis, Journal d'Analyse Mathématique 86 (2002), 105–138, namely that there exist  $\frac{4}{3}$ -Rider sets which are sets of uniform convergence and  $\Lambda(q)$ -sets for all  $q < \infty$ , but which are not Rosenthal sets. In a second part, we show, using an older result of Kashin and Tzafriri that, for  $p > \frac{4}{3}$ , the  $p$ -Rider sets which we had constructed in that paper are almost surely not of uniform convergence.*

**2000 MSC :** *primary : 43 A 46 ; secondary : 42 A 55 ; 42 A 61*

**Key words :** *Boucheron-Lugosi-Massart's deviation inequality;  $\Lambda(q)$ -sets;  $p$ -Rider sets; Rosenthal sets; selectors; sets of uniform convergence*

## 1 Introduction

It is well-known that the Fourier series  $S_n(f, x) = \sum_{-n}^n \hat{f}(k) e^{ikx}$  of a  $2\pi$ -periodic continuous function  $f$  may be badly behaved: for example, it may diverge on a prescribed set of values of  $x$  with measure zero. Similarly, the Fourier series of an integrable function may diverge everywhere. But it is equally well-known that, as soon as the spectrum  $Sp(f)$  of  $f$  (the set of integers  $k$  at which the Fourier coefficients of  $f$  do not vanish, *i.e.*  $\hat{f}(k) \neq 0$ ) is sufficiently “lacunary”, in the sense of Hadamard *e.g.*, then the Fourier series of  $f$  is absolutely convergent if  $f$  is continuous and almost everywhere convergent if  $f$  is merely integrable (and in this latter case  $f \in L^p$  for every  $p < \infty$ ). Those facts have given birth to the theory of thin sets  $\Lambda$  of integers, initiated by Rudin [15]: those sets  $\Lambda$  such that, if  $Sp(f) \subseteq \Lambda$  (we shall write  $f \in \mathcal{B}_\Lambda$  when  $f$  is in some Banach function space  $\mathcal{B}$  contained in  $L^1(\mathbb{T})$  and  $Sp(f) \subseteq \Lambda$ ), then  $S_n(f)$ , or  $f$  itself, is better behaved than in the general case. Let us for example recall that the set  $\Lambda$  is said to be:

- a  *$p$ -Sidon set* ( $1 \leq p < 2$ ) if  $\hat{f} \in l_p$  (and not only  $\hat{f} \in l_2$ ) as soon as  $f$  is continuous and  $Sp(f) \subseteq \Lambda$ ; this amounts to an “*a priori* inequality”  $\|\hat{f}\|_p \leq C \|f\|_\infty$ , for each  $f \in \mathcal{C}_\Lambda$ ; the case  $p = 1$  is the celebrated case of Sidon (= 1-Sidon) sets;
- a  *$p$ -Rider set* ( $1 \leq p < 2$ ) if we have an *a priori* inequality  $\|\hat{f}\|_p \leq C \llbracket f \rrbracket$ , for every trigonometric polynomial with spectrum in  $\Lambda$ ; here  $\llbracket f \rrbracket$  is the so-called

Pisier norm of  $f = \sum \hat{f}(n)e_n$ , where  $e_n(x) = e^{inx}$ , i.e.  $\|f\| = \mathbb{E}\|f_\omega\|_\infty$ , where  $f_\omega = \sum \varepsilon_n(\omega)\hat{f}(n)e_n$ ,  $(\varepsilon_n)$  being an *i.i.d.* sequence of centered,  $\pm 1$ -valued, random variables defined on some probability space (a Rademacher sequence), and where  $\mathbb{E}$  denotes the expectation on that space; this apparently exotic notion (weaker than  $p$ -Sidonicity) turned out to be very useful when Rider [12] reformulated a result of Drury (proved in the course of the result that the union of two Sidon set sets is a Sidon set) under the form: 1-Rider sets and Sidon sets are the same (in spite of some partial results, it is not yet known whether a  $p$ -Rider set is a  $p$ -Sidon set: see [5] however, for a partial result);

- a *set of uniform convergence* (in short a *UC-set*) if the Fourier series of each  $f \in \mathcal{C}_\Lambda$  converges uniformly, which amounts to the inequality  $\|S_n(f)\|_\infty \leq C\|f\|_\infty$ ,  $\forall f \in \mathcal{C}_\Lambda$ ; Sidon sets are *UC*, but the converse is false;
- a  $\Lambda(q)$ -set,  $1 < q < \infty$ , if every  $f \in L_\Lambda^1$  is in fact in  $L^q$ , which amounts to the inequality  $\|f\|_q \leq C_q\|f\|_1$ ,  $\forall f \in L_\Lambda^1$ . Sidon sets are  $\Lambda(q)$  for every  $q < \infty$  (and even  $C_q \leq C\sqrt{q}$ ); the converse is false, except when we require  $C_q \leq C\sqrt{q}$  ([11]);
- a *Rosenthal set* if every  $f \in L_\Lambda^\infty$  is almost everywhere equal to a continuous function. Sidon sets are Rosenthal, but the converse is false.

This theory has long suffered from a severe lack of examples: those examples were always, more or less, sums of Hadamard sets, and in that case the banachic properties of the corresponding  $\mathcal{C}_\Lambda$ -spaces were very rigid. The use of random sets (in the sense of the selectors method) of integers has significantly changed the situation (see [8], and our paper [9]). Let us recall more in detail the notation and setting of our previous work [9]. The method of selectors consists in the following: let  $(\varepsilon_k)_{k \geq 1}$  be a sequence of independent,  $(0, 1)$ -valued random variables, with respective means  $\delta_k$ , defined on a probability space  $\Omega$ , and to which we attach the random set of integers  $\Lambda = \Lambda(\omega)$ ,  $\omega \in \Omega$ , defined by  $\Lambda(\omega) = \{k \geq 1; \varepsilon_k(\omega) = 1\}$ .

The properties of  $\Lambda(\omega)$  of course highly depend on the  $\delta_k$ 's, and roughly speaking the smaller the  $\delta_k$ 's, the better  $\mathcal{C}_\Lambda$ ,  $L_\Lambda^1$ ,  $\dots$ . In [7], and then, in a much deeper way, in [9], relying on a probabilistic result of J. Bourgain on ergodic means, and on a deterministic result of F. Lust-Piquard ([10]) on those ergodic means, we had randomly built new examples of sets  $\Lambda$  of integers which were both: locally thin from the point of view of harmonic analysis (their traces on big segments  $[M_n, M_{n+1}]$  of integers were uniformly Sidon sets); regularly distributed from the point of view of number theory, and therefore globally big from the point of view of Banach space theory, in that the space  $\mathcal{C}_\Lambda$  contained an isomorphic copy of the Banach space  $c_0$  of sequences vanishing at infinity. More precisely, we have constructed subsets  $\Lambda \subseteq \mathbb{N}$  which are thin in the following respects:  $\Lambda$  is a *UC-set*, a  $p$ -Rider set for various  $p \in [1, 2[$ , a  $\Lambda(q)$ -set for every  $q < \infty$ , and large in two respects: the space  $\mathcal{C}_\Lambda$  contains an isomorphic copy of  $c_0$ , and, most often,  $\Lambda$  is dense in the integers equipped with the Bohr topology.

Now, taking  $\delta_k$  bigger and bigger, we had obtained sets  $\Lambda$  which were less and less thin ( $p$ -Sidon for every  $p > 1$ ,  $q$ -Rider, but  $s$ -Rider for no  $s < q$ ,  $s$ -Rider for every  $s > q$ , but not  $q$ -Rider), and, in any case  $\Lambda(q)$  for every  $q < \infty$ , and such

that  $\mathcal{C}_\Lambda$  contains a subspace isomorphic to  $c_0$ . In particular, in Theorem II.7, page 124, and Theorem II.10, page 130, we take respectively  $\delta_k \approx \frac{\log k}{k}$  and  $\delta_k \approx \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$ , where  $\alpha = \frac{2(p-1)}{2-p}$  is an increasing function of  $p \in [1, 2)$ , and which becomes  $\geq 1$  as  $p$  becomes  $\geq 4/3$ . The case  $\delta_k = \frac{1}{k}$  would correspond (randomly) to Sidon sets (*i.e.* 1-Sidon sets).

After the proofs of Theorem II.7 and Theorem II.10, we were asking two questions:

1) (p. 129) Our construction is very complicated and needs a second random construction of a set  $E$  inside the random set  $\Lambda$ . Is it possible to give a simpler proof?

2) (p. 130) In Theorem II.10, can we keep the property for the random set  $\Lambda$  to be a  $UC$ -set, with high probability, when  $\alpha > 1$  (equivalently when  $p > \frac{4}{3}$ )?

The goal of this work is to answer affirmatively the first question (relying on a recent deviation inequality of Boucheron, Lugosi and Massart [1]) and negatively the second one (relying on an older result of Kashin and Tzafriri [3]). This work is accordingly divided into three parts. In Section 2, we prove a (one-sided) concentration inequality for norms of Rademacher sums. In Section 3, we apply the concentration inequality to get a substantially simplified proof of Theorem II.7 in [9]. Finally, in Section 4, we give a (stochastically) negative answer to question 2 when  $p > \frac{4}{3}$ : almost surely,  $\Lambda$  will not be a  $UC$ -set; here, we use the above mentioned result of Kashin and Tzafriri [3] on the non- $UC$  character of big random subsets of integers.

## 2 A one-sided inequality for norms of Rademacher sums

Let  $E$  be a (real or complex) Banach space,  $v_1, \dots, v_n$  be vectors of  $E$ ,  $X_1, \dots, X_n$  be independent, real-valued, centered, random variables, and let  $Z = \left\| \sum_1^n X_j v_j \right\|$ .

If  $|X_j| \leq 1$  *a.s.*, it is well-known (see [6]) that:

$$\mathbb{P}(|Z - \mathbb{E}(Z)| > t) \leq 2 \exp\left(-\frac{t^2}{8 \sum_1^n \|v_j\|^2}\right), \quad \forall t > 0. \quad (2.1)$$

But often, the “strong”  $l_2$ -norm of the  $n$ -tuple  $v = (v_1, \dots, v_n)$ , namely  $\|v\|_{strong} = (\sum_{j=1}^n \|v_j\|^2)^{1/2}$ , is too large for (2.1) to be interesting, and it is advisable to work with the “weak”  $l_2$ -norm of  $v$ , defined by:

$$\sigma = \|v\|_{weak} = \sup_{\varphi \in B_{E^*}} \left( \sum_1^n |\varphi(v_j)|^2 \right)^{1/2} = \sup_{\sum |a_j|^2 \leq 1} \left\| \sum_1^n a_j v_j \right\|, \quad (2.2)$$

where  $B_{E^*}$  denotes the closed unit ball of the dual space  $E^*$ .

If  $(X_j)_j$  is a standard gaussian sequence ( $\mathbb{E} X_j = 0, \mathbb{E} X_j^2 = 1$ ), this is what Maurey and Pisier succeeded in doing, using either the Itô formula or the

rotational invariance of the  $X_j$ 's; they proved the following (see [8], Chapitre 8, Théorème I.4):

$$\mathbb{P}(|Z - \mathbb{E} Z| > t) \leq 2 \exp\left(-\frac{t^2}{C\sigma^2}\right), \quad \forall t > 0, \quad (2.3)$$

where  $\sigma$  is as in (2.2), and  $C$  is a numerical constant, *e.g.*  $C = \pi^2/2$ .

To the best of our knowledge, no inequality as simple and direct as (2.3) is available for non-gaussian (*e.g.* for Rademacher variables) variables, although several more complicated deviation inequalities are known: see *e.g.* [2], [6].

For the applications to Harmonic analysis which we have in view, where we use the so-called “selectors method”, we precisely need an analogue of (2.3), in the non-gaussian, uniformly bounded (and centered) case; we shall prove that at least a one-sided version of (2.3) holds in this case, by showing the following result, which is interesting for itself.

**Theorem 2.1** *With the previous notations, assume that  $|X_j| \leq 1$  a.s. . Then, we have the one-sided estimate:*

$$\mathbb{P}(Z - \mathbb{E} Z > t) \leq \exp\left(-\frac{t^2}{C\sigma^2}\right), \quad \forall t > 0, \quad (2.4)$$

where  $C > 0$  is a numerical constant ( $C = 32$ , for example).

The proof of (2.4) will make use of a recent deviation inequality due to Boucheron, Lugosi and Massart [1]. Before stating this inequality, we need some notation.

Let  $X_1, \dots, X_n$  be independent, real-valued random variables (here, we temporarily forget the assumptions of the previous Theorem), and let  $(X'_1, \dots, X'_n)$  be an independent copy of  $(X_1, \dots, X_n)$ .

If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a given measurable function, we set  $Z = f(X_1, \dots, X_n)$  and  $Z'_i = f(X_1, \dots, X_{i-1}, X'_i, X_{i+1}, \dots, X_n)$ ,  $1 \leq i \leq n$ . With those notations, the Boucheron-Lugosi-Massart Theorem goes as follows:

**Theorem 2.2** *Assume that there is some constant  $a, b \geq 0$ , not both zero, such that:*

$$\sum_{i=1}^n (Z - Z'_i)^2 \mathbf{1}_{(Z > Z'_i)} \leq aZ + b \quad \text{a.s.} \quad (2.5)$$

*Then, we have the following one-sided deviation inequality:*

$$\mathbb{P}(Z > \mathbb{E} Z + t) \leq \exp\left(-\frac{t^2}{4a\mathbb{E} Z + 4b + 2at}\right), \quad \forall t > 0. \quad (2.6)$$

**Proof of Theorem 2.1.** We shall in fact use a very special case of Theorem 2.2, the case when  $a = 0$ ; but, as the three fore-named authors remark, this special case is already very useful, and far from trivial to prove! To prove (2.4), we are going to check that, for  $f(X_1, \dots, X_n) = \|\sum_1^n X_j v_j\| = Z$ , the assumption

(2.5) holds for  $a = 0$  and  $b = 4\sigma^2$ . In fact, fix  $\omega \in \Omega$  and denote by  $I = I_\omega$  the set of indices  $i$  such that  $Z(\omega) > Z'_i(\omega)$ . For simplicity of notation, we assume that the Banach space  $E$  is real. Let  $\varphi = \varphi_\omega \in E^*$  such that  $\|\varphi\| = 1$  and  $Z = \varphi(\sum_{j=1}^n X_j v_j) = \sum_{j=1}^n X_j \varphi(v_j)$ .

For  $i \in I$ , we have  $Z'_i(\omega) = Z'_i \geq \varphi(\sum_{j \neq i} X_j v_j + X'_i v_i)$ , so that  $0 \leq Z - Z'_i \leq \sum_{j=1}^n X_j \varphi(v_j) - \sum_{j \neq i} X_j \varphi(v_j) - X'_i \varphi(v_i) = (X_i - X'_i) \varphi(v_i)$ , implying  $(Z - Z'_i)^2 \leq 4|\varphi(v_i)|^2$ . By summing those inequalities, we get:

$$\begin{aligned} \sum_{i=1}^n (Z - Z'_i)^2 \mathbf{1}_{(Z > Z'_i)} &= \sum_{i \in I} (Z - Z'_i)^2 \leq 4 \sum_{i \in I} |\varphi(v_i)|^2 \leq 4 \sum_{i=1}^n |\varphi(v_i)|^2 \leq 4\sigma^2 \\ &= 0 \cdot Z + 4\sigma^2. \end{aligned}$$

Let us observe the crucial role of the ‘‘conditioning’’  $Z > Z'_i$  when we want to check that (2.5) holds. Now, (2.4) is an immediate consequence of (2.6).  $\square$

### 3 Construction of 4/3-Rider sets

We first recall some notations of [9].  $\Psi_2$  denotes the Orlicz function  $\Psi_2(x) = e^{x^2} - 1$ , and  $\|\cdot\|_{\Psi_2}$  is the corresponding Luxemburg norm. If  $A$  is a finite subset of the integers,  $\Psi_A$  denotes the quantity  $\|\sum_{n \in A} e_n\|_{\Psi_2}$ , where  $e_n(t) = e^{int}$ ,  $t \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$ , and  $\mathbb{T}$  is equipped with its Haar measure  $m$ .  $\Lambda$  will always be a subset of the positive integers  $\mathbb{N}$ . Recall that  $\Lambda$  is *uniformly distributed* if the ergodic means  $A_N(t) = \frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} e_n(t)$  tend to zero as  $N \rightarrow \infty$ , for each  $t \in \mathbb{T}$ ,  $t \neq 0$ . Here,  $\Lambda_N = \Lambda \cap [1, N]$ . If  $\Lambda$  is uniformly distributed,  $\mathcal{C}_\Lambda$  contains  $c_0$ , and if  $\mathcal{C}_\Lambda$  contains  $c_0$ ,  $\Lambda$  cannot be a Rosenthal set (see [9]). According to results of J. Bourgain (see [9]) and F. Lust-Piquard ([10]), respectively, a random set  $\Lambda$  corresponding to selectors of mean  $\delta_k$  with  $k\delta_k \rightarrow \infty$  is almost surely uniformly distributed and if a subset  $E$  of a uniformly distributed set  $\Lambda$  has positive upper density in  $\Lambda$ , *i.e.* if  $\limsup_N \frac{|E \cap [1, N]|}{|\Lambda \cap [1, N]|} > 0$ , then  $\mathcal{C}_E$  contains  $c_0$ , and  $E$  is non-Rosenthal.

In [9], we had given a fairly complicated proof of the following theorem (labelled as Theorem II.7):

**Theorem 3.1** *There exists a subset  $\Lambda$  of the integers, which is uniformly distributed, and contains a subset  $E$  of positive integers with the following properties:*

- 1)  $E$  is a  $\frac{4}{3}$ -Rider set, but is not  $q$ -Rider for  $q < 4/3$ , a UC-set, and a  $\Lambda(q)$ -set for all  $q < \infty$ ;
- 2)  $E$  is of positive upper density inside  $\Lambda$ ; in particular,  $\mathcal{C}_E$  contains  $c_0$  and  $E$  is not a Rosenthal set.

We shall show here that the use of Theorem 2.1 allows a substantially simplified proof, which avoids a double random selection. We first need the following simple lemma.

**Lemma 3.2** *Let  $A$  be a finite subset of the integers, of cardinality  $n \geq 2$ ; let  $v = (e_j)_{j \in A}$ , considered as an  $n$ -tuple of elements of the Banach space  $E = L^{\Psi_2} = L^{\Psi_2}(\mathbb{T}, m)$ , and let  $\sigma$  be its weak  $l_2$ -norm. Then:*

$$\sigma \leq C_0 \sqrt{\frac{n}{\log n}}, \quad (3.1)$$

where  $C_0$  is a numerical constant.

**Proof.** Let  $a = (a_j)_{j \in A}$  be such that  $\sum_{j \in A} |a_j|^2 = 1$ . Let  $f = f_a = \sum_{j \in A} a_j e_j$ , and  $M = \|f\|_\infty$ . By Hölder's inequality, we have  $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{M}{\sqrt{p} M^{2/p}}$  for  $2 < p < \infty$ . Since  $M \leq \sqrt{n}$ , we get  $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{\sqrt{n}}{\sqrt{p} n^{1/p}} \leq C \sqrt{\frac{n}{\log n}}$ . By Stirling's formula,  $\|f\|_{\Psi_2} \approx \sup_{p>2} \frac{\|f\|_p}{\sqrt{p}}$ , so the lemma is proved, since  $\sigma = \sup_a \|f_a\|_{\Psi_2}$   $\square$

We now turn to the shortened proof of Theorem 3.1.

Let  $I_n = [2^n, 2^{n+1}[$ ,  $n \geq 2$ ;  $\delta_k = c \frac{n}{2^n}$  if  $k \in I_n$  ( $c > 0$ ).

Let  $(\varepsilon_k)_k$  be a sequence of "selectors", *i.e.* independent,  $(0, 1)$ -valued, random variables of expectation  $\mathbb{E} \varepsilon_k = \delta_k$ , and let  $\Lambda = \Lambda(\omega)$  be the random set of positive integers defined by  $\Lambda = \{k \geq 1; \varepsilon_k = 1\}$ . We set also  $\Lambda_n = \Lambda \cap I_n$  and  $\sigma_n = \mathbb{E} |\Lambda_n| = \sum_{k \in I_n} \delta_k = cn$ .

We shall now need the following lemma (the notation  $\Psi_A$  is defined at the beginning of the section).

**Lemma 3.3** *Almost surely, for  $n$  large enough:*

$$\frac{c}{2} n \leq |\Lambda_n| \leq 2cn; \quad (3.2)$$

$$\Psi_{\Lambda_n} \leq C'' |\Lambda_n|^{1/2}. \quad (3.3)$$

**Proof:** (3.2) is the easier part of Lemma II.9 in [9]. To prove (3.3), we recall an inequality due to G. Pisier [11]: if  $(X_k)$  is a sequence of independent, centered and square-integrable, random variables of respective variances  $V(X_k)$ , we have:

$$\mathbb{E} \left\| \sum_k X_k e_k \right\|_{\Psi_2} \leq C_1 \left( \sum_k V(X_k) \right)^{1/2}. \quad (3.4)$$

Applying (3.4) to the centered variables  $X_k = \varepsilon_k - \delta_k$ , we get, assuming  $c \leq 1$ :

$$\mathbb{E} \left\| \sum_{k \in I_n} (\varepsilon_k - \delta_k) e_k \right\|_{\Psi_2} \leq C_1 \left( \sum_{k \in I_n} \delta_k (1 - \delta_k) \right)^{1/2} \leq C_1 \left( \sum_{k \in I_n} \delta_k \right)^{1/2} \leq C_1 \sqrt{n}.$$

Now, set  $Z_n = \left\| \sum_{k \in I_n} (\varepsilon_k - \delta_k) e_k \right\|_{\Psi_2}$ . Let  $\lambda$  be a fixed real number  $> 1$ , and  $C_0$  be as in Lemma 3.2. Applying Theorem 2.1 with  $C = 32$ , and  $t_n = \lambda \sqrt{32 C_0^2 n}$ , we get, using Lemma 3.2:

$$\mathbb{P}(Z_n - \mathbb{E} Z_n > t_n) \leq \exp\left(-\frac{t_n^2}{32\sigma^2}\right) \leq \exp\left(-\frac{32\lambda^2 C_0^2 n \log n}{32 C_0^2 n}\right) = n^{-\lambda^2}.$$

By the Borel-Cantelli Lemma, we have almost surely, for  $n$  large enough:

$$Z_n \leq \mathbb{E} Z_n + t_n \leq (C_1 + 4C_0\lambda)\sqrt{n} = C_2\sqrt{n}.$$

For such  $\omega$ 's and  $n$ 's, it follows that:

$$\begin{aligned} \Psi_{\Lambda_n} &= \left\| \sum_{k \in I_n} \varepsilon_k e_k \right\|_{\Psi_2} \leq Z_n + \left\| \sum_{k \in I_n} \delta_k e_k \right\|_{\Psi_2} \leq Z_n + \frac{n}{2^n} \left\| \sum_{k \in I_n} e_k \right\|_{\Psi_2} \\ &\leq C_2\sqrt{n} + \frac{n}{2^n} C_0 \frac{2^n}{\sqrt{\log 2^n}} =: C_3\sqrt{n}, \end{aligned}$$

because, with the notations of Lemma 3.2, we have:

$$\left\| \sum_{k \in I_n} e_k \right\|_{\Psi_2} \leq \sqrt{|I_n|} \sigma \leq 2^{n/2} C_0 \frac{2^{\frac{n}{2}}}{\sqrt{\log 2^n}}.$$

This ends the proof of Lemma 3.3, because we know that  $n \leq \frac{2}{c} |\Lambda_n|$  for large  $n$ , almost surely, and therefore  $\Psi_{\Lambda_n} \leq C_3 \sqrt{\frac{2}{c} |\Lambda_n|} =: c'' |\Lambda_n|^{1/2}$ , *a.s.*  $\square$

We now prove Theorem 3.1 as follows: let us fix a point  $\omega \in \Omega$  in such a way that  $\Lambda = \Lambda(\omega)$  is uniformly distributed and that  $\Lambda_n$  verifies (3.2) and (3.3) for  $n \geq n_0$ ; this is possible from [9] and from Lemma 3.3. We then use a result of the third-named author ([13]), asserting that there is a numerical constant  $\delta > 0$  such that each finite subset  $A$  of  $\mathbb{Z}^*$  contains a quasi-independent subset  $B$  such that  $|B| \geq \delta \left( \frac{|A|}{\Psi_A} \right)^2$  (recall that a subset  $Q$  of  $\mathbb{Z}$  is said to be quasi-independent if, whenever  $n_1, \dots, n_k \in Q$ , the equality  $\sum_{j=1}^k \theta_j n_j = 0$  with  $\theta_j = 0, -1, +1$  holds only when  $\theta_j = 0$  for all  $j$ ). This allows us to select inside each  $\Lambda_n$  a quasi-independent subset  $E_n$  such that:

$$|E_n| \geq \delta \left( \frac{|\Lambda_n|}{\Psi_{\Lambda_n}} \right)^2 \geq \frac{\delta}{c''^2} |\Lambda_n| =: \delta' |\Lambda_n|. \quad (3.5)$$

A combinatorial argument (see [9], p. 128–129) shows that, if  $E = \cup_{n > n_0} E_n$ , then each finite  $A \subset E$  contains a quasi-independent subset  $B \subseteq A$  such that  $|B| \geq \delta |A|^{1/2}$ . By [13],  $E$  is a  $\frac{4}{3}$ -Rider set. The set  $E$  has all the required properties. Indeed, it follows from Lemma 3.2, a) that  $|E \cap [1, N]| \geq \delta (\log N)^2$ . If now  $E$  is  $p$ -Rider, we must have  $|E \cap [1, N]| \leq C (\log N)^{\frac{p}{2-p}}$ ; therefore  $2 \leq \frac{p}{2-p}$ , so  $p \geq 4/3$ . The fact that  $E$  is both  $UC$  and  $\Lambda(q)$  is due to the local character of these notions, and to the fact that the sets  $E \cap [2^n, 2^{n+1}[ = E_n$  are by construction quasi-independent (as detailed in [9]). On the other hand, since each  $E_n$  is approximately proportional to  $\Lambda_n$ ,  $E$  is of positive upper density in  $\Lambda$ . Now  $\Lambda$  is uniformly distributed (by Bourgain's criterion: see [9], p. 115). Therefore, by the result of F. Lust-Piquard ([10], and see Theorem I.9, p. 114 in [9]),  $\mathcal{C}_E$  contains  $c_0$ , which prevents  $E$  from being a Rosenthal set.  $\square$



## 4 $p$ -Rider sets, with $p > 4/3$ , which are not UC-sets

Let  $p \in ]\frac{4}{3}, 2[$ , so that  $\alpha = \frac{2(p-1)}{2-p} > 1$ . As we mentioned in the Introduction, the random set  $\Lambda = \Lambda(\omega)$  of integers in Theorem II.10 of [9] corresponds to selectors  $\varepsilon_k$  with mean  $\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$ . We shall prove the following:

**Theorem 4.1** *The random set  $\Lambda$  corresponding to selectors of mean  $\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$  has almost surely the following properties:*

- a)  $\Lambda$  is  $p$ -Rider, but  $q$ -Rider for no  $q < p$ ;
- b)  $\Lambda$  is  $\Lambda(q)$  for all  $q < \infty$ ;
- c)  $\Lambda$  is uniformly distributed; in particular, it is dense in the Bohr group and  $\mathcal{C}_\Lambda$  contains  $c_0$ ;
- d)  $\Lambda$  is **not** a UC-set.

**Remark.** This supports the conjecture that  $p$ -Rider sets with  $p > 4/3$  are not of the same nature as  $p$ -Rider sets for  $p < 4/3$  (see also [4], Theorem 3.1. and [5]).

The novelty here is *d*), which answers in the negative a question of [9] and we shall mainly concentrate on it, although we shall add some details for *a*), *b*), *c*), since the proof of Theorem II.10 in [9] is too sketchy and contains two small misprints (namely (\*) and (\*\*), p. 130).

Recall that the UC-constant  $U(E)$  of a set  $E$  of positive integers is the smallest constant  $M$  such that  $\|S_N f\|_\infty \leq M \|f\|_\infty$  for every  $f \in \mathcal{C}_E$  and every non-negative integer  $N$ , where  $S_N f = \sum_{-N}^N \hat{f}(k) e_k$ . We shall use the following result of Kashin and Tzafriri [3]:

**Theorem 4.2** *Let  $N \geq 1$  be an integer and  $\varepsilon'_1, \dots, \varepsilon'_N$  be selectors of equal mean  $\delta$ . Set  $\sigma(\omega) = \{k \leq N; \varepsilon'_k(\omega) = 1\}$ . Then:*

$$\mathbb{P} \left( U(\sigma(\omega)) \leq \gamma \log \left( 2 + \frac{\delta N}{\log N} \right) \right) \leq \frac{5}{N^3}, \quad (4.1)$$

where  $\gamma$  is a positive numerical constant.

We now turn to the proof of Theorem 4.1. As in [9], we set, for a fixed  $\beta > \alpha$ :

$$M_n = n^{\beta n}; \quad \Lambda_n = \Lambda \cap [1, n]; \quad \Lambda_n^* = \Lambda \cap [M_n, M_{n+1}[. \quad (4.2)$$

We need the following technical lemma, whose proof is postponed (and is needed only for *a*), *b*), *c*)).

**Lemma 4.3** *We have almost surely for large  $n$*

$$|\Lambda_{M_n}| \approx n^{\alpha+1}; \quad |\Lambda_n^*| \approx n^\alpha. \quad (4.3)$$

Observe that, for  $k \in \Lambda_n^*$ , one has:

$$\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}} \gg \frac{(n \log n)^\alpha}{M_{n+1}(\log n)^{\alpha+1}} = \frac{n^\alpha}{M_{n+1} \log n} =: \frac{q_n}{N_n},$$

where  $N_n = M_{n+1} - M_n$  is the number of elements of the support of  $\Lambda_n^*$  (note that  $N_n \sim M_{n+1}$ ), and where  $q_n$  is such that

$$q_n \approx \frac{n^\alpha}{\log n}. \quad (4.4)$$

We can adjust the constants so as to have  $\delta_k \geq q_n/N_n$  for  $k \in \Lambda_n^*$ . Now, we introduce selectors  $(\varepsilon_k'')$  independent of the  $\varepsilon_j$ 's, of respective means  $\delta_k'' = q_n/(N_n \delta_k)$ . Then the selectors  $\varepsilon_k' = \varepsilon_k \varepsilon_k''$  have means  $\delta_k' = q_n/N_n$  for  $k \in \Lambda_n^*$ , and we have  $\delta_k \geq \delta_k'$  for each  $k \geq 1$ .

Let  $\Lambda' = \{k; \varepsilon_k' = 1\}$  and  $\Lambda_n'^* = \Lambda' \cap [M_n, M_{n+1}[$ . It follows from (4.1) and the fact that  $U(E+a) = U(E)$  for any set  $E$  of positive integers and any non-negative integer  $a$  that:

$$\mathbb{P}\left(U(\Lambda_n'^*) \leq \gamma \log\left(2 + \frac{q_n}{\log N_n}\right)\right) \leq 5N_n^{-3}.$$

By the Borel-Cantelli Lemma, we have almost surely  $U(\Lambda_n'^*) > \gamma \log\left(2 + \frac{q_n}{\log N_n}\right)$  for  $n$  large enough. But we see from (4.3) and (4.2) that:

$$\frac{q_n}{\log N_n} \approx \frac{n^\alpha}{(\log n)(n \log n)} = \frac{n^{\alpha-1}}{(\log n)^2},$$

and this tends to infinity since  $\alpha > 1$ . This shows that  $\Lambda'$  is almost surely non- $UC$ . And due to the construction of the  $\varepsilon_k'$ 's, we have:  $\Lambda \supseteq \Lambda'$  almost surely. This of course implies that  $\Lambda$  is not a  $UC$ -set either (almost surely), ending the proof of  $d$ ) in Theorem 4.1.  $\square$

We now indicate a proof of the lemma. Almost surely,  $|\Lambda_{M_n}|$  behaves for large  $n$  as:

$$\begin{aligned} \mathbb{E}(|\Lambda_{M_n}|) &= \sum_1^{M_n} \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}} \approx \int_{e^2}^{M_n} \frac{(\log t)^\alpha}{t(\log \log t)^{\alpha+1}} dt \\ &= \int_2^{\log M_n} \frac{x^\alpha dx}{(\log x)^{\alpha+1}} \approx \frac{1}{(\log n)^{\alpha+1}} \int_2^{\log M_n} x^\alpha dx \approx \frac{(\log M_n)^{\alpha+1}}{(\log n)^{\alpha+1}} \approx n^{\alpha+1}. \end{aligned}$$

Similarly,  $|\Lambda_n^*|$  behaves almost surely as:

$$\begin{aligned} \int_{M_n}^{M_{n+1}} \frac{(\log t)^\alpha}{t(\log \log t)^{\alpha+1}} dt &= \int_{\log M_n}^{\log M_{n+1}} \frac{x^\alpha}{(\log x)^{\alpha+1}} dx \approx \frac{1}{(\log n)^{\alpha+1}} x^\alpha dx \\ &\approx \frac{1}{(\log n)^{\alpha+1}} (\log M_{n+1} - \log M_n) (\log M_n)^\alpha \\ &\approx \frac{1}{(\log n)^{\alpha+1}} \log n (n \log n)^\alpha \approx n^\alpha. \quad \square \end{aligned}$$

To finish the proof, we shall use a lemma of [9] (recall that a *relation of length*  $n$  in  $A \subseteq \mathbb{Z}^*$  is a  $(-1, 0, +1)$ -valued sequence  $(\theta_k)_{k \in A}$  such that  $\sum_{k \in A} \theta_k k = 0$  and  $\sum_{k \in A} |\theta_k| = n$ ):

**Lemma 4.4** *Let  $n \geq 2$  and  $M$  be integers. Set*

$$\Omega_n(M) = \{\omega \mid \Lambda(\omega) \cap [M, \infty[ \text{ contains at least a relation of length } n\}.$$

*Then:*

$$\mathbb{P}[\Omega_n(M)] \leq \frac{C^n}{n^n} \sum_{j>M} \delta_j^2 \sigma_j^{n-2},$$

*where  $\sigma_j = \delta_1 + \dots + \delta_j$ , and  $C$  is a numerical constant.*

In our case, with  $M = M_n$ , this lemma gives :

$$\begin{aligned} \mathbb{P}[\Omega_n(M)] &\ll \frac{C^n}{n^n} \sum_{j>M} \frac{(\log j)^{2\alpha}}{j^2 (\log \log j)^{2\alpha+2}} \left[ \frac{(\log j)^{\alpha+1}}{(\log \log j)^{\alpha+1}} \right]^{n-2} \\ &\ll \frac{C^n}{n^n} \int_M^\infty \frac{(\log t)^{(\alpha+1)n+2\alpha}}{(\log \log t)^{(\alpha+1)n+2\alpha+2}} \frac{dt}{t^2} \end{aligned}$$

and an integration by parts (see [9], p. 117–118) now gives:

$$\begin{aligned} \mathbb{P}[\Omega_n(M)] &\ll \frac{C^n}{n^n} \frac{1}{M} \frac{(\log M)^{(\alpha+1)n+2\alpha}}{(\log \log M)^{(\alpha+1)n+2\alpha+2}} \\ &\ll \frac{C^n}{n^n} \frac{1}{n^{\beta n}} \frac{(n \log n)^{(\alpha+1)n+2\alpha}}{(\log n)^{(\alpha+1)n+2\alpha+2}} \ll \frac{n^{2\alpha} C^n}{n^{(\beta-\alpha)n} (\log n)^2}; \end{aligned}$$

then the assumption  $\beta > \alpha$  (which reveals its importance here!) shows that  $\sum_n \mathbb{P}[\Omega_n(M_n)] < \infty$ , so that, almost surely  $\Lambda(\omega) \cap [M_n, \infty[$  contains no relation of length  $n$ , for  $n \geq n_0$ . Having this property at our disposal, we prove (exactly as in [9], p. 119–120) that  $\Lambda$  is  $p$ -Rider. It is not  $q$ -Rider for  $q < p$ , because then  $|\Lambda_{M_n}| \ll (\log M_n)^{\frac{q}{2-q}} \ll (n \log n)^{\frac{q}{2-q}}$ , whereas (4.3) of Lemma 4.3 shows that  $|\Lambda_{M_n}| \gg n^{\alpha+1}$ , with  $\alpha + 1 = \frac{p}{2-p} > \frac{q}{2-q}$ . This proves *a*). Conditions *b*), *c*) are clearly explained in [9].  $\square$

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