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Nevanlinna counting function and Carleson function of analytic maps

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Abstract. *We show that the maximal Nevanlinna counting function and the Carleson function of analytic self-maps of the unit disk are equivalent, up to constants.*

Mathematics Subject Classification. Primary: 30C80 – Secondary: 47B33; 47B10

Key-words. Carleson function – Carleson measure – composition operator – Nevanlinna counting function

1 Introduction

Carleson measures and Nevanlinna counting function are two classical concepts in Complex analysis. Carleson measures appeared when L. Carleson ([1], [2]) characterized measures μ on the closed unit disk $\overline{\mathbb{D}}$ for which there is a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \leq C \|f\|_{H^p}$, for every f in the Hardy space H^p , $0 < p < \infty$: μ is such a measure if and only if $\sup_{|\xi|=1} \mu[W(\xi, h)] = O(h)$, where $W(\xi, h)$ is the Carleson window of size h centered at ξ . This supremum is called the Carleson function ρ_μ of μ . If φ is an analytic self-map of \mathbb{D} and $\mu = m_\varphi$ is the image by φ^* , the boundary values function of φ , of the Lebesgue measure on the circle, $\rho_\varphi = \rho_\mu$ is said to be the Carleson function of φ . Nevanlinna counting function traces back earlier, and Littlewood ([9]) used it when he showed that, for every analytic self-map φ of \mathbb{D} , the composition operator $f \mapsto f \circ \varphi$ is continuous on H^p (which means that m_φ is a Carleson measure).

Compact composition operators on H^2 have been characterized in terms of the Carleson function of their symbol φ : $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, by B. McCluer ([10]), and in terms of the Nevanlinna counting function N_φ of this symbol: $N_\varphi(w) = o(1 - |w|)$, as $|w| \rightarrow 1$, by J. Shapiro ([13]). There should therefore exist a link between these two quantities. Some results in this direction had been given: B. R. Choe ([4]) showed that $\limsup_{h \rightarrow 0} (\rho_\varphi(h)/h)^{1/2}$ is equivalent, up to constants, to the distance of C_φ to the space of compact operators on H^2 ; since J. Shapiro proved ([13]) that this distance is $\limsup_{|w| \rightarrow 1} (N_\varphi(w)/\log |w|)^{1/2}$, one gets that $\limsup_{|w| \rightarrow 1} N_\varphi(w)/\log |w| \approx \limsup_{h \rightarrow 0} \rho_\varphi(h)/h$. Later, J. S.

Choa and H. O. Kim ([3]) gave a somewhat direct proof of the equivalence of the two above conditions, without using the properties of the composition operator, but without giving explicitly a direct relation between them.

In this paper, we show that the Nevanlinna counting function and the Carleson function are equivalent in the following sense:

Theorem 1.1 *There exist some universal constants $C, c > 1$, such that, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has:*

$$(1.1) \quad (1/C) \rho_\varphi(h/c) \leq \sup_{|w| \geq 1-h} N_\varphi(w) \leq C \rho_\varphi(ch),$$

for $0 < h < 1$ small enough.

More precisely, for every $\xi \in \partial\mathbb{D}$, one has:

$$(1.2) \quad (1/64) m_\varphi[W(\xi, h/64)] \leq \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi[W(\xi, 24h)],$$

for $0 < h < (1 - |\varphi(0)|)/16$.

2 Notation

We shall denote by $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ the open unit disc of the complex plane and by $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ its boundary; m will be the normalized Lebesgue measure $dt/2\pi$ on \mathbb{T} , and A the normalized Lebesgue measure $dxdy/\pi$ on $\overline{\mathbb{D}}$. For every analytic self-map φ of \mathbb{D} , m_φ will be the pull-back measure of m by φ^* , where φ^* is the boundary values function of φ .

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ is the set

$$(2.1) \quad W(\xi, h) = \{z \in \overline{\mathbb{D}}; |z| \geq 1-h \text{ and } |\arg(z\bar{\xi})| \leq h\}.$$

For convenience, we shall set $W(\xi, h) = \overline{\mathbb{D}}$ for $h \geq 1$.

For every analytic self-map φ of \mathbb{D} , one defines the maximal function of m_φ , for $0 < h < 1$, by:

$$(2.2) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m(\{\zeta \in \mathbb{T}; \varphi^*(\zeta) \in W(\xi, h)\}) = \sup_{\xi \in \mathbb{T}} m_\varphi(W(\xi, h)).$$

We have $\rho_\varphi(h) = 1$ for $h \geq 1$. We shall call this function ρ_φ the *Carleson function* of φ . For convenience, we shall often also use, instead of the Carleson window $W(\xi, h)$, the set $S(\xi, h) = \{z \in \overline{\mathbb{D}}; |z - \xi| \leq h\}$, which has an equivalent measure.

The Nevanlinna counting function N_φ is defined, for $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, by

$$(2.3) \quad N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

each term $\log \frac{1}{|z|}$ being repeated according to the multiplicity of z , and $N_\varphi(w) = 0$ for the other $w \in \mathbb{D}$. Its maximal function will be denoted by

$$(2.4) \quad \nu_\varphi(t) = \sup_{|w| \geq 1-t} N_\varphi(w).$$

3 Majorizing the Nevanlinna counting function by the Carleson function

The goal of this section is to prove:

Theorem 3.1 *For every analytic self-map φ of \mathbb{D} , one has, for every $a \in \mathbb{D}$:*

$$(3.1) \quad N_\varphi(a) \leq 196 m_\varphi(W(\xi, 12h)),$$

for $0 < h < (1 - |\varphi(0)|)/4$, where $\xi = \frac{a}{|a|}$ and $h = 1 - |a|$.

In particular, for $0 < h < (1 - |\varphi(0)|)/4$:

$$(3.2) \quad \nu_\varphi(h) = \sup_{|a| \geq 1-h} N_\varphi(a) \leq 196 \rho_\varphi(12h).$$

Let us note that, since $W(\zeta, s) \subseteq W(\xi, 2t)$ whenever $0 < s \leq t$ and $\zeta \in W(\xi, t) \cap \partial\mathbb{D}$, we get from (3.1) that

$$(3.3) \quad \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w) \leq 196 m_\varphi(W(\xi, 24h)).$$

We shall first prove the following lemma.

Lemma 3.2 *Let φ be an analytic self map of \mathbb{D} . For every $z \in \mathbb{D}$, one has, if $w = \varphi(z)$, $\xi = w/|w|$ and $h = 1 - |w| \leq 1/4$:*

$$(3.4) \quad m_\varphi(W(\xi, 12h)) \geq m_\varphi(S(\xi, 6h)) \geq \frac{|w|}{8} (1 - |z|).$$

Proof. We may assume, by making a rotation, that w is real and positive: $3/4 \leq w < 1$.

Let:

$$(3.5) \quad T(u) = \frac{au + 1}{u + a},$$

where

$$a = w - \frac{2}{w} < -1,$$

so that $T: \mathbb{D} \rightarrow \mathbb{D}$ is analytic, and $T(w) = w/2$.

If P_z is the Poisson kernel at z , one has:

$$\frac{w}{2} = T[\varphi(z)] = \int_{\mathbb{T}} (T \circ \varphi)^* P_z dm = \int_{\mathbb{T}} \Re [(T \circ \varphi)^*] P_z dm.$$

Hence, if one sets:

$$E = \{\Re (T \circ \varphi^*) \geq w/4\} = \{\Re [(T \circ \varphi)^*] \geq w/4\},$$

one has:

$$\frac{w}{2} \leq \int_E P_z dm + \frac{w}{4} \int_{E^c} P_z dm \leq \int_E P_z dm + \frac{w}{4} \int_{\mathbb{D}} P_z dm = \int_E P_z dm + \frac{w}{4};$$

therefore:

$$\int_E P_z dm \geq \frac{w}{4}.$$

Since

$$\|P_z\|_\infty = \frac{1+|z|}{1-|z|} \leq \frac{2}{1-|z|},$$

we get:

$$(3.6) \quad m(E) \geq \frac{w}{8} (1-|z|).$$

On the other hand, (3.5) writes

$$(3.7) \quad u = T^{-1}(U) = \frac{aU-1}{a-U};$$

hence:

$$|1-u| = |a+1| \frac{|1-U|}{|a-U|} \leq \frac{2|a+1|}{|a-U|}.$$

But $a < -1$ is negative, so $\Re U \geq w/4$ implies that

$$|a-U| \geq \Re(U-a) \geq \frac{w}{4} - a = \frac{2}{w} - \frac{3}{4}w \geq \frac{5}{4}.$$

Moreover, for $w \geq 3/4$:

$$|a+1| = (1-w) \left(\frac{2}{w} + 1 \right) \leq \frac{11}{3} (1-w).$$

We get hence $|1-u| \leq 6h$ when (3.7) holds and $\Re U \geq w/4$.

It follows that:

$$(3.8) \quad \varphi^*(E) \subseteq T^{-1}(\{\Re U \geq w/4\}) \subseteq S(1, 6h),$$

giving $m_\varphi(W(1, 12h)) \geq m_\varphi(S(1, 6h)) \geq m(E)$.

Combining this with (3.6), that finishes the proof. \square

Remark. Theorem 3.1 follows immediately when φ is univalent since then, for $|w| \geq 3/4$ and $\varphi(z) = w$:

$$N_\varphi(w) = \log \frac{1}{|z|} \approx (1-|z|) \lesssim m_\varphi(W(1, 12h)).$$

When proving the equivalence between the conditions $\rho_\varphi(h) = o(h)$, as $h \rightarrow 0$, and $N_\varphi(w) = o(1-|w|)$, as $|w| \rightarrow 1$, J. S. Choa and H. O. Kim proved

(see [3], page 112) the following inequality, for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every $w \in \mathbb{D}$, close enough to 1:

$$(3.9) \quad N_\varphi(w) \leq \frac{(1-|w|^2)^2}{8|w|^2} \int_{\partial\mathbb{D}} \frac{1}{|1-\bar{w}\varphi(z)|^2} dm(z).$$

This result follows from an Hilbertian method, *viz.* Littlewood-Paley's identity:

$$(3.10) \quad \|f \circ \varphi\|_2^2 = |f \circ \varphi(0)|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w)$$

for every $f \in H^2$. With (3.9), one cannot go beyond the order 2; for instance, we can deduce from (3.9) (see the proof of Theorem 3.1 below), that, for $0 < h \leq 1/2$:

$$(3.11) \quad \sup_{|w|=1-h} N_\varphi(w) \lesssim h^2 \int_0^{1/h^2} \rho_\varphi\left(\frac{1}{\sqrt{t}}\right) dt \lesssim h^2 + h^2 \int_h^1 \frac{\rho_\varphi(u)}{u^3} du.$$

This is of course interesting only when the second term in the last sum is at most of order h^2 , so, when the integral is bounded. Nevertheless, this result suffices to show that Shapiro's criterion of compactness for $C_\varphi: H^2 \rightarrow H^2$ is implied by McCluer's one. Moreover, when the pull-back measure m_φ is an α -Carleson measure (*i.e.* $\rho_\varphi(h) \leq Ch^\alpha$ for some constant $C > 0$), with $1 \leq \alpha \leq 2$, we get

$$N_\varphi(w) \lesssim h^2 + h^2 \int_h^1 \frac{u^\alpha}{u^3} du \lesssim h^2 + h^2 h^{\alpha-2} \lesssim h^\alpha.$$

Recall ([7], Corollary 3.2) that, when m_φ is an α -Carleson measure, the composition operator C_φ is in the Schatten class S_p on the Hardy space H^2 , for every $p > 2/(\alpha - 1)$, and that m_φ is α -Carleson for every $\alpha \geq 1$ when $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact, if Ψ is an Orlicz function satisfying the growth condition Δ^2 ([8], Theorem 5.2).

But (3.11) does not suffice for the compactness of $C_\varphi: H^\Psi \rightarrow H^\Psi$ on general Hardy-Orlicz spaces (see [6]).

In order to prove Theorem 3.1, we shall replace the Littlewood-Paley identity, by a more general formula, deduced from Stanton's formula (see [5], Theorem 2).

Theorem 3.3 (Stanton's formula) *For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ and every subharmonic function $G: \mathbb{D} \rightarrow \mathbb{R}$, one has:*

$$(3.12) \quad \lim_{r \uparrow 1} \int_{\partial\mathbb{D}} G[\varphi(r\xi)] dm(\xi) = G[\varphi(0)] + \frac{1}{2} \int_{\mathbb{D}} \Delta G(w) N_\varphi(w) dA(w),$$

where Δ is the distributional Laplacian.

Proof of Theorem 3.1. If $a \notin \varphi(\mathbb{D})$, one has $N_\varphi(a) = 0$, and the result is trivial. We shall hence assume that $a \in \varphi(\mathbb{D})$.

Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function, that is a non-decreasing convex function such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, and we assume that Φ' is also an Orlicz function. In other words, Φ'' is an arbitrary non-negative and non-decreasing function and $\Phi'(x) = \int_0^x \Phi''(t) dt$ and $\Phi(x) = \int_0^x \Phi'(t) dt$.

Let now $f: \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. We have, outside the zeroes of f , in writing $\Delta\Phi(|f|) = 4\partial\bar{\partial}\Phi(\sqrt{|f|^2})$:

$$(3.13) \quad \Delta\Phi(|f|) = \left[\Phi''(|f|) + \frac{\Phi'(|f|)}{|f|} \right] |f'|^2.$$

We shall only use here that:

$$(3.14) \quad \Delta\Phi(|f|) \geq \Phi''(|f|) |f'|^2$$

(this is a not too crude estimate, since, Φ' being an Orlicz function, Φ'' is non-negative and non-decreasing, and hence $\Phi'(x) = \int_0^x \Phi''(t) dt \leq x\Phi''(x)$ and $\Phi'(x) = \int_0^x \Phi''(t) dt \geq \int_{x/2}^x \Phi''(t) dt \geq (x/2)\Phi''(x/2)$).

Set now, for $a \in \mathbb{D}$:

$$(3.15) \quad f_a(z) = \frac{1 - |a|}{1 - \bar{a}z}, \quad z \in \overline{\mathbb{D}}.$$

Since $\Phi(|f_a|)$ is subharmonic (Φ being convex and non-decreasing) and bounded, we can use Stanton's formula as:

$$(3.16) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \int_{\mathbb{D}} \Phi''(|f_a|) |f'_a|^2 N_\varphi dA.$$

Let $h = 1 - |a|$. For $|z - a| < h$, one has

$$|1 - \bar{a}z| = |(1 - |a|^2) + \bar{a}(a - z)| \leq (1 - |a|^2) + |a - z| \leq 2h + h = 3h;$$

Hence $|f_a(z)| \geq \frac{h}{3h} = \frac{1}{3}$ for $|z - a| < h$. It follows, since Φ'' is non-decreasing:

$$(3.17) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \int_{D(a,h)} |f'_a|^2 N_\varphi dA.$$

Now, if $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, one has $|f'_a(z)| = \frac{|a|}{1+|a|} |\varphi'_a(z)| \geq \frac{3}{7} |\varphi'_a(z)|$ (we may, and do, assume that $1 - |a| = h \leq 1/4$); hence:

$$\begin{aligned} \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &\geq \frac{1}{2} \Phi''\left(\frac{1}{3}\right) \frac{9}{49} \int_{D(a,h)} |\varphi'_a|^2 N_\varphi dA \\ &= \frac{9}{98} \Phi''\left(\frac{1}{3}\right) \int_{\varphi_a(D(a,h))} N_{\varphi_a \circ \varphi} dA \end{aligned}$$

(because $N_{\varphi_a \circ \varphi}(\varphi_a(w)) = N_\varphi(w)$ and $\varphi_a^{-1} = \varphi_a$).

But $\varphi_a(D(a, h)) \supseteq D(0, 1/3)$: indeed, if $|w| < 1/3$, then $w = \varphi_a(z)$, with

$$|a - z| = \left| \frac{(1 - |a|^2)w}{1 - \bar{a}w} \right| \leq (1 - |a|^2) \frac{|w|}{1 - |w|} < 2h \frac{1/3}{1 - 1/3} = h.$$

We are going now to use the sub-averaging property of the Nevanlinna function ([12], page 190, [13], § 4.6, or [14], Proposition 10.2.4): for every analytic self-map $\psi: \mathbb{D} \rightarrow \mathbb{D}$, one has

$$N_\psi(w_0) \leq \frac{1}{A(\Delta)} \int_\Delta N_\psi(w) dA(w),$$

for every disk Δ of center w_0 which does not contain $\psi(0)$.

This will be possible thanks to the following:

Lemma 3.4 *For $1 - |a| < (1 - |\varphi(0)|)/4$, one has $|(\varphi_a \circ \varphi)(0)| > 1/3$.*

Proof. One has $|1 - \bar{a}\varphi(0)| \leq (1 - |a|^2) + |\bar{a}| |a - \varphi(0)| \leq (1 - |a|^2) + |a - \varphi(0)|$; hence:

$$\begin{aligned} |\varphi_a(\varphi(0))| &\geq \frac{|a - \varphi(0)|}{(1 - |a|^2) + |a - \varphi(0)|} \geq 1 - \frac{1 - |a|^2}{(1 - |a|^2) + |a - \varphi(0)|} \\ &\geq 1 - \frac{1 - |a|^2}{|a - \varphi(0)|} \geq 1 - 2 \frac{1 - |a|}{|a - \varphi(0)|}. \end{aligned}$$

But when $1 - |a| < (1 - |\varphi(0)|)/4$, one has:

$$|a - \varphi(0)| \geq |a| - |\varphi(0)| = (1 - |\varphi(0)|) - (1 - |a|) > 3(1 - |a|),$$

and the result follows. \square

Hence:

$$\int_{D(0,1/3)} N_{\varphi_a \circ \varphi} dA \geq \frac{1}{9} N_{\varphi_a \circ \varphi}(0) = \frac{1}{9} N_\varphi(a),$$

and

$$(3.18) \quad \int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \geq \frac{1}{98} \Phi''\left(\frac{1}{3}\right) N_\varphi(a).$$

We now have to estimate from above $\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm$. For that, we shall use the following easy lemma.

Lemma 3.5 *For every $\xi \in \partial\mathbb{D}$ and every $h \in (0, 1/2]$, one has:*

$$(3.19) \quad |1 - \bar{a}z|^2 \geq \frac{1}{4}(h^2 + |z - \xi|^2), \quad \forall z \in \overline{\mathbb{D}},$$

where $a = (1 - h)\xi$.

Proof. The result is rotation-invariant; so we may assume that $\xi = 1$ (and hence $a > 0$). Write $z = 1 - re^{i\theta}$. Since $|z| \leq 1$ if and only if $r \leq 2 \cos \theta$, one has $\cos \theta \geq 0$ and hence $|\theta| \leq \pi/2$. Then:

$$\begin{aligned} |1 - \bar{a}z|^2 &= |1 - a(1 - re^{i\theta})|^2 = |1 - a + are^{i\theta}|^2 \\ &= (1 - a)^2 + a^2 r^2 + 2ar(1 - a) \cos \theta \\ &\geq (1 - a)^2 + a^2 r^2 \geq \frac{1}{4}(h^2 + r^2) = \frac{1}{4}(h^2 + |z - 1|^2). \end{aligned} \quad \square$$

Then:

$$\begin{aligned}
\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm &= \int_{\mathbb{D}} \Phi\left(\frac{1-|a|}{|1-\bar{a}z|}\right) dm_\varphi(z) \\
&\leq \int_{\mathbb{D}} \Phi\left(\frac{2h}{(h^2+|z-\xi|^2)^{1/2}}\right) dm_\varphi(z), \quad \text{by (3.19)} \\
&= \int_0^{+\infty} m_\varphi\left(\Phi\left(\frac{2h}{(h^2+|z-\xi|^2)^{1/2}}\right) \geq t\right) dt \\
&= \int_0^{+\infty} m_\varphi((h^2+|z-\xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt \\
&= \int_0^{\Phi(2)} m_\varphi((h^2+|z-\xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)) dt,
\end{aligned}$$

since $h \leq (h^2 + |z - \xi|^2)^{1/2} \leq 2h/\Phi^{-1}(t)$ implies $t \leq \Phi(2)$. We get:

$$\int_{\partial\mathbb{D}} \Phi(|f_a \circ \varphi|) dm \leq \int_0^{\Phi(2)} m_\varphi(|z - \xi| \leq 2h/\Phi^{-1}(t)) dt.$$

We obtain from (3.18), by setting $u = 2h/\Phi^{-1}(t)$:

$$N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{2h}{u^2} \Phi'\left(\frac{2h}{u}\right) du.$$

Since $\Phi'(x) \leq x\Phi''(x)$, we get:

$$(3.20) \quad N_\varphi(a) \leq \frac{98}{\Phi''(1/3)} \int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi''\left(\frac{2h}{u}\right) du.$$

We are going now to choose suitably the Orlicz function Φ . It suffices to define Φ'' , for $a \in \mathbb{D}$ given (with $\xi = a/|a|$ and $h = 1-|a| \leq 1/4$). By Lemma 3.2, since $a \in \varphi(\mathbb{D})$, there is a constant $c_0 > 0$, such that $m_\varphi(S(\xi, c_0h)) > 0$; we can hence set (note that $m_\varphi(S(\xi, u)) \leq 1$):

$$(3.21) \quad \Phi''(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq h, \\ \frac{1}{m_\varphi(S(\xi, 2h/v))} & \text{if } h \leq v \leq 2/c_0, \\ \frac{1}{m_\varphi(S(\xi, c_0h))} & \text{if } v \geq 2/c_0. \end{cases}$$

It is a non-negative non-decreasing function, so the assumptions made on Φ at the beginning are satisfied. One has, since $m_\varphi(S(\xi, u)) \Phi''(2h/u) \leq 1$:

$$\int_h^\infty m_\varphi(S(\xi, u)) \frac{4h^2}{u^3} \Phi''\left(\frac{2h}{u}\right) du \leq \int_h^\infty \frac{4h^2}{u^3} du = 2.$$

Since $c_0 \leq 6$, one has $h \leq 1/3 \leq 2/c_0$ and hence $\Phi''(1/3) = 1/m_\varphi(S(\xi, 6h))$;

therefore (3.20) gives, for $h \leq (1 - |\varphi(0)|)/4$:

$$(3.22) \quad N_\varphi(a) \leq 196 m_\varphi(S(\xi, 6h)),$$

finishing the proof since $S(\xi, 6h) \subseteq W(\xi, 12h)$. \square

4 Domination of the Carleson function by the Nevanlinna function

We cannot expect to estimate individually from above the m_φ -measure of Carleson windows centered at $\xi = w/|w|$ by $N_\varphi(w)$, as in Theorem 3.1. In fact, consider a conformal mapping φ from \mathbb{D} onto $\mathbb{D} \setminus [0, 1[$. One has $N_\varphi(t) = 0$ for every $t \in [0, 1[$, though $m_\varphi(W(1, h)) > 0$ for every $h > 0$ (because $W(1, h) \supset W(e^{ih/2}, h/2)$ and $m_\varphi(W(e^{ih/2}, h/2)) > 0$ by Lemma 3.2).

Let us give another example. Let $\varphi(z) = (1 + z)/2$. Then:

a) One has $\varphi(e^{i\theta}) = (\cos(\theta/2) e^{i\theta/2})$ (with $|\theta| \leq \pi$). Hence $\varphi(e^{i\theta}) \in W(e^{i\theta_0}, h)$ if and only if $\cos(\theta/2) \geq 1 - h$ and $|(\theta/2) - \theta_0| \leq h$, *i.e.* $2(\theta_0 - h) \leq \theta \leq 2(\theta_0 + h)$.

Now, $1 - \cos(\theta/2) \leq \theta^2/8$, so the modulus condition is satisfied when $\theta^2 \leq 8h$; in particular when $|\theta| \leq 2\sqrt{h}$.

For $\theta_0 = \sqrt{h}$, $m_\varphi(W(e^{i\theta_0}, h))$ is bigger than the length of the interval

$$[-2\sqrt{h}, 2\sqrt{h}] \cap [2(\sqrt{h} - h), 2(\sqrt{h} + h)] = [2\sqrt{h} - 2h, 2\sqrt{h}],$$

that is $2h$. Therefore $m_\varphi(W(e^{i\theta_0}, h)) \geq 2h$.

b) Let now $w = \varphi(z)$. Write $w = \frac{1}{2} + r e^{i\zeta}$ with $0 \leq r < 1/2$. Then, writing $r = \frac{1}{2} - s$, one has $|z| = |2w - 1| = 2r$ and

$$N_\varphi(w) = \log \frac{1}{|z|} = \log \frac{1}{2r} = \log \frac{1}{1 - 2s} \approx s.$$

Now, $|w|^2 = \frac{1}{4} + r^2 + r \cos \zeta$ and

$$h \approx 1 - |w|^2 = \frac{1}{2}(1 - \cos \zeta) + s(1 + \cos \zeta) - s^2 \approx \frac{\zeta^2}{4} + 2s.$$

Writing $\zeta = s^{1/2\alpha}$, one gets:

- (i) for “small” ζ (*i.e.* $0 < \alpha \leq 1$): $h \approx s$, and so $N_\varphi(w) \approx h$;
- (ii) for “large” (*i.e.* $\alpha \geq 1$): $h \approx s^{1/\alpha}$, and so $N_\varphi(w) \approx h^\alpha$.

On the other hand, $w = e^{i\zeta/2}[(1 - s) \cos(\zeta/2) - is \sin(\zeta/2)]$; hence, when s goes to 0, one has

$$\theta_w := \arg w = \frac{\zeta}{2} + \arctan \left[\frac{s \sin(\zeta/2)}{(1 - s) \cos(\zeta/2)} \right] \sim \frac{\zeta}{2} \approx \zeta.$$

For $\alpha \geq 1$, one has $h \approx s^{1/\alpha} = \zeta^2$, i.e. $\zeta \approx \sqrt{h}$. Then, choosing $\alpha > 1$ such that $\zeta = \theta_0$, one has $m_\varphi(W(w/|w|, h)) \approx h$, though $N_\varphi(w) \approx h^\alpha \ll h$.

One cannot hence dominate $m_\varphi(W(w/|w|, h))$ by $N_\varphi(w)$.

We can remark that, nevertheless, in either case, one has $\rho_\varphi(h) \approx h$ and $\nu_\varphi(h) \approx h$.

We shall prove:

Theorem 4.1 *For every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, one has, for every $\xi \in \partial\mathbb{D}$:*

$$(4.1) \quad m_\varphi(W(\xi, h)) \leq 64 \sup_{w \in W(\xi, 64h) \cap \mathbb{D}} N_\varphi(w),$$

for $0 < h < (1 - |\varphi(0)|)/16$.

Proof. We shall set:

$$(4.2) \quad \nu_\varphi(\xi, h) = \sup_{w \in W(\xi, h) \cap \mathbb{D}} N_\varphi(w).$$

If for some $h_0 > 0$, one has $\nu_\varphi(\xi, h_0) = 0$, then $\varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus W(\xi, h_0)$, and hence $m_\varphi(W(\xi, h)) = 0$ for $0 < h < h_0$. Therefore we shall assume that $\nu_\varphi(\xi, h) > 0$. We may, and do, also assume that $h \leq 1/4$. By replacing φ by $e^{i\theta}\varphi$, it suffices to estimate $m_\varphi(S(1, h))$ (recall that $S(1, t) = \{z \in \overline{\mathbb{D}}; |1 - z| \leq t\}$).

We shall use the same functions f_a as in the proof of Theorem 3.1, but, for convenience, with a different notation. We set, for $0 < r < 1$:

$$(4.3) \quad u(z) = \frac{1-r}{1-rz}.$$

Let us take an Orlicz function Φ as in the beginning of the proof of Theorem 3.1, which will be precised later. We shall take this function in such a way that $\Phi(|u(\varphi(0))|) = 0$.

Since $\Phi'(x) \leq x\Phi''(x)$, (3.13) becomes:

$$(4.4) \quad \Delta\Phi(|u|) \leq 2\Phi''(|u|)|u'|^2,$$

and Stanton's formula writes, since $\Phi(|u(\varphi(0))|) = 0$:

$$(4.5) \quad \int_{\partial\mathbb{D}} \Phi(|u \circ \varphi|) dm \leq \int_{\mathbb{D}} \Phi''(|u(w)|)|u'(w)|^2 N_\varphi(w) dA(w).$$

In all the sequel, we shall fix h , $0 < h \leq 1/4$, and take $r = 1 - h$.

For $|z| \leq 1$ and $|1 - z| \leq h$, one has $|1 - rz| = |(1 - z) + hz| \leq |1 - z| + h \leq 2h$, so:

$$|u(z)| \geq \frac{(1-r)}{2h} = \frac{1}{2}.$$

Hence:

$$\begin{aligned} m_\varphi(S(1, h)) &\leq \frac{1}{\Phi(1/2)} \int_{S(1, h)} \Phi(|u(z)|) dm_\varphi(z) \\ &\leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi(|u(z)|) dm_\varphi(z) \\ &= \frac{1}{\Phi(1/2)} \int_{\mathbb{T}} \Phi(|(u \circ \varphi)(z)|) dm(z), \end{aligned}$$

and so, by (4.5):

$$(4.6) \quad m_\varphi(S(1, h)) \leq \frac{1}{\Phi(1/2)} \int_{\mathbb{D}} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z).$$

We are going to estimate this integral by separating two cases: $|1 - z| \leq h$ and $|1 - z| > h$.

For convenience, we shall set:

$$(4.7) \quad \tilde{\nu}(t) = \sup_{w \in S(1, t) \cap \mathbb{D}} N_\varphi(w).$$

1) Remark first that

$$u'(z) = \frac{rh}{(1 - rz)^2},$$

and so:

$$|u'(z)| \leq \frac{h}{(1 - r)^2} = \frac{1}{h}.$$

Since $|u(z)| \leq 1$, we get hence:

$$\int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1, h)} \Phi''(1) \frac{1}{h^2} \tilde{\nu}(h) dA(z),$$

giving, since $A(S(1, h)) \leq h^2$:

$$(4.8) \quad \int_{|1-z| \leq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \Phi''(1) \tilde{\nu}(h).$$

2) For $0 < h \leq 1/4$, one has:

$$|u(z)| \leq \frac{2h}{|1 - z|} \quad \text{and} \quad |u'(z)| \leq \frac{2h}{|1 - z|^2};$$

indeed, we have (this is obvious, by drawing a picture):

$$|1 - rz| = r \left| \frac{1}{r} - z \right| \geq r |1 - z|,$$

and hence $|1 - rz| \geq \frac{3}{4}|1 - z|$, since $r = 1 - h \geq 3/4$. We obtain:

$$\begin{aligned} \int_{|1-z|>h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \int_{|1-z|>h} \Phi''\left(\frac{2h}{|1-z|}\right) \frac{h^2}{|1-z|^4} N_\varphi(z) dA(z). \end{aligned}$$

Then, using polar coordinates centered at 1 (note that we only have to integrate over an arc of length less than π), and the obvious inequality $N_\varphi(z) \leq \tilde{\nu}(|1 - z|)$, we get:

$$(4.9) \quad \int_{|1-z|>h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq 4 \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt.$$

We now choose the Orlicz function as follows (with $a = \varphi(0)$):

$$(4.10) \quad \Phi''(v) = \begin{cases} \frac{1}{\tilde{\nu}(h)} & \text{if } v \geq 2, \\ \frac{1}{\tilde{\nu}(2h/v)} & \text{if } h/(1 - |a|) < v < 2, \\ 0 & \text{if } 0 \leq v \leq h/(1 - |a|). \end{cases}$$

This function is non-negative and non-decreasing. Moreover, one has $\Phi(x) = 0$ for $0 \leq x \leq h/(1 - |a|)$. Hence, since $|u(a)| \leq \frac{h}{1 - |a|}$, one has $\Phi(|u(a)|) = 0$.

Then

$$(4.11) \quad \begin{aligned} \int_h^2 \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt &= \int_h^{2^{2(1-|a|)}} \Phi''\left(\frac{2h}{t}\right) \frac{h^2}{t^3} \tilde{\nu}(t) dt \\ &\leq \int_h^\infty \frac{h^2}{t^3} dt = \frac{1}{2}. \end{aligned}$$

Now,

$$\begin{aligned} \Phi\left(\frac{1}{2}\right) &= \int_0^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \Phi'(t) dt \geq \int_{1/4}^{1/2} \frac{t}{2} \Phi''\left(\frac{t}{2}\right) dt \\ &\geq \Phi''\left(\frac{1}{8}\right) \int_{1/4}^{1/2} \frac{t}{2} dt = \frac{3}{64} \Phi''\left(\frac{1}{8}\right). \end{aligned}$$

When $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\Phi''(1/8) = 1/\tilde{\nu}(16h)$, and $\Phi''(1) = 1/\tilde{\nu}(2h)$. We get hence, from (4.6), (4.8), (4.9) and (4.11):

$$(4.12) \quad m_\varphi(S(1, h)) \leq \frac{64}{3} \tilde{\nu}(16h) \left[\frac{\tilde{\nu}(h)}{\tilde{\nu}(2h)} + 2 \right] \leq 64 \tilde{\nu}(16h).$$

Since $W(1, t) \subseteq S(1, 2t)$, we get $m_\varphi(W(1, h)) \leq 64 \sup_{w \in S(1, 32h)} N_\varphi(w)$ for $0 < h < (1 - |\varphi(0)|)/16$, and that ends the proof of Theorem 4.1, since $S(1, 32h) \subseteq W(1, 64h)$. \square

Remark. A slight modification of the proof gives the following improvement, if one allows a (much) bigger constant.

Theorem 4.2 *There are universal constants $C, c > 1$ such that*

$$m_\varphi(S(\xi, h)) \leq C \frac{1}{A(S(\xi, ch))} \int_{S(\xi, ch)} N_\varphi(z) dA(z)$$

for every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, every $\xi \in \partial\mathbb{D}$, and $0 < h < (1 - |\varphi(0)|)/8$.

Proof. We are going to follow the proof of Theorem 4.1. We shall assume that $\xi = 1$ and we set:

$$(4.13) \quad I(t) = \int_{S(1, t)} N_\varphi(z) dA(z).$$

Then:

1) When $|1 - z| < h$, , we have, instead of (4.8):

$$(4.14) \quad \int_{|1-z|<h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \int_{S(1, h)} \Phi''(1) \frac{1}{h^2} N_\varphi(z) dA(z) \\ = \Phi''(1) \frac{1}{h^2} I(h).$$

2) For $|z - 1| \geq h$, we write:

$$\int_{|1-z|\geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ = \sum_{k=1}^{\infty} \int_{kh \leq |1-z| < (k+1)h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \\ \leq 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2h}{kh}\right) \frac{h^2}{k^4 h^4} I((k+1)h) \\ = 4 \sum_{k=1}^{\infty} \Phi''\left(\frac{2}{k}\right) \frac{1}{k^4 h^2} I((k+1)h).$$

We take, with $a = \varphi(0)$:

$$(4.15) \quad \Phi''(v) = \begin{cases} \frac{1}{I\left(\left(\frac{2}{v} + 1\right)h\right)} & \text{if } v > h/(1 - |a|), \\ 0 & \text{if } 0 \leq v \leq h/(1 - |a|). \end{cases}$$

Then

$$(4.16) \quad \int_{|1-z| \geq h} \Phi''(|u(z)|) |u'(z)|^2 N_\varphi(z) dA(z) \leq \frac{4}{h^2} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{4}{h^2} \frac{\pi^4}{90} \leq \frac{5}{h^2}.$$

Since $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\Phi''(1/8) = \frac{1}{I(17h)}$ and $\Phi''(1) = \frac{1}{I(3h)}$. Therefore:

$$\begin{aligned} m_\varphi(S(1, h)) &\leq \frac{64}{3} I(17h) \left[\frac{1}{h^2} \frac{I(h)}{I(3h)} + \frac{5}{h^2} \right] \\ &\leq \frac{64}{3} I(17h) \frac{6}{h^2} = 128 \frac{I(17h)}{h^2} \\ &\leq 128 \times 17^2 \frac{I(17h)}{A(S(1, 17h))}, \end{aligned}$$

ending the proof of Theorem 4.2. \square

5 Some consequences

In [6], we proved (Theorem 4.19) that the Carleson function of an analytic self-map φ has the following property of regularity: $m_\varphi(S(\xi, \varepsilon h)) \leq K \varepsilon m_\varphi(S(\xi, h))$ for $0 < h < 1 - |\varphi(0)|$, $0 < \varepsilon < 1$ and $\xi \in \partial\mathbb{D}$, where K is a universal constant. It follows from Theorem 1.1, (actually Theorem 3.1 and Theorem 4.1) that:

Theorem 5.1 *There exist a universal constants $K > 0$ such that, for every analytic self-map φ of \mathbb{D} , one has, for $0 < \varepsilon < 1$:*

$$(5.1) \quad \nu_\varphi(\varepsilon t) \leq K \varepsilon \nu_\varphi(t),$$

for t small enough.

More precisely, for every $\xi \in \partial\mathbb{D}$, one has, for t small enough:

$$(5.2) \quad \sup_{w \in W(\xi, \varepsilon t) \cap \mathbb{D}} N_\varphi(w) \leq K \varepsilon \sup_{w \in W(\xi, t) \cap \mathbb{D}} N_\varphi(w).$$

We shall end this paper with some consequences of Theorem 1.1 for composition operators. Recall that if Ψ is an Orlicz function, the Hardy-Orlicz space is the space of functions $f \in H^1$ whose boundary values are in the Orlicz space $L^\Psi(\partial\mathbb{D}, m)$. We proved in [6], Theorem 4.18 that, if $\frac{\Psi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty$, the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if, for every $A > 0$, one has $\rho_\varphi(h) = o[1/\Psi(A\Psi^{-1}(1/h))]$ when h goes to 0. This remains true when $H^\Psi = H^1$. Hence Theorem 1.1 gives:

Theorem 5.2 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map and Ψ be an Orlicz function. Then the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact if and only if*

$$(5.3) \quad \sup_{|w| \geq 1-h} N_\varphi(w) = o\left(\frac{1}{\Psi(A\Psi^{-1}(1/h))}\right), \quad \text{as } h \rightarrow 0, \quad \forall A > 0.$$

It should be noted, due to the arbitrary $A > 0$, that (5.3) may be replaced by

$$(5.4) \quad \sup_{|w| \geq 1-h} N_\varphi(w) \leq \frac{1}{\Psi(A\Psi^{-1}(1/h))}, \quad \forall A > 0,$$

for $h \leq h_A$.

It is known that if $C_\varphi: H^2 \rightarrow H^2$ is compact, then $\lim_{|z| \rightarrow 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty$, and that this condition is sufficient when φ is univalent, or p -valent, but not sufficient in general (see [11] and [12], § 3.2). It follows from Theorem 5.2 that an analogous result holds for Hardy-Orlicz spaces:

Theorem 5.3 *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map, and Ψ be an Orlicz function. Assume that the composition operator $C_\varphi: H^\Psi \rightarrow H^\Psi$ is compact. Then:*

$$(5.5) \quad \lim_{|z| \rightarrow 1} \frac{\Psi^{-1}\left(\frac{1}{1-|z|}\right)}{\Psi^{-1}\left(\frac{1}{1-|\varphi(z)|}\right)} = \infty.$$

Conversely, if φ is p -valent, then (5.5) suffices to $C_\varphi: H^\Psi \rightarrow H^\Psi$ be compact.

Proof. For the necessity, we could use Theorem 5.2 and the fact that $1-|z| \leq \log \frac{1}{|z|} \leq N_\varphi(\varphi(z))$; but we shall give a more elementary proof. Recall that H^Ψ is the bidual of HM^Ψ , the closure of H^∞ . Since $C_\varphi(H^\infty) \subseteq H^\infty$, C_φ maps HM^Ψ into itself and $C_\varphi: H^\Psi \rightarrow H^\Psi$ is the bi-adjoint of $C_\varphi: HM^\Psi \rightarrow HM^\Psi$. We know that the evaluation $\delta_a: f \in HM^\Psi \mapsto f(a) \in \mathbb{C}$ has norm $\approx \Psi^{-1}\left(\frac{1}{1-|a|}\right)$ ([6], Lemma 3.11); hence $\delta_a/\|\delta_a\| \xrightarrow[|a| \rightarrow 1]{} 0$ weak-star (because $|\delta_a(f)| = |f(a)| \leq \|f\|_\infty$ for $f \in H^\infty$). If C_φ is compact, its adjoint C_φ^* also; we get hence $\|C_\varphi^*(\delta_a/\|\delta_a\|)\| \xrightarrow[|a| \rightarrow 1]{} 0$. But $C_\varphi^*\delta_a = \delta_{\varphi(a)}$. Therefore

$$\frac{\Psi^{-1}\left(\frac{1}{1-|\varphi(a)|}\right)}{\Psi^{-1}\left(\frac{1}{1-|a|}\right)} \xrightarrow[|a| \rightarrow 1]{} 0.$$

Conversely, assume that (5.5) holds. For every $A > 0$, one has, for $|z|$ close enough to 1: $\Psi^{-1}\left(\frac{1}{1-|z|}\right) \geq A\Psi^{-1}\left(\frac{1}{1-|\varphi(z)|}\right)$; in other words, one has: $1/\Psi(A\Psi^{-1}(1/(1-|\varphi(z)|))) \geq 1-|z|$. But, when φ is p -valent, and if $w = \varphi(z)$

with $|z| > 0$ minimal, one has $N_\varphi(w) \leq p \log \frac{1}{|z|} \approx 1 - |z|$. Since $|z| \rightarrow 1$ when $|w| = |\varphi(z)| \rightarrow 1$ (otherwise, we should have a sequence (z_n) converging to some $z_0 \in \mathbb{D}$ and $\varphi(z_n)$ would converge to $\varphi(z_0) \in \mathbb{D}$), we get $\sup_{|w| \geq 1-h} N_\varphi(w) \lesssim 1/\Psi(A\Psi^{-1}(1/1 - |w|)) \leq 1/\Psi(A\Psi^{-1}(1/1 - h))$, for h small enough. By Theorem 5.2, with (5.4), that means that C_φ is compact on H^Ψ . \square

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