



HAL
open science

Nevanlinna counting function and Carleson function of analytic maps

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza

► **To cite this version:**

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza. Nevanlinna counting function and Carleson function of analytic maps. 2009. hal-00375955v2

HAL Id: hal-00375955

<https://hal.science/hal-00375955v2>

Preprint submitted on 9 Jul 2009 (v2), last revised 17 Dec 2009 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Nevanlinna counting function and Carleson function of analytic maps

*Pascal Lefèvre, Daniel Li,
Hervé Queffélec, Luis Rodríguez-Piazza*

July 9, 2009

Abstract. *We show that the maximal Nevanlinna counting function and the Carleson function of analytic self-maps of the unit disk are equivalent, up to constants. We give some applications to composition operators on Hardy spaces and on Hardy-Orlicz spaces.*

Mathematics Subject Classification. Primary: 47B33 – Secondary: 30C80; 47B10

Key-words. Blaschke product – Carleson function – Carleson measure – composition operator – Hardy-Orlicz spaces – Nevanlinna counting function – Schatten classes

1 Introduction

Carleson measures and the Nevanlinna counting function are two classical concepts in complex analysis. Carleson measures emerged (under the initial form $\mu = \sum_{n=1}^{\infty} (1 - |z_n|) \delta_{z_n}$) when L. Carleson ([1], [2]) tried to characterize the interpolation sequences $(z_n)_{n \geq 1}$ of the open unit disk \mathbb{D} , and showed at this occasion his famous *embedding theorem*: For any positive finite measure μ on the closed unit disk $\overline{\mathbb{D}}$, the following two conditions, analytic and geometric, are equivalent:

- (1) There exists a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \leq C \|f\|_{H^p}$, for every f in the Hardy space H^p , for some (or all) $0 < p < \infty$;
- (2) The measure μ satisfies the geometric condition $\sup_{|\xi|=1} \mu[W(\xi, h)] = O(h)$, where $W(\xi, h)$ is the *Carleson window* of size h centered at ξ .

A measure μ satisfying one of those two equivalent properties is now called a *Carleson measure* and the supremum in (2) is called the *Carleson function* ρ_μ of μ . Later, it was realized that a quite different type of Carleson measures naturally appears in the context of composition operators: If φ is an analytic self-map of \mathbb{D} and $\mu = m_\varphi$ is the image by φ^* , the boundary values function of φ , of the normalized Lebesgue measure on the circle, then μ is a Carleson measure, whatever φ is. One says then that $\rho_\varphi = \rho_\mu$ is the *Carleson function* of φ .

The Nevanlinna counting function N_φ traces back earlier, in connection with the Jensen formula and the Nevanlinna theory of defect ([22] or [18]). In a slightly different context, Littlewood used it implicitly ([13], see Theorem 4) when he showed that, for every analytic self-map φ of \mathbb{D} , we have $N_\varphi(z) = O(1 - |z|)$ as $|z| \rightarrow 1$. This turns out to imply ([25], [24]) that the composition operator $f \mapsto f \circ \varphi = C_\varphi(f)$ is continuous on H^p (which precisely means, in present language, that m_φ is a Carleson measure).

Later, and till now, the regularity of composition operators C_φ on H^2 (their compactness, or membership in a Schatten class) in terms of their “symbol” φ has been studied either from the point of view of Carleson measures or from the point of view of the Nevanlinna counting function, those two points of view being completely separated. For example, the compactness of $C_\varphi: H^2 \rightarrow H^2$ has been characterized in terms of the Carleson function of the symbol $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, by B. McCluer ([16] – see also [20]). In another paper, it was characterized in terms of the Nevanlinna counting function N_φ of the symbol: $N_\varphi(w) = o(1 - |w|)$, as $|w| \rightarrow 1$, by J. Shapiro ([25]). A similar situation exists for the characterization of the membership of C_φ in a prescribed Schatten class ([14] and [15]). There should therefore exist a direct link between these two parameters (the Carleson measure m_φ and the Nevanlinna counting function N_φ), but results of this type are still very sparse and rather disconnected in the literature. Let us quote two of them:

- B. R. Choe ([4]) showed that $\limsup_{h \rightarrow 0} (\rho_\varphi(h)/h)^{1/2}$ is equivalent, up to constants, to the distance of C_φ to the space of compact operators on H^2 ; since J. Shapiro had proved ([25]) that this distance is exactly

$$\limsup_{|w| \rightarrow 1} (N_\varphi(w)/\log |w|)^{1/2},$$

one gets, in an indirect way, the fact that

$$\limsup_{|w| \rightarrow 1} N_\varphi(w)/\log |w| \approx \limsup_{h \rightarrow 0} \rho_\varphi(h)/h.$$

- Later, J. S. Choa and H. O. Kim ([3]) gave a somewhat direct proof of the equivalence of the above two conditions, without using the properties of the associated composition operator. But they did not give any explicit relation between the two functions ρ_φ and N_φ .

The aim of this paper is precisely to establish, on the basis of Green’s formula (under a form due to Stanton) and of Orlicz functions, a general equivalence between those two functions, from a “complex variable” point of view which seems to have an independent interest, and without help of the associated composition operator. More specifically, the main result of this work is that the Nevanlinna counting function and the Carleson function of an analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ are “equivalent” in the following sense:

Theorem 1.1 *There exist some universal constants $C, c > 1$, such that, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has:*

$$(1.1) \quad (1/C) \rho_\varphi(h/c) \leq \sup_{|w| \geq 1-h} N_\varphi(w) \leq C \rho_\varphi(ch),$$

for $0 < h < 1$ small enough.

More precisely, for every $\xi \in \partial\mathbb{D}$, one has:

$$(1.2) \quad (1/64) m_\varphi[W(\xi, h/64)] \leq \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi[W(\xi, 24h)],$$

for $0 < h < (1 - |\varphi(0)|)/16$.

Actually the above explicit constants are not relevant and we did not try to have “best” constants. It can be shown that for every $\alpha > 1$, there is a constant $C_\alpha > 0$ such that $m_\varphi(S(\xi, h)) \leq C_\alpha \tilde{\nu}_\varphi(\xi, \alpha h)$ and $\tilde{\nu}_\varphi(\xi, h) \leq C_\alpha m_\varphi(S(\xi, \alpha h))$ for $0 < h < (1 - |\varphi(0)|)/\alpha$, where $S(\xi, h)$ is defined in (2.3) and $\tilde{\nu}(\xi, h) = \sup_{w \in S(\xi, h) \cap \mathbb{D}} N_\varphi(w)$ (see (3.29)).

We end the paper with some applications to composition operators on Hardy-Orlicz spaces (which were introduced in [10]): we give generalizations of results known in the classical case of Hardy spaces H^p , $1 \leq p < \infty$, some of which have already been mentioned in this introduction. We shall see that in the general case of Hardy-Orlicz spaces, new difficulties occur with regard to the classical case.

First, we characterize their compactness with the Nevalinna counting function and show the necessity of a boundary condition, analogous to the *angular derivative* criterion of McCluer and Shapiro ([17]), which is sufficient when the symbol is finitely-valent. We construct then a “slow” Blaschke product (generalizing [24], § 10.2 and [10], Proposition 5.5) showing that this condition is not sufficient in general. We also construct a compact composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ with surjective symbol φ , generalizing a result of B. McCluer and J. Shapiro ([17], Example 3.12; see also the survey [21], § 2). We show directly the equivalence of Luecking’s and Luecking-Zhu’s criteria ([14], [15]) for the membership of $C_\varphi: H^2 \rightarrow H^2$ in the Schatten classes. We finally give a characterization of composition operators $C_\varphi: H^p \rightarrow H^p$, $1 \leq p < \infty$, with a closed range, simpler than the former ones (see [5] and [28]).

2 Notation

We shall denote by $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ the open unit disk of the complex plane and by $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ its boundary; m will be the normalized Lebesgue measure $dt/2\pi$ on \mathbb{T} , and A the normalized Lebesgue measure $dxdy/\pi$ on $\overline{\mathbb{D}}$. For every analytic self-map φ of \mathbb{D} , m_φ will be the pull-back measure of m by φ^* , where φ^* is the boundary values function of φ .

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ centered at ξ and of size h is the set

$$(2.1) \quad W(\xi, h) = \{z \in \overline{\mathbb{D}}; |z| \geq 1 - h \text{ and } |\arg(z\bar{\xi})| \leq h\}.$$

For convenience, we shall set $W(\xi, h) = \overline{\mathbb{D}}$ for $h \geq 1$.

For every analytic self-map φ of \mathbb{D} , one defines the maximal function of m_φ , for $0 < h < 1$, by:

$$(2.2) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m(\{\zeta \in \mathbb{T}; \varphi^*(\zeta) \in W(\xi, h)\}) = \sup_{\xi \in \mathbb{T}} m_\varphi(W(\xi, h)).$$

We have $\rho_\varphi(h) = 1$ for $h \geq 1$. We shall call this function ρ_φ the *Carleson function* of φ . For convenience, we shall often also use, instead of the Carleson window $W(\xi, h)$, the set

$$(2.3) \quad S(\xi, h) = \{z \in \overline{\mathbb{D}}; |z - \xi| \leq h\},$$

which has an equivalent size.

The Nevanlinna counting function N_φ is defined, for $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, by

$$(2.4) \quad N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

each term $\log \frac{1}{|z|}$ being repeated according to the multiplicity of z , and $N_\varphi(w) = 0$ for the other $w \in \mathbb{D}$. Its maximal function will be denoted by

$$(2.5) \quad \nu_\varphi(t) = \sup_{|w| \geq 1-t} N_\varphi(w).$$

3 Proof of the main theorem

3.1 Majorizing the Nevanlinna counting function by the Carleson function

The goal of this section is to prove:

Theorem 3.1 *For every analytic self-map φ of \mathbb{D} , one has, for every $a \in \mathbb{D}$:*

$$(3.1) \quad N_\varphi(a) \leq 196 m_\varphi(W(\xi, 12h)),$$

for $0 < h < (1 - |\varphi(0)|)/4$, where $\xi = \frac{a}{|a|}$ and $h = 1 - |a|$.

In particular, for $0 < h < (1 - |\varphi(0)|)/4$:

$$(3.2) \quad \nu_\varphi(h) = \sup_{|a| \geq 1-h} N_\varphi(a) \leq 196 \rho_\varphi(12h).$$

Let us note that, since $W(\zeta, s) \subseteq W(\xi, 2t)$ whenever $0 < s \leq t$ and $\zeta \in W(\xi, t) \cap \partial\mathbb{D}$, we get from (3.1) that

$$(3.3) \quad \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi(W(\xi, 24h)).$$

We shall first prove the following lemma.

Lemma 3.2 *Let φ be an analytic self map of \mathbb{D} . For every $z \in \mathbb{D}$, one has, if $w = \varphi(z)$, $\xi = w/|w|$ and $h = 1 - |w| \leq 1/4$:*

$$(3.4) \quad m_\varphi(W(\xi, 12h)) \geq m_\varphi(S(\xi, 6h)) \geq \frac{|w|}{8} (1 - |z|).$$

Proof. We may assume, by making a rotation, that w is real and positive: $3/4 \leq w < 1$.

Let:

$$(3.5) \quad T(u) = \frac{au + 1}{u + a},$$

where

$$a = w - \frac{2}{w} < -1,$$

so that $T: \mathbb{D} \rightarrow \mathbb{D}$ is analytic, and $T(w) = w/2$.

If P_z is the Poisson kernel at z , one has:

$$\frac{w}{2} = T[\varphi(z)] = \int_{\mathbb{T}} (T \circ \varphi)^* P_z dm = \int_{\mathbb{T}} \Re [(T \circ \varphi)^*] P_z dm.$$

Hence, if one sets:

$$E = \{\Re (T \circ \varphi^*) \geq w/4\} = \{\Re [(T \circ \varphi)^*] \geq w/4\},$$

one has:

$$\frac{w}{2} \leq \int_E P_z dm + \frac{w}{4} \int_{E^c} P_z dm \leq \int_E P_z dm + \frac{w}{4} \int_{\mathbb{D}} P_z dm = \int_E P_z dm + \frac{w}{4};$$

therefore:

$$\int_E P_z dm \geq \frac{w}{4}.$$

Since

$$\|P_z\|_\infty = \frac{1 + |z|}{1 - |z|} \leq \frac{2}{1 - |z|},$$

we get:

$$(3.6) \quad m(E) \geq \frac{w}{8} (1 - |z|).$$

On the other hand, (3.5) writes

$$(3.7) \quad u = T^{-1}(U) = \frac{aU - 1}{a - U};$$

hence:

$$|1 - u| = |a + 1| \frac{|1 - U|}{|a - U|} \leq \frac{2|a + 1|}{|a - U|}.$$

But $a < -1$ is negative, so $\Re U \geq w/4$ implies that

$$|a - U| \geq \Re(U - a) \geq \frac{w}{4} - a = \frac{2}{w} - \frac{3}{4}w \geq \frac{5}{4}.$$

Moreover, for $w \geq 3/4$:

$$|a + 1| = (1 - w) \left(\frac{2}{w} + 1 \right) \leq \frac{11}{3}(1 - w).$$

We get hence $|1 - u| \leq 6h$ when (3.7) holds and $\Re U \geq w/4$.

It follows that:

$$(3.8) \quad \varphi^*(E) \subseteq T^{-1}(\{\Re U \geq w/4\}) \subseteq S(1, 6h),$$

giving $m_\varphi(W(1, 12h)) \geq m_\varphi(S(1, 6h)) \geq m(E)$.

Combining this with (3.6), that finishes the proof. \square

Remark. Theorem 3.1 follows immediately when φ is univalent since then, for $|w| \geq 3/4$ and $\varphi(z) = w$:

$$N_\varphi(w) = \log \frac{1}{|z|} \approx (1 - |z|) \lesssim m_\varphi(W(1, 12h)).$$

When proving the equivalence between the conditions $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, and $N_\varphi(w) = o(1 - |w|)$, as $|w| \rightarrow 1$, J. S. Choa and H. O. Kim proved (see [3], page 112) the following inequality, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every $w \in \mathbb{D}$, close enough to 1:

$$(3.9) \quad N_\varphi(w) \leq \frac{(1 - |w|^2)^2}{8|w|^2} \int_{\partial\mathbb{D}} \frac{1}{|1 - \bar{w}\varphi(z)|^2} dm(z).$$

This result follows from an Hilbertian method, *viz.* Littlewood-Paley's identity:

$$(3.10) \quad \|f \circ \varphi\|_2^2 = |f \circ \varphi(0)|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w)$$

for every $f \in H^2$. With (3.9), one cannot go beyond the order 2; for instance, we can deduce from (3.9) (see the proof of Theorem 3.1 below), that, for $0 < h \leq 1/2$:

$$(3.11) \quad \sup_{|w|=1-h} N_\varphi(w) \lesssim h^2 \int_0^{1/h^2} \rho_\varphi\left(\frac{1}{\sqrt{t}}\right) dt \lesssim h^2 + h^2 \int_h^1 \frac{\rho_\varphi(u)}{u^3} du.$$

This is of course interesting only when the second term in the last sum is at most of order h^2 , so, when the integral is bounded. Nevertheless, this result suffices to show that Shapiro's criterion of compactness for $C_\varphi: H^2 \rightarrow H^2$ is implied by McCluer's one. Moreover, when the pull-back measure m_φ is an α -Carleson measure (i.e. $\rho_\varphi(h) \leq Ch^\alpha$ for some constant $C > 0$), with $1 \leq \alpha \leq 2$, we get

$$N_\varphi(w) \lesssim h^2 + h^2 \int_h^1 \frac{u^\alpha}{u^3} du \lesssim h^2 + h^2 h^{\alpha-2} \lesssim h^\alpha.$$

Recall ([11], Corollary 3.2) that, when m_φ is an α -Carleson measure, the composition operator C_φ is in the Schatten class S_p on the Hardy space H^2 , for every $p > 2/(\alpha - 1)$, and that m_φ is α -Carleson for every $\alpha \geq 1$ when $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact, if Ψ is an Orlicz function satisfying the growth condition Δ^2 ([12], Theorem 5.2).

But (3.11) does not suffice for the compactness of $C_\varphi: H^\Psi \rightarrow H^\Psi$ on general Hardy-Orlicz spaces (see [10]).

In order to prove Theorem 3.1, we shall replace the Littlewood-Paley identity, by a more general formula, deduced from Stanton's formula (see [7], Theorem 2).

Theorem 3.3 (Stanton's formula) *For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every subharmonic function $G: \mathbb{D} \rightarrow \mathbb{R}$, one has:*

$$(3.12) \quad \lim_{r \uparrow 1} \int_{\partial \mathbb{D}} G[\varphi(r\xi)] dm(\xi) = G[\varphi(0)] + \frac{1}{2} \int_{\mathbb{D}} \Delta G(w) N_\varphi(w) dA(w),$$

where Δ is the distributional Laplacian.

Proof of Theorem 3.1. If $a \notin \varphi(\mathbb{D})$, one has $N_\varphi(a) = 0$, and the result is trivial. We shall hence assume that $a \in \varphi(\mathbb{D})$.

Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function, that is a non-decreasing convex function such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, and we assume that Φ' is also an Orlicz function. In other words, Φ'' is an arbitrary non-negative and non-decreasing function and $\Phi'(x) = \int_0^x \Phi''(t) dt$ and $\Phi(x) = \int_0^x \Phi'(t) dt$.

Let now $f: \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. We have, outside the zeroes of f , by writing $\Delta \Phi(|f|) = 4\partial\bar{\partial}\Phi(\sqrt{|f|^2})$:

$$(3.13) \quad \Delta \Phi(|f|) = \left[\Phi''(|f|) + \frac{\Phi'(|f|)}{|f|} \right] |f'|^2.$$

We shall only use here that:

$$(3.14) \quad \Delta \Phi(|f|) \geq \Phi''(|f|) |f'|^2$$

(this is a not too crude estimate, since, Φ' being an Orlicz function, Φ'' is non-negative and non-decreasing, and hence $\Phi'(x) = \int_0^x \Phi''(t) dt \leq x\Phi''(x)$ and $\Phi'(x) = \int_0^x \Phi''(t) dt \geq \int_{x/2}^x \Phi''(t) dt \geq (x/2)\Phi''(x/2)$).

Set now, for $a \in \mathbb{D}$:

$$(3.15) \quad f_a(z) = \frac{1 - |a|}{1 - \bar{a}z}, \quad z \in \bar{\mathbb{D}}.$$

Since $\Phi(|f_a|)$ is subharmonic (Φ being convex and non-decreasing) and bounded, we can use Stanton's formula as:

$$(3.16) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \int_{\mathbb{D}} \Phi''(|f_a|) |f'_a|^2 N_\varphi dA.$$

Let $h = 1 - |a|$. For $|z - a| < h$, one has

$$|1 - \bar{a}z| = |(1 - |a|^2) + \bar{a}(a - z)| \leq (1 - |a|^2) + |a - z| \leq 2h + h = 3h;$$

Hence $|f_a(z)| \geq \frac{h}{3h} = \frac{1}{3}$ for $|z - a| < h$. It follows, since Φ'' is non-decreasing:

$$(3.17) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \int_{D(a,h)} |f'_a|^2 N_\varphi dA.$$

Now, if $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, one has $|f'_a(z)| = \frac{|a|}{1+|a|} |\varphi'_a(z)| \geq \frac{3}{7} |\varphi'_a(z)|$ (we may, and do, assume that $1 - |a| = h \leq 1/4$); hence:

$$\begin{aligned} \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &\geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \frac{9}{49} \int_{D(a,h)} |\varphi'_a|^2 N_\varphi dA \\ &= \frac{9}{98} \Phi''\left(\frac{1}{3}\right) \int_{\varphi_a(D(a,h))} N_{\varphi_a \circ \varphi} dA \end{aligned}$$

(because $N_{\varphi_a \circ \varphi}(\varphi_a(w)) = N_\varphi(w)$ and $\varphi_a^{-1} = \varphi_a$).

But $\varphi_a(D(a, h)) \supseteq D(0, 1/3)$: indeed, if $|w| < 1/3$, then $w = \varphi_a(z)$, with

$$|a - z| = \left| \frac{(1 - |a|^2)w}{1 - \bar{a}w} \right| \leq (1 - |a|^2) \frac{|w|}{1 - |w|} < 2h \frac{1/3}{1 - 1/3} = h.$$

We are going now to use the sub-averaging property of the Nevanlinna function ([24], page 190, [25], § 4.6, or [27], Proposition 10.2.4): for every analytic self-map $\psi: \mathbb{D} \rightarrow \mathbb{D}$, one has

$$N_\psi(w_0) \leq \frac{1}{A(\Delta)} \int_{\Delta} N_\psi(w) dA(w),$$

for every disk Δ of center w_0 which does not contain $\psi(0)$.

This will be possible thanks to the following:

Lemma 3.4 *For $1 - |a| < (1 - |\varphi(0)|)/4$, one has $|(\varphi_a \circ \varphi)(0)| > 1/3$.*

Proof. One has $|1 - \bar{a}\varphi(0)| \leq (1 - |a|^2) + |\bar{a}| |a - \varphi(0)| \leq (1 - |a|^2) + |a - \varphi(0)|$; hence:

$$\begin{aligned} |\varphi_a(\varphi(0))| &\geq \frac{|a - \varphi(0)|}{(1 - |a|^2) + |a - \varphi(0)|} \geq 1 - \frac{1 - |a|^2}{(1 - |a|^2) + |a - \varphi(0)|} \\ &\geq 1 - \frac{1 - |a|^2}{|a - \varphi(0)|} \geq 1 - 2 \frac{1 - |a|}{|a - \varphi(0)|}. \end{aligned}$$

But when $1 - |a| < (1 - |\varphi(0)|)/4$, one has:

$$|a - \varphi(0)| \geq |a| - |\varphi(0)| = (1 - |\varphi(0)|) - (1 - |a|) > 3(1 - |a|),$$

and the result follows. \square

Hence:

$$\int_{D(0,1/3)} N_{\varphi_a \circ \varphi} dA \geq \frac{1}{9} N_{\varphi_a \circ \varphi}(0) = \frac{1}{9} N_{\varphi}(a),$$

and

$$(3.18) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{98} \Phi''\left(\frac{1}{3}\right) N_{\varphi}(a).$$

We now have to estimate from above $\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm$. For that, we shall use the following easy lemma.

Lemma 3.5 *For every $\xi \in \partial\mathbb{D}$ and every $h \in (0, 1/2]$, one has:*

$$(3.19) \quad |1 - \bar{a}z|^2 \geq \frac{1}{4}(h^2 + |z - \xi|^2), \quad \forall z \in \bar{\mathbb{D}},$$

where $a = (1 - h)\xi$.

Proof. The result is rotation-invariant; so we may assume that $\xi = 1$ (and hence $a > 0$). Write $z = 1 - re^{i\theta}$. Since $|z| \leq 1$ if and only if $r \leq 2 \cos \theta$, one has $\cos \theta \geq 0$ and hence $|\theta| \leq \pi/2$. Then:

$$\begin{aligned} |1 - \bar{a}z|^2 &= |1 - a(1 - re^{i\theta})|^2 = |1 - a + are^{i\theta}|^2 \\ &= (1 - a)^2 + a^2 r^2 + 2ar(1 - a) \cos \theta \\ &\geq (1 - a)^2 + a^2 r^2 \geq \frac{1}{4}(h^2 + r^2) = \frac{1}{4}(h^2 + |z - 1|^2). \end{aligned} \quad \square$$

Then:

$$\begin{aligned} \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &= \int_{\bar{\mathbb{D}}} \Phi\left(\frac{1 - |a|}{|1 - \bar{a}z|}\right) dm_{\varphi}(z) \\ &\leq \int_{\bar{\mathbb{D}}} \Phi\left(\frac{2h}{(h^2 + |z - \xi|^2)^{1/2}}\right) dm_{\varphi}(z), \quad \text{by (3.19)} \\ &= \int_0^{+\infty} m_{\varphi}\left(\Phi\left(\frac{2h}{(h^2 + |z - \xi|^2)^{1/2}}\right) \geq t\right) dt \\ &= \int_0^{+\infty} m_{\varphi}((h^2 + |z - \xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt \\ &= \int_0^{\Phi(2)} m_{\varphi}((h^2 + |z - \xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt, \end{aligned}$$

since $h \leq (h^2 + |z - \xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)$ implies $t \leq \Phi(2)$. We get:

$$\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \leq \int_0^{\Phi(2)} m_\varphi(|z - \xi| \leq 2h/\Phi^{-1}(t)) dt.$$

We obtain from (3.18), by setting $u = 2h/\Phi^{-1}(t)$:

$$N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{2h}{u^2} \Phi' \left(\frac{2h}{u} \right) du.$$

Since $\Phi'(x) \leq x\Phi''(x)$, we get:

$$(3.20) \quad N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi'' \left(\frac{2h}{u} \right) du.$$

We are going now to choose suitably the Orlicz function Φ . It suffices to define Φ'' , for $a \in \mathbb{D}$ given (with $\xi = a/|a|$ and $h = 1 - |a| \leq 1/4$). By Lemma 3.2, since $a \in \varphi(\mathbb{D})$, there is a constant $c_0 > 0$, such that $m_\varphi(S(\xi, c_0h)) > 0$; we can hence set (note that $m_\varphi(S(\xi, u)) \leq 1$):

$$(3.21) \quad \Phi''(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq h, \\ \frac{1}{m_\varphi(S(\xi, 2h/v))} & \text{if } h \leq v \leq 2/c_0, \\ \frac{1}{m_\varphi(S(\xi, c_0h))} & \text{if } v \geq 2/c_0. \end{cases}$$

It is a non-negative non-decreasing function, so the assumptions made on Φ at the beginning are satisfied. One has, since $m_\varphi(S(\xi, u)) \Phi''(2h/u) \leq 1$:

$$\int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi'' \left(\frac{2h}{u} \right) du \leq \int_h^\infty \frac{4h^2}{u^3} du = 2.$$

Since $c_0 \leq 6$, one has $h \leq 1/3 \leq 2/c_0$ and hence $\Phi''(1/3) = 1/m_\varphi(S(\xi, 6h))$; therefore (3.20) gives, for $h \leq (1 - |\varphi(0)|)/4$:

$$(3.22) \quad N_\varphi(a) \leq 196 m_\varphi(S(\xi, 6h)),$$

finishing the proof since $S(\xi, 6h) \subseteq W(\xi, 12h)$. \square

3.2 Domination of the Carleson function by the Nevanlinna function

We cannot expect to estimate individually from above the m_φ -measure of Carleson windows centered at $\xi = w/|w|$ by $N_\varphi(w)$, as in Theorem 3.1. In fact, consider a conformal mapping φ from \mathbb{D} onto $\mathbb{D} \setminus [0, 1[$. One has $N_\varphi(t) = 0$ for every $t \in [0, 1[$, though $m_\varphi(W(1, h)) > 0$ for every $h > 0$ (because $W(1, h) \supset W(e^{ih/2}, h/2)$ and $m_\varphi(W(e^{ih/2}, h/2)) > 0$ by Lemma 3.2).

Let us give another example. Let $\varphi(z) = (1+z)/2$. Then:

a) One has $\varphi(e^{i\theta}) = (\cos \theta/2) e^{i\theta/2}$ (with $|\theta| \leq \pi$). Hence $\varphi(e^{i\theta_0}) \in W(e^{i\theta_0}, h)$ if and only if $\cos(\theta/2) \geq 1-h$ and $|(\theta/2) - \theta_0| \leq h$, i.e. $2(\theta_0 - h) \leq \theta \leq 2(\theta_0 + h)$. Now, $1 - \cos(\theta/2) \leq \theta^2/8$, so the modulus condition is satisfied when $\theta^2 \leq 8h$; in particular when $|\theta| \leq 2\sqrt{h}$.

For $\theta_0 = \sqrt{h}$, $m_\varphi(W(e^{i\theta_0}, h))$ is bigger than the length of the interval

$$[-2\sqrt{h}, 2\sqrt{h}] \cap [2(\sqrt{h} - h), 2(\sqrt{h} + h)] = [2\sqrt{h} - 2h, 2\sqrt{h}],$$

that is $2h$. Therefore $m_\varphi(W(e^{i\theta_0}, h)) \geq 2h$.

b) Let now $w = \varphi(z)$. Write $w = \frac{1}{2} + r e^{i\zeta}$ with $0 \leq r < 1/2$. Then, writing $r = \frac{1}{2} - s$, one has $|z| = |2w - 1| = 2r$ and

$$N_\varphi(w) = \log \frac{1}{|z|} = \log \frac{1}{2r} = \log \frac{1}{1-2s} \approx s.$$

Now, $|w|^2 = \frac{1}{4} + r^2 + r \cos \zeta$ and

$$h \approx 1 - |w|^2 = \frac{1}{2}(1 - \cos \zeta) + s(1 + \cos \zeta) - s^2 \approx \frac{\zeta^2}{4} + 2s.$$

Writing $\zeta = s^{1/2\alpha}$, one gets:

- (i) for “small” ζ (i.e. $0 < \alpha \leq 1$): $h \approx s$, and so $N_\varphi(w) \approx h$;
- (ii) for “large” (i.e. $\alpha \geq 1$): $h \approx s^{1/\alpha}$, and so $N_\varphi(w) \approx h^\alpha$.

On the other hand, $w = e^{i\zeta/2}[(1-s)\cos(\zeta/2) - is\sin(\zeta/2)]$; hence, when s goes to 0, one has

$$\theta_w := \arg w = \frac{\zeta}{2} + \arctan \left[\frac{s \sin(\zeta/2)}{(1-s)\cos(\zeta/2)} \right] \sim \frac{\zeta}{2} \approx \zeta.$$

For $\alpha \geq 1$, we have $h \approx s^{1/\alpha} = \zeta^2$, i.e. $\zeta \approx \sqrt{h}$. Then, choosing $\alpha > 1$ such that $\zeta = \theta_0$, we obtain $m_\varphi(W(w/|w|, h)) \approx h$, though $N_\varphi(w) \approx h^\alpha \ll h$.

One cannot hence dominate $m_\varphi(W(w/|w|, h))$ by $N_\varphi(w)$.

We can remark that, nevertheless, in either case, one has $\rho_\varphi(h) \approx h$ and $\nu_\varphi(h) \approx h$.

We shall prove:

Theorem 3.6 *For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has, for every $\xi \in \partial\mathbb{D}$:*

$$(3.23) \quad m_\varphi(W(\xi, h)) \leq 64 \sup_{w \in W(\xi, 64h) \cap \mathbb{D}} N_\varphi(w),$$

for $0 < h < (1 - |\varphi(0)|)/16$.

Proof. We shall set:

$$(3.24) \quad \nu_\varphi(\xi, h) = \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w).$$

Note that

$$\nu_\varphi(h) = \sup_{|\xi|=1} \nu_\varphi(\xi, h),$$

where ν_φ is defined in (2.5)

If for some $h_0 > 0$, one has $\nu_\varphi(\xi, h_0) = 0$, then $\varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus W(\xi, h_0)$, and hence $m_\varphi(W(\xi, h)) = 0$ for $0 < h < h_0$. Therefore we shall assume that $\nu_\varphi(\xi, h) > 0$. We may, and do, also assume that $h \leq 1/4$. By replacing φ by $e^{i\theta}\varphi$, it suffices to estimate $m_\varphi(S(1, h))$ (recall that $S(1, t) = \{z \in \overline{\mathbb{D}}; |1 - z| \leq t\}$).

We shall use the same functions f_a as in the proof of Theorem 3.1, but, for convenience, with a different notation. We set, for $0 < r < 1$:

$$(3.25) \quad u(z) = \frac{1-r}{1-rz}.$$

Let us take an Orlicz function Φ as in the beginning of the proof of Theorem 3.1, which will be precised later. We shall take this function in such a way that $\Phi(|u(\varphi(0))|) = 0$.

Since $\Phi'(x) \leq x \Phi''(x)$, (3.13) becomes:

$$(3.26) \quad \Delta\Phi(|u|) \leq 2\Phi''(|u|) |u'|^2,$$

and Stanton's formula writes, since $\Phi(|u(\varphi(0))|) = 0$:

$$(3.27) \quad \int_{\partial\mathbb{D}} \Phi(|u \circ \varphi|) dm \leq \int_{\mathbb{D}} \Phi''(|u(w)|) |u'(w)|^2 N_\varphi(w) dA(w).$$

In all the sequel, we shall fix h , $0 < h \leq 1/4$, and take $r = 1 - h$.

For $|z| \leq 1$ and $|1 - z| \leq h$, one has $|1 - rz| = |(1 - z) + hz| \leq |1 - z| + h \leq 2h$, so:

$$|u(z)| \geq \frac{(1-r)}{2h} = \frac{1}{2}.$$

Hence:

$$\begin{aligned} m_\varphi(S(1, h)) &\leq \frac{1}{\Phi(1/2)} \int_{S(1, h)} \Phi(|u(z)|) dm_\varphi(z) \\ &\leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi(|u(z)|) dm_\varphi(z) \\ &= \frac{1}{\Phi(1/2)} \int_{\mathbb{T}} \Phi(|(u \circ \varphi)(z)|) dm(z), \end{aligned}$$

and so, by (3.27):

$$(3.28) \quad m_\varphi(S(1, h)) \leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z).$$

We are going to estimate this integral by separating two cases: $|1 - z| \leq h$ and $|1 - z| > h$.

For convenience, we shall set:

$$(3.29) \quad \tilde{\nu}(t) = \sup_{w \in S(1,t) \cap \mathbb{D}} N_\varphi(w).$$

1) Remark first that

$$u'(z) = \frac{rh}{(1-rz)^2},$$

and so:

$$|u'(z)| \leq \frac{h}{(1-r)^2} = \frac{1}{h}.$$

Since $|u(z)| \leq 1$, we get hence:

$$\int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1,h)} \Phi''(1) \frac{1}{h^2} \tilde{\nu}(h) dA(z),$$

giving, since $A(S(1,h)) \leq h^2$:

$$(3.30) \quad \int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \Phi''(1) \tilde{\nu}(h).$$

2) For $0 < h \leq 1/4$, one has:

$$|u(z)| \leq \frac{2h}{|1-z|} \quad \text{and} \quad |u'(z)| \leq \frac{2h}{|1-z|^2};$$

indeed, we have (this is obvious, by drawing a picture):

$$|1 - rz| = r \left| \frac{1}{r} - z \right| \geq r |1 - z|,$$

and hence $|1 - rz| \geq \frac{3}{4} |1 - z|$, since $r = 1 - h \geq 3/4$. We obtain:

$$\begin{aligned} \int_{|1-z| > h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \int_{|1-z| > h} \Phi''\left(\frac{2h}{|1-z|}\right) \frac{h^2}{|1-z|^4} N_\varphi(z) dA(z). \end{aligned}$$

Then, using polar coordinates centered at 1 (note that we only have to integrate over an arc of length less than π), and the obvious inequality $N_\varphi(z) \leq \tilde{\nu}(|1 - z|)$, we get:

$$(3.31) \quad \begin{aligned} \int_{|1-z| > h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt. \end{aligned}$$

We now choose the Orlicz function as follows (with $a = \varphi(0)$):

$$(3.32) \quad \Phi''(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq h/(1 - |a|), \\ \frac{1}{\tilde{\nu}(2h/v)} & \text{if } h/(1 - |a|) < v < 2, \\ \frac{1}{\tilde{\nu}(h)} & \text{if } v \geq 2. \end{cases}$$

This function is non-negative and non-decreasing. Moreover, one has $\Phi(x) = 0$ for $0 \leq x \leq h/(1 - |a|)$. Hence, since $|u(a)| \leq \frac{h}{1 - |a|}$, one has $\Phi(|u(a)|) = 0$.

Then

$$(3.33) \quad \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt = \int_h^{2^{1-|a|}} \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt \\ \leq \int_h^\infty \frac{h^2}{t^3} dt = \frac{1}{2}.$$

Now,

$$\Phi\left(\frac{1}{2}\right) = \int_0^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \frac{t}{2} \Phi''\left(\frac{t}{2}\right) dt \\ \geq \Phi''\left(\frac{1}{8}\right) \int_{1/4}^{1/2} \frac{t}{2} dt = \frac{3}{64} \Phi''\left(\frac{1}{8}\right).$$

When $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\Phi''(1/8) = 1/\tilde{\nu}(16h)$, and $\Phi''(1) = 1/\tilde{\nu}(2h)$. We get hence, from (3.28), (3.30), (3.31) and (3.33):

$$(3.34) \quad m_\varphi(S(1, h)) \leq \frac{64}{3} \tilde{\nu}(16h) \left[\frac{\tilde{\nu}(h)}{\tilde{\nu}(2h)} + 2 \right] \leq 64 \tilde{\nu}(16h).$$

Since $W(1, t) \subseteq S(1, 2t)$, we get $m_\varphi(W(1, h)) \leq 64 \sup_{w \in S(1, 32h)} N_\varphi(w)$ for $0 < h < (1 - |\varphi(0)|)/16$, and that ends the proof of Theorem 3.6, since $S(1, 32h) \subseteq W(1, 64h)$. \square

Remark. A slight modification of the proof gives the following improvement, if one allows a (much) bigger constant.

Theorem 3.7 *There are universal constants $C, c > 1$ such that*

$$m_\varphi(S(\xi, h)) \leq C \frac{1}{A(S(\xi, ch))} \int_{S(\xi, ch)} N_\varphi(z) dA(z)$$

for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, every $\xi \in \partial\mathbb{D}$, and $0 < h < (1 - |\varphi(0)|)/8$.

Proof. We are going to follow the proof of Theorem 3.6. We shall assume that $\xi = 1$ and we set:

$$(3.35) \quad I(t) = \int_{S(1,t)} N_\varphi(z) dA(z).$$

Then:

1) When $|1 - z| < h$, , we have, instead of (3.30):

$$(3.36) \quad \int_{|1-z|<h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1,h)} \Phi''(1) \frac{1}{h^2} N_\varphi(z) dA(z) \\ = \Phi''(1) \frac{1}{h^2} I(h).$$

2) For $|z - 1| \geq h$, we write:

$$\int_{|1-z|\geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ = \sum_{k=1}^{\infty} \int_{kh \leq |1-z| < (k+1)h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2h}{kh}\right) \frac{h^2}{k^4 h^4} I((k+1)h) \\ = 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2}{k}\right) \frac{1}{k^4 h^2} I((k+1)h).$$

We take, with $a = \varphi(0)$:

$$(3.37) \quad \Phi''(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq h/(1 - |a|), \\ \frac{1}{I\left(\left(\frac{2}{v} + 1\right)h\right)} & \text{if } v > h/(1 - |a|). \end{cases}$$

Then

$$(3.38) \quad \int_{|1-z|\geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \frac{4}{h^2} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{4}{h^2} \frac{\pi^4}{90} \leq \frac{5}{h^2}.$$

Since $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\Phi''(1/8) = \frac{1}{I(17h)}$ and $\Phi''(1) = \frac{1}{I(3h)}$. Therefore:

$$m_\varphi(S(1, h)) \leq \frac{64}{3} I(17h) \left[\frac{1}{h^2} \frac{I(h)}{I(3h)} + \frac{5}{h^2} \right] \\ \leq \frac{64}{3} I(17h) \frac{6}{h^2} = 128 \frac{I(17h)}{h^2} \\ \leq 128 \times 17^2 \frac{I(17h)}{A(S(1, 17h))},$$

ending the proof of Theorem 3.7. \square

4 Some consequences

4.1 Behaviour of the Nevanlinna counting function

In [10], we proved (Theorem 4.19) that the Carleson function of an analytic self-map φ has the following property of regularity: $m_\varphi(S(\xi, \varepsilon h)) \leq K \varepsilon m_\varphi(S(\xi, h))$ for $0 < h < 1 - |\varphi(0)|$, $0 < \varepsilon < 1$ and $\xi \in \partial\mathbb{D}$, where K is a universal constant. It follows from Theorem 1.1, (actually Theorem 3.1 and Theorem 3.6) that:

Theorem 4.1 *There exist a universal constant $K > 0$ such that, for every analytic self-map φ of \mathbb{D} , one has, for $0 < \varepsilon < 1$:*

$$(4.1) \quad \nu_\varphi(\varepsilon t) \leq K \varepsilon \nu_\varphi(t),$$

for t small enough.

More precisely, for t small enough, one has, for every $\xi \in \partial\mathbb{D}$:

$$(4.2) \quad \nu_\varphi(\xi, \varepsilon t) \leq K \varepsilon \nu_\varphi(\xi, t),$$

where $\nu_\varphi(\xi, s) = \sup_{w \in W(\xi, s) \cap \mathbb{D}} N_\varphi(w)$.

Note that the two above quoted theorems give Theorem 4.1 *a priori* only for $0 < \varepsilon < 1/K$; but if $1/K \leq \varepsilon < 1$, one has $\nu_\varphi(\xi, \varepsilon t) \leq \nu_\varphi(\xi, t) \leq K \varepsilon \nu_\varphi(\xi, t)$.

4.2 Compact composition operators

We shall end this paper with some consequences of Theorem 1.1 for composition operators. Recall that if Ψ is an Orlicz function, the Hardy-Orlicz space is the space of functions $f \in H^1$ whose boundary values are in the Orlicz space $L^\Psi(\partial\mathbb{D}, m)$. We proved in [10], Theorem 4.18 that, if $\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty$, the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if, for every $A > 0$, one has $\rho_\varphi(h) = o[1/\Psi(A\Psi^{-1}(1/h))]$ when h goes to 0. This remains true when $H^\Psi = H^1$. Hence Theorem 1.1 gives:

Theorem 4.2 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map and Ψ be an Orlicz function. Then the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if*

$$(4.3) \quad \sup_{|w| \geq 1-h} N_\varphi(w) = o\left(\frac{1}{\Psi(A\Psi^{-1}(1/h))}\right), \quad \text{as } h \rightarrow 0, \quad \forall A > 0.$$

It should be noted, due to the arbitrary choice of $A > 0$, that (4.3) may be replaced by

$$(4.4) \quad \sup_{|w| \geq 1-h} N_\varphi(w) \leq \frac{1}{\Psi(A\Psi^{-1}(1/h))}, \quad \forall A > 0,$$

for $h \leq h_A$, and this condition also writes, setting $\nu_\varphi(h) = \sup_{|w| \geq 1-h} N_\varphi(w)$ (see (2.5)):

$$(4.5) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\nu_\varphi(h))} = 0.$$

It is known that if $C_\varphi: H^2 \rightarrow H^2$ is compact, then $\lim_{|z| \rightarrow 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty$, and that this condition is sufficient when φ is univalent, or finitely-valent, but not sufficient in general (see [17] and [24], § 3.2). It follows from Theorem 4.2 that an analogous result holds for Hardy-Orlicz spaces:

Theorem 4.3 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map, and Ψ be an Orlicz function. Assume that the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact. Then:*

$$(4.6) \quad \lim_{|z| \rightarrow 1} \frac{\Psi^{-1}\left[\frac{1}{1-|\varphi(z)|}\right]}{\Psi^{-1}\left[\frac{1}{1-|z|}\right]} = 0.$$

Conversely, if φ is finitely-valent, then (4.6) suffices for $C_\varphi: H^\Psi \rightarrow H^\Psi$ to be compact.

Recall that the assumption “ φ is finitely-valent” means that there is an integer $p \geq 1$ such that each $w \in \varphi(\mathbb{D})$ is the image by φ of at most p elements of \mathbb{D} .

Proof. To get the necessity, we could use Theorem 4.2 and the fact that $1-|z| \leq \log \frac{1}{|z|} \leq N_\varphi(\varphi(z))$; but we shall give a more elementary proof. Recall that H^Ψ is the bidual of HM^Ψ , the closure of H^∞ in H^Ψ . Since $C_\varphi(H^\infty) \subseteq H^\infty$, C_φ maps HM^Ψ into itself and $C_\varphi: H^\Psi \rightarrow H^\Psi$ is the bi-adjoint of $C_\varphi: HM^\Psi \rightarrow HM^\Psi$. We know that the evaluation $\delta_a: f \in HM^\Psi \mapsto f(a) \in \mathbb{C}$ has norm $\approx \Psi^{-1}\left(\frac{1}{1-|a|}\right)$ ([10], Lemma 3.11). This norm tends to infinity as $|a|$ goes to 1; hence $\delta_a/\|\delta_a\| \xrightarrow{|a| \rightarrow 1} 0$ weak-star (because $|\delta_a(f)| = |f(a)| \leq \|f\|_\infty$ for $f \in H^\infty$). If C_φ is compact, its adjoint C_φ^* also; we get hence $\|C_\varphi^*(\delta_a/\|\delta_a\|)\| \xrightarrow{|a| \rightarrow 1} 0$. But $C_\varphi^*\delta_a = \delta_{\varphi(a)}$. Therefore

$$\frac{\Psi^{-1}\left[\frac{1}{1-|\varphi(a)|}\right]}{\Psi^{-1}\left[\frac{1}{1-|a|}\right]} \xrightarrow{|a| \rightarrow 1} 0.$$

Conversely, assume that (4.6) holds. For every $A > 0$, one has, for $|z|$ close enough to 1: $\Psi^{-1}\left(\frac{1}{1-|z|}\right) \geq A \Psi^{-1}\left(\frac{1}{1-|\varphi(z)|}\right)$; in other words, one has: $1/\Psi(A\Psi^{-1}(1/1-|\varphi(z)|)) \geq 1-|z|$. But, when φ is p -valent, and if $w = \varphi(z)$

with $|z| > 0$ minimal, one has $N_\varphi(w) \leq p \log \frac{1}{|z|} \approx 1 - |z|$. Since $|z| \rightarrow 1$ as $|w| = |\varphi(z)| \rightarrow 1$ (otherwise, we should have a sequence (z_n) converging to some $z_0 \in \mathbb{D}$ and $\varphi(z_n)$ would converge to $\varphi(z_0) \in \mathbb{D}$), we get $\sup_{|w| \geq 1-h} N_\varphi(w) \lesssim 1/\Psi(A\Psi^{-1}(1/1-|w|)) \leq 1/\Psi(A\Psi^{-1}(1/h))$, for h small enough. By Theorem 4.2, with (4.4), that means that C_φ is compact on H^Ψ . \square

4.3 Slow Blaschke products

Without the assumption that φ is p -valent, condition (4.6) is no longer sufficient to ensure the compactness of $C_\varphi: H^\Psi \rightarrow H^\Psi$. Indeed, we are going to construct a Blaschke product satisfying (4.6), but whose associated composition operator is of course not compact on H^Ψ , as this is the case for every inner function.

Theorem 4.4 *Let $\delta: (0, 1) \rightarrow (0, 1/2]$ be any function such that $\lim_{t \rightarrow 0} \delta(t) = 0$. Then, there exists a Blaschke product B such that:*

$$(4.7) \quad 1 - |B(z)| \geq \delta(1 - |z|), \quad \text{for all } z \in \mathbb{D}.$$

Corollary 4.5 *For every Orlicz function Ψ there exists a Blaschke product B which satisfies*

$$\lim_{|z| \rightarrow 1} \frac{\Psi^{-1} \left[\frac{1}{1 - |B(z)|} \right]}{\Psi^{-1} \left[\frac{1}{1 - |z|} \right]} = 0.$$

though the composition operator $C_B: H^\Psi \rightarrow H^\Psi$ is not compact.

Proof. C_B is not compact since every compact composition operator should satisfy $|\varphi^*| < 1$ a.e. (see [10], Lemma 4.8). It suffices then to chose $\delta(t) = 1/\Psi(\sqrt{\Psi^{-1}(1/t)})$, which satisfies the hypothesis of Theorem 4.4. Moreover:

$$\frac{\Psi^{-1}(1/\delta(t))}{\Psi^{-1}(1/t)} = \frac{1}{\sqrt{\Psi^{-1}(1/t)}} \xrightarrow{t \rightarrow 0} 0,$$

and condition (4.7) gives the result. \square

Proof of Theorem 4.4. We shall essentially construct our Blaschke product B as an infinite product of finite Blaschke products

$$\prod_n B_n ;$$

where each finite Blaschke product B_n has p_n zeros equidistributed in the circumference of radius r_n . That is, we will have, writing $\theta_k = 2\pi k/p_n$ and $z_k = r_n e^{i\theta_k}$, for $k = 1, 2, \dots, p_n$:

$$(4.8) \quad B_n(z) = \prod_{k=1}^{p_n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} = \prod_{k=1}^{p_n} \frac{r_n - e^{-i\theta_k} z}{1 - r_n e^{-i\theta_k} z}.$$

We shall need the following estimate for the finite Blaschke product in (4.8).

Lemma 4.6 *Let $p \in \mathbb{N}$, and $0 < r < 1$. Consider the finite Blaschke product*

$$(4.9) \quad G(z) = \prod_{k=1}^p \frac{r - e^{-i\theta_k} z}{1 - r e^{-i\theta_k} z},$$

where $\theta_k = \frac{2k\pi}{p}$, for $k = 1, 2, \dots, p$.

(a) *Then, for every $z \in \mathbb{D}$ with $|z| = r$,*

$$(4.10) \quad |G(z)| \leq \frac{2r^p}{1 + r^{2p}} = 1 - \frac{(1 - r^p)^2}{1 + r^{2p}}.$$

(b) *If besides we have $ph \leq 1/2$, where $h = 1 - r$, we also have, for every $z \in \mathbb{D}$ with $|z| = r$,*

$$(4.11) \quad |G(z)| \leq 1 - \frac{(ph)^2}{2e}.$$

Let us continue the proof of the theorem. Define $\chi: (0, 1) \rightarrow (0, 1]$ by:

$$(4.12) \quad \chi(x) = \sup_{t \leq x} [\max\{2\delta(t), \sqrt{t}\}].$$

Then χ is non-decreasing, $\lim_{x \rightarrow 0} \chi(x) = 0$ and $\lim_{x \rightarrow 1} \chi(x) = 1$. We can find a decreasing sequence $(h_n)_{n \geq 0}$ of point $h_n \in (0, 1)$, such that $\chi(h_n) \leq 2^{-n}$. This sequence converges to 0; in fact, $\sqrt{h_n} \leq \chi(h_n) \leq 2^{-n}$, by (4.12), and hence:

$$(4.13) \quad h_n \leq 2^{-2n}.$$

We now define, for every $n \in \mathbb{N}$, a positive integer p_n , by:

$$(4.14) \quad p_n = \min\{p \in \mathbb{N}; \frac{p^2 h_n^2}{2e} > 2^{-n}\}.$$

We have $p_n > 1$ because $h_n^2/2e < h_n^2 \leq 2^{-4n}$. So, for every n , we have $4(p_n - 1)^2 \geq p_n^2$, and then:

$$(4.15) \quad 4 \cdot 2^{-n} \geq \frac{4(p_n - 1)^2 h_n^2}{2e} \geq \frac{p_n^2 h_n^2}{2e}.$$

This yields, for $n \geq 7$, that $(p_n h_n)^2 \leq 8e 2^{-n} \leq 1/4$. Therefore $p_n h_n \leq 1/2$, and we can use the estimate in part (b) of Lemma 4.6.

Now, for $n \geq 7$, let B_n be the finite Blaschke product defined by (4.8), where $r_n = 1 - h_n$. Using (b) in Lemma 4.6, the Maximum Modulus Principle and the definition of p_n in (4.14), we have:

$$(4.16) \quad |B_n(z)| \leq 1 - \frac{p_n^2 h_n^2}{2e} < 1 - 2^{-n}, \quad \text{for } |z| \leq r_n.$$

Consider then the Blaschke product D defined by:

$$(4.17) \quad D(z) = \prod_{n=7}^{\infty} B_n(z).$$

This product is convergent since, by (4.15), we have

$$\sum p_n(1 - r_n) = \sum p_n h_n \leq \sum \sqrt{8e} 2^{-n} < +\infty.$$

Finally, take $N \in \mathbb{N}$ big enough to have $r_6^N < 1/2$, and define

$$(4.18) \quad B(z) = z^N D(z).$$

Thus B is a Blaschke product, and, if $|z| \leq r_6$, we have, since $\delta(t) \leq 1/2$:

$$(4.19) \quad |B(z)| \leq |z^N| \leq r_6^N < 1/2 \leq 1 - \delta(1 - |z|).$$

If $1 > |z| > r_6$, there exists $k \geq 7$ such that $r_k \geq |z| > r_{k-1}$. Therefore, thanks to (4.16),

$$(4.20) \quad |B(z)| \leq |D(z)| \leq |B_k(z)| \leq 1 - 2^{-k}.$$

On the other hand $r_k \geq |z| > r_{k-1}$ implies $h_k \leq 1 - |z| < h_{k-1}$, and so

$$(4.21) \quad \delta(1 - |z|) \leq \frac{1}{2}\chi(1 - |z|) \leq \frac{1}{2}\chi(h_{k-1}) \leq 2^{-k}.$$

Combining (4.20) and (4.21) we get $|B(z)| \leq 1 - \delta(1 - |z|)$, when $1 > |z| > r_6$. From this and (4.19), Theorem 4.4 follows. \square

Proof of Lemma 4.6. It is obvious that, for all $a, z \in \mathbb{C}$,

$$\prod_{k=1}^p (z - ae^{i\theta_k}) = z^p - a^p.$$

Using this we have:

$$(4.22) \quad G(z) = \prod_{k=1}^p \frac{r - e^{-i\theta_k} z}{1 - re^{-i\theta_k} z} = \prod_{k=1}^p \frac{z - re^{i\theta_k}}{rz - e^{i\theta_k}} = \frac{z^p - r^p}{(rz)^p - 1}.$$

Now, if $|z| = r$, we can write $z^p = r^p u$, for some u with $|u| = 1$. Then $|G(z)| = |T(u)|$, where T is the Moebius transformation

$$T(u) = \frac{r^p(u - 1)}{r^{2p}u - 1}.$$

This transformation T maps the unit circle $\partial\mathbb{D}$ onto a circumference C . As T maps the extended real line \mathbb{R}_∞ to itself, and $\partial\mathbb{D}$ is orthogonal to \mathbb{R}_∞ at the intersection points 1 and -1 , C is the circumference orthogonal to \mathbb{R}_∞ crossing through the points $T(1) = 0$ and $T(-1) = \alpha$. It is easy to see that $|w| \leq |\alpha|$, for every $w \in C$; consequently:

$$|G(z)| \leq \sup_{u \in \partial\mathbb{D}} |T(u)| = |T(-1)| = \frac{2r^p}{1 + r^{2p}}.$$

This finishes the proof of the statement (a).

To prove part (b), observe that, $1 + r^{2p} \leq 2$, and so, for $|z| = r$,

$$(4.23) \quad |G(z)| \leq 1 - \frac{(1 - r^p)^2}{1 + r^{2p}} \leq 1 - \frac{(1 - r^p)^2}{2}.$$

Remember that $r = 1 - h$, so $r \leq e^{-h}$, and $r^p \leq e^{-ph}$. Thus $1 - r^p \geq 1 - e^{-ph}$. Now, if $x \in [0, 1/2]$, we have, by the Mean Value theorem:

$$1 - e^{-x} \geq \frac{x}{\sqrt{e}}.$$

Since $ph \leq 1/2$, we can apply this last estimate to (4.23) to get, as promised,

$$|G(z)| \leq 1 - \frac{(1 - e^{-ph})^2}{2} \leq 1 - \frac{p^2 h^2}{2e},$$

and ending the proof of Lemma 4.6. \square

Remark. The key point in the proof of Theorem 4.4 is the inequality (4.10) in Lemma 4.6. This inequality may be viewed as a consequence of the strong triangle inequality (applied to $a = z^p$, $b = r^p$ and $c = 0$):

$$(4.24) \quad d(a, b) \leq \frac{d(a, c) + d(c, b)}{1 + d(a, c)d(c, b)}$$

for the pseudo-hyperbolic distance $d(u, v) = \frac{|u-v|}{|1-\bar{u}v|}$ on \mathbb{D} . Let us recall a proof for the convenience of the reader: by conformal invariance, we may assume that $c = 0$; then

$$1 - [d(a, b)]^2 = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \bar{a}b|^2} \geq \frac{(1 - |a|^2)(1 - |b|^2)}{(1 + |a||b|)^2} = 1 - [d(|a|, -|b|)]^2,$$

so that

$$d(a, b) \leq d(|a|, -|b|) = \frac{|a| + |b|}{1 + |a||b|},$$

proving (4.24), since $d(a, 0) = |a|$ and $d(0, b) = |b|$.

4.4 A Compact composition operator with a surjective symbol

A well-known result of J. H. Schwartz ([23], Theorem 2.8) asserts that $C_\varphi: H^\infty \rightarrow H^\infty$ is compact if and only if $\|\varphi\|_\infty < 1$. In particular, the compactness of $C_\varphi: H^\infty \rightarrow H^\infty$ prevents the surjectivity of φ . It may be therefore to be expected that, the bigger Ψ , the more difficult it will be to obtain both the compactness of $C_\varphi: H^\Psi \rightarrow H^\Psi$ and the surjectivity of φ . Nevertheless, this is possible, as says the following theorem, and the case H^∞ appears really as a singular case (corresponding to an ‘‘Orlicz’’ function which is discontinuous and can take infinite values).

Theorem 4.7 *For every Orlicz function Ψ , there exists a symbol $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ which is 4-valent and surjective and such that $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact.*

In the case of H^2 ($\Psi(x) = x^2$), B. McCluer and J. Shapiro ([17], Example 3.12) gave an example based on the Riemann mapping theorem and on the fact that, for a finitely valent symbol φ , we have the equivalence:

$$(4.25) \quad C_\varphi: H^2 \rightarrow H^2 \text{ compact} \iff \lim_{|z| \lesssim 1} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty.$$

A specific example is as follows. Take

$$(4.26) \quad R = \left\{ z = x + iy \in \mathbb{C}; x > 0 \text{ and } \frac{1}{x} < y < \frac{1}{x} + 4\pi \right\},$$

let $g: \mathbb{D} \rightarrow R$ be a Riemann map and set $\varphi = e^{-g}$. Then, φ is 2-valent, $\varphi(\mathbb{D}) = \mathbb{D}^*$ (where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$), and the validity of (4.25) is tested through the use of the Julia-Carathéodory theorem (see [21] for details). To get a fully surjective mapping φ_1 , just compose φ with a Blaschke product B of length 2:

$$\varphi_1(z) = B \circ \varphi, \quad \text{with } B(z) = z \left(\frac{z - \alpha}{1 - \bar{\alpha}z} \right), \quad \alpha \in \mathbb{D} \setminus \{0\}.$$

Since $C_{\varphi_1} = C_\varphi \circ C_B$, we see that C_{φ_1} is compact as well and we are done.

Here, we can no longer rely on the Julia-Carathéodory theorem. But we shall use the following necessary and sufficient condition, in terms of the maximal Carleson function ρ_φ , which is valid for any symbol, finitely-valent or not (see [10], Theorem 4.18, where a different, but equivalent, formulation is given):

$$(4.27) \quad C_\varphi: H^\Psi \rightarrow H^\Psi \text{ compact} \iff \lim_{h \geq 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} = 0.$$

For the sequel, we shall set:

$$(4.28) \quad \Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))}.$$

Our strategy will be to elaborate on the previous example to produce a (nearly) surjective φ such that $\rho_\varphi(h)$ is very small (depending on Ψ) for small h . The tool will be the notion of harmonic measure for certain open sets of the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called *hyperbolic* (see [6], Definition 19.9.3); for example, every conformal image of \mathbb{D} is hyperbolic (see [6], Proposition 19.9.2 (d) and Theorem 19.9.7). If G is a hyperbolic domain and $a \in G$, the *harmonic measure* of G at a is the probability measure $\omega_G(a, \cdot)$ supported by ∂G (in this section, the boundaries and the closures will be taken in $\hat{\mathbb{C}}$) such that:

$$u(a) = \int_{\partial G} u(z) d\omega_G(a, z)$$

for each bounded and continuous function u on \overline{G} , which is harmonic in G (see [6], Definition 21.1.3). The harmonic measure at a of a Borel set $A \subseteq \partial G$ will be denoted by $\omega_G(a, A)$. Clearly,

$$\omega_{\mathbb{D}}(0, \cdot) = m,$$

the Haar measure (*i.e.* normalized Lebesgue measure) of $\partial\mathbb{D}$.

R. Nevanlinna (see [6], Proposition 21.1.6) showed that harmonic measures share a *conformal invariance property*. Namely, assume that G is a simply connected domain, in which the Dirichlet problem can be solved (a *Dirichlet domain*), and that there is $\tau: \overline{\mathbb{D}} \rightarrow \overline{G}$ which maps conformally \mathbb{D} onto G and maps $\partial\mathbb{D}$ onto ∂G , one has, if $\tau(0) = a$:

$$(4.29) \quad \omega_G(a, A) = m(\tau^{-1}(A))$$

for every Borel set $A \subseteq \partial G$. This explains why harmonic measures enter the matter when we consider composition operators C_φ for which $m_\varphi = \varphi^*(m)$: m_φ is, by (4.29), nothing but the harmonic measure of G at a .

A useful alternative way of defining the harmonic measure, due to S. Kakutani, and completed by J. Doob (see [26], page 454, and [9], Appendix F, page 477) is the following: Let $(B_t)_{t>0}$ be the 2-dimensional Brownian motion starting at $a \in G$ (*i.e.* $B_0 = a$), and τ be the stopping time defined by

$$(4.30) \quad \tau = \inf\{t > 0; B_t \notin G\};$$

we have:

$$(4.31) \quad \omega_G(a, A) = \mathbb{P}_a(B_\tau \in A),$$

i.e. the harmonic measure of A at a is the probability that the Brownian motion starting at a exits from G through the Borel set $A \subseteq \partial G$. The following lemma will be basic for the construction of our example. We shall provide two proofs, the second one being more illuminating.

Lemma 4.8 (Hole principle) *Let G_0 and G_1 be two hyperbolic open sets and $H \subseteq \partial G_0$ a Borel set such that*

$$G_0 \subseteq G_1 \quad \text{and} \quad \partial G_0 \subseteq \partial G_1 \cup H.$$

Then, for every $a \in G_0$, we have the following inequality:

$$(4.32) \quad \omega_{G_1}(a, \partial G_1 \setminus \partial G_0) \leq \omega_{G_0}(a, H).$$

Proof 1. From [6], Corollary 21.1.14, with $\Delta = \partial G_0 \cap \partial G_1$, one has $\omega_{G_0}(a, \delta) \leq \omega_{G_1}(a, \Delta)$. But $\partial G_1 \setminus \Delta = \partial G_1 \setminus \partial G_0$, and hence, since harmonic measures are probability measures, $\omega_{G_1}(a, \partial G_1 \setminus \partial G_0) = \omega_{G_1}(a, \partial G_1 \setminus \Delta) = 1 - \omega_{G_1}(a, \Delta) \leq \omega_{G_0}(a, \Delta)$; we get the result since $\partial G_0 = H \cup \Delta$, which implies $1 \leq \omega_{G_0}(a, H) + \omega_{G_0}(a, \Delta)$. \square

Proof 2. Let us define

$$(4.33) \quad \tau_0 = \inf\{t > 0; B_t \notin G_0\}, \quad \tau_1 = \inf\{t > 0; B_t \notin G_1\}$$

and

$$(4.34) \quad E = \{B_{\tau_1} \in \partial G_1 \setminus \partial G_0\}, \quad F = \{B_{\tau_0} \in H\}.$$

Inequality (4.32) amounts to proving that $\mathbb{P}_a(E) \leq \mathbb{P}_a(F)$, which will follow from the inclusion $E \subseteq F$. Suppose that the event E holds. Since $G_0 \subseteq G_1$, one has $\tau_0 \leq \tau_1$. The Brownian path $(B_s)_{0 \leq s \leq \tau_1}$ being continuous with $B_0 = a \in G_0$, one has $B_{\tau_0} \in \partial G_0 \subseteq \partial G_1 \cup H$. If we had $B_{\tau_0} \in \partial G_1$, we should have $B_{\tau_0} \notin G_1$, since G_1 is open, and hence $\tau_0 = \tau_1$, since we know that $\tau_0 \leq \tau_1$. But then $B_{\tau_1} = B_{\tau_0} \in \partial G_0$, contrary to the definition of E . Therefore, $B_{\tau_0} \in H$ and F holds. \square

We also shall need the following result (see [6], Proposition 21.1.17).

Proposition 4.9 (Continuity principle) *If G is a hyperbolic open set and $a \in G$, then the harmonic measure $\omega_G(a, \cdot)$ is atomless.*

Proof of Theorem 4.7. It will be enough to construct a 2-valent mapping $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ such that $\varphi(\mathbb{D}) = \mathbb{D}^*$ and $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact. We can then modify φ by the same trick as the one used by B. McCluer and J. Shapiro. Note that every point in \mathbb{D}^* is the image by e^{-z} of two distinct points of R , except those which are the image of points of the hyperbola $y = (1/x) + 2\pi$, which have only one pre-image.

For a positive integer n , set

$$(4.35) \quad b_n = \frac{1}{4n\pi},$$

and let $\varepsilon_n > 0$ such that

$$(4.36) \quad \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n}.$$

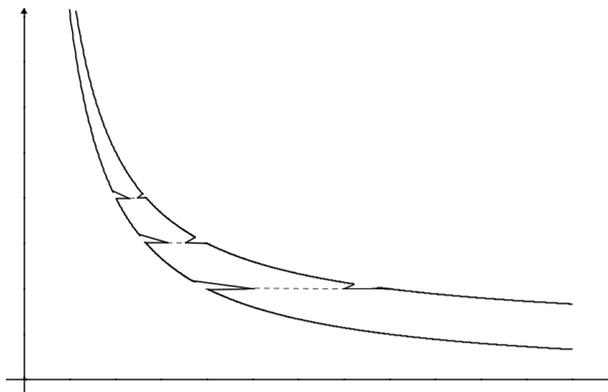
We now modify the domain R , including “barriers” in it (not in the sense of potential theory, nor of Perron!) in the following way.

Let, for every $n \geq 1$, M_n be the intersection point of the horizontal line $y = 4\pi n$ and of the hyperbola $y = (1/x) + 2\pi$, that is $M_n = \frac{1}{4\pi n - 2\pi} + 4\pi ni$.

Define inductively closed sets P_n^+ and P_n^- , which are like small points of swords (two segments and a piece of hyperbola), in the following way:

- The lower part of P_n^+ or P_n^- is a horizontal segment of altitude $4n\pi$.
- Those two horizontal segments are separated by a small open horizontal segment H_n whose middle is M_n .

- The upper part of P_n^+ is a slant segment whose upper extremity c_n^+ lies on the hyperbola $y = 1/x$.
- The upper part of P_n^- is a slant segment whose upper extremity c_n^- lies on the hyperbola $y = (1/x) + 4\pi$.
- The curvilinear part of P_n^+ is supported by the hyperbola $y = 1/x$.
- The curvilinear part of P_n^- is supported by the hyperbola $y = (1/x) + 4\pi$.
- One has $4(n+1)\pi - \Im c_n^\pm > 2\pi$.



The size of the small horizontal holes will be determined inductively in the following way. Fix once and for all $a \in \mathbb{R}$ such that $\Im a < 4\pi$. Suppose that H_1, H_2, \dots, H_{n-1} have already been determined. Set

$$(4.37) \quad \Omega_n = \left\{ z \in \mathbb{R} \setminus \bigcup_{j < n} (P_j^+ \cup P_j^-); \Im z < 4n\pi \right\}.$$

We can adjust H_n so small that

$$(4.38) \quad \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n.$$

Indeed, Ω_n is bounded above by the horizontal segment $[b_n + 4in\pi, b_{n-1} + 4in\pi]$ of middle, say, M_n , and if $H_n = [M_n - \delta, M_n + \delta]$, we see that H_n decreases to the singleton $\{M_n\}$ as δ decreases to zero. Therefore, by Proposition 4.9, we can adjust δ so as to realize (4.38).

We now define our modified open set Ω by the formula

$$(4.39) \quad \Omega = \mathbb{R} \setminus \bigcup_{n \geq 1} (P_n^+ \cup P_n^-) = \bigcup_{n \geq 1} \Omega_n.$$

It is useful to observe that

$$(4.40) \quad \inf_{w \in \partial \Omega_n} \Re w = b_n.$$

This is obvious by the way we defined the upper part of $\partial\Omega_n$.

Now, we can easily finish the proof. Fix $h \leq b_1/2$ and let n be the integer such that

$$b_{n+1} < 2h \leq b_n.$$

Let $g: \mathbb{D} \rightarrow \Omega$ be a conformal mapping such that $g(0) = a$. Since $\partial_\infty\Omega$ is connected, Caratheodory's Theorem (see [19]) ensures that g can be continuously extended from $\overline{\mathbb{D}}$ onto $\hat{\Omega}$. Set $\varphi = e^{-g}$. It is clear from the remark made at the beginning of the proof that $\varphi(\mathbb{D}) = \mathbb{D}^*$ and that φ is 2-valent. Moreover, Ω is a Dirichlet domain (because each component of $\partial\Omega$ has more than one point: see the comment after Definition 19.7.1 in [6]), so we can use the conformal invariance. Then by (4.29), (4.38), (4.40) and by the hole principle, we see that, if $A = \{\Re g^*(e^{it}) < 2h\}$:

$$\begin{aligned} \rho_\varphi(h) &\leq m_\varphi(\{|z| > 1 - h\}) = m(\{e^{-\Re g^*(e^{it})} > 1 - h\}) \\ &= m(\{\Re g^*(e^{it}) < \log(1/(1-h))\}) \\ &\leq m(\{\Re g^*(e^{it}) < 2h\}) = \omega_{\mathbb{D}}(0, A) \\ &= \omega_{g(\mathbb{D})}(g(0), g(A)) = \omega_\Omega(a, \{\Re w < 2h\}) \\ &\leq \omega_\Omega(a, \{\Re w \leq b_n\}) \\ &\leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n. \end{aligned}$$

It remains to observe that

$$\Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} \leq \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n} \leq Ch,$$

in view of (4.36) and of the choice of n , C being a numerical constant. We should point out the fact that we applied the hole principle to the domains $G_0 = \Omega_n$ and $G_1 = \Omega$ and that this was licit because the assumptions of the hole principle (in particular the inclusion $\partial\Omega_n \subseteq \partial\Omega \cup H_n$) are satisfied. We have therefore proved that:

$$\lim_{h \searrow 0} \Delta(h) = 0,$$

and this ends, as we already explained, the proof of Theorem 4.7. \square

4.5 Composition operators in Schatten classes

In [14], D. Luecking characterized composition operators $C_\varphi: H^2 \rightarrow H^2$ which are in the Schatten classes, by using, essentially, the m_φ -measure of Carleson windows. Five years later, D. Luecking and K. Zhu ([15]) characterized them by using the Nevanlinna counting function of φ . We shall see how our principal result makes these two characterizations directly equivalent.

It will be convenient here to work with *modified* Carleson windows, namely:

$$W_{n,j} = \left\{ z \in \overline{\mathbb{D}}; 1 - 2^{-n} \leq |z| \leq 1 \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

($j = 0, 1, \dots, 2^n - 1$, $n = 1, 2, \dots$). We shall say that $W_{n,j}$ is the Carleson window centered at $e^{2\pi ij/2^n}$ with size 2^{-n} .

Theorem 4.10 *For $p > 0$ the two following conditions are equivalent:*

- a) $\frac{N_\varphi(z)}{\log(1/|z|)} \in L^{p/2}(\lambda)$, where $d\lambda(z) = (1 - |z|)^{-2} dA(z)$ and A is the normalized area measure on \mathbb{D} ;
- b) $\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} [2^n m_\varphi(W_{n,j})]^{p/2} < \infty$.

Condition b) in the last theorem yields that $\lim_{n \rightarrow \infty} \max_j 2^n m_\varphi(W_{n,j}) = 0$, and it is not difficult to see that this implies that $m_\varphi(\partial\mathbb{D}) = 0$, or equivalently, that $|\varphi^*| < 1$ almost everywhere on $\partial\mathbb{D}$. In this situation we know ([11], Proposition 3.3) that b) in Theorem 4.10 is equivalent to Luecking's condition in [14]. In fact the characterization of belonging to a Schatten class in [14] includes the requirement $m_\varphi(\partial\mathbb{D}) = 0$.

Proof. We may, and do, assume that $\varphi(0) = 0$.

1) Assume first that condition b) is satisfied. Let

$$R_{n,j} = \left\{ z \in \mathbb{D}; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

be the (disjoint) Luecking windows ($0 \leq j \leq 2^n - 1$, $n \geq 0$). One has $R_{n,j} \subseteq W_{n,j}$.

By Theorem 3.1, there are a constant $C > 0$ and an integer K such that $N_\varphi(z) \leq C m_\varphi(\widetilde{W}_{n,j})$, for every $z \in R_{n,j}$, where $\widetilde{W}_{n,j}$ is the window centered at $e^{2\pi ij/2^n}$, as $W_{n,j}$, but with size 2^{K-n} . The windows $W_{n-K,j}$, $j = 0, 1, \dots, 2^{n-K} - 1$, have the same size as the windows $\widetilde{W}_{n,j}$, but may have a different center; nevertheless, each $\widetilde{W}_{n,j}$ can be covered with two windows $W_{n-K,l}$: for $n > K$, $\widetilde{W}_{n,j} \subseteq W_{n-K,l} \cup W_{n-K,l+1}$, for some $l = 1, 2, \dots, 2^{n-K}$ (where $l+1$ is understood as 0 if $l = 2^{n-K} - 1$), we get (we shall use \lesssim to mean \leq up to a constant):

$$\begin{aligned} \int_{\mathbb{D}} \frac{(N_\varphi(z))^{p/2}}{(1 - |z|)^{\frac{p}{2}+2}} dA(z) &\leq \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} (N_\varphi(z))^{p/2} dA(z) \\ &\lesssim \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} (m_\varphi(\widetilde{W}_{n,j}))^{p/2} dA(z) \\ &\lesssim \sum_{n,j} (2^n)^{p/2} (m_\varphi(\widetilde{W}_{n,j}))^{p/2} \\ &\lesssim \sum_{\nu,l} (2^\nu)^{p/2} (m_\varphi(W_{\nu,l}))^{p/2} < \infty, \end{aligned}$$

and a) holds.

2) Conversely, assume that $a)$ is satisfied. We shall use the following inequality, whose proof will be postponed (for $p \geq 2$, (4.41) follows directly from Theorem 3.7 and Hölder's inequality):

$$(4.41) \quad [m_\varphi(W_{n,j})]^{p/2} \lesssim \frac{1}{A(\widetilde{W}_{n,j})} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} dA(z),$$

where $\widetilde{W}_{n,j}$ is a window with the same center as $W_{n,j}$ but with a bigger proportional size; say of size 2^{-n+L} . We get

$$\begin{aligned} \sum_{n,j} [2^n m_\varphi(W_{n,j})]^{p/2} &\lesssim \sum_{n,j} 2^{np/2} 2^{2n} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} dA(z) \\ &= \int_{\mathbb{D}} \left(\sum_n 2^{n(2+\frac{p}{2})} \left[\sum_j \mathbf{1}_{\widetilde{W}_{n,j}}(z) \right] \right) [N_\varphi(z)]^{p/2} dA(z). \end{aligned}$$

Let $k = 0, 1, \dots$ such that $1 - 2^{-k+1} < |z| \leq 1 - 2^{-k}$. One has $z \in \widetilde{W}_{n,j}$ only if $n \leq k + L$, and then, for each such n , z is at most in 2^L windows $\widetilde{W}_{n,j}$. It follows that

$$\sum_n 2^{n(2+\frac{p}{2})} \sum_j \mathbf{1}_{\widetilde{W}_{n,j}}(z) \leq 2^{(k+L+1)(2+\frac{p}{2})} \times 2^L.$$

But $|z| \geq 1 - 2^{-k+1}$ implies $2^{(k+L+1)(2+\frac{p}{2})} \leq C_p / (1 - |z|)^{2+\frac{p}{2}}$; hence:

$$\sum_{n,j} [2^n m_\varphi(W_{n,j})]^{p/2} \lesssim \int_{\mathbb{D}} \frac{[N_\varphi(z)]^{p/2}}{(1 - |z|)^{\frac{p}{2}+2}} dA(z) < \infty,$$

and $b)$ holds.

It remains to show (4.41).

By Theorem 3.6, we can find a window W with the same center as $W_{n,j}$, but with greater size ch ($h = 2^{-n}$ is the size of the window $W_{n,j}$), such that

$$m_\varphi(W_{n,j}) \lesssim \sup_{w \in W} N_\varphi(w).$$

There is hence some $w_0 \in W$ such that

$$m_\varphi(W_{n,j}) \lesssim N_\varphi(w_0).$$

Take $R = |w_0| + ch$ (one has $R \geq 1$ since $w_0 \in W$ and W has size ch) and set $\varphi_0(z) = \varphi(z)/R$. One has $N_{\varphi_0}(z) = N_\varphi(Rz)$ for $|z| < 1/R$ and $N_{\varphi_0}(z) = 0$ if $|z| \geq 1/R$.

Let now u be the upper subharmonic regularization of N_{φ_0} ([15], Lemma 1, and its proof page 1140): u is a subharmonic function on $\mathbb{D} \setminus \{0\}$ such that $u \geq N_{\varphi_0}$ and $u = N_{\varphi_0}$ almost everywhere, with respect to dA .

A result of C. Fefferman and E. M. Stein ([8], Lemma 2), generously attributed by them to Hardy and Littlewood, asserts that for any $q > 0$, there exists a constant $C = C(q)$ such that

$$(4.42) \quad [u(a)]^q \leq \frac{C}{A(D(a, r))} \int_{D(a, r)} [u(z)]^q dA(z)$$

for every nonnegative subharmonic function u on a domain G and every disk $D(a, r) \subseteq G$ (see also [15], Lemma 3).

If Δ is the disk centered at w_0/R and of radius $1 - |w_0|/R$ (which is contained in $\mathbb{D} \setminus \{0\}$ since $R > |w_0|$), one has, by (4.42):

$$\begin{aligned} [N_\varphi(w_0)]^{p/2} &= [N_{\varphi_0}(w_0/R)]^{p/2} \leq [u(w_0/R)]^{p/2} \\ &\leq \frac{C}{A(\Delta)} \int_{\Delta} [u(z)]^{p/2} dA(z) \\ &= \frac{C}{A(\Delta)} \int_{\Delta} [N_{\varphi_0}(z)]^{p/2} dA(z) \\ &= \frac{C}{A(\Delta)} \int_{\Delta \cap D(0, 1/R)} [N_\varphi(Rz)]^{p/2} dA(z) \\ &= \frac{C}{A(\tilde{\Delta})} \int_{\tilde{\Delta} \cap \mathbb{D}} [N_\varphi(w)]^{p/2} dA(w), \end{aligned}$$

where $\tilde{\Delta} = D(w_0, R - |w_0|) = D(w_0, ch)$.

Since the center w_0 of $\tilde{\Delta}$ is in \mathbb{D} , $\tilde{\Delta} \cap \mathbb{D}$ contains more than a quarter of $\tilde{\Delta}$ (at least for $ch \leq 1$), and hence $A(\tilde{\Delta} \cap \mathbb{D}) \geq A(\tilde{\Delta})/4 = c^2 h^2 / 4\pi$. Now, let $\tilde{W}_{n,j}$ be the window with the same center as $W_{n,j}$ and of size $2ch$. Since $2ch \geq ch + (1 - |w_0|)$, $\tilde{W}_{n,j}$ contains $\tilde{\Delta} \cap \mathbb{D}$ and $A(\tilde{W}_{n,j}) \approx h^2 \approx A(\tilde{\Delta})$ (\approx meaning that the ratio is between two absolute constants). We therefore get:

$$[N_\varphi(w_0)]^{p/2} \lesssim \frac{1}{A(\tilde{W}_{n,j})} \int_{\tilde{W}_{n,j}} [N_\varphi(w)]^{p/2} dA(w),$$

proving (4.41). □

4.6 Composition operators with closed range

In [5], J. Cima, J. Thomson and W. Wogen gave a characterization of composition operators $C_\varphi: H^p \rightarrow H^p$ with closed range. This characterization involves the Radon-Nikodym derivative of the restriction to $\partial\mathbb{D}$ of m_φ . They found it not satisfactory, and asked a characterization with the range of φ itself. N. Zorboska ([28]) gave such a characterization, but her statement is somewhat complicated. We shall give here more explicit characterizations, either in terms of the Nevanlinna counting function N_φ , or in terms of the Carleson measure m_φ .

Theorem 4.11 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be a non-constant analytic self map. Then the composition operator $C_\varphi: H^p \rightarrow H^p$, $1 \leq p < \infty$, has a closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(4.43) \quad \frac{1}{A(S(\xi, h))} \int_{S(\xi, h)} N_\varphi(z) dA(z) \geq ch, \quad \forall \xi \in \partial\mathbb{D}.$$

Theorem 4.11 will follow immediately from the next theorem, applied to $\mu = m_\varphi$, and from Theorem 3.7.

Theorem 4.12 *Let μ be a finite positive measure on $\overline{\mathbb{D}}$. Assume that the canonical map $J: H^p \rightarrow L^p(\mu)$ is continuous, $1 \leq p < \infty$. Then J is one-to-one and has a closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(4.44) \quad \mu[W(\xi, h)] \geq ch, \quad \forall \xi \in \partial\mathbb{D}.$$

Proof. 1) Assume that J has a closed range. By making a rotation on the variable z , we only have to find a constant $c > 0$ such that

$$(4.45) \quad \mu(S_h) \geq ch,$$

for $h > 0$ small enough, where $S_h = S(1, h)$.

Since J is one-to-one, there is a constant $C > 0$ such that

$$(4.46) \quad \|f\|_{L^p(\mu)}^p \geq C^p \|f\|_p^p, \quad \forall f \in H^p.$$

We are going to test (4.46) on

$$(4.47) \quad f_N(z) = \left(\frac{1+z}{2} \right)^N.$$

It is classical that there is a constant $c_p > 0$ such that

$$(4.48) \quad \|f_N\|_p^p = \int_{-\pi}^{\pi} \left| \cos \frac{t}{2} \right|^{pN} dt \geq \frac{c_p}{\sqrt{N}}.$$

Now, since $|z+1|^2 + |z-1|^2 = 2(|z|^2 + 1) \leq 4$ for every $z \in \overline{\mathbb{D}}$, one has

$$|f_N(z)| \leq \left(1 - \frac{|z-1|^2}{4} \right)^{N/2} \leq e^{-\frac{N}{8}|z-1|^2}.$$

Hence, using $|f_N(z)| \leq 1$ when $|z-1| \leq h$, one has

$$\begin{aligned} \|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_{|z-1|>h} e^{-p\frac{N}{8}|z-1|^2} d\mu \\ &= \mu(S_h) + \int_0^{e^{-pNh^2/8}} \mu(\{e^{-p\frac{N}{8}|z-1|^2} > u\}) du, \end{aligned}$$

that is, making the change of variable $u = e^{-p\frac{N}{8}x^2}$,

$$\|f_N\|_{L^p(\mu)}^p \leq \mu(S_h) + \int_h^\infty \mu(\{|z-1| \leq x\}) \frac{pN}{4} x e^{-p\frac{N}{8}x^2} dx.$$

Now, the continuity of J means, by Carleson's Theorem, that there is a constant $K > 0$ such that

$$(4.49) \quad \sup_{|\xi|=1} \mu(S(\xi, x)) \leq Kx, \quad 0 \leq x < 1.$$

We get hence:

$$\begin{aligned} \|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_h^\infty Kx \frac{pN}{4} x e^{-p\frac{N}{8}x^2} dx \\ &= \mu(S_h) + \frac{K\sqrt{8}}{\sqrt{p}} \frac{1}{\sqrt{N}} \int_{h\sqrt{\frac{pN}{8}}}^\infty y^2 e^{-y^2} dy. \end{aligned}$$

We take now for N the smaller integer $> 1/h^2$, multiplied by some constant integer a_p , large enough to have

$$\frac{K\sqrt{8}}{\sqrt{p}} \int_{\sqrt{\frac{p a_p}{8}}}^\infty y^2 e^{-y^2} dy \leq \frac{c_p C^p}{2}.$$

We get then, from (4.46) and (4.48):

$$\mu(S_h) \geq \frac{C^p c_p}{2} \frac{1}{\sqrt{N}},$$

which gives (4.45).

2) Conversely, assume that (4.44) holds. Since the disk algebra $A(\mathbb{D})$ is dense in H^p , it suffices to show that there exists a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \geq C \|f\|_p$ for every $f \in A(\mathbb{D})$.

Let $f \in A(\mathbb{D})$ such that $\|f\|_p = 1$. Choose an integer N such that

$$\frac{1}{N} \sum_{n=1}^N |f(e^{2\pi i n/N})|^p \geq \frac{1}{2} \int_{\partial\mathbb{D}} |f(\xi)|^p dm(\xi) = \frac{1}{2},$$

and such that, due to the uniform continuity of f ,

$$z, z' \in \overline{\mathbb{D}} \quad \text{and} \quad |z - z'| \leq \frac{2\pi}{N} \quad \implies \quad |f(z) - f(z')| \leq \frac{1}{2^{(p+1)/p}}.$$

Then, setting $W_n = W(e^{2\pi i n/N}, \pi/N)$, $1 \leq n \leq N$, one has

$$\|f\|_{L^p(\mu)}^p = \int_{\overline{\mathbb{D}}} |f|^p d\mu \geq \sum_{n=1}^N \int_{W_n} |f|^p d\mu.$$

If we choose $z_n \in W_n$ such that $|f(z_n)| = \min_{z \in W_n} |f(z)|$, we get, using (4.44):

$$\|f\|_{L^p(\mu)}^p \geq \sum_{n=1}^N |f(z_n)|^p \mu(W_n) \geq \frac{c\pi}{N} \sum_{n=1}^N |f(z_n)|^p.$$

Since $A^p \leq 2^{p-1}[(A-B)^p + B^p]$, by Hölder's inequality, one has

$$|f(z_n)|^p \geq \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - |f(z_n) - f(e^{2\pi in/N})|^p$$

and hence

$$\|f\|_{L^p(\mu)}^p \geq \frac{c\pi}{N} \sum_{n=1}^N \left[\frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - |f(z_n) - f(e^{2\pi in/N})|^p \right].$$

Now, since $z_n \in W_n$, one has

$$|z_n - e^{2\pi in/N}| \leq \left| z_n - \frac{z_n}{|z_n|} \right| + \left| \frac{z_n}{|z_n|} - e^{2\pi in/N} \right| \leq \frac{\pi}{N} + \frac{\pi}{N} = \frac{2\pi}{N};$$

therefore $|f(z_n) - f(e^{2\pi in/N})| \leq 1/2^{p+1}$ and we get:

$$\begin{aligned} \|f\|_{L^p(\mu)}^p &\geq c\pi \left[\frac{1}{N} \sum_{n=1}^N \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - \frac{1}{2^{p+1}} \right] \\ &\geq c\pi \left(\frac{1}{2^{p-1}} \frac{1}{2} - \frac{1}{2^{p+1}} \right) = \frac{c\pi}{2^{p+1}}. \end{aligned}$$

That ends the proof of Theorem 4.12. \square

Remark. To make the link with Cima-Thomson-Wogen's criterion, we shall see that condition 4.44 implies that the restriction of μ to the boundary $\mathbb{T} = \partial\mathbb{D}$ of the disk dominates the Lebesgue measure m . In fact, let I be an arc of \mathbb{T} . If $m(I) = h$, we can write

$$I = \bigcap_{n \geq 1} \bigcup_{j=1}^n W(\xi_{n,j}, h/2n),$$

with disjoint windows $W(\xi_{n,1}, h/2n), \dots, W(\xi_{n,n}, h/2n)$; hence

$$\mu(I) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu[W(\xi_{n,j}, h/2n)] \geq c \sum_{j=1}^n \frac{h}{2n} = \frac{c}{2} h.$$

Acknowledgement. This work has been initiated when the second-named author was invited by the Departamento de Análisis Matemático of the Universidad de Sevilla, in April 2007; it is a pleasure to thanks all its members for their warm hospitality. Part of this work was also made during the fourth-named author visited the University of Lille 1 and the University of Artois (Lens) in June 2007 and in June 2009. This fourth-named author is partially supported by a Spanish research project MTM2006-05622.

References

- [1] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 (1958), 921–930.
- [2] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Annals of Math. (2) 76 (1962), 547–559.
- [3] J. S. Choa and H.O. Kim, On function-theoretic conditions characterizing compact composition operators on H^2 , Proc. Japan Acad. 75, Ser. A (1999), 109–112.
- [4] B. R. Choe, The essential norms of composition operators, Glasg. Math. J. 34, No. 2 (1992), 143–155.
- [5] J. Cima, J. Thomson and W. Wogen, On some properties of composition operators, Indiana Univ. Math. J. 24 (3) (1974), 215–220.
- [6] J. B. Conway, Functions of One Complex Variable II, Graduate Texts in Math. 159, Springer-Verlag (1995)
- [7] M. Essén, D. F. Shea, and C. S. Stanton, A value-distribution criterion for the class $L \log L$ and some related questions, Ann. Inst. Fourier (Grenoble), 35 (1985), 125–150.
- [8] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–193.
- [9] J. B. Garnett and D. E. Marshall, Harmonic measure, New Mathematical Monographs 2, Cambridge University Press, Cambridge (2005).
- [10] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *preprint*, math.FA/0610905, to appear in Memoirs Amer. Math. Soc.
- [11] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some examples of compact composition operators on H^2 , J. Funct. Anal. 255, No. 11 (2008), 3098–3124.
- [12] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on H^2 and Hardy-Orlicz spaces, J. Math. Anal. Appl. 354 (2009), 360–371.
- [13] J. E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc. (2) 23 (1925), 481–519.
- [14] D. H. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. 73 (1987), 345–368.
- [15] D. H. Luecking and K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992), 878–906.

- [16] B. McCluer, Compact composition operators on $H^p(B_N)$, Michigan Math. J. 32 (1985), 237–248.
- [17] B. McCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38, no. 4 (1986), 878–906.
- [18] R. Nevanlinna, Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes, second edition, Chelsea Publishing Company, New York (1974).
- [19] C. Pommerenke, Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften 299, Springer-Verlag (1992).
- [20] S. C. Power, Vanishing Carleson measures, Bull. London Math. Soc. 12 (1980), no. 3, 207–210.
- [21] H. Queffélec, Carleson measures and composition operators, Proceed. 2008 OT Conference in Timisoara, *to appear*.
- [22] L. A. Rubel, Entire and Meromorphic Functions, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1996).
- [23] H. J. Schwartz, Composition operators on H^p , Thesis, University of Toledo (1969).
- [24] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1993).
- [25] J. H. Shapiro, The essential norm of a composition operator, Annals of Math. 125 (1987), 375–404.
- [26] D. W. Stroock, Probability Theory, An Analytic View, Cambridge University Press, Cambridge (1994).
- [27] K. Zhu, Operator Theory in Function Spaces, Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, Inc. (1990).
- [28] N. Zorboska, Composition operators with closed range, Trans. Amer. Math. Soc. 334 (2) (1994), 791–801.

Pascal Lefèvre, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
UArtois, Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
F-62 300 LENS, FRANCE
pascal.lefevre@euler.univ-artois.fr

Daniel Li, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
UArtois, Laboratoire de Mathématiques de Lens EA 2462,
Fédération CNRS Nord-Pas-de-Calais FR 2956,
Faculté des Sciences Jean Perrin,
Rue Jean Souvraz, S.P. 18,
F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec, *Univ Lille Nord de France F-59 000 LILLE, FRANCE*
USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524,
F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
queff@math.univ-lille1.fr

Luis Rodríguez-Piazza, *Universidad de Sevilla,*
Facultad de Matemáticas, Departamento de Análisis Matemático,
Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es