Some structural properties of planar graphs and their applications to 3-choosability

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Some structural properties of planar graphs and their applications to 3-choosability

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Abstract

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. We prove that a planar graph G is 3-choosable if it is satisfied one of the following conditions:

(1) each vertex x is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle.

(2) each vertex x is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$.

This work implies five results already published [13, 3, 7, 12, 4].

1 Introduction

Only simple graphs are considered in this paper unless otherwise stated. A *plane graph* is a particular drawing of a planar graph in the euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. A *proper vertex coloring* of G is an assignment c of integers (or labels) to the vertices of G such that $c(u) \neq c(v)$ if the vertices u and v are adjacent in G. A graph G is L*-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring c of the vertices such that $\forall v \in V(G), c(v) \in L(v)$. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be k*-choosable*.

Thomassen [8] proved that every planar graph is 5-choosable, whereas Voigt [9] proved that there exist planar graphs which are not 4-choosable. On the other hand, in 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [6]). This conjecture remains widely open. In 1990, Erdős suggested the following relaxation of Steinberg's conjecture: What is the smallest integer i such that every graph without j-cycles for $4 \leq j \leq i$ is 3-colorable. The best known upper bound is $i \leq 7$ [2]. It is natural to ask the same question for choosability:

Problem 1 *What is the smallest integer i such that every graph without j-cycles for $4 \leq j \leq i$ is 3-choosable?*

Voigt [10] proved that it is not possible to extend Steinberg's conjecture to list coloring: she gave a planar graph without 4-cycles and 5-cycles which is not 3-choosable ; hence $i \geq 6$. The best known upper bound is $i \leq 9$: this bound is obtained by using a structural lemma of Borodin [1].

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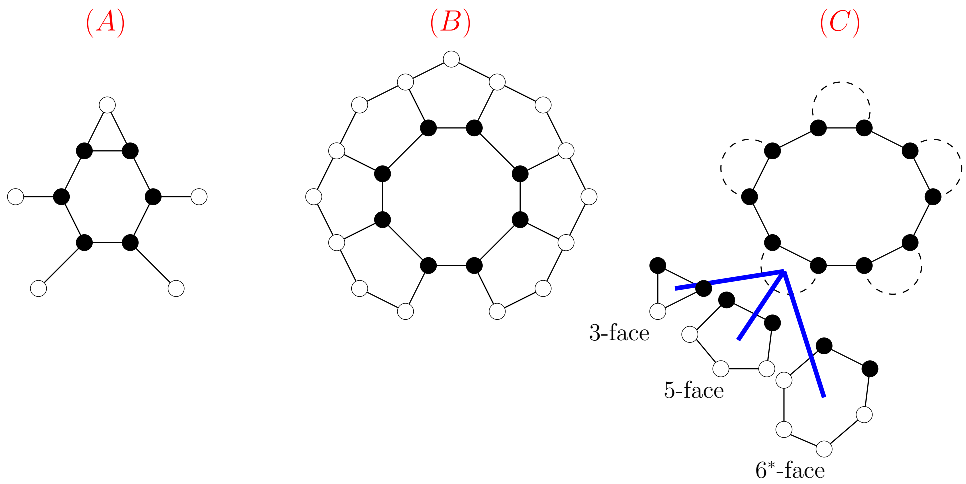

Figure 1: (A) Orchid, (B) sunflower, and (C) lotus.

Lemma 1 *[1] Let G be a planar graph with minimum degree at least 3. If G does not contain cycles of lengths 4 to 9, then G contains a 10-face incident to ten 3-vertices and adjacent to five 3-faces.*

It follows by Erdős, Rubin and Taylor [5] that every planar graph without cycles of lengths 4 to 9 is 3-choosable. Zhang and Wu [13] improved Borodin's result by proving that:

Lemma 2 *[13] Let G be a planar graph with minimum degree at least 3. If G does not contain cycles of lengths 4, 5, 6, and 9, then G contains a 10-face incident to ten 3-vertices and adjacent to five 3-faces.*

It implies that every planar graph without cycles of lengths 4, 5, 6, 9 is 3-choosable. Chen, Lu, and Wang [3] proved that every planar graph without cycles of lengths 4, 6, 7, 9 is 3-choosable. Their result is based on the following lemma:

Lemma 3 *[3] Let G be a planar graph with minimum degree at least 3. If G contains neither cycles of lengths 4, 7, 9, nor 6-cycle with a chord, then G contains a 10-face incident to ten 3-vertices or an 8-face incident to eight 3-vertices.*

Shen and Wang [7] proved that every planar graph without cycles of lengths 4, 6, 8, 9 is 3-choosable by showing that:

Lemma 4 *[7] Let G be a planar graph with minimum degree at least 3. If G does not contain cycles of lengths 4, 6, 8, and 9, then G contains a 10-face incident to ten 3-vertices.*

Moreover every planar graph without cycles of lengths 4, 5, 7, 9 (resp. 4, 5, 8, 9, and 4, 7, 8, 9) is 3-choosable [12] (resp. [11], [4]).

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. Let us introduce some notations which will allow to present our main result.

Some notation: The degree of a face is the length of its boundary walk. We will write $d(x)$ for $d_G(x)$ the degree of the vertex x in G when no confusion can arise. A k-*vertex*, k^+-*vertex*, or k^--*vertex* is a vertex of degree k, at least k, or at most k. Similarly, we can define k-face, k^+-face, k^--face, etc. We say that two cycles (or faces) are *adjacent* if they share at least one common edge. Suppose that f and f' are two adjacent faces by sharing a common edge e. We say that f and f' are *normally adjacent* if $|V(f) \cap V(f')| = 2$. A *triangle* is synonymous with a 3-face. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are the vertices of $b(f)$ appearing in a boundary walk of f.

A cycle C or a face f is called *nontriangular* if it is not adjacent to any 3-cycles. We say an i-face f is an i^*-*face* if f is adjacent to exactly one 3-face normally. Moreover, we call such i^*-face is *heavy*. Similarly, we say an i-cycle C is an i^*-*cycle* if C is adjacent to exactly one 3-cycle normally. For simpleness, we call such i^*-cycle is *heavy*. Two i^*-cycles (or i^*-faces) are *normally adjacent* if these two i-cycles (or i-faces) are normally adjacent.

An *orchid* is a 6^*-face incident to six 3-vertices and adjacent to a 3-face. A *sunflower* is an 8-face incident to eight 3-vertices and adjacent to at least seven 5-faces. A *lotus* is a 10-face incident to ten 3-vertices and adjacent to five clusters, where a cluster is either a 3-face, or a 5-face, or a 6^*-face (see Figure 1).

The following theorem is our main result which implies Lemmas 1-4.

Theorem 1 *Let G be a planar graph with minimum degree at least 3 and G does not contain 4-cycles and 9-cycles. If G further satisfies the following structural properties:*

(C1) a 5-cycle or 6-cycle is adjacent to at most one 3-cycle;

(C2) a 5^*-cycle is neither adjacent to a 5^*-cycle normally, nor adjacent to an i-cycle with $i \in \{7, 8\}$;

(C3) a 6^*-cycle is neither adjacent to a 6-cycle, nor incident to an i-cycle C with $i \in \{3, 5\}$, where C is opposite to such 6^*-cycle by a 4-vertex;

(C4) a nontriangular 7-cycle is not adjacent to two 5-cycles which are normally adjacent;

(C5) a 7^*-cycle is neither adjacent to a 5-cycle nor a 6^*-cycle.
Then G contains an orchid or a sunflower or a lotus.

We obtain the following Corollary 1 and Corollary 2 by Theorem 1.

Corollary 1 *Let G be a planar graph in which each vertex x is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle. Then G is 3-choosable.*

Corollary 2 *Let G be a planar graph in which every vertex x is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$. Then G is 3-choosable.*

Assuming Theorem 1, we can easily prove Corollary 1 and Corollary 2.

Proofs of Corollary 1 and Corollary 2: Suppose that G_1, G_2 is a counterexample to Corollary 1, Corollary 2 with the smallest number of vertices respectively. Thus, G_i is connected ($i = 1, 2$). Obviously, for each $i \in \{1, 2\}$, we observe that $\delta(G_i) \geq 3$. Otherwise, let u_i be a vertex of minimum degree in G_i. By the minimality of G_i, $G_i - u_i$ is 3-choosable. Obviously, we can extend any L-coloring such that $\forall x \in V(G) : |L(x)| \geq 3$ of $G_i - u_i$ to G_i and ensure G_i is 3-choosable. Next, in each case, we will show that each G_i contains either an orchid, or a sunflower, or a lotus. Denote N_A, N_B, N_C be the set of black vertices of (A), (B) and (C) in Figure 1, respectively. For each $j \in \{1, 2, 3\}$, one can easily observe that we can extend any L-coloring such that for all $x \in V(G) : |L(x)| \geq 3$ of $G_i - N_j$ to N_j and make sure G_i is 3-choosable. Thus, G_1 and G_2 are both 3-choosable. A contradiction.

Since G_i does not contain 4-cycles and 9-cycles, we only need to verify if G_i satisfies all the structural properties (C1) to (C5), where $i \in \{1, 2\}$.

(1) For G_1, since each vertex x is not incident to 6-cycles adjacent to a 3-cycle, each 5-cycle or 6-cycle only can be nontriangular cycles. This implies that there is neither 5^*-face nor 6^*-face in G_1. Thus, (C1), (C2) and (C3) are satisfied. Then we only need to consider (C4) and (C5). If (C4) is not satisfied, then there appears a vertex x which is incident to an i_x-cycle with $i_x \in \{5, 7, 8\}$, which contradicts the assumption of G_1. If (C5) is not satisfied, then a vertex y is appeared such that y is incident to an i_y-cycle with $i_y \in \{5, 7, 8\}$. A contradiction.

(2) For G_2, because it does not contain 7-cycles, we confirm that there is no 6^*-cycle and 7^*-cycle in G_2. Thus, we only need to check properties (C1) and (C2). It is easy to establish a 7-cycle or a 4-cycle if a 5-cycle or 6-cycle is adjacent to at least two 3-cycles. Thus, (C1) is satisfied. Let us check (C2). If there exist two 5^*-cycles adjacent normally, then a 7-cycle or a 9-cycle is produced, contradicting the absence of 7-cycles and 9-cycles. If a 5^*-cycle is adjacent to an 8-cycle, then there is a vertex incident to a 5-cycle, a 6-cycle and an 8-cycle, which is impossible. Therefore, (C2) is satisfied.

This completes the proofs of Corollary 1 and Corollary 2. □

By Corollary 1, it is easy to deduce Corollary 3:

Corollary 3 *Every planar graph G in which every vertex v is not incident to cycles of lengths $4, 6, 9, i_x$ with $i_x \in \{5, 7, 8\}$ is 3-choosable.*

Thus, by Corollary 2 and Corollary 3, we deduce Corollary 4 which covers five results mentioned before [13, 3, 7, 12, 4].

Corollary 4 *Every planar graph G without $\{4, i, j, 9\}$-cycles with $5 \leq i < j \leq 8$ and $(i, j) \neq (5, 8)$ is 3-choosable.*

Section 2 is dedicated to the proof of Theorem 1.

2 Proof of Theorem 1

Let G be a counterexample to Theorem 1, i.e., an embedded plane graph G with $\delta(G) \geq 3$, no cycles of lengths 4 and 9, satisfying the structural properties (C1) to (C5), and containing no orchid, no sunflower, and no lotus (i.e., none of the configurations depicted by Figure 1).

First, we suppose G is 2-connected. Thus, the boundary of each face f of G forms a cycle. Besides, the following assertions (O1) to (O7) hold naturally by the assumption of G.

(O1) A 5-face or a 6-face is adjacent to at most one 3-face;

(O2) A 5^*-face is neither adjacent to a 5^*-face normally, nor adjacent to an i-face with $i \in \{7, 8\}$;

(O3) A 6^*-face is neither adjacent to a 6-face, nor incident to an i-face f with $i \in \{3, 5\}$, where f is opposite to such 6^*-face by a 4-vertex;

(O4) A nontriangular 7-face is not adjacent to two 5-faces which are normally adjacent (there is no 3-vertex incident to a nontriangular 7-face and to two 5-faces);

(O5) A 7^*-face is neither adjacent to a 5-face nor a 6^*-face;

(O6) G does not contain 4-faces and 9-faces;

(O7) Each vertex v is incident to at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-faces.

Moreover, the following additional properties hold:

Claim 1 *For some fixed $i \in \{5,6,7,8\}$, if an i-face is adjacent to a 3-face, then they are normally adjacent.*

Proof. Suppose the claim is false. Let $f_i = [v_1v_2\cdots v_i]$ be an i-face and $f_2 = [v_1v_2u]$ be a 3-face such that f_1 is adjacent to f_2 and $|V(f_1)\cap V(f_2)| \geq 3$. It means that u is equal to some v_j with $j \in \{3,4,\cdots,i\}$. According to the value of i, one can easily observe that if u is a vertex v_j with $3 \leq j \leq i$, the G contains either a 2-vertex or a 4-cycle, a contradiction. This completes the proof of Claim 1. □

Since G does not contain 9-cycles, we obtain the following Claims 2 and 3 easily by Claim 1:

Claim 2 *Each 7-face is adjacent to at most one 3-face.*

Claim 3 *No 8-face is adjacent to a 3-face.*

Claim 4 *If two 5-faces are adjacent to each other, then they can only be normally adjacent.*

Proof. Suppose that there are two adjacent 5-faces $f_1 = [v_1v_2\cdots v_5]$ and $f_2 = [v_1v_2uvw]$ with v_1v_2 as a common edge. If $|V(f_1)\cap V(f_2)| = 2$, then Claim 4 follows. Otherwise, by symmetry, we only need to consider the following cases. If $w = v_5$, then $d(v_1) = 2$ which is impossible. If $w = v_4$, then G contains a 4-cycle $v_1v_2v_3v_4v_1$, a contradiction. This implies $u \notin b(f_1)$ and $w \notin b(f_1)$. If $v = v_5$ or $v = v_4$, then a 4-cycle $uv_2v_1v_5u$ or $wv_1v_5v_4w$ can be easily established, a contradiction, that completes the proof of Claim 4. □

Claim 5 *A nontriangular 5-face can not be adjacent to a 5^*-face in G.*

Proof. Suppose on the contrary that a nontriangular 5-face $f_1 = [v_1v_2\cdots v_5]$ is adjacent to a 5^*-face $f_2 = [v_1v_2u_3u_4u_5]$ by a common edge v_1v_2. By definition, f_1 is not adjacent to any 3-face. By Claim 4, each u_i can not be equal to some v_j with $i,j \in \{3,4,5\}$. By symmetry, we have to handle the following two properties:

• Assume that v_1u_5u is a 3-face. By Claim 1, $u \neq v_2, u_3, u_4$. Moreover, $u \neq v_5$ by choice of f_1. If $u = v_4$ or $u = v_3$, then G contains a 4-cycle, which is impossible. Thus, u does not belong to $b(f_1)\cup b(f_2)$ and G contains a 9-cycle $uv_1v_5v_4v_3v_2u_3u_4u_5u$, a contradiction.

• Assume that u_5u_4u is a 3-face. Notice that $u \neq v_1, v_2, u_3$ by Claim 1. If $u = v_3$ or v_4 or v_5, then G contains a 4-cycle which is impossible. Thus, u does not belong to $b(f_1)\cup b(f_2)$ and G contains a 9-cycle $uu_5v_1v_5v_4v_3v_2u_3u_4u$, a contradiction, that completes the proof of Claim 5. □

By Claim 4 and assertion (O2), we have:

Claim 6 *There is no adjacent two 5^*-faces in G.*

Claim 7 *No 3-vertex is incident to three 5-faces.*

Proof. Assume to the contrary that G contains a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to three 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and $f_3 = [uv_3z_1z_2v_1]$. By Claim 4, f_i and f_j are normally adjacent for each pair $\{i,j\} \subset \{1,2,3\}$. It implies that all vertices in $(V(f_1)\cup V(f_2)\cup V(f_3)) \setminus \{u\}$ are mutually distinct. However, a 9-cycle $v_1x_1x_2v_2y_1y_2v_3z_1z_2v_1$ is established, contradicting the assumption on G. Thus, we complete the proof of Claim 7. □

Claim 8 *Under isomorphism, a 6-face can be adjacent to a 5-face in an unique way as depicted by Figure 2.*

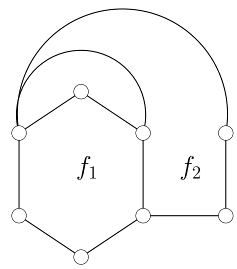

Figure 2: A 6-face f_1 is adjacent to a 5-face f_2.

Proof. Assuming that a 6-face $f_1 = [v_1v_2\cdots v_6]$ is adjacent to a 5-face $f_2 = [v_1v_2uvw]$ with v_1v_2 as a common edge. If $\{u, v, w\}\cap\{v_3, v_4, v_5, v_6\} = \emptyset$, then a 9-cycle $v_2v_3v_4v_5v_6v_1wvuv_2$ is formed, which is a contradiction. Thus, we confirm that $|V(f_1)\cap V(f_2)| \geq 3$. By symmetry of f_1, it suffices to consider the following cases.

• If $w = v_5$ and $u = v_4$, then a 4-cycle $v_1v_5v_4v_2v_1$ is established, which is impossible.

• Assuming that $w \neq v_5$ and $u \neq v_4$. Notice that $w \neq v_6$. Or else, a 2-vertex v_1 is produced. If $w = v_4$, then a 4-cycle $v_1v_6v_5v_4v_1$ is made. If $w = v_3$, then a 4-cycle v_2uvv_3 is constructed. A contradiction is always produced. By symmetry, we see that w and u do not belong to $b(f_1)$. Moreover, $v \neq v_3$ and $v \neq v_4$. Otherwise, a 4-cycle $wv_1v_2v_3w$ or $v_4v_3v_2uv_4$ is established, contradicting the assumption on G. Thus, by symmetry, $v \notin V(f_1)$, which means that $|V(f_1) \cap V(f_2)| = 2$, which is impossible.

• Without loss of generality, we may suppose that $w = v_5$ and $u \neq v_4$. Clearly, $u \neq v_3$ since $d(v_2) \geq 3$. So $u \notin V(f_1)$. One can easily observe that $v \neq v_3$ and $v \neq v_4$ by the absence of 4-cycles. Thus, we ensure that $v \notin b(f_1)$. It implies that f_1 is adjacent to f_2 in an unique way as Figure 2 shown. This completes the proof of Claim 8. □

Claim 9 *No 3-vertex is incident to two 5-faces and one 6-face.*

Proof. Suppose the claim is not true. We assume that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and one 6-face $f_3 = [uv_3z_1z_2z_3v_1]$.

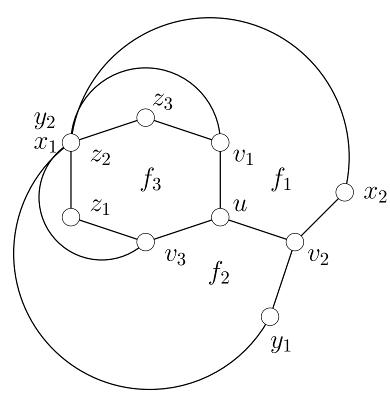

Figure 3: A 3-vertex u incident to two 5-faces f_1 and f_2 and to one 6-face f_3.

By Claim 8, $z_2 = y_2 = x_1$. Hence a 4-cycle $z_2v_1uv_3z_2$ exists which is a contradiction. Thus, we complete the proof of Claim 9. □

Claim 10 *No 3-vertex is incident to one 5-face and two 6-faces.*

Proof. Suppose on the contrary that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 6-faces $f_1 = [uv_3y_1y_2y_3v_1]$, $f_2 = [uv_2z_1z_2z_3v_3]$, and one 5-face

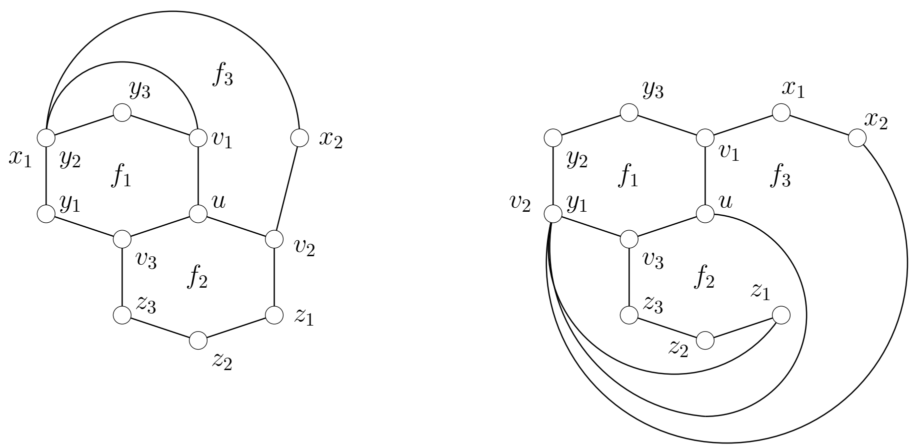

Figure 4: A 3-vertex u incident to one 5-face f_3 and two 6-faces f_1 and f_2.

$f_3 = [uv_1x_1x_2v_2]$. By Claim 8, we see that f_1 and f_3 can only be adjacent to each other in an unique way as depicted by Figure 2. One can easily observe that $x_1 = y_2$ or $v_2 = y_1$. Next, we will make use of contradictions to show that f_2 can not exist in G. We have to deal with the following two cases.

Case 1 $x_1 = y_2$.

For simpleness, denote $x^* = x_1 = y_2$. By Claim 8, we see that $x_2 = z_2$. It is easy to see that a 5-face $x^*v_1uv_2x_2x^*$ adjacent to two 3-cycles $x^*y_3v_1x^*$ and $v_2z_1x_2v_2$ is produced. This contradicts (C1).

Case 2 $v_2 = y_1$.

Clearly, uv_3y_1 is a 3-cycle which is not a 3-face. For simpleness, let $y^* = v_2 = y_1$. Obviously, $\{z_1, z_2, z_3\} \cap \{y_2, y_3, x_1, x_2\} = \emptyset$ since G is a plane graph. However, a 9-cycle $y^*z_1z_2z_3v_3uv_1x_1x_2y^*$ is easily established, which is impossible. This completes the proof of Claim 10. □

Claim 11 *No* 6^**-face is adjacent to a 5-face in* G*.*

Proof. Suppose on the contrary that there exists a 6-face $f_1 = [v_1v_2\cdots v_6]$ adjacent to a 5-face $f_2 = [v_1v_2uvw]$ by a common edge v_1v_2. By Claim 8, f_1 and f_2 can only be adjacent in an unique way depicted by Figure 2, which means that $w = v_5$. Note that f_1 is adjacent to a 3-cycle $v_1v_5v_6v_1$ which is not a 3-face. Thus, f_1 can not be adjacent to any other 3-face by (C1), which means that f_1 can not be a 6^*-face. This completes the proof of Claim 11. □

It is easy to derive Claim 12 by Claim 11.

Claim 12 *No* 6^**-face is adjacent to a* 5^**-face in* G*.*

By (C1), similarly as the proof of Claim 11 we have:

Claim 13 *No* 5^**-face is adjacent to a 6-face in* G*.*

Furthermore, assertion (O3) implies the following claim:

Claim 14 *There is no adjacent* 6^**-faces in* G*.*

Claim 15 *Let* G *be a connected plane graph with* n *vertices,* m *edges and* r *faces. Then using Euler's formula we have:*

$$\sum_{v\in V(G)} (2d(v) - 6) + \sum_{f\in F(G)} (d(f) - 6) = -12 \quad (1)$$

Proof. Euler's formula $n - m + r = 2$ yields $(4m - 6n) + (2m - 6r) = -12$. This identity and the relation $\sum_{v \in V} d(v) = \sum_{f \in F} d(f) = 2m$ imply (1). □

We now use a discharging procedure. We first assign to each vertex v an initial charge $\omega(v)$ such that for all $v \in V(G), \omega(v) = 2d(v) - 6$ and to each face f an initial charge such that for all $f \in F(G), \omega(f) = d(f) - 6$. In the following, we define discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function ω^* is produced. However, the total sum of charges is kept fixed when the discharging is in process. Nevertheless, we can show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Using (1), this leads to the following obvious contradiction:

$$-12 = \sum_{v \in V(G) \cup F(G)} \omega(v) = \sum_{v \in V(G) \cup F(G)} \omega^*(v) \geq 0$$

and hence demonstrates that no such counterexample can exist.

Before stating the discharging rules, we first give some notations which will be used frequently in the following argument. Let $x, y \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the charge transferred from x to y. For a vertex $v \in V(G)$ and for an integer $i \geq 5$, let $m_3(v)$, $m_i(v)$, and $m_{i^*}(v)$ denote the number of 3-faces, nontriangular i-faces, and heavy i-faces incident to v, respectively. Furthermore, we denote $M_i(v) = m_i(v) + m_{i^*}(v)$ and call a face f a *non-3-face* if $d(f) \neq 3$.

For simpleness, we write an edge uv is a (b_1, b_2)*-edge* if $d(u) = b_1$ and $d(v) = b_2$. Let f_1 and f_2 be two faces adjacent to each other by a common edge uv. If u and v are both not incident to any 3-face, then we call uv a *good common edge*. We further say such uv is a *good common* (b_1, b_2)*-edge* if uv is a (b_1, b_2)-edge.

The discharging rules are defined as follows:

(R1) Each 5^+-face sends 1 to its adjacent 3-face.

(R2) Let v be a 4-vertex.

- (R2a) If $m_3(v) = 2$, then for each non-3-face f, $\tau(v \to f) = 1$.
- (R2b) If $m_3(v) = 1$, then let f_1 denote the incident 3-face and f' be the opposite face of f_1.
  - (R2b1) If f' is a nontriangular 5-face, then v sends $\frac{2}{3}$ to each incident face different of f_1.
  - (R2b2) Otherwise, v sends 1 to each incident face which is adjacent to f_1.
- (R2c) If $m_3(v) = 0$, let f_1, f_2, f_3, and f_4 denote the faces of G incident to v in a cyclic order such that the degree of f_1 is the smallest one among all faces incident to v, then we do like this:
  - (R2c1) if $M_5(v) = 0$, then v sends $\frac{1}{2}$ to each incident face.
  - (R2c2) if $M_5(v) = 1$, then v sends $\frac{2}{3}$ to each of f_1, f_2, and f_4 when f_1 is a nontriangular 5-face; or v sends 1 to each of f_2 and f_4 when f_1 is a 5^*-face.
  - (R2c3) if $M_5(v) = 2$, then
    - (R2c3.1) v sends $\frac{2}{3}$ to each nontriangular 5-face and $\frac{1}{3}$ to each other incident face when $m_5(v) = 2$.
    - (R2c3.2) v sends $\frac{2}{3}$ to each incident face of v except the unique 5^*-face when $m_5(v) = 1$ and $m_{5^*}(v) = 1$.
    - (R2c3.3) v sends 1 to each incident face which is not a 5^*-face when $m_{5^*}(v) = 2$.
  - (R2c4) if $M_5(v) = 3$, then v gives $\frac{2}{3}$ to each incident nontriangular 5-face.
  - (R2c5) if $M_5(v) = 4$, then v gives $\frac{1}{2}$ to each incident nontriangular 5-face.

(R3) Let v be a 5-vertex and f be a non-3-face incident to v. Then

- (R3a) $\tau(v \to f) = \frac{4}{3}$ if $m_3(v) = 2$.
- (R3b) $\tau(v \to f) = 1$ if $m_3(v) = 1$.
- (R3c) if $m_3(v) = 0$, v sends 1 to each incident face different from 5^*-faces when $m_{5^*}(v) \geq 1$; or sends $\frac{5}{6}$ to each incident 6^*-face and sends $\frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}$ to each other incident face when $m_{5^*}(v) = 0$.

(R4) Let f be a 7^+-face. If f' is adjacent to f by a good common edge e, then

- (R4a) $\tau(f \to f') = \frac{1}{3}$ if f' is a nontriangular 5-face and e is a $(3,3)$-edge.
- (R4b) $\tau(f \to f') = \frac{1}{6}$ if f' is a 6^*-face and e is a $(3,3)$-edge or a $(3,4)$-edge.

(R5) Each 10^+-face sends 1 to each adjacent 5^*-face by a good common $(3^+, 3^+)$-edge.

(R6) Each 6^+-vertex sends 1 to each incident face.

Let us check that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Since $\delta(G) \geq 3$, $d(v) \geq 3$ for each vertex $v \in V(G)$. We have to handle the following cases, depending on the size of $d(v)$.

Case 1 $d(v) = 3$.

It is easy to see that $\omega^*(v) = \omega(v) = 2 \times 3 - 6 = 0$ by (R1) to (R6).

Case 2 $d(v) = 4$.

Clearly, $\omega(v) = 2$ and v is incident to at most two 3-faces by (O7). If $m_3(v) = 2$, then we deduce that $\omega^*(v) = 2 - 2 \times 1 = 0$ by (R2a). If $m_3(v) = 1$ (v is incident to exactly one 3-face), then depending on the opposite face of such 3-face, v gives either $\frac{2}{3} \times 3 = 2$, or $1 \times 2 = 2$ by (R2b1) or (R2b2). Hence, $\omega^*(v) = 0$. Finally, we only need to consider the case of $m_3(v) = 0$. We divide the discussion into five subcases in the light of the value of $M_5(v)$.

Subcase 2.1 $M_5(v) = 0$.

This implies that the degree of each face incident to v is at least 6 by the absence of 4-faces. According to (R2c1), $\omega^*(v) \geq 2 - \frac{1}{2} \times 4 = 0$.

Subcase 2.2 $M_5(v) = 1$.

It is easy to observe that v sends either $\frac{2}{3} \times 3 = 2$ if $m_5(v) = 1$, or $1 \times 2 = 2$ if $m_{5^*}(v) = 1$ by (R2c2). Thus, v gives totally at most 2 to incident faces. Hence, $\omega^*(v) \geq 2 - 2 = 0$.

Subcase 2.3 $M_5(v) = 2$.

If $m_5(v) = 2$, then $\omega^*(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R2c3.1). If $m_5(v) = m_{5^*}(v) = 1$, then such nontriangular 5-face and 5^*-face can not be adjacent to each other by Claim 5. Thus, applying (R2c3.2), $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$. Otherwise, suppose $m_{5^*}(v) = 2$. Notice that v is incident to two 5^*-faces which are opposite to each other by Claim 6. Thus, $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R2c3.3).

Subcase 2.4 $M_5(v) = 3$.

We first notice that $m_{5^*}(v) \neq 3$ since there are no adjacent 5^*-faces in G by Claim 6. If $1 \leq m_{5^*}(v) \leq 2$, then there exists at least one nontriangular 5-face adjacent to one 5^*-face, contradicting the Claim 5. Thus, $m_{5^*}(v) = 0$, and so $m_5(v) = 3$. According to (R2c4), we have that $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$.

Subcase 2.5 $M_5(v) = 4$.

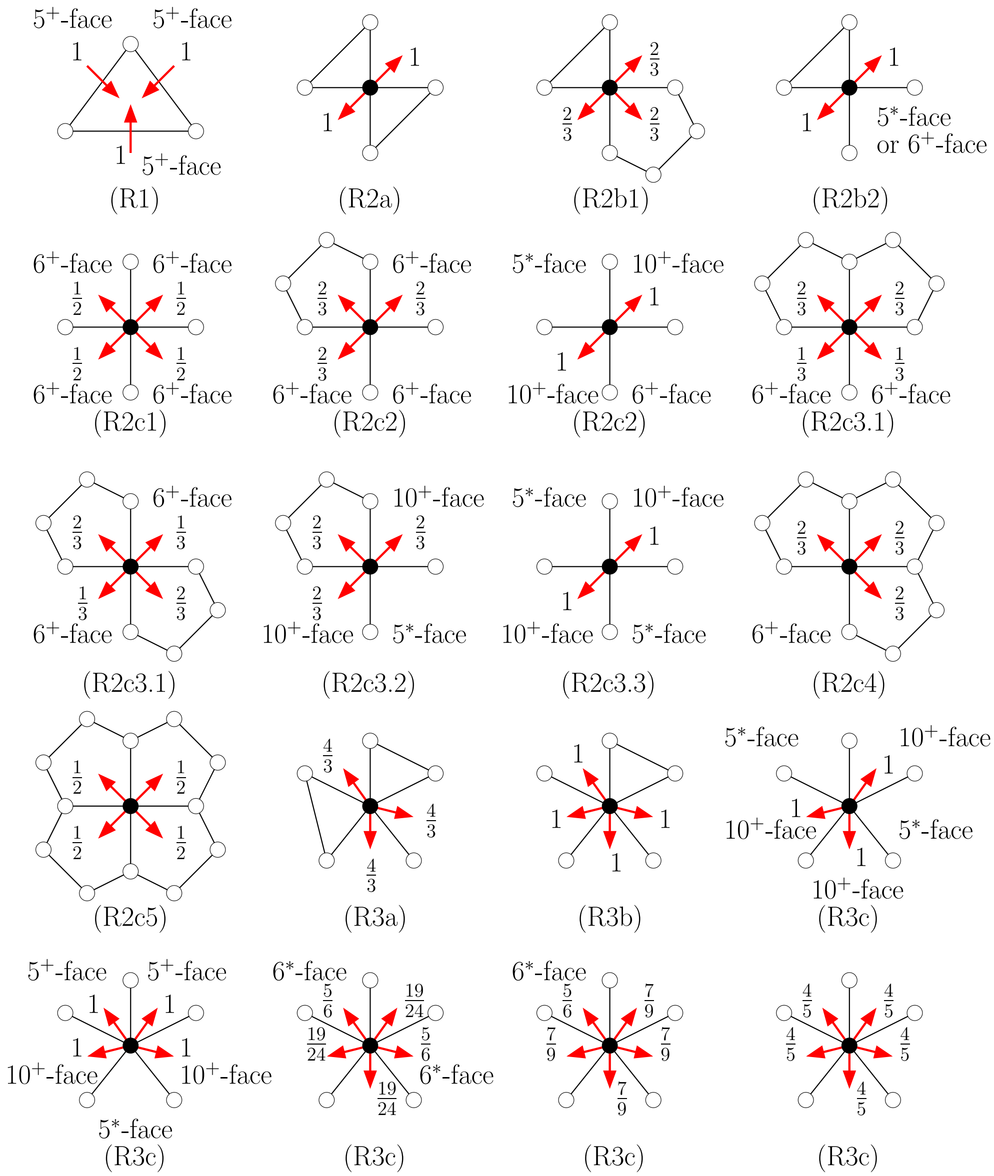

Figure 5: Some of discharging rules (R1) to (R3).

One can observe that $m_{5^*}(v) = 0$ by Claim 5 and Claim 6. It implies that v is incident to exactly four nontriangular 5-faces. Consequently, we have that $\omega^*(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by (R2c5).

Case 3 $d(v) = 5$.

Obviously, $\omega(v) = 4$ and $m_3(v) \leq 2$ by (O7). It is easy to observe that v sends either $\frac{4}{3} \times 3 = 4$ by (R3a) if $m_3(v) = 2$, or $1 \times 4 = 4$ by (R3b) if $m_3(v) = 1$. Therefore, $\omega^*(v) \geq 4 - 4 = 0$ if $m_3(v) > 0$. Now we may assume that $m_3(v) = 0$. This implies that each face incident to v is a 5^+-face combining the fact that G does not contain any 4-faces. By Claim 6, we have that $m_{5^*}(v) \leq 2$. Moreover, each face adjacent to a 5^*-face must be a 10^+-face by Claim 5, Claim 6, Claim 12, Claim 13, (O2) and the absence of 9-faces. So by (R3c), $\omega^*(v) \geq 4 - 1 \times 4 = 0$ if $m_{5^*}(v) \geq 1$; or $\omega^*(v) \geq 4 - \frac{5}{6} m_{6^*}(v) - \frac{4 - \frac{5}{6} m_{6^*}(v)}{5 - m_{6^*}(v)}(5 - m_{6^*}(v)) = 0$ if $m_{5^*}(v) = 0$.

Case 4 $d(v) \geq 6$.

According to (R6), we have that $\omega^*(v) \geq (2d(v) - 6) - 1 \times d(v) = d(v) - 6 \geq 0$.

Let $f \in F(G)$. Then $b(f)$ is a cycle since G is 2-connected. We write $f = [v_1 v_2 \cdots v_{d(f)}]$ and suppose that f_i is the face of G adjacent to f with $v_i v_{i+1} \in b(f) \cap b(f_i)$ for $i = 1, 2, \cdots, d(f)$, where (and in the following discussion) all indices are taken modulo $d(f)$ plus 1. We observe that $d(f) \neq 4$ and $d(f) \neq 9$ by (O6). For $i \geq 3$, let $n_i(f)$ denote the number of i-vertices incident to f. Let $m_5(f)$, $m_{5^*}(f)$, and $m_{6^*}(f)$ denote the number of nontriangular 5-faces, heavy 5-faces, and heavy 6^*-faces adjacent to f.

Case 5 $d(f) = 3$.

Let f be a 3-face and then $\omega(f) = -3$. Since $\delta(G) \geq 3$, f is adjacent to three faces and each adjacent face is neither a 3-face nor a 4-face by the absence of 4-cycles in G. It implies that f gets 3×1 from its adjacent faces by (R1). Thus, $\omega^*(f) \geq -3 + 1 \times 3 = 0$.

Case 6 $d(f) = 5$.

Let $f = [v_1 \cdots v_5]$ and then $\omega(f) = -1$. Clearly, f is adjacent to at most one 3-face by (O1).

• We first assume that f is a nontriangular 5-face, which means that there is no 3-face adjacent to f. Thus, f sends nothing to all its adjacent faces. Moreover, each f_i can not be a 5^*-face by Claim 5. We only have to deal with the the following three possibilities, depending on the value of $n_3(f)$.

Subcase 6.1 $n_3(f) = 5$.

It means that v_i is a 3-vertex for all $i = 1, \ldots, 5$. If there exists a 6-face adjacent to f, then by Claim 8 we see that they must be adjacent to each other in an unique way as depicted by Figure 2. It is easy to see that there is one 4^+-vertex appeared on $b(f)$, which contradicts $n_3(f) = 5$. Thus, each face adjacent to f is either a nontriangular 5-face or a 7^+-face by the absence of 4-faces. Furthermore, we notice that f is adjacent to at most two nontriangular 5-faces which are not adjacent by Claim 7. So f is adjacent to at least three 7^+-faces such that each 7^+-face is adjacent to f by a good common $(3, 3)$-edge. Therefore, applying (R4a), we obtain that $\omega^*(f) \geq -1 + 3 \times \frac{1}{3} = 0$.

Subcase 6.2 $n_3(f) = 4$.

Let v_1 be such a 4^+-vertex and v_j be a 3-vertex for all $j = 2, 3, 4, 5$. Clearly, v_1 gives at least $\frac{1}{2}$ to f by (R2) and (R3). Moreover, f_1 and f_5 can not be any 6-face by Claim 8. If $d(f_1) = 5$ and $d(f_5) = 5$, then $d(f_j) \notin \{5, 6\}$ with $j \in \{2, 4\}$ according to Claim 7 and Claim 9. Thus, for $j \in \{2, 4\}$, f_j is a 7^+-face by the absence of 4-faces and each f_j is adjacent to f by a good common $(3, 3)$-edge. By (R4a), we see that $\tau(f_2 \to f) = \frac{1}{3}$ and $\tau(f_4 \to f) = \frac{1}{3}$. So we obtain that $\omega^*(f) \geq -1 + \frac{1}{2} + \frac{1}{3} \times 2 = \frac{1}{6} > 0$.

Now we may suppose that there exists at least one face of f_1 and f_5 which is a 7^+-face, i.e., $d(f_1) \geq 7$. Then by (R2) and (R3), we see that $\tau(v_1 \to f) \geq \frac{2}{3}$. Clearly, for each $i \in \{2, 3, 4\}$, f_i is adjacent to f by a good common $(3, 3)$-edge. According to Claim 7, Claim 9 and Claim 10, we see that there exists at least one face of f_2, f_3, f_4 which is a 7^+-face. Hence, $\omega^*(f) \geq -1 + \frac{1}{3} + \frac{2}{3} = 0$ by (R4a).

Subcase 6.3 $n_3(f) \leq 3$.

It means that there are at least two vertices whose degree are both at least 4. By (R2), we derive that $\omega^*(f) \geq -1 + \frac{1}{2} \times 2 = 0$.

• Now, we may suppose that f is a 5^*-face. It implies that f is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such 3-face that it is adjacent to f. By Claim 1, $v \neq v_i$ for all $i = 3, 4, 5$. Since $\delta(G) \geq 3$, $d(v_i) \geq 3$ with $i \in \{1, 2, \cdots, 5\}$. By Claim 6 and (O2), for each $j \in \{2, 3, 4, 5\}$, f_j is neither a 5^*-face nor an i-face with $i = 7, 8$. Furthermore, we observe that $d(f_j) \neq 3$, $d(f_j) \neq 4$, $d(f_j) \neq 5$, $d(f_j) \neq 6$, and $d(f_j) \neq 9$ by (O1), (O6), Claim 5, and Claim 13, respectively. Thus, we confirm that $d(f_j) \geq 10$. Therefore, we derive that $\tau(f_3 \to f) = 1$ and $\tau(f_4 \to f) = 1$ by (R5). Hence, $\omega^*(f) \geq -1 - 1 + 1 \times 2 = 0$ by (R1).

Case 7 $d(f) = 6$.

Let $f = [v_1 \cdots v_6]$ and then $\omega(f) = 0$. If f is a nontriangular 6-face, then it is easy to deduce that $\omega^*(f) = \omega(f) = 0$ by (R1) to (R6). Now, we assume that f is a 6^*-face. Without loss of generality, assume $f_1 = [vv_1v_2]$ is a 3-face adjacent to f. It is obvious that $v \notin b(f)$ by Claim 1. Furthermore, f is adjacent to at most one 3-face by (O1). So f only need to send 1 to the unique 3-face f_1. Obviously, for each $j \in \{2, \cdots, 6\}$, $d(f_j) \notin \{3, 4, 5, 6\}$ by (O1), (O6), Claim 11 and (O3). Noting that $v_3v_5 \notin E(G)$ and $v_3v_6 \notin E(G)$ by (C1) and the absence of 4-cycles. This implies that each v_i has at least one outgoing neighbor which is not lied on $b(f)$. Since there is no orchid in G, f is incident to at least one 4^+-vertex. It implies that $n_3(f) \leq 5$. Next, in each case, we will show that the total charge f obtained is at least 1 and thus $\omega^*(f) \geq -1 + 1 = 0$.

Subcase 7.1 $n_3(f) = 5$.

It means that there is exactly one 4^+-vertex incident to f. If $d(v_2) \geq 4$, then $\tau(v_2 \to f) \geq 1$ by (R2a), (R2b2), (R3a), (R3b) and (R6) since $d(f_2) \neq 5$. Otherwise, by symmetry, suppose some v_i is a 4^+-vertex where $i \in \{3, 4\}$. Denote v^* be such a 4^+-vertex. First, we observe that each adjacent face different from f_1 is a 7^+-face by the discussion above. If $d(v^*) \geq 5$, then $\tau(v^* \to f) \geq \frac{5}{6}$ by (R3) and (R6). Noting that there is at least one f_j sends $\frac{1}{6}$ to f, where $j \in \{3, 4, 5\}$. Thus, f gets at least $\frac{5}{6} + \frac{1}{6} = 1$ from v^* and its adjacent 7^+-faces. If $d(v^*) = 4$, then the opposite face of f, which is incident to f by v^*, can not be a 3-face or a 5-face by (O3). So v^* is incident to four 6^+-faces and thus v^* gives $\frac{1}{2}$ to f by (R2c1). Consequently, f gets at least $\frac{1}{2} + \frac{1}{6} \times 3 = 1$ by (R4b).

Subcase 7.2 $0 \leq n_3(f) \leq 4$.

It implies that there are at least two 4^+-vertices incident to f. It is easy to see that every 5^+-vertex sends at least $\frac{5}{6}$ to f by (R3) and (R6). Moreover, every 4-vertex sends at least $\frac{1}{2}$ to f since it is not incident to any 3-face or 5-face. Hence, f receives at least $\frac{1}{2} \times 2 = 1$ from its incident 4^+-vertices.

In what follows, for simpleness, let $p_5(f)$, $p_{5^*}(f)$, and $p_{6^*}(f)$ denote the number of nontriangular 5-face, 5^*-face, and 6^*-face receiving a charge $\frac{1}{3}$, 1, $\frac{1}{5}$ from f, respectively. Clearly, $p_5(f) \leq m_5(f)$, $p_{5^*}(f) \leq m_{5^*}(f)$ and $p_{6^*}(f) \leq m_{6^*}(f)$.

Case 8 $d(f) = 7$.

Then $\omega(f) = 1$. Let $m_3(f)$ be the number of 3-faces adjacent to f. Clearly, $m_3(f) \leq 1$ by Claim 2.

• We first assume f is a nontriangular 7-face. Noting that $d(f_i) \geq 5$ since G contains no 4-faces. By (O2), $m_{5^*}(f) = 0$. By (O4), $p_5(f) \leq 3$. We will divide the argument into four subcases according to the value of $p_5(f)$.

Subcase 8.1 $p_5(f) = 3$.

Suppose f_1, f_3, f_5 are such three 5-faces that each of them takes a charge $\frac{1}{3}$ from f. By (R4a), we see that all common edges (v_1v_2), (v_3, v_4) and (v_5, v_6) are good $(3, 3)$-edges. This implies that $d(v_i) = 3$ with $i \in \{1, \cdots, 6\}$. By Claim 11, one can easily defer that none of f_2, f_4, f_6, f_7 can be a 6^*-face. Thus, $p_{6^*}(f) \leq m_{6^*}(f) = 0$. Consequently, we deduce that $\omega^*(f) \geq 1 - \frac{1}{3} \times 3 = 0$ by (R4a).

Subcase 8.2 $p_5(f) = 2$.

We may suppose that f_i is a 5-face which takes $\frac{1}{3}$ from f. It means that $d(v_i) = d(v_{i+1}) = 3$ and (v_i, v_{i+1}) is a good common edge. Thus, f_{i-1} and f_{i+1} can not be any 6^*-face by Claim 11. It follows immediately that $p_{6^*}(f) \leq 7 - (2 + 3) = 2$ since $p_5(f) = 2$. Consequently, we have that $\omega^*(f) \geq 1 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = 0$ by (R4).

Subcase 8.3 $p_5(f) = 1$.

Without loss of generality, let f_1 be such a nontriangular 5-face that (v_1, v_2) be a good common $(3, 3)$-edge. This implies that neither f_2 nor f_7 can be a 6^*-face. Thus, $p_{6^*}(f) \leq 7 - 3 = 4$. Hence, we have $\omega^*(f) \geq 1 - \frac{1}{3} - \frac{1}{6} \times 4 = 0$ by (R4a) and (R4b).

Subcase 8.4 $p_5(f) = 0$.

If $p_{6^*}(f) = 0$, then according to (R4), we obtain that $\omega^*(f) \geq 1 - 0 = 1$. Otherwise, we may let f_1 is a 6^*-face, which takes a charge $\frac{1}{6}$ from f. It is obvious that f_1 must be adjacent to f by a good common $(3, 3)$-edge or $(3, 4)$-edge, i.e., $d(v_1) = 3$ and $d(v_2) \in \{3, 4\}$. It is easy to observe that f_7 can not be any 6^*-face because of Claim 14. Thus, $p_{6^*}(f) \leq 6$ and $\omega^*(f) \geq 1 - \frac{1}{6} \times 6 = 0$ by (R4b).

• Now we may assume $m_3(f) = 1$, which implies that f is a 7^*-face and it is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such a 3-face that f sends 1 to f_1. By Claim 1, we notice that v is not lied on $b(f)$. Moreover, for each $j \in \{2, \cdots, 7\}$, we deduce that f_j is neither a 5-face nor a 6^*-face by (O5). It implies that f sends nothing to each f_j with $j \in \{2, \cdots, 7\}$. Applying (R1), we deduce that $\omega^*(f) \geq 1 - 1 = 0$.

Case 9 $d(f) = 8$.

Clearly, $\omega(f) = 2$ and f can not be adjacent to any 3-face by Claim 3. So we only need to consider the size of $p_5(f)$ and $p_{6^*}(f)$ since they may take charge from f. It is easy to calculate that $p_5(f) \leq 6$ by the fact that there is no sunflower in G. We have to consider the following possibilities by the value of $p_5(f)$.

Subcase 9.1 $p_5(f) = 6$.

It implies that there are at least seven vertices in $V(f)$ are 3-vertices. Thus, the remaining two faces adjacent to f, which are not nontriangular 5-faces, can not be any 6^*-faces by Claim 11. So $\omega^*(f) \geq 2 - 6 \times \frac{1}{3} = 0$ by (R4).

Subcase 9.2 $p_5(f) = 5$.

One can easily notice that there is at most one of f_i with $i \in \{1, \cdots, 8\}$ which is a 6^*-face because no 5-face can be adjacent to a 6^*-face by Claim 11 again. Therefore, $\omega^*(f) \geq 2 - 5 \times \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$.

Subcase 9.3 $0 \leq p_5(f) \leq 4$.

By (R4), we derive that

$$
\begin{aligned}
\omega^*(f) &\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}p_{6^*}(f) \\
&\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}(8 - p_5(f)) \\
&= \frac{2}{3} - \frac{1}{6}p_5(f) \\
&\geq \frac{2}{3} - \frac{1}{6}\cdot 4 \\
&= 0.
\end{aligned}
$$

Next, we will discuss several cases where $d(f) \geq 10$. Let f be such a 10^+-face that f' is adjacent to f. We call f' is *special* if it takes charge 1 from f. Let $|F_1(f)|$ denote the number of adjacent special faces. Let S_i be a face adjacent to f by an edge e_i for $i = 1, 2$. If e_1 is not incident to e_2, then we call S_1 and S_2 are *mutually disjoint*. According to (R1) and (R5), we see that only 3-face and 5^*-face may take charge 1 from f, respectively. It implies that each special face is either a 3-face or a 5^*-face. We first observe that:

Observation 1 *If f is adjacent to two special faces which share at least one common vertex v that is lied on $b(f)$, then $\tau(v \to f) \geq 1$.*

Proof. Without loss of generality, assume f_1 and f_2 are both such two special faces that $v_2 \in V(f_1) \cap V(f_2)$ and $v_2 \in V(f)$. Since each 5^*-face taking charge 1 from a 10^+-face must be adjacent to f by a good common $(3^+, 3^+)$-edge, we see that f_1 and f_2 are either both 3-faces or both 5^*-faces. By the absence of two adjacent 3-faces and two adjacent 5^*-faces, we confirm that $d(v_2) \geq 4$. If $d(f_1) = d(f_2) = 3$, then by (R2a), (R3a) and (R6), it is easy to deduce that $\tau(v_2 \to f) \geq 1$. Otherwise, we may suppose f_1 and f_2 are both 5^*-faces. According to (R2c3.3), (R3c) and (R6), we derive that $\tau(v_2 \to f) = 1$. This complete the proof of Observation 1. □

If there exist two special faces which share at least one common vertex v that is lied on $b(f)$, i.e., let f_i and f_{i+1} be such two special faces that $v_{i+1} \in V(f_i) \cap V(f_{i+1})$ and $v_{i+1} \in V(f)$, then we see that $\tau(v_{i+1} \to f) \geq 1$ by Observation 1 and f sends at most 2×1 to f_i and f_{i+1}. It means that f takes charge 1 from v_{i+1} and then sends it to f_{i+1}. Thus, we can consider that f_{i+1} takes nothing from f. Therefore, we may suppose that all of the special faces adjacent to f are mutually disjoint, which implies $|F_1(f)| \leq \left\lfloor \frac{d(f)}{2} \right\rfloor$.

Observation 2 *If f_i is a special face with a (good) common $(3^+, 3^+)$-edge v_iv_{i+1}, then f sends nothing to f_{i-1} and f_{i+1}.*

Proof. If f_i is a 3-face, then f_{i-1} and f_{i+1} can not be any special faces by the assumption that special faces adjacent to f are mutually disjoint. Since $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ are not good common edges, we conclude that f sends nothing to f_{i-1} and f_{i+1} by (R4) and (R5).

Now we may assume that f_i is a 5^*-face. By symmetry, we only need to consider v_i, depending on $d(v_i)$.

• If $d(v_i) = 3$, then f_{i-1} can not be a 3-face since v_iv_{i+1} is a good common edge. Moreover, f_{i-1} can not be any nontriangular 5-face, 5^*-face or 6^*-face by Claim 5, Claim 6 and Claim 12. So $\tau(f \to f_{i-1}) = 0$.

• Next, we may suppose $d(v_i) \geq 4$. Clearly, f_{i-1} can not be any special faces by the assumption that special faces adjacent to f are mutually disjoint. Furthermore, if f_{i-1} is a nontriangular 5-face, then f sends nothing to it because $v_{i-1}v_i$ is not a $(3,3)$-edge. If f_{i-1} is a 6^*-face, then we discuss as follows: when v_i is a 5^+-vertex, then $\tau(f \to f_{i-1}) = 0$ since $v_{i-1}v_i$ is neither a $(3,3)$-edge nor a $(3,4)$-edge; when v_i is a 4-vertex, then f_i is the opposite face of f_{i-1} by a 4-vertex, which contradicts (O3). Thus, we prove that $\tau(f \to f_{i-1}) = 0$ and $\tau(f \to f_{i+1}) = 0$. This completes the proof of Observation 2. □

By using Observation 2, one can easily deduce Observation 3:

Observation 3 $p_5(f) + p_{6^*}(f) \le d(f) - 2|F_1(f)| - 1$.

Case 10 $d(f) = 10$.

Then $\omega(f) = 4$ and $|F_1(f)| \le 5$. We divide the argument into the following three subcases in light of $|F_1(f)|$.

Case 10.1 $|F_1(f)| = 5$.

It implies that there exist five mutually disjoint special faces that are adjacent to f. Since G does not contain lotus, there exists at least one 4^+-vertex on $b(f)$. Without loss of generality, suppose v_1 is such a vertex that f_1, f_3, f_5, f_7, f_9 are all special faces. If v_1 is a 5^+-vertex, then v_1 sends at least 1 to f by (R3) and (R6). Now we assume that v_1 is a 4-vertex. If $d(v_{10}) = 3$, then f_{10} is not a nontriangular 5-face since f_9 is a special face. So $\tau(v_1 \to f) = 1$ by (R2b2), (R2c2) and (R2c3.3). Otherwise, $d(v_{10}) \ge 4$ and f receives at least $\frac{2}{3} \times 2 = \frac{4}{3}$ from v_1 and v_{10} totally by (R2b1), (R2b2), (R2c2), (R2c3.2) and (R2c3.3). Thus, $\omega^*(f) \ge 4 - 1 \times 5 + 1 = 0$.

Case 10.2 $|F_1(f)| = 4$.

It implies that f is adjacent to exactly four special faces by four (good) common edges which are disjoint each other. Denote S_i be such a special face adjacent to f by a common edge e_i, where $i = 1, 2, 3, 4$. Noting that e_i can not be incident to e_j for each pair $(i, j) \subset \{1, \cdots, 4\}$. Thus, it follows that there are exist two vertices lied on $b(f)$ which are not incident to any common edge e_i with $i \in \{1, \cdots, 4\}$. W.l.o.g., assume $i < j$. If $j = i + 1$, then $v_i v_j$ is an edge of $b(f)$. Notice that f_i can not be a special face, i.e., f_i is neither a 3-face nor a 5^*-face. Furthermore, if f_i is nontriangular 5-face or a 6^*-face, then there exists at least one vertex in $V(f)$ whose degree is at least 4 by the absence of lotus. Let v^* be such 4^+-vertex. If $d(v^*) \ge 5$, then $\tau(v^* \to f) \ge \frac{4}{5}$ by (R3) and (R6). Now we may suppose $d(v^*) = 4$. According to (R2c2), (R2c3.2) and (R2c3.3), it is obvious that each 4-vertex sends at least $\frac{2}{3}$ to its incident face which is adjacent to a special face. Thus, we have that $\omega^*(f) \ge 4 - 1 \times 4 - \frac{1}{3} + \frac{2}{3} = \frac{1}{3} > 0$.

Case 10.3 $0 \le |F_1(f)| \le 3$.

By Observation 3, $p_5(f) + p_{6^*}(f) \le 9 - 2|F_1(f)|$. Therefore, $\omega^*(f) \ge 4 - |F_1(f)| - \frac{1}{3}(9 - 2|F_1(f)|) = 1 - \frac{1}{3}|F_1(f)| \ge 1 - \frac{1}{3} \times 3 = 0$.

Case 11 $d(f) = 11$.

Clearly, $\omega(f) = 5$ and $|F_1(f)| \le 5$. By Observation 3, $p_5(f) + p_{6^*}(f) \le 10 - 2|F_1(f)|$. Then $\omega^*(f) \ge 5 - |F_1(f)| - \frac{1}{3}(10 - 2|F_1(f)|) = \frac{5}{3} - \frac{1}{3}|F_1(f)| \ge \frac{5}{3} - \frac{1}{3} \times 5 = 0$.

Case 12 $d(f) \ge 12$.

By Observation 3, $p_5(f) + p_{6^*}(f) \le d(f) - 2|F_1(f)| - 1$. Moreover, $|F_1(f)| \le \lfloor \frac{1}{2} d(f) \rfloor$. Thus, we have that

$$
\begin{aligned}
\omega^*(f) &\geq (d(f)-6)-|F_1(f)|-\frac{1}{3}(d(f)-2|F_1(f)|-1) \\
&= \frac{2}{3}d(f)-\frac{17}{3}-\frac{1}{3}|F_1(f)| \\
&\geq \frac{2}{3}d(f)-\frac{17}{3}-\frac{1}{3}\cdot\frac{d(f)}{2} \\
&= \frac{1}{2}d(f)-\frac{17}{3} \\
&\geq \frac{1}{2}\cdot 12-\frac{17}{3} \\
&= \frac{1}{3}.
\end{aligned}
$$

Up to now, we proved Theorem 1 for 2-connected graphs.

Suppose now that G is not a 2-connected planar graph and we will construct a 2-connected plane graph G^* with $\delta(G^*) \geq 3$ having neither 4-cycles nor 9-cycles and satisfying structural properties (C1) to (C5). This obviously contradicts the result just established before.

We remark that the following proof is stimulated by the technique used in [3].

Let B be an end block of G with the unique cut-vertex x. Let f be the outside face of G. Notice that $d_B(x) \geq 2$ and $d_B(v) \geq 3$ for each $v \in V(B)\backslash\{x\}$. Choosing another vertex y of B such that $y \neq x$ and y lies on the boundary of B. Obviously, x and y are both belonging to $b(f)$. Then we take ten copies of B, i.e., B_k with $k = 1, \cdots, 10$. In each copy B_k, the vertices corresponding to x and y are denoted by x_k and y_k, respectively. Then one can embed B_k, $k = 1, \cdots, 10$, into f in the following way: first, let $B = B_1$. Next, for each $k = 2, \cdots, 10$, consecutively embed B_k into f by identifying x_k with y_{k-1}. Finally, identify y_{10} with a vertex $u \in V(f)\backslash V(B)$. Then the first resulting graph, denoted by G_1.

Obviously, in the processing of constructing G_1, we confirm that there are no new adjacent cycles established. Furthermore, no 4-cycles and 9-cycles are formed. Thus, it is easy to deduce that G_1 satisfies the following structural properties.

(A1) Fewer end blocks than G;

(A2) The minimum degree is at least 3;

(A3) Neither 4-cycles nor 9-cycles;

(A4) A 5-cycle or a 6-cycle is adjacent to at most one 3-cycle;

(A5) A 5^*-cycle is neither adjacent to a 5^*-cycle normally, nor adjacent to an i-cycle with $i \in \{7, 8\}$;

(A6) A 6^*-cycle is not adjacent to a 6-cycle;

(A7) A nontriangular 7-cycle is not adjacent to two 5-cycles which are normally adjacent;

(A8) A 7^*-cycle is neither adjacent to a 5-cycle nor a 6^*-cycle.

Furthermore, we confirm that G_1 also satisfies the following two structural properties:

(P1) G_1 has neither orchid, nor sunflower, nor lotus;

(P2) A 6^*-cycle is not incident to an i-cycle C with $i \in \{3, 5\}$, where C is opposite to such 6^*-cycle by a 4-vertex.

(P1) For some $k \in \{2, \cdots, 10\}$, notice that we just identify some vertex x_k with y_{k-1}. It implies that any new cycle, which is not completely belong to some B_k, must be an 11^+-cycles, i.e., $C^* = x_1 \cdots x_{10} u \cdots x_1$. Thus, any orchid, sunflower, or lotus can not be established.

(P2) Assume to the contrary that G_1 contains a 6^*-cycle, denoted by C_6^*, which is incident to a 3-cycle C_3 or a 5-cycle C_5 by a 4-vertex v^*. Clearly, v^* must be equal to u or some vertex x_k with $k \in \{2, \cdots, 10\}$. However, $d_{G_1}(u) = d_{B_{10}}(u) + d_{G \setminus B_1}(u) \geq 2 + 3 = 5$ or $d_{G_1}(x_k) = d_{B_{k-1}}(x_k) + d_{B_k}(x_k) \geq 3 + 2 = 5$ for all $k \in \{2, \cdots, 10\}$. We always get a contradiction to $d_{G_1}(v^*) = 4$.

Now, if G_1 is 2-connected, then we well done. Otherwise, we may repeat the process described above and finally obtain a desired G^*.

Thus, we complete the proof of Theorem 1. □

References

[1] O. V. Borodin. Structural properties of plane graphs without adjacent triangles and an application to 3-colorings. *J. Graph Theory*, 21:183–186, 1996.

[2] O. V. Borodin, A. N. Glebov, A. Raspaud, and M. R. Salavatipour. Planar graphs without cycles of length from 4 to 7 are 3-colorable. *J. Comb. Theory, Ser. B*, 93:303–311, 2005.

[3] M. Chen, H. Lu, and Y. Wang. A note on 3-choosability of planar graphs. *Inform. Process. Lett.*, 105(5):206–211, 2008.

[4] M. Chen, L. Shen, and Y. Wang. Planar graphs without cycles of length 4, 7, 8 or 9 are 3-choosable. accepted.

[5] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs *Congr. Numer.*, 26:125–157, 1979.

[6] T. R. Jensen and B. Toft. Graph coloring problems. Wiley-Interscience, New York, 1995.

[7] L. Shen and Y. Wang. A sufficient condition for a planar graph to be 3-choosable. *Inform. Process. Lett.*, 104:146–151, 2007.

[8] C. Thomassen. Every planar graph is 5-choosable *J. Combin. Theory Ser. B*, 62:180–181, 1994.

[9] M. Voigt. List colourings of planar graphs *Discrete Math.*, 120:215–219, 1993.

[10] M. Voigt. A non-3-choosable planar graph without cycles of length 4 and 5. *Discrete Math.*, 307(7-8):1013–1015, 2007.

[11] Y. Wang *et al.*. Planar graphs without cycles of length 4, 5, 8 or 9 are 3-choosable. Submitted, 2007.

[12] L. Zhang and B. Wu. Three-coloring planar graphs without certain small cycles. *Graph Theory Notes of New York*, 46:27–30, 2004.

[13] L. Zhang and B. Wu. A note on 3-choosability of planar graphs without certain cycles. *Discrete Math.*, 297:206–209, 2005.