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Nevanlinna counting function and Carleson function of analytic maps

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Abstract. *We show that the maximal Nevanlinna counting function and the Carleson function of analytic self-maps of the unit disk are equivalent, up to constants.*

Mathematics Subject Classification. Primary: 30C80 – Secondary: 47B33; 47B10

Key-words. Analytic self-map of the unit disk – Carleson function – Carleson measure – composition operator – Nevanlinna counting function

1 Introduction

Carleson measures and the Nevanlinna counting function are two classical concepts in Complex Analysis. Carleson measures emerged in 1958 when L. Carleson ([1], [2]) showed his famous embedding theorem: For any positive finite measure μ on the closed unit disk \mathbb{D} , the identity map from the Hardy space H^2 into $L^2(\mu)$ is bounded if and only if this measure satisfies the following geometric condition: $\sup_{|\xi|=1} \mu[W(\xi, h)] = O(h)$, where $W(\xi, h)$ is the Carleson window of size h centered at ξ . This supremum is called the Carleson function ρ_μ of μ .

If φ is an analytic self-map of \mathbb{D} (such a function is sometimes called a Schur function), φ induces a composition operator $C_\varphi: f \in H^2 \mapsto f \circ \varphi \in H^2$, which may be seen as the identity from H^2 into $L^2(m_\varphi)$, where $\mu = m_\varphi$ is the image of the Lebesgue measure on the unit circle by φ^* , the boundary values function of φ . We say that $\rho_\varphi = \rho_{m_\varphi}$ is the Carleson function of φ .

Nevanlinna counting function traces back earlier, in the thirties of the last century, in connection with the Jensen formula and the Nevanlinna theory of defect ([18] or [16]). It is defined, for $w \in \varphi(\mathbb{D})$ and $w \neq \varphi(0)$, by $N_\varphi(w) = \sum_{\varphi(z)=w} \log 1/|z|$ (see (2.4)).

In a slightly different context, Littlewood used it implicitly ([11], see Theorem 4) when he showed that, for every analytic self-map φ of \mathbb{D} , we have $N_\varphi(z) = O(1 - |z|)$ as $|z| \rightarrow 1$. This turns out to imply ([20], [19]) that the

composition operator $f \mapsto f \circ \varphi = C_\varphi(f)$ is continuous on H^2 (which precisely means, in present language, that m_φ is a Carleson measure).

Later, and till now, the regularity of composition operators C_φ on H^2 (their compactness, or membership in a Schatten class) in terms of their “symbol” φ has been studied either from the point of view of Carleson measures or from the point of view of the Nevanlinna counting function, those two points of view being completely separated. For example, the compactness of $C_\varphi: H^2 \rightarrow H^2$ has been characterized in terms of the Carleson function of the symbol $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, by B. McCluer ([14] – see also [17]). In another paper, it was characterized in terms of the Nevanlinna counting function N_φ of the symbol: $N_\varphi(w) = o(1 - |w|)$, as $|w| \rightarrow 1$, by J. Shapiro ([20]). A similar situation exists for the characterization of the membership of C_φ in a prescribed Schatten class ([12] and [13]).

Though the definition the Carleson measure m_φ and that of the Nevanlinna counting function N_φ are of different nature, there should therefore exist a direct link between these two quantities.

Some results in this direction had been given: B. R. Choe ([4]) showed that $\limsup_{h \rightarrow 0} (\rho_\varphi(h)/h)^{1/2}$ is equivalent, up to constants, to the distance of C_φ to the space of compact operators on H^2 ; since J. Shapiro proved ([20]) that this distance is $\limsup_{|w| \rightarrow 1} (N_\varphi(w)/\log |w|)^{1/2}$, one gets that

$$\limsup_{|w| \rightarrow 1} N_\varphi(w)/\log |w| \approx \limsup_{h \rightarrow 0} \rho_\varphi(h)/h.$$

Later, J. S. Choa and H. O. Kim ([3]) gave a somewhat direct proof of the equivalence of the two above conditions, without using the properties of the composition operator, but without giving explicitly a direct relation between the two functions ρ_φ and N_φ .

The aim of this paper is to show the surprising fact that the Nevanlinna counting function and the Carleson function are actually equivalent, in the following sense:

Theorem 1.1 *There exists a universal constant $C > 1$, such that, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has:*

$$(1.1) \quad (1/C) \rho_\varphi(h/C) \leq \sup_{|w| \geq 1-h} N_\varphi(w) \leq C \rho_\varphi(C h),$$

for $0 < h < 1$ small enough.

More precisely, for every $\xi \in \partial\mathbb{D}$, one has:

$$(1.2) \quad (1/64) m_\varphi[W(\xi, h/64)] \leq \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi[W(\xi, 24 h)],$$

for $0 < h < (1 - |\varphi(0)|)/16$.

Actually the above explicit constants are not relevant and we did not try to have “best” constants. It can be shown that for every $\alpha > 1$, there is a constant $C_\alpha > 0$ such that $m_\varphi(S(\xi, h)) \leq C_\alpha \tilde{\nu}_\varphi(\xi, \alpha h)$ and $\tilde{\nu}_\varphi(\xi, h) \leq C_\alpha m_\varphi(S(\xi, \alpha h))$ for $0 < h < (1 - |\varphi(0)|)/\alpha$, where $S(\xi, h)$ is defined in (2.3) and $\tilde{\nu}(\xi, h) = \sup_{w \in S(\xi, h) \cap \mathbb{D}} N_\varphi(w)$ (see (4.7)).

2 Notation

We shall denote by $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ the open unit disc of the complex plane and by $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ its boundary; m will be the normalized Lebesgue measure $dt/2\pi$ on \mathbb{T} , and A the normalized Lebesgue measure $dxdy/\pi$ on $\overline{\mathbb{D}}$. For every analytic self-map φ of \mathbb{D} , m_φ will be the pull-back measure of m by φ^* , where φ^* is the boundary values function of φ .

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ centered at ξ and of size h is the set

$$(2.1) \quad W(\xi, h) = \{z \in \overline{\mathbb{D}}; |z| \geq 1 - h \quad \text{and} \quad |\arg(z\bar{\xi})| \leq h\}.$$

For convenience, we shall set $W(\xi, h) = \overline{\mathbb{D}}$ for $h \geq 1$.

For every analytic self-map φ of \mathbb{D} , one defines the maximal function of m_φ , for $0 < h < 1$, by:

$$(2.2) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m(\{\zeta \in \mathbb{T}; \varphi^*(\zeta) \in W(\xi, h)\}) = \sup_{\xi \in \mathbb{T}} m_\varphi(W(\xi, h)).$$

We have $\rho_\varphi(h) = 1$ for $h \geq 1$. We shall call this function ρ_φ the *Carleson function* of φ . For convenience, we shall often also use, instead of the Carleson window $W(\xi, h)$, the set

$$(2.3) \quad S(\xi, h) = \{z \in \overline{\mathbb{D}}; |z - \xi| \leq h\},$$

which has an equivalent size.

The Nevanlinna counting function N_φ is defined, for $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, by

$$(2.4) \quad N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

each term $\log \frac{1}{|z|}$ being repeated according to the multiplicity of z , and $N_\varphi(w) = 0$ for the other $w \in \mathbb{D}$. Its maximal function will be denoted by

$$(2.5) \quad \nu_\varphi(t) = \sup_{|w| \geq 1-t} N_\varphi(w).$$

3 Majorizing the Nevanlinna counting function by the Carleson function

The goal of this section is to prove:

Theorem 3.1 For every analytic self-map φ of \mathbb{D} , one has, for every $a \in \mathbb{D}$:

$$(3.1) \quad N_\varphi(a) \leq 196 m_\varphi(W(\xi, 12h)),$$

for $0 < h < (1 - |\varphi(0)|)/4$, where $\xi = \frac{a}{|a|}$ and $h = 1 - |a|$.

In particular, for $0 < h < (1 - |\varphi(0)|)/4$:

$$(3.2) \quad \nu_\varphi(h) = \sup_{|a| \geq 1-h} N_\varphi(a) \leq 196 \rho_\varphi(12h).$$

Let us note that, since $W(\zeta, s) \subseteq W(\zeta, 2t)$ whenever $0 < s \leq t$ and $\zeta \in W(\xi, t) \cap \partial\mathbb{D}$, we get from (3.1) that

$$(3.3) \quad \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi(W(\xi, 24h)).$$

We shall first prove the following lemma.

Lemma 3.2 Let φ be an analytic self map of \mathbb{D} . For every $z \in \mathbb{D}$, one has, if $w = \varphi(z)$, $\xi = w/|w|$ and $h = 1 - |w| \leq 1/4$:

$$(3.4) \quad m_\varphi(W(\xi, 12h)) \geq m_\varphi(S(\xi, 6h)) \geq \frac{|w|}{8} (1 - |z|).$$

Proof. We may assume, by making a rotation, that w is real and positive: $3/4 \leq w < 1$.

Let:

$$(3.5) \quad T(u) = \frac{au + 1}{u + a},$$

where

$$a = w - \frac{2}{w} < -1,$$

so that $T: \mathbb{D} \rightarrow \mathbb{D}$ is analytic, and $T(w) = w/2$.

If P_z is the Poisson kernel at z , one has:

$$\frac{w}{2} = T[\varphi(z)] = \int_{\mathbb{T}} (T \circ \varphi)^* P_z dm = \int_{\mathbb{T}} \Re[(T \circ \varphi)^*] P_z dm.$$

Hence, if one sets:

$$E = \{\Re(T \circ \varphi^*) \geq w/4\} = \{\Re[(T \circ \varphi)^*] \geq w/4\},$$

one has:

$$\frac{w}{2} \leq \int_E P_z dm + \frac{w}{4} \int_{E^c} P_z dm \leq \int_E P_z dm + \frac{w}{4} \int_{\mathbb{D}} P_z dm = \int_E P_z dm + \frac{w}{4};$$

therefore:

$$\int_E P_z dm \geq \frac{w}{4}.$$

Since

$$\|P_z\|_\infty = \frac{1+|z|}{1-|z|} \leq \frac{2}{1-|z|},$$

we get:

$$(3.6) \quad m(E) \geq \frac{w}{8}(1-|z|).$$

On the other hand, (3.5) writes

$$(3.7) \quad u = T^{-1}(U) = \frac{aU-1}{a-U};$$

hence:

$$|1-u| = |a+1| \frac{|1-U|}{|a-U|} \leq \frac{2|a+1|}{|a-U|}.$$

But $a < -1$ is negative, so $\Re U \geq w/4$ implies that

$$|a-U| \geq \Re(U-a) \geq \frac{w}{4} - a = \frac{2}{w} - \frac{3}{4}w \geq \frac{5}{4}.$$

Moreover, for $w \geq 3/4$:

$$|a+1| = (1-w) \left(\frac{2}{w} + 1 \right) \leq \frac{11}{3}(1-w).$$

We get hence $|1-u| \leq 6h$ when (3.7) holds and $\Re U \geq w/4$.

It follows that:

$$(3.8) \quad \varphi^*(E) \subseteq T^{-1}(\{\Re U \geq w/4\}) \subseteq S(1, 6h),$$

giving $m_\varphi(W(1, 12h)) \geq m_\varphi(S(1, 6h)) \geq m(E)$.

Combining this with (3.6), that finishes the proof. \square

Remark. Theorem 3.1 follows immediately when φ is univalent since then, for $|w| \geq 3/4$ and $\varphi(z) = w$:

$$N_\varphi(w) = \log \frac{1}{|z|} \approx (1-|z|) \lesssim m_\varphi(W(1, 12h)).$$

When proving the equivalence between the conditions $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, and $N_\varphi(w) = o(1-|w|)$, as $|w| \rightarrow 1$, J. S. Choa and H. O. Kim proved (see [3], page 112) the following inequality, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every $w \in \mathbb{D}$, close enough to 1:

$$(3.9) \quad N_\varphi(w) \leq \frac{(1-|w|^2)^2}{8|w|^2} \int_{\partial\mathbb{D}} \frac{1}{|1-\bar{w}\varphi(z)|^2} dm(z).$$

This result follows from an Hilbertian method, *viz.* Littlewood-Paley's identity:

$$(3.10) \quad \|f \circ \varphi\|_2^2 = |f \circ \varphi(0)|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w)$$

for every $f \in H^2$. With (3.9), one cannot go beyond the order 2; for instance, we can deduce from (3.9) (see the proof of Theorem 3.1 below), that, for $0 < h \leq 1/2$:

$$(3.11) \quad \sup_{|w|=1-h} N_\varphi(w) \lesssim h^2 \int_0^{1/h^2} \rho_\varphi\left(\frac{1}{\sqrt{t}}\right) dt \lesssim h^2 + h^2 \int_h^1 \frac{\rho_\varphi(u)}{u^3} du.$$

This is of course interesting only when the second term in the last sum is at most of order h^2 , so, when the integral is bounded. Nevertheless, this result suffices to show that Shapiro's criterion of compactness for $C_\varphi: H^2 \rightarrow H^2$ is implied by McCluer's one. Moreover, when the pull-back measure m_φ is an α -Carleson measure (*i.e.* $\rho_\varphi(h) \leq C h^\alpha$ for some constant $C > 0$), with $1 \leq \alpha \leq 2$, we get

$$N_\varphi(w) \lesssim h^2 + h^2 \int_h^1 \frac{u^\alpha}{u^3} du \lesssim h^2 + h^2 h^{\alpha-2} \lesssim h^\alpha.$$

Recall ([8], Corollary 3.2) that, when m_φ is an α -Carleson measure, the composition operator C_φ is in the Schatten class S_p on the Hardy space H^2 , for every $p > 2/(\alpha - 1)$, and that m_φ is α -Carleson for every $\alpha \geq 1$ when $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact, if Ψ is an Orlicz function satisfying the growth condition Δ^2 ([9], Theorem 5.2).

But (3.11) does not suffice for the compactness of $C_\varphi: H^\Psi \rightarrow H^\Psi$ on general Hardy-Orlicz spaces (see [7] or [6]).

In order to prove Theorem 3.1, we shall replace the Littlewood-Paley identity, by a more general formula, deduced from Stanton's formula (see [5], Theorem 2).

Theorem 3.3 (Stanton's formula) *For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every subharmonic function $G: \mathbb{D} \rightarrow \mathbb{R}$, one has:*

$$(3.12) \quad \lim_{r \uparrow 1} \int_{\partial \mathbb{D}} G[\varphi(r\xi)] dm(\xi) = G[\varphi(0)] + \frac{1}{2} \int_{\mathbb{D}} \Delta G(w) N_\varphi(w) dA(w),$$

where Δ is the distributional Laplacian.

Proof of Theorem 3.1. If $a \notin \varphi(\mathbb{D})$, one has $N_\varphi(a) = 0$, and the result is trivial. We shall hence assume that $a \in \varphi(\mathbb{D})$.

Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function, that is a non-decreasing convex function such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, and we assume that Φ' is also an Orlicz function. In other words, Φ'' is an arbitrary non-negative and non-decreasing function and $\Phi'(x) = \int_0^x \Phi''(t) dt$ and $\Phi(x) = \int_0^x \Phi'(t) dt$.

Let now $f: \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. We have, outside the zeroes of f , in writing $\Delta\Phi(|f|) = 4\partial\bar{\partial}\Phi(\sqrt{|f|^2})$:

$$(3.13) \quad \Delta\Phi(|f|) = \left[\Phi''(|f|) + \frac{\Phi'(|f|)}{|f|} \right] |f'|^2.$$

We shall only use here that:

$$(3.14) \quad \Delta\Phi(|f|) \geq \Phi''(|f|) |f'|^2$$

(this is a not too crude estimate, since, Φ' being an Orlicz function, Φ'' is non-negative and non-decreasing, and hence $\Phi'(x) = \int_0^x \Phi''(t) dt \leq x\Phi''(x)$ and $\Phi'(x) = \int_0^x \Phi''(t) dt \geq \int_{x/2}^x \Phi''(t) dt \geq (x/2)\Phi''(x/2)$).

Set now, for $a \in \mathbb{D}$:

$$(3.15) \quad f_a(z) = \frac{1 - |a|}{1 - \bar{a}z}, \quad z \in \bar{\mathbb{D}}.$$

Since $\Phi(|f_a|)$ is subharmonic (Φ being convex and non-decreasing) and bounded, we can use Stanton's formula as:

$$(3.16) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \int_{\mathbb{D}} \Phi''(|f_a|) |f'_a|^2 N_\varphi dA.$$

Let $h = 1 - |a|$. For $|z - a| < h$, one has

$$|1 - \bar{a}z| = |(1 - |a|^2) + \bar{a}(a - z)| \leq (1 - |a|^2) + |a - z| \leq 2h + h = 3h;$$

Hence $|f_a(z)| \geq \frac{h}{3h} = \frac{1}{3}$ for $|z - a| < h$. It follows, since Φ'' is non-decreasing:

$$(3.17) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \int_{D(a,h)} |f'_a|^2 N_\varphi dA.$$

Now, if $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, one has $|f'_a(z)| = \frac{|a|}{1+|a|} |\varphi'_a(z)| \geq \frac{3}{7} |\varphi'_a(z)|$ (we may, and do, assume that $1 - |a| = h \leq 1/4$); hence:

$$\begin{aligned} \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &\geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \frac{9}{49} \int_{D(a,h)} |\varphi'_a|^2 N_\varphi dA \\ &= \frac{9}{98} \Phi''\left(\frac{1}{3}\right) \int_{\varphi_a(D(a,h))} N_{\varphi_a \circ \varphi} dA \end{aligned}$$

(because $N_{\varphi_a \circ \varphi}(\varphi_a(w)) = N_\varphi(w)$ and $\varphi_a^{-1} = \varphi_a$).

But $\varphi_a(D(a, h)) \supseteq D(0, 1/3)$: indeed, if $|w| < 1/3$, then $w = \varphi_a(z)$, with

$$|a - z| = \left| \frac{(1 - |a|^2)w}{1 - \bar{a}w} \right| \leq (1 - |a|^2) \frac{|w|}{1 - |w|} < 2h \frac{1/3}{1 - 1/3} = h.$$

We are going now to use the sub-averaging property of the Nevanlinna function ([19], page 190, [20], § 4.6, or [21], Proposition 10.2.4): for every analytic self-map $\psi: \mathbb{D} \rightarrow \mathbb{D}$, one has

$$N_\psi(w_0) \leq \frac{1}{A(\Delta)} \int_\Delta N_\psi(w) dA(w),$$

for every disk Δ of center w_0 which does not contain $\psi(0)$.

This will be possible thanks to the following:

Lemma 3.4 *For $1 - |a| < (1 - |\varphi(0)|)/4$, one has $|(\varphi_a \circ \varphi)(0)| > 1/3$.*

Proof. One has $|1 - \bar{a}\varphi(0)| \leq (1 - |a|^2) + |\bar{a}| |a - \varphi(0)| \leq (1 - |a|^2) + |a - \varphi(0)|$; hence:

$$\begin{aligned} |\varphi_a(\varphi(0))| &\geq \frac{|a - \varphi(0)|}{(1 - |a|^2) + |a - \varphi(0)|} \geq 1 - \frac{1 - |a|^2}{(1 - |a|^2) + |a - \varphi(0)|} \\ &\geq 1 - \frac{1 - |a|^2}{|a - \varphi(0)|} \geq 1 - 2 \frac{1 - |a|}{|a - \varphi(0)|}. \end{aligned}$$

But when $1 - |a| < (1 - |\varphi(0)|)/4$, one has:

$$|a - \varphi(0)| \geq |a| - |\varphi(0)| = (1 - |\varphi(0)|) - (1 - |a|) > 3(1 - |a|),$$

and the result follows. \square

Hence:

$$\int_{D(0,1/3)} N_{\varphi_a \circ \varphi} dA \geq \frac{1}{9} N_{\varphi_a \circ \varphi}(0) = \frac{1}{9} N_\varphi(a),$$

and

$$(3.18) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{98} \Phi''\left(\frac{1}{3}\right) N_\varphi(a).$$

We now have to estimate from above $\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm$. For that, we shall use the following easy lemma.

Lemma 3.5 *For every $\xi \in \partial\mathbb{D}$ and every $h \in (0, 1/2]$, one has:*

$$(3.19) \quad |1 - \bar{a}z|^2 \geq \frac{1}{4}(h^2 + |z - \xi|^2), \quad \forall z \in \overline{\mathbb{D}},$$

where $a = (1 - h)\xi$.

Proof. The result is rotation-invariant; so we may assume that $\xi = 1$ (and hence $a > 0$). Write $z = 1 - re^{i\theta}$. Since $|z| \leq 1$ if and only if $r \leq 2 \cos \theta$, one has $\cos \theta \geq 0$ and hence $|\theta| \leq \pi/2$. Then:

$$\begin{aligned} |1 - \bar{a}z|^2 &= |1 - a(1 - re^{i\theta})|^2 = |1 - a + are^{i\theta}|^2 \\ &= (1 - a)^2 + a^2 r^2 + 2ar(1 - a) \cos \theta \\ &\geq (1 - a)^2 + a^2 r^2 \geq \frac{1}{4}(h^2 + r^2) = \frac{1}{4}(h^2 + |z - 1|^2). \end{aligned} \quad \square$$

Then:

$$\begin{aligned}
\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &= \int_{\mathbb{D}} \Phi\left(\frac{1-|a|}{|1-\bar{a}z|}\right) dm_\varphi(z) \\
&\leq \int_{\mathbb{D}} \Phi\left(\frac{2h}{(h^2+|z-\xi|^2)^{1/2}}\right) dm_\varphi(z), \quad \text{by (3.19)} \\
&= \int_0^{+\infty} m_\varphi\left(\Phi\left(\frac{2h}{(h^2+|z-\xi|^2)^{1/2}}\right) \geq t\right) dt \\
&= \int_0^{+\infty} m_\varphi((h^2+|z-\xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt \\
&= \int_0^{\Phi(2)} m_\varphi((h^2+|z-\xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt,
\end{aligned}$$

since $h \leq (h^2 + |z - \xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)$ implies $t \leq \Phi(2)$. We get:

$$\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \leq \int_0^{\Phi(2)} m_\varphi(|z - \xi| \leq 2h/\Phi^{-1}(t)) dt.$$

We obtain from (3.18), by setting $u = 2h/\Phi^{-1}(t)$:

$$N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{2h}{u^2} \Phi'\left(\frac{2h}{u}\right) du.$$

Since $\Phi'(x) \leq x\Phi''(x)$, we get:

$$(3.20) \quad N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi''\left(\frac{2h}{u}\right) du.$$

We are going now to choose suitably the Orlicz function Φ . It suffices to define Φ'' , for $a \in \mathbb{D}$ given (with $\xi = a/|a|$ and $h = 1 - |a| \leq 1/4$). By Lemma 3.2, since $a \in \varphi(\mathbb{D})$, there is a constant $c_0 > 0$, such that $m_\varphi(S(\xi, c_0h)) > 0$; we can hence set (note that $m_\varphi(S(\xi, u)) \leq 1$):

$$(3.21) \quad \Phi''(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq h, \\ \frac{1}{m_\varphi(S(\xi, 2h/v))} & \text{if } h \leq v \leq 2/c_0, \\ \frac{1}{m_\varphi(S(\xi, c_0h))} & \text{if } v \geq 2/c_0. \end{cases}$$

It is a non-negative non-decreasing function, so the assumptions made on Φ at the beginning are satisfied. One has, since $m_\varphi(S(\xi, u)) \Phi''(2h/u) \leq 1$:

$$\int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi''\left(\frac{2h}{u}\right) du \leq \int_h^\infty \frac{4h^2}{u^3} du = 2.$$

Since $c_0 \leq 6$, one has $h \leq 1/3 \leq 2/c_0$ and hence $\Phi''(1/3) = 1/m_\varphi(S(\xi, 6h))$; therefore (3.20) gives, for $h \leq (1 - |\varphi(0)|)/4$:

$$(3.22) \quad N_\varphi(a) \leq 196 m_\varphi(S(\xi, 6h)),$$

finishing the proof since $S(\xi, 6h) \subseteq W(\xi, 12h)$. \square

4 Domination of the Carleson function by the Nevanlinna function

We cannot expect to estimate individually from above the m_φ -measure of Carleson windows centered at $\xi = w/|w|$ by $N_\varphi(w)$, as in Theorem 3.1. In fact, consider a conformal mapping φ from \mathbb{D} onto $\mathbb{D} \setminus [0, 1[$. One has $N_\varphi(t) = 0$ for every $t \in [0, 1[$, though $m_\varphi(W(1, h)) > 0$ for every $h > 0$ (because $W(1, h) \supset W(e^{i\theta/2}, h/2)$ and $m_\varphi(W(e^{i\theta/2}, h/2)) > 0$ by Lemma 3.2).

Let us give another example. Let $\varphi(z) = (1+z)/2$. Then:

a) One has $\varphi(e^{i\theta}) = (\cos(\theta/2) + i\sin(\theta/2))e^{i\theta/2}$ (with $|\theta| \leq \pi$). Hence $\varphi(e^{i\theta}) \in W(e^{i\theta_0}, h)$ if and only if $\cos(\theta/2) \geq 1-h$ and $|(\theta/2) - \theta_0| \leq h$, *i.e.* $2(\theta_0 - h) \leq \theta \leq 2(\theta_0 + h)$.

Now, $1 - \cos(\theta/2) \leq \theta^2/8$, so the modulus condition is satisfied when $\theta^2 \leq 8h$; in particular when $|\theta| \leq 2\sqrt{h}$.

For $\theta_0 = \sqrt{h}$, $m_\varphi(W(e^{i\theta_0}, h))$ is bigger than the length of the interval

$$[-2\sqrt{h}, 2\sqrt{h}] \cap [2(\sqrt{h} - h), 2(\sqrt{h} + h)] = [2\sqrt{h} - 2h, 2\sqrt{h}],$$

that is $2h$. Therefore $m_\varphi(W(e^{i\theta_0}, h)) \geq 2h$.

b) Let now $w = \varphi(z)$. Write $w = \frac{1}{2} + r e^{i\zeta}$ with $0 \leq r < 1/2$. Then, writing $r = \frac{1}{2} - s$, one has $|z| = |2w - 1| = 2r$ and

$$N_\varphi(w) = \log \frac{1}{|z|} = \log \frac{1}{2r} = \log \frac{1}{1-2s} \approx s.$$

Now, $|w|^2 = \frac{1}{4} + r^2 + r \cos \zeta$ and

$$h \approx 1 - |w|^2 = \frac{1}{2}(1 - \cos \zeta) + s(1 + \cos \zeta) - s^2 \approx \frac{\zeta^2}{4} + 2s.$$

Writing $\zeta = s^{1/2\alpha}$, one gets:

- (i) for “small” ζ (*i.e.* $0 < \alpha \leq 1$): $h \approx s$, and so $N_\varphi(w) \approx h$;
- (ii) for “large” (*i.e.* $\alpha \geq 1$): $h \approx s^{1/\alpha}$, and so $N_\varphi(w) \approx h^\alpha$.

On the other hand, $w = e^{i\zeta/2}[(1-s)\cos(\zeta/2) - is\sin(\zeta/2)]$; hence, when s goes to 0, one has

$$\theta_w := \arg w = \frac{\zeta}{2} + \arctan \left[\frac{s \sin(\zeta/2)}{(1-s)\cos(\zeta/2)} \right] \sim \frac{\zeta}{2} \approx \zeta.$$

For $\alpha \geq 1$, one has $h \approx s^{1/\alpha} = \zeta^2$, *i.e.* $\zeta \approx \sqrt{h}$. Then, choosing $\alpha > 1$ such that $\zeta = \theta_0$, one has $m_\varphi(W(w/|w|, h)) \approx h$, though $N_\varphi(w) \approx h^\alpha \ll h$.

One cannot hence dominate $m_\varphi(W(w/|w|, h))$ by $N_\varphi(w)$.

We can remark that, nevertheless, in either case, one has $\rho_\varphi(h) \approx h$ and $\nu_\varphi(h) \approx h$.

We shall prove:

Theorem 4.1 For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has, for every $\xi \in \partial\mathbb{D}$:

$$(4.1) \quad m_\varphi(W(\xi, h)) \leq 64 \sup_{w \in W(\xi, 64h) \cap \mathbb{D}} N_\varphi(w),$$

for $0 < h < (1 - |\varphi(0)|)/16$.

Proof. We shall set:

$$(4.2) \quad \nu_\varphi(\xi, h) = \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w).$$

Note that

$$\nu_\varphi(h) = \sup_{|\xi|=1} \nu_\varphi(\xi, h),$$

where ν_φ is defined in (2.5)

If for some $h_0 > 0$, one has $\nu_\varphi(\xi, h_0) = 0$, then $\varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus W(\xi, h_0)$, and hence $m_\varphi(W(\xi, h)) = 0$ for $0 < h < h_0$. Therefore we shall assume that $\nu_\varphi(\xi, h) > 0$. We may, and do, also assume that $h \leq 1/4$. By replacing φ by $e^{i\theta}\varphi$, it suffices to estimate $m_\varphi(S(1, h))$ (recall that $S(1, t) = \{z \in \overline{\mathbb{D}}; |1 - z| \leq t\}$).

We shall use the same functions f_a as in the proof of Theorem 3.1, but, for convenience, with a different notation. We set, for $0 < r < 1$:

$$(4.3) \quad u(z) = \frac{1-r}{1-rz}.$$

Let us take an Orlicz function Φ as in the beginning of the proof of Theorem 3.1, which will be precised later. We shall take this function in such a way that $\Phi(|u(\varphi(0))|) = 0$.

Since $\Phi'(x) \leq x\Phi''(x)$, (3.13) becomes:

$$(4.4) \quad \Delta\Phi(|u|) \leq 2\Phi''(|u|)|u'|^2,$$

and Stanton's formula writes, since $\Phi(|u(\varphi(0))|) = 0$:

$$(4.5) \quad \int_{\partial\mathbb{D}} \Phi(|u \circ \varphi|) dm \leq \int_{\mathbb{D}} \Phi''(|u(w)|)|u'(w)|^2 N_\varphi(w) dA(w).$$

In all the sequel, we shall fix h , $0 < h \leq 1/4$, and take $r = 1 - h$.

For $|z| \leq 1$ and $|1 - z| \leq h$, one has $|1 - rz| = |(1 - z) + hz| \leq |1 - z| + h \leq 2h$, so:

$$|u(z)| \geq \frac{(1-r)}{2h} = \frac{1}{2}.$$

Hence:

$$\begin{aligned} m_\varphi(S(1, h)) &\leq \frac{1}{\Phi(1/2)} \int_{S(1, h)} \Phi(|u(z)|) dm_\varphi(z) \\ &\leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi(|u(z)|) dm_\varphi(z) \\ &= \frac{1}{\Phi(1/2)} \int_{\mathbb{T}} \Phi(|(u \circ \varphi)(z)|) dm(z), \end{aligned}$$

and so, by (4.5):

$$(4.6) \quad m_\varphi(S(1, h)) \leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z).$$

We are going to estimate this integral by separating two cases: $|1 - z| \leq h$ and $|1 - z| > h$.

For convenience, we shall set:

$$(4.7) \quad \tilde{\nu}(t) = \sup_{w \in S(1, t) \cap \mathbb{D}} N_\varphi(w).$$

1) Remark first that

$$u'(z) = \frac{rh}{(1 - rz)^2},$$

and so:

$$|u'(z)| \leq \frac{h}{(1 - r)^2} = \frac{1}{h}.$$

Since $|u(z)| \leq 1$, we get hence:

$$\int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1, h)} \Phi''(1) \frac{1}{h^2} \tilde{\nu}(h) dA(z),$$

giving, since $A(S(1, h)) \leq h^2$:

$$(4.8) \quad \int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \Phi''(1) \tilde{\nu}(h).$$

2) For $0 < h \leq 1/4$, one has:

$$|u(z)| \leq \frac{2h}{|1 - z|} \quad \text{and} \quad |u'(z)| \leq \frac{2h}{|1 - z|^2};$$

indeed, we have (this is obvious, by drawing a picture):

$$|1 - rz| = r \left| \frac{1}{r} - z \right| \geq r |1 - z|,$$

and hence $|1 - rz| \geq \frac{3}{4} |1 - z|$, since $r = 1 - h \geq 3/4$. We obtain:

$$\begin{aligned} \int_{|1-z| > h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \int_{|1-z| > h} \Phi''\left(\frac{2h}{|1-z|}\right) \frac{h^2}{|1-z|^4} N_\varphi(z) dA(z). \end{aligned}$$

Then, using polar coordinates centered at 1 (note that we only have to integrate over an arc of length less than π), and the obvious inequality $N_\varphi(z) \leq$

$\tilde{\nu}(|1-z|)$, we get:

$$(4.9) \quad \int_{|1-z|>h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq 4 \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt.$$

We now choose the Orlicz function as follows (with $a = \varphi(0)$):

$$(4.10) \quad \Phi''(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq h/(1-|a|), \\ \frac{1}{\tilde{\nu}(2h/v)} & \text{if } h/(1-|a|) < v < 2, \\ \frac{1}{\tilde{\nu}(h)} & \text{if } v \geq 2. \end{cases}$$

This function is non-negative and non-decreasing. Moreover, one has $\Phi(x) = 0$ for $0 \leq x \leq h/(1-|a|)$. Hence, since $|u(a)| \leq \frac{h}{1-|a|}$, one has $\Phi(|u(a)|) = 0$.

Then

$$(4.11) \quad \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt = \int_h^{2(1-|a|)} \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt \leq \int_h^\infty \frac{h^2}{t^3} dt = \frac{1}{2}.$$

Now,

$$\begin{aligned} \Phi\left(\frac{1}{2}\right) &= \int_0^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \frac{t}{2} \Phi''\left(\frac{t}{2}\right) dt \\ &\geq \Phi''\left(\frac{1}{8}\right) \int_{1/4}^{1/2} \frac{t}{2} dt = \frac{3}{64} \Phi''\left(\frac{1}{8}\right). \end{aligned}$$

When $h < (1-|a|)/8$, one has $1/8 > h/(1-|a|)$; hence $\Phi''(1/8) = 1/\tilde{\nu}(16h)$, and $\Phi''(1) = 1/\tilde{\nu}(2h)$. We get hence, from (4.6), (4.8), (4.9) and (4.11):

$$(4.12) \quad m_\varphi(S(1, h)) \leq \frac{64}{3} \tilde{\nu}(16h) \left[\frac{\tilde{\nu}(h)}{\tilde{\nu}(2h)} + 2 \right] \leq 64 \tilde{\nu}(16h).$$

Since $W(1, t) \subseteq S(1, 2t)$, we get $m_\varphi(W(1, h)) \leq 64 \sup_{w \in S(1, 32h)} N_\varphi(w)$ for $0 < h < (1-|\varphi(0)|)/16$, and that ends the proof of Theorem 4.1, since $S(1, 32h) \subseteq W(1, 64h)$. \square

Remark. A slight modification of the proof gives the following improvement, if one allows a (much) bigger constant.

Theorem 4.2 *There are universal constants $C, c > 1$ such that*

$$m_\varphi(S(\xi, h)) \leq C \frac{1}{A(S(\xi, ch))} \int_{S(\xi, ch)} N_\varphi(z) dA(z)$$

for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, every $\xi \in \partial\mathbb{D}$, and $0 < h < (1 - |\varphi(0)|)/8$.

Proof. We are going to follow the proof of Theorem 4.1. We shall assume that $\xi = 1$ and we set:

$$(4.13) \quad I(t) = \int_{S(1, t)} N_\varphi(z) dA(z).$$

Then:

1) When $|1 - z| < h$, we have, instead of (4.8):

$$(4.14) \quad \int_{|1-z|<h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1, h)} \Phi''(1) \frac{1}{h^2} N_\varphi(z) dA(z) \\ = \Phi''(1) \frac{1}{h^2} I(h).$$

2) For $|z - 1| \geq h$, we write:

$$\int_{|1-z|\geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ = \sum_{k=1}^{\infty} \int_{kh \leq |1-z| < (k+1)h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2h}{kh}\right) \frac{h^2}{k^4 h^4} I((k+1)h) \\ = 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2}{k}\right) \frac{1}{k^4 h^2} I((k+1)h).$$

We take, with $a = \varphi(0)$:

$$(4.15) \quad \Phi''(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq h/(1 - |a|), \\ \frac{1}{I\left(\left(\frac{2}{v} + 1\right)h\right)} & \text{if } v > h/(1 - |a|). \end{cases}$$

Then

$$(4.16) \quad \int_{|1-z|\geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \frac{4}{h^2} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{4}{h^2} \frac{\pi^4}{90} \leq \frac{5}{h^2}.$$

Since $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\Phi''(1/8) = \frac{1}{I(17h)}$ and

$\Phi''(1) = \frac{1}{I(3h)}$. Therefore:

$$\begin{aligned} m_\varphi(S(1, h)) &\leq \frac{64}{3} I(17h) \left[\frac{1}{h^2} \frac{I(h)}{I(3h)} + \frac{5}{h^2} \right] \\ &\leq \frac{64}{3} I(17h) \frac{6}{h^2} = 128 \frac{I(17h)}{h^2} \\ &\leq 128 \times 17^2 \frac{I(17h)}{A(S(1, 17h))}, \end{aligned}$$

ending the proof of Theorem 4.2. \square

5 Some consequences

In [7] (see also [6], Théorème 4.2), we proved (Theorem 4.19) that the Carleson function of an analytic self-map φ has the following property of homogeneity, improving that m_φ is a Carleson measure: $m_\varphi(S(\xi, \varepsilon h)) \leq K \varepsilon m_\varphi(S(\xi, h))$ for $0 < h < 1 - |\varphi(0)|$, $0 < \varepsilon < 1$ and $\xi \in \partial\mathbb{D}$, where K is a universal constant. It follows from Theorem 1.1, (actually Theorem 3.1 and Theorem 4.1) that:

Theorem 5.1 *There exist a universal constant $K > 0$ such that, for every analytic self-map φ of \mathbb{D} , one has, for $0 < \varepsilon < 1$:*

$$(5.1) \quad \nu_\varphi(\varepsilon t) \leq K \varepsilon \nu_\varphi(t),$$

for t small enough.

More precisely, for t small enough, one has, for every $\xi \in \partial\mathbb{D}$:

$$(5.2) \quad \nu_\varphi(\xi, \varepsilon t) \leq K \varepsilon \nu_\varphi(\xi, t),$$

where $\nu_\varphi(\xi, s) = \sup_{w \in W(\xi, s) \cap \mathbb{D}} N_\varphi(w)$.

Note that the two above quoted theorems give Theorem 5.1 *a priori* only for $0 < \varepsilon < 1/K$; but if $1/K \leq \varepsilon < 1$, one has $\nu_\varphi(\xi, \varepsilon t) \leq \nu_\varphi(\xi, t) \leq K \varepsilon \nu_\varphi(\xi, t)$.

We shall end this paper with a consequences of Theorem 1.1 for composition operators. Recall that if Ψ is an Orlicz function, the Hardy-Orlicz space is the space of functions $f \in H^1$ whose boundary values are in the Orlicz space $L^\Psi(\partial\mathbb{D}, m)$. We proved in [7], Theorem 4.18 (see also [6], Théorème 4.2) that, if $\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty$, the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if, for every $A > 0$, one has $\rho_\varphi(h) = o[1/\Psi(A\Psi^{-1}(1/h))]$ when h goes to 0; in other words, if and only if

$$\lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} = 0.$$

This remains true when $H^\Psi = H^1$. Hence Theorem 1.1 gives:

Theorem 5.2 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map and Ψ be an Orlicz function. Then the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if*

$$(5.3) \quad \sup_{|w| \geq 1-h} N_\varphi(w) = o\left(\frac{1}{\Psi(A\Psi^{-1}(1/h))}\right), \quad \text{as } h \rightarrow 0, \quad \forall A > 0.$$

It should be noted, due to the arbitrary $A > 0$, that (5.3) may be replaced by

$$(5.4) \quad \sup_{|w| \geq 1-h} N_\varphi(w) \leq \frac{1}{\Psi(A\Psi^{-1}(1/h))}, \quad \forall A > 0,$$

for $h \leq h_A$, and this condition also writes, setting $\nu_\varphi(h) = \sup_{|w| \geq 1-h} N_\varphi(w)$ (see (2.5)):

$$(5.5) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\nu_\varphi(h))} = 0.$$

It is known that if $C_\varphi: H^2 \rightarrow H^2$ is compact, then $\lim_{|z| \rightarrow 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty$, and that this condition is sufficient when φ is univalent, or finitely-valent, but not sufficient in general (see [15] and [19], § 3.2). It follows from Theorem 5.2 that an analogous result holds for Hardy-Orlicz spaces:

Theorem 5.3 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map, and Ψ be an Orlicz function. Assume that the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact. Then:*

$$(5.6) \quad \lim_{|z| \rightarrow 1} \frac{\Psi^{-1}\left(\frac{1}{1-|z|}\right)}{\Psi^{-1}\left(\frac{1}{1-|\varphi(z)|}\right)} = \infty.$$

Conversely, if φ is finitely-valent, then (5.6) suffices for $C_\varphi: H^\Psi \rightarrow H^\Psi$ to be compact.

Recall that the assumption “ φ is finitely-valent” means that there is an integer $p \geq 1$ such that each $w \in \varphi(\mathbb{D})$ is the image by φ of at most p elements of \mathbb{D} .

Proof. To get the necessity, we could use Theorem 5.2 and the fact that $1 - |z| \leq \log \frac{1}{|z|} \leq N_\varphi(\varphi(z))$; but we shall give a more elementary proof. Let HM^Ψ be the closure of H^∞ in H^Ψ . Since $C_\varphi(H^\infty) \subseteq H^\infty$, C_φ maps HM^Ψ into itself and $C_\varphi: H^\Psi \rightarrow H^\Psi$ being compact, its restriction $C_\varphi: HM^\Psi \rightarrow HM^\Psi$ is compact too. We know that the evaluation $\delta_a: f \in HM^\Psi \mapsto f(a) \in \mathbb{C}$ has norm $\approx \Psi^{-1}\left(\frac{1}{1-|a|}\right)$ ([7], Lemma 3.11); hence $\delta_a/\|\delta_a\| \xrightarrow{|a| \rightarrow 1} 0$ weak-star (because

$|\delta_a(f)| = |f(a)| \leq \|f\|_\infty$ for $f \in H^\infty$). If C_φ is compact, its adjoint C_φ^* also; we get hence $\|C_\varphi^*(\delta_a/\|\delta_a\|)\| \xrightarrow{|a| \rightarrow 1} 0$. But $C_\varphi^*\delta_a = \delta_{\varphi(a)}$. Therefore

$$\frac{\Psi^{-1}\left(\frac{1}{1-|\varphi(a)|}\right)}{\Psi^{-1}\left(\frac{1}{1-|a|}\right)} \xrightarrow{|a| \rightarrow 1} 0.$$

Conversely, assume that (5.6) holds. For every $A > 0$, one has, for $|z|$ close enough to 1: $\Psi^{-1}\left(\frac{1}{1-|z|}\right) \geq A \Psi^{-1}\left(\frac{1}{1-|\varphi(z)|}\right)$; in other words, one has: $1/\Psi(A\Psi^{-1}(1/1-|\varphi(z)|)) \geq 1-|z|$. But, when φ is p -valent, and if $w = \varphi(z)$ with $|z| > 0$ minimal, one has $N_\varphi(w) \leq p \log \frac{1}{|z|} \approx 1-|z|$. Since $|z| \rightarrow 1$ when $|w| = |\varphi(z)| \rightarrow 1$ (otherwise, we should have a sequence (z_n) converging to some $z_0 \in \mathbb{D}$ and $\varphi(z_n)$ would converge to $\varphi(z_0) \in \mathbb{D}$), we get $\sup_{|w| \geq 1-h} N_\varphi(w) \lesssim 1/\Psi(A\Psi^{-1}(1/1-|w|)) \leq 1/\Psi(A\Psi^{-1}(1/1-h))$, for h small enough. By Theorem 5.2, with (5.4), that means that C_φ is compact on H^Ψ . \square

Other consequences will be given in the subsequent paper [10].

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