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THE CANONICAL STRIP PHENOMENON FOR COMPLETE INTERSECTIONS IN HOMOGENEOUS SPACES

L. MANIVEL

ABSTRACT. We show that a refined version of Golyshev's canonical strip hypothesis does hold for the Hilbert polynomials of complete intersections in rational homogeneous spaces.

1. INTRODUCTION

Golyshev [Go] recently made some intriguing observations about Hilbert polynomials of canonically or anti-canonically polarized complex projective varieties. For a Fano variety X , the polynomial defined by $H_{-K_X}(k) = \chi(X, -kK_X)$ for $k \in \mathbb{Z}$ has the symmetry

$$H_{K_X}(z) = (-1)^{\dim X} H_{K_X}(-z - 1).$$

In particular its zeroes are symmetric with respect to the vertical line $Re(z) = -\frac{1}{2}$. Golyshev suggested to consider the following series of hypothesis:

- (CS) The roots of $H_{-K_X}(z)$ belong to the *canonical strip* $-1 < Re(z) < 0$.
- (NCS) The roots of $H_{-K_X}(z)$ belong to the *narrow canonical strip*.
- (CL) The roots of $H_{-K_X}(z)$ belong to the *canonical line* $Re(z) = -\frac{1}{2}$.

The above-mentioned *narrow canonical strip* is defined by the conditions that

$$-1 + \frac{1}{\dim X + 1} < Re(z) < -\frac{1}{\dim X + 1}.$$

We introduce the following variant for a Fano variety X of index ι_X . We define the *tight canonical strip* by the conditions

$$-1 + \frac{1}{\iota_X} < Re(z) < -\frac{1}{\iota_X},$$

and the corresponding hypothesis

- (TCS) The roots of $H_{-K_X}(z)$ belong to the *tight canonical strip*.

Golyshev noticed that (NCS) holds for Fano threefolds and that (CS) holds for minimal threefolds of general type, as a consequence of well-known estimates of their characteristic numbers. Shramov checked that (CL) holds for smooth Fano toric manifolds up to dimension four [Sh].

Note that we can express $H_{-K_X}(z)$ as follows. Consider the entire function

$$\varphi(t) = \frac{t/2}{sh(t/2)} = \sum_{t=0}^{\infty} B_n\left(\frac{1}{2}\right) \frac{t^n}{n!},$$

where $B_n(u)$ denotes the n -th Bernoulli polynomial. This function defines a characteristic class $\varphi(X) \in H^*(X, \mathbb{Q})$, and we can let

$$\psi_k(X) = \int_X \varphi_k(X) \cup c_1(X)^{\dim X - k}.$$

Note that this is zero if k is odd. It is then a direct consequence of the Grothendieck-Riemann-Roch theorem that

$$H_{-K_X}(z - \frac{1}{2}) = \sum_{k=0}^{\dim X} \psi_{\dim X - k}(X) \frac{z^k}{k!}.$$

Write $\dim X = 2m + \epsilon$, with $\epsilon \in \{0, 1\}$. Then $H_{-K_X}(z - \frac{1}{2}) = z^\epsilon P_X(z^2)$ for some polynomial of degree m , and the canonical line hypothesis (CL) can be translated into the hypothesis that the polynomial P_X has only real and non-positive roots.

In this note we prove that:

- (1) (TCS) holds for rational homogeneous spaces of Picard number one;
- (2) (TCS) holds for all Fano complete intersections in rational homogeneous spaces of Picard number one;
- (3) (CL) holds for all general type complete intersections in rational homogeneous spaces of Picard number one.

The first statement is a relatively straightforward application of the Weyl dimension formula and of the combinatorics of root systems. The next two statements can be deduced from the first one by induction.

2. THE TIGHT CANONICAL STRIP FOR RATIONAL HOMOGENEOUS SPACES

Let $X = G/P$ be a complex rational homogeneous space with Picard number one. Here G denotes a simple affine algebraic group and P a maximal parabolic subgroup. Once we choose a maximal torus T in P , we get a root system Φ and its decomposition into positive and negative roots, $\Phi = \Phi_+ \cup (-\Phi_+)$. Our (slightly unusual) convention will be that negative roots are roots of P . Recall that the choice of P (up to conjugation) is equivalent to the choice of a fundamental weight ω_0 , or equivalently, of a simple root α_0 .

Let L be the ample line bundle generating $\text{Pic}(X)$. It can be defined as the line bundle L_{ω_0} associated to the fundamental weight ω_0 , considered as a character of P . The index ι_X of X is defined by the identity $-K_X = \iota_X L$.

By the Bott-Borel-Weil theorem, we know that $\Gamma(X, L^k) = V_{k\omega_0}$ is the irreducible G -module of highest weight $k\omega_0$, and that the higher cohomology groups vanish. We can then use Weyl's dimension formula (see e.g. [Se]) to express the Hilbert polynomial of L as

$$H_L(z) = \prod_{\alpha \in \Phi_+} \frac{(z\omega_0 + \rho, \alpha)}{(\rho, \alpha)}.$$

Here we have used an invariant pairing $(\ , \)$ on the weight lattice. We will use the normalization defined by the condition that $(\omega_0, \alpha_0) = 1$, or equivalently, that $(\alpha_0, \alpha_0) = 2$.

We can decompose H_L as the product of the polynomials H_L^ℓ defined as

$$H_L^\ell(z) = \prod_{\substack{\alpha \in \Phi_+, \\ (\omega_0, \alpha) = \ell}} \frac{\ell z + (\rho, \alpha)}{(\rho, \alpha)} = \prod_j \left(\frac{\ell z + j}{j} \right)^{h_{\ell, j}},$$

where $h_{\ell, j}$ denotes the number of positive roots $\alpha \in \Phi_+$ such that $(\omega_0, \alpha) = \ell$ and $(\rho, \alpha) = j$.

Remark. It is usually more natural to write down Weyl's dimension formula in terms of the coroots α^\vee attached to the roots α . Then (ρ, α^\vee) is an integer called the height of α and sometimes denoted $ht(\alpha)$. Since ρ is equal to the sum of the fundamental weights, whose dual basis is that of the simple coroots, $ht(\alpha)$ can be computed as the sum of the coefficients of α over the basis of simple coroots. See for example [GW].

For our purposes, it seems necessary to work with roots rather than coroots. In the simply laced case this makes no difference, but in the non simply laced case we need to be careful with the fact that (ρ, α) can take non integer values. To avoid notational complications, we will suppose in the sequel that \mathfrak{g} is simply laced, but our results also hold in the non simply laced case, with essentially the same proofs.

Proposition 1. *For each ℓ , the sequence $h_{\ell, j}$ is symmetric and unimodal. That is*

$$\begin{aligned} (S) \quad h_{\ell, j} &= h_{\ell, \ell_X - j}, \\ (U) \quad h_{\ell, j} &\leq h_{\ell, j+1} \quad \text{if } 2j + 1 \leq \ell_X. \end{aligned}$$

To prove this statement, we observe that there is a close connection between the decomposition of H_L into the product of the H_L^ℓ 's, and the \mathbb{Z} -grading on $\mathfrak{g} = \text{Lie}(G)$ induced by ω_0 :

$$\mathfrak{g} = \mathfrak{g}_{-\ell_{max}} \oplus \cdots \oplus \mathfrak{g}_{-i} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{\ell_{max}}.$$

Here \mathfrak{g}_i denotes the sum of the root spaces associated to the roots having coefficient i on α_0 (plus the Cartan subalgebra for $i = 0$). So the roots that contribute to H_L^ℓ are precisely those appearing in \mathfrak{g}_ℓ , for $\ell \geq 1$. Now, we know (see e.g. [Ru]) that

- \mathfrak{g}_0 is a reductive subalgebra of \mathfrak{g} of maximal rank, with rank one center;
- the Dynkin diagram encoding the semi-simple part of \mathfrak{g}_0 can be deduced from the Dynkin diagram of \mathfrak{g} by erasing the node corresponding to the fundamental weight ω_0 , and the edges attached to this node;
- each \mathfrak{g}_ℓ is a simple \mathfrak{g}_0 -module.

This implies that for each ℓ such that \mathfrak{g}_ℓ is non zero, there is a unique root γ_ℓ of \mathfrak{g} which is a highest weight of \mathfrak{g}_ℓ considered as a \mathfrak{g}_0 -module. Symmetrically, there is also a unique root β_ℓ of \mathfrak{g} which is a lowest weight of \mathfrak{g}_ℓ . Moreover,

$$\beta_\ell = w_{00}(\gamma_\ell),$$

where w_{00} denotes the longest element in the Weyl group of \mathfrak{g}_0 (not to be confused with the longest element in the Weyl group of \mathfrak{g} , usually denoted w_0), which is the subgroup of the Weyl group of \mathfrak{g} generated by the simple reflections other than s_{α_0} . Note in particular that

$$(\rho, \beta_\ell + \gamma_\ell) = (\rho + w_{00}(\rho), \gamma_\ell).$$

Note also that ℓ_{max} is the coefficient of the highest root ψ of \mathfrak{g} on α_0 .

Lemma 1. *We have $\rho + w_{00}(\rho) = \iota_X \omega_0$.*

Proof. Recall that 2ρ is the sum of the positive roots. The tangent bundle of $X = G/P$ is the homogeneous bundle defined by the P -module $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{p} = \text{Lie}(P)$. But

$$\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\ell_{max}}.$$

This implies that the anticanonical bundle of X , being the determinant of the tangent bundle, is defined by the weight $2\rho_X$ equal to the sum of the roots α such that $(\omega_0, \alpha) > 0$.

We can conclude that $2\rho - 2\rho_X$ is the sum of the positive roots in \mathfrak{g}_0 . In particular the action of w_{00} takes it to its opposite. On the contrary, $2\rho_X = \iota_X \omega_0$ is not affected by the action of w_{00} . Therefore

$$w_{00}(2\rho - 2\rho_X) = w_{00}(2\rho) - 2\rho_X = 2\rho_X - 2\rho,$$

and our claim follows. \square

Remark. Recall that 2ρ is also the sum of the fundamental weights. For the root system of \mathfrak{g}_0 , which is the set of roots of \mathfrak{g} with zero coefficient on α_0 , the fundamental weights are of the form $\omega_i^0 = \omega_i - a_i \omega_0$. Having coefficient zero on α_0 is equivalent to being orthogonal

to ω_0 , so $a_i = (\omega_i, \omega_0)/(\omega_0, \omega_0)$ and therefore $2\rho_0 = \sum_i \omega_i^0 = 2\rho - \frac{(2\rho, \omega_0)}{(\omega_0, \omega_0)}\omega_0$. Applying w_{00} , which maps ρ_0 to $-\rho_0$, we deduce that ι_X is also given by the simple formula

$$\iota_X = \frac{(2\rho, \omega_0)}{(\omega_0, \omega_0)}.$$

Observe that $(2\rho, \omega_0) = \sum_{\ell \geq 1} \ell \dim \mathfrak{g}_\ell$.

We deduce from the previous lemma, and for each ℓ , the identity

$$(\rho, \beta_\ell + \gamma_\ell) = (\iota_X \omega_0, \gamma_\ell) = \iota_X \ell.$$

Now, w_{00} acts on the roots in \mathfrak{g}_ℓ , sending β_ℓ to γ_ℓ , and more generally a root of height k to a root of height $ht(\beta_\ell) + ht(\gamma_\ell) - k = \iota_X \ell - k$. This implies the symmetry property (S).

Finally, the unimodality property (U) is a special case of a general property of weights of \mathfrak{g} -modules. Indeed, if V_λ is the irreducible \mathfrak{g} -module of highest weight λ , then any weight of V_λ is of the form $\lambda - \theta$ for some θ in the weight lattice, and the numbers of weights (counted with multiplicities) $\lambda - \theta$ such that (ρ, θ) is a given integer ℓ , form a unimodular sequence. This general property gives (U) when applied to the irreducible and multiplicity-free \mathfrak{g}_0 -module \mathfrak{g}_ℓ (see e.g. [St]). \square

We can summarize our discussion by the following statement. Let b_ℓ denote the height of β_ℓ , the unique smallest root such that $(\omega_0, \beta_\ell) = \ell$.

Proposition 2. *The Hilbert polynomial of the ample generator L of the Picard group of $X = G/P$ can be expressed as*

$$H_L(z) = \prod_{\ell=1}^{\ell_{max}} \prod_{k=b_\ell}^{\ell_{max} - \ell_{max} - b_\ell} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k}},$$

where for each ℓ the sequence $h_{\ell,k}$ is symmetric and unimodal.

This implies that the (TCS) hypothesis holds for X . More precisely:

Corollary 1. *The zeroes of $H_{-K_X}(z)$ are contained in the real segment*

$$\left[-1 + \frac{1}{\iota_X}, -\frac{1}{\iota_X} \right].$$

Example. Consider $X = E_6/P_4$ where, according to the notations of Bourbaki [Bou], P_4 is the maximal parabolic subgroup associated with the simple root α_4 . This simple root corresponds to the triple node of the Dynkin diagram of type E_6 . In this case we have $\ell_{max} = 3$ and the contributions of the three levels $\ell = 1, 2, 3$ can be analyzed as follows.

$\ell = 1$: the extremal roots are

$$\beta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & & & & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

In particular $b_1 = ht(\beta_1) = 1$, and $ht(\beta_1) + ht(\gamma_1) = 7 = \iota_X$. There are eighteen roots contributing to H_X^1 , and

$$H_X^1(z) = \left(\frac{z+1}{1} \right) \left(\frac{z+2}{2} \right)^3 \left(\frac{z+3}{3} \right)^5 \left(\frac{z+4}{4} \right)^5 \left(\frac{z+5}{5} \right)^3 \left(\frac{z+6}{6} \right).$$

$\ell = 2$: the extremal roots are

$$\beta_2 = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ & & & & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ & & & & 1 \end{pmatrix}.$$

In particular $b_2 = ht(\beta_2) = 5$, and $ht(\beta_2) + ht(\gamma_2) = 14 = 2\iota_X$. There are nine roots contributing to H_X^2 , and

$$H_X^2(z) = \left(\frac{2z+5}{5}\right) \left(\frac{2z+6}{6}\right)^2 \left(\frac{2z+7}{7}\right)^3 \left(\frac{2z+8}{8}\right)^2 \left(\frac{2z+9}{9}\right).$$

$\ell = 3$: the extremal roots are

$$\beta_3 = \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & 1 & & \end{matrix}, \quad \gamma_3 = \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & & 2 & \end{matrix}.$$

In particular $b_3 = ht(\beta_3) = 10$, and $ht(\beta_2) + ht(\gamma_2) = 21 = 3\iota_X$. These are the only two roots contributing to H_X^3 , and

$$H_X^3(z) = \left(\frac{3z+10}{10}\right) \left(\frac{3z+11}{11}\right).$$

We conclude in particular that X has dimension 29 and degree 996584151214080.

Remarks.

1. The simplest situation is when $\ell_{max} = 1$, which occurs when X is *cominuscule*. Then $\beta_1 = \alpha_0$, and $\gamma_1 = \psi$ is the highest root. In particular we recover the known fact that $\iota_X = h + 1$, where h denotes the Coxeter number. Moreover we get the relation

$$\dim X = (\omega_0, \omega_0)\iota_X.$$

The zeroes of the Hilbert polynomial in that case are just the $-j/\iota_X$, with $0 < j < \iota_X$.

2. The Hilbert polynomials of the adjoint varieties (this is the special case where $X = G/P \subset \mathbb{P}(\mathfrak{g})$ is the projectivization of a minimal nilpotent orbit) have been investigated in [LM1, LM3] from the perspective of Vogel and Deligne works on the *universal Lie algebra*. The related work [LM2] explores some other cases related to certain series generalizing the lines of Freudenthal's magic square.

3. COMPLETE INTERSECTIONS

In this section we prove that the (TCS) hypothesis holds for any complete intersection Y in the rational homogeneous space $X = G/P$. More precisely, we will show that the Hilbert polynomial of L_Y on Y can be expressed as

$$H_{L_Y}(z) = H_{L_Y}^0(z) \prod_{\ell=1}^{\ell_{max}^Y} \prod_{k=b_\ell(Y)}^{\ell_{\iota_Y - b_\ell(Y)}} \left(\frac{\ell z + k}{k}\right)^{h_{\ell,k}(Y)},$$

with the additional properties that:

- (1) for each ℓ , the sequence $h_\ell, k(Y)$ is symmetric and unimodal;
- (2) the zeroes of the polynomial $H_{L_Y}^0(z)$ all belong to the line $Re(z) = -\frac{1}{2}$.

Note that when $\iota_Y \leq 0$, the product in the right hand side of the previous identity is empty, since the integers $b_\ell(Y)$ will always be positive.

We proceed by induction. Let $Z = Y \cap H_d$ be the transverse intersection of Y with a hypersurface of degree d . Then $\iota_Z = \iota_Y - d$ and the Hilbert polynomial of L_Z is simply

$$H_{L_Z}(z) = H_{L_Y}(z) - H_{L_Y}(z - d).$$

For a given ℓ , consider the polynomial

$$H_{L_Y}^\ell(z) = \prod_{k=b_\ell(Y)}^{\ell_{\iota_Y - b_\ell(Y)}} \left(\frac{\ell z + k}{k}\right)^{h_{\ell,k}(Y)}.$$

Then we can write $P_{LZ}^\ell(z) = H_{LY}^\ell(z) - H_{LY}^\ell(z-d)$ as a product $A^\ell(z)(B_+^\ell(z) - B_-^\ell(z))$, where

$$\begin{aligned} A^\ell(z) &= \prod_{k=b_\ell(Y)}^{\ell_Y - b_\ell(Y) - \ell d} \left(\frac{\ell z + k}{k} \right)^{\min(h_{\ell,k}(Y), h_{\ell,k+\ell d}(Y))}, \\ B_+^\ell(z) &= \prod_{k=b_\ell(Y)}^{\ell_Y - b_\ell(Y)} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k}(Y) - \min(h_{\ell,k}(Y), h_{\ell,k+\ell d}(Y))}, \\ B_-^\ell(z) &= \prod_{k=b_\ell(Y)}^{\ell_Y - b_\ell(Y)} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k+\ell d}(Y) - \min(h_{\ell,k}(Y), h_{\ell,k+\ell d}(Y))}. \end{aligned}$$

Let $h_{\ell,k}(Z) = \min(h_{\ell,k}(Y), h_{\ell,k+\ell d}(Y))$. First observe that the symmetry of the sequence $h_{\ell,k}(Y)$ relative to the substitution $k \mapsto \ell_Y - k$ implies that of the sequence $h_{\ell,k}(Z)$ relative to $k \mapsto \ell_Z - k$. Moreover, the sequence $h_{\ell,k}(Z)$ is again unimodal. Indeed, for $k \leq \ell_Y/2 - d\ell/2 = \ell_Z/2$,

$$(1) \quad h_{\ell,k}(Z) = h_{\ell,k}(Y).$$

This is clear for $k \leq \ell_Y/2 - d\ell$ since then $h_{\ell,k}(Y) \leq h_{\ell,k+d\ell}(Y)$ by the unimodality for Y . For $\ell_Y/2 - d\ell \leq k \leq \ell_Y/2 - d\ell/2$, we observe that by symmetry,

$$h_{\ell,k+d\ell}(Y) = h_{\ell,\ell_Y - k - d\ell}(Y) \geq h_{\ell,k}(Y)$$

since $k \leq \ell_Y - k - d\ell \leq \ell_Y/2$. We can therefore write

$$A^\ell(z) = \prod_{k=b_\ell(Z)}^{\ell_Z - b_\ell(Z)} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k}(Z)},$$

where $b_\ell(Z) = b_\ell(Y)$ and the sequence $h_{\ell,k}(Z)$ is symmetric and unimodal. Beware that we can have $b_\ell(Z) > \ell_Z - b_\ell(Z)$, in which case $A^\ell(z) = 1$ by convention.

Now, using (1) we can rewrite B_+^ℓ and B_-^ℓ as

$$\begin{aligned} B_+^\ell(z) &= \prod_{2k > \ell_Z} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k}(Y) - h_{\ell,k+\ell d}(Y)}, \\ B_-^\ell(z) &= \prod_{2k < \ell_Z} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k+\ell d}(Y) - h_{\ell,k}(Y)}. \end{aligned}$$

Observe that by symmetry $B_-^\ell(z) = \pm B_+^\ell(-\ell_Z - z)$. Moreover the zeroes of $B_+^\ell(z)$ (resp. $B_-^\ell(z)$) are located to the left (resp. to the right) of the line $\operatorname{Re}(z) = -\ell_Z/2$. This implies that $|B_+^\ell(z)| > |B_-^\ell(z)|$ if $\operatorname{Re}(z) > -\ell_Z/2$, and $|B_+^\ell(z)| < |B_-^\ell(z)|$ if $\operatorname{Re}(z) < -\ell_Z/2$ (see the proof of Lemma 2.2 in [Go]).

The same conclusion holds for the polynomial $Q(z) = H_{LY}^0(z) - H_{LY}^0(z-d)$. Since the zeroes of $H_{LY}^0(z)$ are all supposed to lie on the line $\operatorname{Re}(z) = -\ell_Y/2$, we have $|H_{LY}^0(z)| > |H_{LY}^0(z-d)|$ if $\operatorname{Re}(z) > -\ell_Z/2$, and $|H_{LY}^0(z)| < |H_{LY}^0(z-d)|$ if $\operatorname{Re}(z) < -\ell_Z/2$.

Now we can conclude our analysis as follows: we can write

$$H_{LZ}(z) = H_{LZ}^0(z) \prod_{\ell=1}^{\ell_{max}^Z} \prod_{k=b_\ell(Z)}^{\ell_Z - b_\ell(Z)} \left(\frac{\ell z + k}{k} \right)^{h_{\ell,k}(Z)},$$

where the polynomial $H_{L_Z}^0(z)$ is equal to

$$H_{L_Y}^0(z) \prod_{\ell} B_+^{\ell}(z) - H_{L_Y}^0(z-d) \prod_{\ell} B_-^{\ell}(z).$$

As we have seen, these two products can have the same moduli only if $\operatorname{Re}(z) = -\iota_Z/2$, so the zeroes of $H_{L_Z}^0(z)$ have to lie on that line.

Note that this analysis is still correct if ι_Z is negative, in which case we simply have $H_{L_Z}(z) = H_{L_Z}^0(z)$. We have proved:

Theorem 1. *Let Y be a smooth complete intersection in a rational homogeneous space $X = G/P$ with Picard number one. Then:*

- (1) *If Y has general type, all the zeroes of $H_{K_Y}(z)$ lie on the line $\operatorname{Re}(z) = -\frac{1}{2}$, so that the (CL) hypothesis holds;*
- (2) *If Y is Fano, the zeroes of $H_{-K_Y}(z)$ lie either on the line $\operatorname{Re}(z) = -\frac{1}{2}$ or on the real segment $[-1 + \frac{1}{\iota_Y}, -\frac{1}{\iota_Y}]$. In particular, the (TCS) hypothesis holds.*
- (3) *If Y is Calabi-Yau, and L_Y denotes the restriction to Y of the ample generator of $\operatorname{Pic}(G/P)$, then the zeroes of the Hilbert polynomial $H_{L_Y}(z)$ are purely imaginary.*

In the Fano case we can be more precise and describe explicitly the real zeros $H_{-K_Y}(z)$, which are all rational. Note that the (CL) hypothesis does hold when the index ι_Y equals one or two.

4. MISCELLANII

4.1. Branched coverings. A variant of the previous remarks allows to draw similar conclusions for double coverings of rational homogeneous spaces, as we did for their hypersurfaces. Let $X = G/P$ be as above, L the ample generator of its Picard group, $H \subset X$ a general hypersurface in the linear system $|2dL|$. Let $\pi : Y \rightarrow X$ be a double-cover branched over H . Then $K_Y = \pi^*(K_X + dL)$, so that Y is Fano of index $\iota_Y = \iota_X - d$ if $d < \iota_X$. Since $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus L^{-d}$, we can easily compute the Hilbert polynomial of $-K_Y$:

$$H_{-K_Y}(z) = H_{-K_X}(z) + H_{-K_X}(z - \frac{d}{\iota_Y}).$$

Up to the sign, this is the same formula as that giving the Hilbert polynomial of the hypersurface H . Since that sign did not matter in our proof of Theorem 1 for hypersurfaces, we can conclude:

Proposition 3. *The zeroes of the Hilbert polynomial of Y lie either on the line $\operatorname{Re}(z) = -\frac{1}{2}$ or on the real segment $[-1 + \frac{1}{\iota_Y}, -\frac{1}{\iota_Y}]$. In particular the (TCS) hypothesis holds for Y .*

This extends to Calabi-Yau double covers (that is, $d = \iota_X$): the (CL) hypothesis does hold for the Hilbert polynomial of the polarization π^*L of Y .

4.2. Complete intersections in abelian varieties. We conclude this note by observing that the (CL) hypothesis is easily seen to hold for varieties of general type which are complete intersections in abelian varieties.

Let A be an abelian variety of dimension $n + c$, and L_1, \dots, L_c be ample line bundles. Consider $X = H_1 \cap \dots \cap H_c \subset A$, a transverse intersection of hypersurfaces $H_1 \in |L_1|, \dots, H_c \in |L_c|$.

Proposition 4. *The canonical line hypothesis (CL) holds for X .*

Proof. The canonical bundle of X is $K_X = (L_1 + \cdots + L_c)|X$. Using the Koszul complex of X , we can compute its Hilbert polynomial as

$$\begin{aligned} H_{-K_X}(z) &= \frac{1}{(n+c)!} \sum_{\epsilon} (-1)^{|\epsilon|} ((z - \epsilon_1)L_1 + \cdots + (z - \epsilon_c)L_c)^{n+c} \\ &= \sum_{\ell} \frac{(L_1^{\ell_1} \cdots L_c^{\ell_c})}{\ell_1! \cdots \ell_c!} \sum_{\epsilon} (-1)^{|\epsilon|} (z - \epsilon_1)^{\ell_1} \cdots (z - \epsilon_c)^{\ell_c}. \end{aligned}$$

In these sums $\epsilon = (\epsilon_1, \dots, \epsilon_c)$ belongs to $\{0, 1\}^c$, $|\epsilon| = \epsilon_1 + \cdots + \epsilon_c$, and ℓ_1, \dots, ℓ_c are non-negative integers of sum $n + c$. The polynomial $P_{\ell_1, \dots, \ell_c}(z)$ defined by the previous sum over ϵ factors as

$$P_{\ell_1, \dots, \ell_c}(z) = P_{\ell_1}(z) \cdots P_{\ell_c}(z),$$

where $P_{\ell}(z) = z^{\ell} - (z - 1)^{\ell}$. There remains to observe that $P_{\ell}(z + \frac{1}{2})$ is a polynomial with non-negative coefficients, of the same parity as ℓ . Since the intersection coefficients $(L_1^{\ell_1} \cdots L_c^{\ell_c})$ are all positive, $H_{-K_X}(z)$ is therefore a polynomial in z^2 with positive coefficients, multiplied by z if n is odd. As we noticed in the introduction, this is enough to imply (CL). \square

REFERENCES

- [Bou] Bourbaki N., Groupes et algèbres de Lie, Hermann 1968.
- [Go] Golyshev V.: *The canonical strip I*, arXiv:0903.2076.
- [GW] Gross B.H., Wallach N.R.: *On the Hilbert polynomials and Hilbert series of homogeneous projective varieties*, preprint 2007.
- [LM1] Landsberg J.M., Manivel L.: *Triality, exceptional Lie algebras and Deligne dimension formulas*, Advances in Mathematics **171** (2002), 59-85.
- [LM2] Landsberg J.M., Manivel L.: *Series of Lie groups*, Michigan Math. Journal **52** (2004), 453-479.
- [LM3] Landsberg J.M., Manivel L.: *A universal dimension formula for complex simple Lie algebras*, Advances in Mathematics **201** (2006), 379-407.
- [Ru] Rubenthaler H.: *Espaces préhomogènes de type parabolique*, in Lectures on harmonic analysis on Lie groups and related topics, pp. 189–221, Lectures in Math. **14**, Kinokuniya 1982.
- [Se] Serre J.P., Algèbres de Lie semi-simples complexes, Benjamin 1966.
- [Sh] Shramov C.: *The canonical strip II*, preprint.
- [St] Stanley R.: *Unimodal sequences arising from Lie algebras*, in Combinatorics, representation theory and statistical methods in groups, pp. 127–136, Lecture Notes in Pure and Appl. Math. **57**, Dekker 1980.

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