# Some structural properties of planar graphs and their applications to 3-choosability

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# Some structural properties of planar graphs and their applications to 3-choosability

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**Abstract**

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. We prove that a planar graph $G$ is 3-choosable if it is satisfied one of the following conditions:

(1) each vertex $x$ is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle.

(2) each vertex $x$ is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$.

This work implies five results already published [13, 3, 7, 12, 4].

## 1 Introduction

Only simple graphs are considered in this paper unless otherwise stated. A *plane graph* is a particular drawing of a planar graph in the euclidean plane. For a plane graph $G$, we denote its vertex set, edge set, face set and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. A *proper vertex coloring* of $G$ is an assignment $c$ of integers (or labels) to the vertices of $G$ such that $c(u) \neq c(v)$ if the vertices $u$ and $v$ are adjacent in $G$. A graph $G$ is $L$*-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring $c$ of the vertices such that $\forall v \in V(G), c(v) \in L(v)$. If $G$ is $L$-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said to be $k$*-choosable*.

Thomassen [8] proved that every planar graph is 5-choosable, whereas Voigt [9] proved that there exist planar graphs which are not 4-choosable. On the other hand, in 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [6]). This conjecture remains widely open. In 1990, Erdős suggested the following relaxation of Steinberg's conjecture: What is the smallest integer $i$ such that every graph without $j$-cycles for $4 \leq j \leq i$ is 3-colorable. The best known upper bound is $i \leq 7$ [2]. It is natural to ask the same question for choosability:

**Problem 1** *What is the smallest integer $i$ such that every graph without $j$-cycles for $4 \leq j \leq i$ is 3-choosable?*

Voigt [10] proved that it is not possible to extend Steinberg's conjecture to list coloring: she gave a planar graph without 4-cycles and 5-cycles which is not 3-choosable ; hence $i \geq 6$. The best known upper bound is $i \leq 9$: this bound is obtained by using a structural lemma of Borodin [1].

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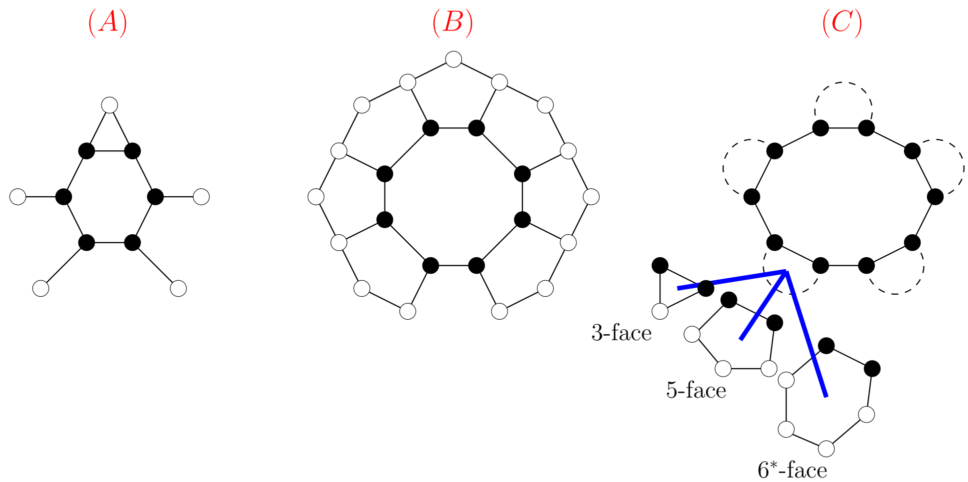


Figure 1: (A) Orchid, (B) sunflower, and (C) lotus.

**Lemma 1** *[1] Let $G$ be a planar graph with minimum degree at least 3. If $G$ does not contain cycles of lengths 4 to 9, then $G$ contains a 10-face incident to ten 3-vertices and adjacent to five 3-faces.*

It follows by Erdős, Rubin and Taylor [5] that every planar graph without cycles of lengths 4 to 9 is 3-choosable. Zhang and Wu [13] improved Borodin's result by proving that:

**Lemma 2** *[13] Let $G$ be a planar graph with minimum degree at least 3. If $G$ does not contain cycles of lengths 4, 5, 6, and 9, then $G$ contains a 10-face incident to ten 3-vertices and adjacent to five 3-faces.*

It implies that every planar graph without cycles of lengths 4, 5, 6, 9 is 3-choosable. Chen, Lu, and Wang [3] proved that every planar graph without cycles of lengths 4, 6, 7, 9 is 3-choosable. Their result is based on the following lemma:

**Lemma 3** *[3] Let $G$ be a planar graph with minimum degree at least 3. If $G$ contains neither cycles of lengths 4, 7, 9, nor 6-cycle with a chord, then $G$ contains a 10-face incident to ten 3-vertices or an 8-face incident to eight 3-vertices.*

Shen and Wang [7] proved that every planar graph without cycles of lengths 4, 6, 8, 9 is 3-choosable by showing that:

**Lemma 4** *[7] Let $G$ be a planar graph with minimum degree at least 3. If $G$ does not contain cycles of lengths 4, 6, 8, and 9, then $G$ contains a 10-face incident to ten 3-vertices.*

Moreover every planar graph without cycles of lengths 4, 5, 7, 9 (resp. 4, 5, 8, 9, and 4, 7, 8, 9) is 3-choosable [12] (resp. [11], [4]).

In this article, we consider planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. Let us introduce some notations which will allow to present our main result.

**Some notation:** The degree of a face is the length of its boundary walk. We will write $d(x)$ for $d_G(x)$ the degree of the vertex $x$ in $G$ when no confusion can arise. A $k$*-vertex*, $k^+$*-vertex*, or $k^-$*-vertex* is a vertex of degree $k$, at least $k$, or at most $k$. Similarly, we can define $k$-face, $k^+$-face, $k^-$-face, etc. We say that two cycles (or faces) are *adjacent* if they share at least one common edge. Suppose that $f$ and $f'$ are two adjacent faces by sharing a common edge $e$. We say that $f$ and $f'$ are *normally adjacent* if $|V(f) \cap V(f')| = 2$. A *triangle* is synonymous with a 3-face. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of $f$ and write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are the vertices of $b(f)$ appearing in a boundary walk of $f$.

A cycle $C$ or a face $f$ is called *nontriangular* if it is not adjacent to any 3-cycles. We say an $i$-face $f$ is an $i^*$*-face* if $f$ is adjacent to exactly one 3-face normally. Moreover, we call such $i^*$-face is *heavy*. Similarly, we say an $i$-cycle $C$ is an $i^*$*-cycle* if $C$ is adjacent to exactly one 3-cycle normally. For simpleness, we call such $i^*$-cycle is *heavy*. Two $i^*$-cycles (or $i^*$-faces) are *normally adjacent* if these two $i$-cycles (or $i$-faces) are normally adjacent.

An *orchid* is a $6^*$-face incident to six 3-vertices and adjacent to a 3-face. A *sunflower* is an 8-face incident to eight 3-vertices and adjacent to at least seven 5-faces. A *lotus* is a 10-face incident to ten 3-vertices and adjacent to five clusters, where a cluster is either a 3-face, or a 5-face, or a $6^*$-face (see Figure 1).

The following theorem is our main result which implies Lemmas 1-4.

**Theorem 1** *Let $G$ be a planar graph with minimum degree at least 3 and $G$ does not contain 4-cycles and 9-cycles. If $G$ further satisfies the following structural properties:*

*(C1) a 5-cycle or 6-cycle is adjacent to at most one 3-cycle;*

*(C2) a $5^*$-cycle is neither adjacent to a $5^*$-cycle normally, nor adjacent to an $i$-cycle with $i \in \{7, 8\}$;*

*(C3) a $6^*$-cycle is neither adjacent to a 6-cycle, nor incident to an $i$-cycle $C$ with $i \in \{3, 5\}$, where $C$ is opposite to such $6^*$-cycle by a 4-vertex;*

*(C4) a nontriangular 7-cycle is not adjacent to two 5-cycles which are normally adjacent;*

*(C5) a $7^*$-cycle is neither adjacent to a 5-cycle nor a $6^*$-cycle.*
*Then $G$ contains an orchid or a sunflower or a lotus.*

We obtain the following Corollary 1 and Corollary 2 by Theorem 1.

**Corollary 1** *Let $G$ be a planar graph in which each vertex $x$ is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle. Then $G$ is 3-choosable.*

**Corollary 2** *Let $G$ be a planar graph in which every vertex $x$ is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$. Then $G$ is 3-choosable.*

Assuming Theorem 1, we can easily prove Corollary 1 and Corollary 2.

**Proofs of Corollary 1 and Corollary 2:** Suppose that $G_1, G_2$ is a counterexample to Corollary 1, Corollary 2 with the smallest number of vertices respectively. Thus, $G_i$ is connected ($i = 1, 2$). Obviously, for each $i \in \{1, 2\}$, we observe that $\delta(G_i) \geq 3$. Otherwise, let $u_i$ be a vertex of minimum degree in $G_i$. By the minimality of $G_i$, $G_i - u_i$ is 3-choosable. Obviously, we can extend any $L$-coloring such that $\forall x \in V(G) : |L(x)| \geq 3$ of $G_i - u_i$ to $G_i$ and ensure $G_i$ is 3-choosable. Next, in each case, we will show that each $G_i$ contains either an orchid, or a sunflower, or a lotus. Denote $N_A$, $N_B$, $N_C$ be the set of black vertices of (A), (B) and (C) in Figure 1, respectively. For each $j \in \{1, 2, 3\}$, one can easily observe that we can extend any $L$-coloring such that for all $x \in V(G) : |L(x)| \geq 3$ of $G_i - N_j$ to $N_j$ and make sure $G_i$ is 3-choosable. Thus, $G_1$ and $G_2$ are both 3-choosable. A contradiction.

Since $G_i$ does not contain 4-cycles and 9-cycles, we only need to verify if $G_i$ satisfies all the structural properties (C1) to (C5), where $i \in \{1, 2\}$.

(1) For $G_1$, since each vertex $x$ is not incident to 6-cycles adjacent to a 3-cycle, each 5-cycle or 6-cycle only can be nontriangular cycles. This implies that there is neither $5^*$-face nor $6^*$-face in $G_1$. Thus, (C1), (C2) and (C3) are satisfied. Then we only need to consider (C4) and (C5). If (C4) is not satisfied, then there appears a vertex $x$ which is incident to an $i_x$-cycle with $i_x \in \{5, 7, 8\}$, which contradicts the assumption of $G_1$. If (C5) is not satisfied, then a vertex $y$ is appeared such that $y$ is incident to an $i_y$-cycle with $i_y \in \{5, 7, 8\}$. A contradiction.

(2) For $G_2$, because it does not contain 7-cycles, we confirm that there is no $6^*$-cycle and $7^*$-cycle in $G_2$. Thus, we only need to check properties (C1) and (C2). It is easy to establish a 7-cycle or a 4-cycle if a 5-cycle or 6-cycle is adjacent to at least two 3-cycles. Thus, (C1) is satisfied. Let us check (C2). If there exist two $5^*$-cycles adjacent normally, then a 7-cycle or a 9-cycle is produced, contradicting the absence of 7-cycles and 9-cycles. If a $5^*$-cycle is adjacent to an $8$-cycle, then there is a vertex incident to a 5-cycle, a 6-cycle and an 8-cycle, which is impossible. Therefore, (C2) is satisfied.

This completes the proofs of Corollary 1 and Corollary 2. □

By Corollary 1, it is easy to deduce Corollary 3:

**Corollary 3** *Every planar graph $G$ in which every vertex $v$ is not incident to cycles of lengths $4, 6, 9, i_x$ with $i_x \in \{5, 7, 8\}$ is 3-choosable.*

Thus, by Corollary 2 and Corollary 3, we deduce Corollary 4 which covers five results mentioned before [13, 3, 7, 12, 4].

**Corollary 4** *Every planar graph $G$ without $\{4, i, j, 9\}$-cycles with $5 \leq i < j \leq 8$ and $(i, j) \neq (5, 8)$ is 3-choosable.*

Section 2 is dedicated to the proof of Theorem 1.

## 2 Proof of Theorem 1

Let $G$ be a counterexample to Theorem 1, i.e., an embedded plane graph $G$ with $\delta(G) \geq 3$, no cycles of lengths 4 and 9, satisfying the structural properties (C1) to (C5), and containing no orchid, no sunflower, and no lotus (i.e., none of the configurations depicted by Figure 1).

First, we suppose $G$ is 2-connected. Thus, the boundary of each face $f$ of $G$ forms a cycle. Besides, the following assertions (O1) to (O7) hold naturally by the assumption of $G$.

(O1) A 5-face or a 6-face is adjacent to at most one 3-face;

(O2) A $5^*$-face is neither adjacent to a $5^*$-face normally, nor adjacent to an $i$-face with $i \in \{7, 8\}$;

(O3) A $6^*$-face is neither adjacent to a 6-face, nor incident to an $i$-face $f$ with $i \in \{3, 5\}$, where $f$ is opposite to such $6^*$-face by a 4-vertex;

(O4) A nontriangular 7-face is not adjacent to two 5-faces which are normally adjacent (there is no 3-vertex incident to a nontriangular 7-face and to two 5-faces);

(O5) A $7^*$-face is neither adjacent to a 5-face nor a $6^*$-face;

(O6) $G$ does not contain 4-faces and 9-faces;

(O7) Each vertex $v$ is incident to at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-faces.

Moreover, the following additional properties hold:

**Claim 1** *For some fixed $i \in \{5, 6, 7, 8\}$, if an $i$-face is adjacent to a 3-face, then they are normally adjacent.*

**Proof.** Suppose the claim is false. Let $f_i = [v_1 v_2 \cdots v_i]$ be an $i$-face and $f_2 = [v_1 v_2 u]$ be a 3-face such that $f_1$ is adjacent to $f_2$ and $|V(f_1) \cap V(f_2)| \geq 3$. It means that $u$ is equal to some $v_j$ with $j \in \{3, 4, \cdots, i\}$. According to the value of $i$, one can easily observe that if $u$ is a vertex $v_j$ with $3 \leq j \leq i$, the $G$ contains either a 2-vertex or a 4-cycle, a contradiction. This completes the proof of Claim 1. □

Since $G$ does not contain 9-cycles, we obtain the following Claims 2 and 3 easily by Claim 1:

**Claim 2** *Each 7-face is adjacent to at most one 3-face.*

**Claim 3** *No 8-face is adjacent to a 3-face.*

**Claim 4** *If two 5-faces are adjacent to each other, then they can only be normally adjacent.*

**Proof.** Suppose that there are two adjacent 5-faces $f_1 = [v_1 v_2 \cdots v_5]$ and $f_2 = [v_1 v_2 u v w]$ with $v_1 v_2$ as a common edge. If $|V(f_1) \cap V(f_2)| = 2$, then Claim 4 follows. Otherwise, by symmetry, we only need to consider the following cases. If $w = v_5$, then $d(v_1) = 2$ which is impossible. If $w = v_4$, then $G$ contains a 4-cycle $v_1 v_2 v_3 v_4 v_1$, a contradiction. This implies $u \notin b(f_1)$ and $w \notin b(f_1)$. If $v = v_5$ or $v = v_4$, then a 4-cycle $u v_2 v_1 v_5 u$ or $w v_1 v_5 v_4 w$ can be easily established, a contradiction, that completes the proof of Claim 4. □

**Claim 5** *A nontriangular 5-face can not be adjacent to a $5^*$-face in $G$.*

**Proof.** Suppose on the contrary that a nontriangular 5-face $f_1 = [v_1 v_2 \cdots v_5]$ is adjacent to a $5^*$-face $f_2 = [v_1 v_2 u_3 u_4 u_5]$ by a common edge $v_1 v_2$. By definition, $f_1$ is not adjacent to any 3-face. By Claim 4, each $u_i$ can not be equal to some $v_j$ with $i, j \in \{3, 4, 5\}$. By symmetry, we have to handle the following two properties:

• Assume that $v_1 u_5 u$ is a 3-face. By Claim 1, $u \neq v_2, u_3, u_4$. Moreover, $u \neq v_5$ by choice of $f_1$. If $u = v_4$ or $u = v_3$, then $G$ contains a 4-cycle, which is impossible. Thus, $u$ does not belong to $b(f_1) \cup b(f_2)$ and $G$ contains a 9-cycle $u v_1 v_5 v_4 v_3 v_2 u_3 u_4 u_5 u$, a contradiction.

• Assume that $u_5 u_4 u$ is a 3-face. Notice that $u \neq v_1, v_2, u_3$ by Claim 1. If $u = v_3$ or $v_4$ or $v_5$, then $G$ contains a 4-cycle which is impossible. Thus, $u$ does not belong to $b(f_1) \cup b(f_2)$ and $G$ contains a 9-cycle $u u_5 v_1 v_5 v_4 v_3 v_2 u_3 u_4 u$, a contradiction, that completes the proof of Claim 5. □

By Claim 4 and assertion (O2), we have:

**Claim 6** *There is no adjacent two $5^*$-faces in $G$.*

**Claim 7** *No 3-vertex is incident to three 5-faces.*

**Proof.** Assume to the contrary that $G$ contains a 3-vertex $u$ adjacent to three vertices $v_1, v_2, v_3$ and incident to three 5-faces $f_1 = [u v_1 x_1 x_2 v_2]$, $f_2 = [u v_2 y_1 y_2 v_3]$, and $f_3 = [u v_3 z_1 z_2 v_1]$. By Claim 4, $f_i$ and $f_j$ are normally adjacent for each pair $\{i, j\} \subset \{1, 2, 3\}$. It implies that all vertices in $(V(f_1) \cup V(f_2) \cup V(f_3)) \setminus \{u\}$ are mutually distinct. However, a 9-cycle $v_1 x_1 x_2 v_2 y_1 y_2 v_3 z_1 z_2 v_1$ is established, contradicting the assumption on $G$. Thus, we complete the proof of Claim 7. □

**Claim 8** *Under isomorphism, a 6-face can be adjacent to a 5-face in an unique way as depicted by Figure 2.*

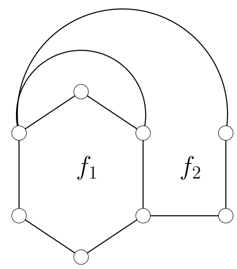


Figure 2: A 6-face $f_1$ is adjacent to a 5-face $f_2$.

**Proof.** Assuming that a 6-face $f_1 = [v_1v_2 \cdots v_6]$ is adjacent to a 5-face $f_2 = [v_1v_2uvw]$ with $v_1v_2$ as a common edge. If $\{u, v, w\} \cap \{v_3, v_4, v_5, v_6\} = \emptyset$, then a 9-cycle $v_2v_3v_4v_5v_6v_1wvuv_2$ is formed, which is a contradiction. Thus, we confirm that $|V(f_1) \cap V(f_2)| \geq 3$. By symmetry of $f_1$, it suffices to consider the following cases.

• If $w = v_5$ and $u = v_4$, then a 4-cycle $v_1v_5v_4v_2v_1$ is established, which is impossible.

• Assuming that $w \neq v_5$ and $u \neq v_4$. Notice that $w \neq v_6$. Or else, a 2-vertex $v_1$ is produced. If $w = v_4$, then a 4-cycle $v_1v_6v_5v_4v_1$ is made. If $w = v_3$, then a 4-cycle $v_2uvv_3$ is constructed. A contradiction is always produced. By symmetry, we see that $w$ and $u$ do not belong to $b(f_1)$. Moreover, $v \neq v_3$ and $v \neq v_4$. Otherwise, a 4-cycle $wv_1v_2v_3w$ or $v_4v_3v_2uv_4$ is established, contradicting the assumption on $G$. Thus, by symmetry, $v \notin V(f_1)$, which means that $|V(f_1) \cap V(f_2)| = 2$, which is impossible.

• Without loss of generality, we may suppose that $w = v_5$ and $u \neq v_4$. Clearly, $u \neq v_3$ since $d(v_2) \geq 3$. So $u \notin V(f_1)$. One can easily observe that $v \neq v_3$ and $v \neq v_4$ by the absence of 4-cycles. Thus, we ensure that $v \notin b(f_1)$. It implies that $f_1$ is adjacent to $f_2$ in an unique way as Figure 2 shown. This completes the proof of Claim 8. $\square$

**Claim 9** *No 3-vertex is incident to two 5-faces and one 6-face.*

**Proof.** Suppose the claim is not true. We assume that there exists a 3-vertex $u$ adjacent to three vertices $v_1, v_2, v_3$ and incident to two 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and one 6-face $f_3 = [uv_3z_1z_2z_3v_1]$.

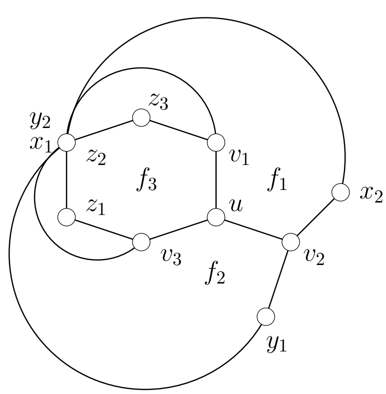


Figure 3: A 3-vertex $u$ incident to two 5-faces $f_1$ and $f_2$ and to one 6-face $f_3$.

By Claim 8, $z_2 = y_2 = x_1$. Hence a 4-cycle $z_2v_1uv_3z_2$ exists which is a contradiction. Thus, we complete the proof of Claim 9. $\square$

**Claim 10** *No 3-vertex is incident to one 5-face and two 6-faces.*

**Proof.** Suppose on the contrary that there exists a 3-vertex $u$ adjacent to three vertices $v_1, v_2, v_3$ and incident to two 6-faces $f_1 = [uv_3y_1y_2y_3v_1]$, $f_2 = [uv_2z_1z_2z_3v_3]$, and one 5-face

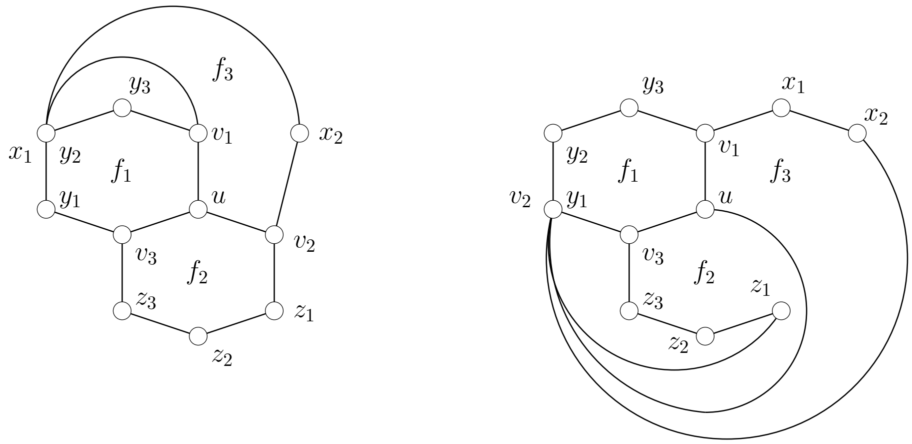


Figure 4: A 3-vertex $u$ incident to one 5-face $f_3$ and two 6-faces $f_1$ and $f_2$.

$f_3 = [uv_1x_1x_2v_2]$. By Claim 8, we see that $f_1$ and $f_3$ can only be adjacent to each other in an unique way as depicted by Figure 2. One can easily observe that $x_1 = y_2$ or $v_2 = y_1$. Next, we will make use of contradictions to show that $f_2$ can not exist in $G$. We have to deal with the following two cases.

**Case 1** $x_1 = y_2$.

For simpleness, denote $x^* = x_1 = y_2$. By Claim 8, we see that $x_2 = z_2$. It is easy to see that a 5-face $x^*v_1uv_2x_2x^*$ adjacent to two 3-cycles $x^*y_3v_1x^*$ and $v_2z_1x_2v_2$ is produced. This contradicts (C1).

**Case 2** $v_2 = y_1$.

Clearly, $uv_3y_1$ is a 3-cycle which is not a 3-face. For simpleness, let $y^* = v_2 = y_1$. Obviously, $\{z_1, z_2, z_3\} \cap \{y_2, y_3, x_1, x_2\} = \emptyset$ since $G$ is a plane graph. However, a 9-cycle $y^*z_1z_2z_3v_3uv_1x_1x_2y^*$ is easily established, which is impossible. This completes the proof of Claim 10. □

**Claim 11** *No $6^*$-face is adjacent to a 5-face in $G$.*

**Proof.** Suppose on the contrary that there exists a 6-face $f_1 = [v_1v_2 \cdots v_6]$ adjacent to a 5-face $f_2 = [v_1v_2uvw]$ by a common edge $v_1v_2$. By Claim 8, $f_1$ and $f_2$ can only be adjacent in an unique way depicted by Figure 2, which means that $w = v_5$. Note that $f_1$ is adjacent to a 3-cycle $v_1v_5v_6v_1$ which is not a 3-face. Thus, $f_1$ can not be adjacent to any other 3-face by (C1), which means that $f_1$ can not be a $6^*$-face. This completes the proof of Claim 11. □

It is easy to derive Claim 12 by Claim 11.

**Claim 12** *No $6^*$-face is adjacent to a $5^*$-face in $G$.*

By (C1), similarly as the proof of Claim 11 we have:

**Claim 13** *No $5^*$-face is adjacent to a 6-face in $G$.*

Furthermore, assertion (O3) implies the following claim:

**Claim 14** *There is no adjacent $6^*$-faces in $G$.*

**Claim 15** *Let $G$ be a connected plane graph with $n$ vertices, $m$ edges and $r$ faces. Then using Euler's formula we have:*

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12 \quad (1)$$

**Proof.** Euler's formula $n - m + r = 2$ yields $(4m - 6n) + (2m - 6r) = -12$. This identity and the relation $\sum_{v \in V} d(v) = \sum_{f \in F} d(f) = 2m$ imply (1). $\square$

We now use a discharging procedure. We first assign to each vertex $v$ an initial charge $\omega(v)$ such that for all $v \in V(G), \omega(v) = 2d(v) - 6$ and to each face $f$ an initial charge such that for all $f \in F(G), \omega(f) = d(f) - 6$. In the following, we define discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function $\omega^*$ is produced. However, the total sum of charges is kept fixed when the discharging is in process. Nevertheless, we can show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Using (1), this leads to the following obvious contradiction:

$$-12 = \sum_{v \in V(G) \cup F(G)} \omega(v) = \sum_{v \in V(G) \cup F(G)} \omega^*(v) \geq 0$$

and hence demonstrates that no such counterexample can exist.

Before stating the discharging rules, we first give some notations which will be used frequently in the following argument. Let $x, y \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the charge transferred from $x$ to $y$. For a vertex $v \in V(G)$ and for an integer $i \geq 5$, let $m_3(v)$, $m_i(v)$, and $m_{i^*}(v)$ denote the number of 3-faces, nontriangular $i$-faces, and heavy $i$-faces incident to $v$, respectively. Furthermore, we denote $M_i(v) = m_i(v) + m_{i^*}(v)$ and call a face $f$ a *non-3-face* if $d(f) \neq 3$.

For simpleness, we write an edge $uv$ is a $(b_1, b_2)$*-edge* if $d(u) = b_1$ and $d(v) = b_2$. Let $f_1$ and $f_2$ be two faces adjacent to each other by a common edge $uv$. If $u$ and $v$ are both not incident to any 3-face, then we call $uv$ a *good common edge*. We further say such $uv$ is a *good common* $(b_1, b_2)$*-edge* if $uv$ is a $(b_1, b_2)$-edge.

The discharging rules are defined as follows:

(R1) Each $5^+$-face sends 1 to its adjacent 3-face.

(R2) Let $v$ be a 4-vertex.

- (R2a) If $m_3(v) = 2$, then for each non-3-face $f$, $\tau(v \to f) = 1$.
- (R2b) If $m_3(v) = 1$, then let $f_1$ denote the incident 3-face and $f'$ be the opposite face of $f_1$.
  - (R2b1) If $f'$ is a nontriangular 5-face, then $v$ sends $\frac{2}{3}$ to each incident face different of $f_1$.
  - (R2b2) Otherwise, $v$ sends 1 to each incident face which is adjacent to $f_1$.
- (R2c) If $m_3(v) = 0$, let $f_1, f_2, f_3$, and $f_4$ denote the faces of $G$ incident to $v$ in a cyclic order such that the degree of $f_1$ is the smallest one among all faces incident to $v$, then we do like this:
  - (R2c1) if $M_5(v) = 0$, then $v$ sends $\frac{1}{2}$ to each incident face.
  - (R2c2) if $M_5(v) = 1$, then $v$ sends $\frac{2}{3}$ to each of $f_1, f_2$, and $f_4$ when $f_1$ is a nontriangular 5-face; or $v$ sends 1 to each of $f_2$ and $f_4$ when $f_1$ is a $5^*$-face.
  - (R2c3) if $M_5(v) = 2$, then
    - (R2c3.1) $v$ sends $\frac{2}{3}$ to each nontriangular 5-face and $\frac{1}{3}$ to each other incident face when $m_5(v) = 2$.
    - (R2c3.2) $v$ sends $\frac{2}{3}$ to each incident face of $v$ except the unique $5^*$-face when $m_5(v) = 1$ and $m_{5^*}(v) = 1$.
    - (R2c3.3) $v$ sends 1 to each incident face which is not a $5^*$-face when $m_{5^*}(v) = 2$.
  - (R2c4) if $M_5(v) = 3$, then $v$ gives $\frac{2}{3}$ to each incident nontriangular 5-face.
  - (R2c5) if $M_5(v) = 4$, then $v$ gives $\frac{1}{2}$ to each incident nontriangular 5-face.

(R3) Let $v$ be a 5-vertex and $f$ be a non-3-face incident to $v$. Then

(R3a) $\tau(v \to f) = \frac{4}{3}$ if $m_3(v) = 2$.

(R3b) $\tau(v \to f) = 1$ if $m_3(v) = 1$.

(R3c) if $m_3(v) = 0$, $v$ sends 1 to each incident face different from $5^*$-faces when $m_{5^*}(v) \geq 1$; or sends $\frac{5}{6}$ to each incident $6^*$-face and sends $\frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}$ to each other incident face when $m_{5^*}(v) = 0$.

(R4) Let $f$ be a $7^+$-face. If $f'$ is adjacent to $f$ by a good common edge $e$, then

(R4a) $\tau(f \to f') = \frac{1}{3}$ if $f'$ is a nontriangular 5-face and $e$ is a $(3,3)$-edge.

(R4b) $\tau(f \to f') = \frac{1}{6}$ if $f'$ is a $6^*$-face and $e$ is a $(3,3)$-edge or a $(3,4)$-edge.

(R5) Each $10^+$-face sends 1 to each adjacent $5^*$-face by a good common $(3^+, 3^+)$-edge.

(R6) Each $6^+$-vertex sends 1 to each incident face.

Let us check that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Since $\delta(G) \geq 3$, $d(v) \geq 3$ for each vertex $v \in V(G)$. We have to handle the following cases, depending on the size of $d(v)$.

**Case 1** $d(v) = 3$.

It is easy to see that $\omega^*(v) = \omega(v) = 2 \times 3 - 6 = 0$ by (R1) to (R6).

**Case 2** $d(v) = 4$.

Clearly, $\omega(v) = 2$ and $v$ is incident to at most two 3-faces by (O7). If $m_3(v) = 2$, then we deduce that $\omega^*(v) = 2 - 2 \times 1 = 0$ by (R2a). If $m_3(v) = 1$ ($v$ is incident to exactly one 3-face), then depending on the opposite face of such 3-face, $v$ gives either $\frac{2}{3} \times 3 = 2$, or $1 \times 2 = 2$ by (R2b1) or (R2b2). Hence, $\omega^*(v) = 0$. Finally, we only need to consider the case of $m_3(v) = 0$. We divide the discussion into five subcases in the light of the value of $M_5(v)$.

**Subcase 2.1** $M_5(v) = 0$.

This implies that the degree of each face incident to $v$ is at least 6 by the absence of 4-faces. According to (R2c1), $\omega^*(v) \geq 2 - \frac{1}{2} \times 4 = 0$.

**Subcase 2.2** $M_5(v) = 1$.

It is easy to observe that $v$ sends either $\frac{2}{3} \times 3 = 2$ if $m_5(v) = 1$, or $1 \times 2 = 2$ if $m_{5^*}(v) = 1$ by (R2c2). Thus, $v$ gives totally at most 2 to incident faces. Hence, $\omega^*(v) \geq 2 - 2 = 0$.

**Subcase 2.3** $M_5(v) = 2$.

If $m_5(v) = 2$, then $\omega^*(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R2c3.1). If $m_5(v) = m_{5^*}(v) = 1$, then such nontriangular 5-face and $5^*$-face can not be adjacent to each other by Claim 5. Thus, applying (R2c3.2), $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$. Otherwise, suppose $m_{5^*}(v) = 2$. Notice that $v$ is incident to two $5^*$-faces which are opposite to each other by Claim 6. Thus, $\omega^*(v) \geq 2 - 1 \times 2 = 0$ by (R2c3.3).

**Subcase 2.4** $M_5(v) = 3$.

We first notice that $m_{5^*}(v) \neq 3$ since there are no adjacent $5^*$-faces in $G$ by Claim 6. If $1 \leq m_{5^*}(v) \leq 2$, then there exists at least one nontriangular 5-face adjacent to one $5^*$-face, contradicting the Claim 5. Thus, $m_{5^*}(v) = 0$, and so $m_5(v) = 3$. According to (R2c4), we have that $\omega^*(v) \geq 2 - \frac{2}{3} \times 3 = 0$.

**Subcase 2.5** $M_5(v) = 4$.

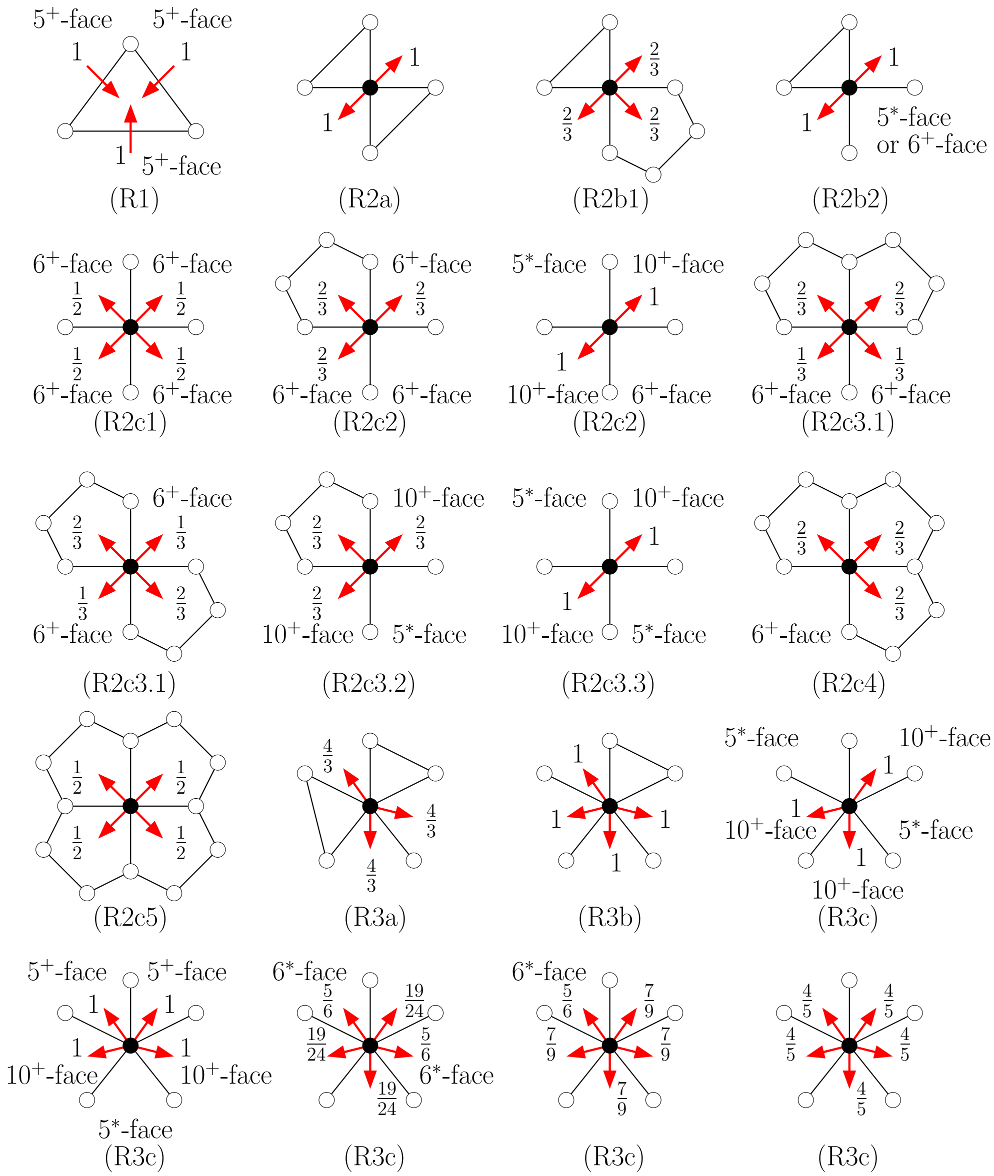


Figure 5: Some of discharging rules (R1) to (R3).

One can observe that $m_{5^*}(v)=0$ by Claim 5 and Claim 6. It implies that $v$ is incident to exactly four nontriangular 5-faces. Consequently, we have that $\omega^*(v)\geq 2-\frac{1}{2}\times 4=0$ by (R2c5).

**Case 3** $d(v)=5$.

Obviously, $\omega(v)=4$ and $m_3(v)\leq 2$ by (O7). It is easy to observe that $v$ sends either $\frac{4}{3}\times 3=4$ by (R3a) if $m_3(v)=2$, or $1\times 4=4$ by (R3b) if $m_3(v)=1$. Therefore, $\omega^*(v)\geq 4-4=0$ if $m_3(v)>0$. Now we may assume that $m_3(v)=0$. This implies that each face incident to $v$ is a $5^+$-face combining the fact that $G$ does not contain any 4-faces. By Claim 6, we have that $m_{5^*}(v)\leq 2$. Moreover, each face adjacent to a $5^*$-face must be a $10^+$-face by Claim 5, Claim 6, Claim 12, Claim 13, (O2) and the absence of 9-faces. So by (R3c), $\omega^*(v)\geq 4-1\times 4=0$ if $m_{5^*}(v)\geq 1$; or $\omega^*(v)\geq 4-\frac{5}{6}m_{6^*}(v)-\frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}(5-m_{6^*}(v))=0$ if $m_{5^*}(v)=0$.

**Case 4** $d(v)\geq 6$.

According to (R6), we have that $\omega^*(v)\geq (2d(v)-6)-1\times d(v)=d(v)-6\geq 0$.

Let $f\in F(G)$. Then $b(f)$ is a cycle since $G$ is 2-connected. We write $f=[v_1v_2\cdots v_{d(f)}]$ and suppose that $f_i$ is the face of $G$ adjacent to $f$ with $v_iv_{i+1}\in b(f)\cap b(f_i)$ for $i=1,2,\cdots,d(f)$, where (and in the following discussion) all indices are taken modulo $d(f)$ plus 1. We observe that $d(f)\neq 4$ and $d(f)\neq 9$ by (O6). For $i\geq 3$, let $n_i(f)$ denote the number of $i$-vertices incident to $f$. Let $m_5(f)$, $m_{5^*}(f)$, and $m_{6^*}(f)$ denote the number of nontriangular 5-faces, heavy 5-faces, and heavy $6^*$-faces adjacent to $f$.

**Case 5** $d(f)=3$.

Let $f$ be a 3-face and then $\omega(f)=-3$. Since $\delta(G)\geq 3$, $f$ is adjacent to three faces and each adjacent face is neither a 3-face nor a 4-face by the absence of 4-cycles in $G$. It implies that $f$ gets $3\times 1$ from its adjacent faces by (R1). Thus, $\omega^*(f)\geq -3+1\times 3=0$.

**Case 6** $d(f)=5$.

Let $f=[v_1\cdots v_5]$ and then $\omega(f)=-1$. Clearly, $f$ is adjacent to at most one 3-face by (O1).

• We first assume that $f$ is a nontriangular 5-face, which means that there is no 3-face adjacent to $f$. Thus, $f$ sends nothing to all its adjacent faces. Moreover, each $f_i$ can not be a $5^*$-face by Claim 5. We only have to deal with the the following three possibilities, depending on the value of $n_3(f)$.

**Subcase 6.1** $n_3(f)=5$.

It means that $v_i$ is a 3-vertex for all $i=1,\ldots,5$. If there exists a 6-face adjacent to $f$, then by Claim 8 we see that they must be adjacent to each other in an unique way as depicted by Figure 2. It is easy to see that there is one $4^+$-vertex appeared on $b(f)$, which contradicts $n_3(f)=5$. Thus, each face adjacent to $f$ is either a nontriangular 5-face or a $7^+$-face by the absence of 4-faces. Furthermore, we notice that $f$ is adjacent to at most two nontriangular 5-faces which are not adjacent by Claim 7. So $f$ is adjacent to at least three $7^+$-faces such that each $7^+$-face is adjacent to $f$ by a good common $(3,3)$-edge. Therefore, applying (R4a), we obtain that $\omega^*(f)\geq -1+3\times\frac{1}{3}=0$.

**Subcase 6.2** $n_3(f)=4$.

Let $v_1$ be such a $4^+$-vertex and $v_j$ be a 3-vertex for all $j=2,3,4,5$. Clearly, $v_1$ gives at least $\frac{1}{2}$ to $f$ by (R2) and (R3). Moreover, $f_1$ and $f_5$ can not be any 6-face by Claim 8. If $d(f_1)=5$ and $d(f_5)=5$, then $d(f_j)\notin\{5,6\}$ with $j\in\{2,4\}$ according to Claim 7 and Claim 9. Thus, for $j\in\{2,4\}$, $f_j$ is a $7^+$-face by the absence of 4-faces and each $f_j$ is adjacent to $f$ by a good common $(3,3)$-edge. By (R4a), we see that $\tau(f_2\rightarrow f)=\frac{1}{3}$ and $\tau(f_4\rightarrow f)=\frac{1}{3}$. So we obtain that $\omega^*(f)\geq -1+\frac{1}{2}+\frac{1}{3}\times 2=\frac{1}{6}>0$.

Now we may suppose that there exists at least one face of $f_1$ and $f_5$ which is a $7^+$-face, i.e., $d(f_1) \geq 7$. Then by (R2) and (R3), we see that $\tau(v_1 \to f) \geq \frac{2}{3}$. Clearly, for each $i \in \{2, 3, 4\}$, $f_i$ is adjacent to $f$ by a good common $(3, 3)$-edge. According to Claim 7, Claim 9 and Claim 10, we see that there exists at least one face of $f_2, f_3, f_4$ which is a $7^+$-face. Hence, $\omega^*(f) \geq -1 + \frac{1}{3} + \frac{2}{3} = 0$ by (R4a).

**Subcase 6.3** $n_3(f) \leq 3$.

It means that there are at least two vertices whose degree are both at least 4. By (R2), we derive that $\omega^*(f) \geq -1 + \frac{1}{2} \times 2 = 0$.

• Now, we may suppose that $f$ is a $5^*$-face. It implies that $f$ is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such 3-face that it is adjacent to $f$. By Claim 1, $v \neq v_i$ for all $i = 3, 4, 5$. Since $\delta(G) \geq 3$, $d(v_i) \geq 3$ with $i \in \{1, 2, \cdots, 5\}$. By Claim 6 and (O2), for each $j \in \{2, 3, 4, 5\}$, $f_j$ is neither a $5^*$-face nor an $i$-face with $i = 7, 8$. Furthermore, we observe that $d(f_j) \neq 3$, $d(f_j) \neq 4$, $d(f_j) \neq 5$, $d(f_j) \neq 6$, and $d(f_j) \neq 9$ by (O1), (O6), Claim 5, and Claim 13, respectively. Thus, we confirm that $d(f_j) \geq 10$. Therefore, we derive that $\tau(f_3 \to f) = 1$ and $\tau(f_4 \to f) = 1$ by (R5). Hence, $\omega^*(f) \geq -1 - 1 + 1 \times 2 = 0$ by (R1).

**Case 7** $d(f) = 6$.

Let $f = [v_1 \cdots v_6]$ and then $\omega(f) = 0$. If $f$ is a nontriangular 6-face, then it is easy to deduce that $\omega^*(f) = \omega(f) = 0$ by (R1) to (R6). Now, we assume that $f$ is a $6^*$-face. Without loss of generality, assume $f_1 = [vv_1v_2]$ is a 3-face adjacent to $f$. It is obvious that $v \notin b(f)$ by Claim 1. Furthermore, $f$ is adjacent to at most one 3-face by (O1). So $f$ only need to send 1 to the unique 3-face $f_1$. Obviously, for each $j \in \{2, \cdots, 6\}$, $d(f_j) \notin \{3, 4, 5, 6\}$ by (O1), (O6), Claim 11 and (O3). Noting that $v_3v_5 \notin E(G)$ and $v_3v_6 \notin E(G)$ by (C1) and the absence of 4-cycles. This implies that each $v_i$ has at least one outgoing neighbor which is not lied on $b(f)$. Since there is no orchid in $G$, $f$ is incident to at least one $4^+$-vertex. It implies that $n_3(f) \leq 5$. Next, in each case, we will show that the total charge $f$ obtained is at least 1 and thus $\omega^*(f) \geq -1 + 1 = 0$.

**Subcase 7.1** $n_3(f) = 5$.

It means that there is exactly one $4^+$-vertex incident to $f$. If $d(v_2) \geq 4$, then $\tau(v_2 \to f) \geq 1$ by (R2a), (R2b2), (R3a), (R3b) and (R6) since $d(f_2) \neq 5$. Otherwise, by symmetry, suppose some $v_i$ is a $4^+$-vertex where $i \in \{3, 4\}$. Denote $v^*$ be such a $4^+$-vertex. First, we observe that each adjacent face different from $f_1$ is a $7^+$-face by the discussion above. If $d(v^*) \geq 5$, then $\tau(v^* \to f) \geq \frac{5}{6}$ by (R3) and (R6). Noting that there is at least one $f_j$ sends $\frac{1}{6}$ to $f$, where $j \in \{3, 4, 5\}$. Thus, $f$ gets at least $\frac{5}{6} + \frac{1}{6} = 1$ from $v^*$ and its adjacent $7^+$-faces. If $d(v^*) = 4$, then the opposite face of $f$, which is incident to $f$ by $v^*$, can not be a 3-face or a 5-face by (O3). So $v^*$ is incident to four $6^+$-faces and thus $v^*$ gives $\frac{1}{2}$ to $f$ by (R2c1). Consequently, $f$ gets at least $\frac{1}{2} + \frac{1}{6} \times 3 = 1$ by (R4b).

**Subcase 7.2** $0 \leq n_3(f) \leq 4$.

It implies that there are at least two $4^+$-vertices incident to $f$. It is easy to see that every $5^+$-vertex sends at least $\frac{5}{6}$ to $f$ by (R3) and (R6). Moreover, every 4-vertex sends at least $\frac{1}{2}$ to $f$ since it is not incident to any 3-face or 5-face. Hence, $f$ receives at least $\frac{1}{2} \times 2 = 1$ from its incident $4^+$-vertices.

In what follows, for simpleness, let $p_5(f)$, $p_{5^*}(f)$, and $p_{6^*}(f)$ denote the number of nontriangular 5-face, $5^*$-face, and $6^*$-face receiving a charge $\frac{1}{3}$, 1, $\frac{1}{5}$ from $f$, respectively. Clearly, $p_5(f) \leq m_5(f)$, $p_{5^*}(f) \leq m_{5^*}(f)$ and $p_{6^*}(f) \leq m_{6^*}(f)$.

**Case 8** $d(f) = 7$.

Then $\omega(f) = 1$. Let $m_3(f)$ be the number of 3-faces adjacent to $f$. Clearly, $m_3(f) \leq 1$ by Claim 2.

• We first assume $f$ is a nontriangular 7-face. Noting that $d(f_i) \geq 5$ since $G$ contains no 4-faces. By (O2), $m_{5^*}(f) = 0$. By (O4), $p_5(f) \leq 3$. We will divide the argument into four subcases according to the value of $p_5(f)$.

**Subcase 8.1** $p_5(f) = 3$.

Suppose $f_1, f_3, f_5$ are such three 5-faces that each of them takes a charge $\frac{1}{3}$ from $f$. By (R4a), we see that all common edges $(v_1 v_2)$, $(v_3, v_4)$ and $(v_5, v_6)$ are good $(3,3)$-edges. This implies that $d(v_i) = 3$ with $i \in \{1, \cdots, 6\}$. By Claim 11, one can easily defer that none of $f_2, f_4, f_6, f_7$ can be a 6*-face. Thus, $p_{6^*}(f) \leq m_{6^*}(f) = 0$. Consequently, we deduce that $\omega^*(f) \geq 1 - \frac{1}{3} \times 3 = 0$ by (R4a).

**Subcase 8.2** $p_5(f) = 2$.

We may suppose that $f_i$ is a 5-face which takes $\frac{1}{3}$ from $f$. It means that $d(v_i) = d(v_{i+1}) = 3$ and $(v_i, v_{i+1})$ is a good common edge. Thus, $f_{i-1}$ and $f_{i+1}$ can not be any 6*-face by Claim 11. It follows immediately that $p_{6^*}(f) \leq 7 - (2+3) = 2$ since $p_5(f) = 2$. Consequently, we have that $\omega^*(f) \geq 1 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = 0$ by (R4).

**Subcase 8.3** $p_5(f) = 1$.

Without loss of generality, let $f_1$ be such a nontriangular 5-face that $(v_1, v_2)$ be a good common $(3,3)$-edge. This implies that neither $f_2$ nor $f_7$ can be a 6*-face. Thus, $p_{6^*}(f) \leq 7 - 3 = 4$. Hence, we have $\omega^*(f) \geq 1 - \frac{1}{3} - \frac{1}{6} \times 4 = 0$ by (R4a) and (R4b).

**Subcase 8.4** $p_5(f) = 0$.

If $p_{6^*}(f) = 0$, then according to (R4), we obtain that $\omega^*(f) \geq 1 - 0 = 1$. Otherwise, we may let $f_1$ is a 6*-face, which takes a charge $\frac{1}{6}$ from $f$. It is obvious that $f_1$ must be adjacent to $f$ by a good common $(3,3)$-edge or $(3,4)$-edge, i.e., $d(v_1) = 3$ and $d(v_2) \in \{3, 4\}$. It is easy to observe that $f_7$ can not be any 6*-face because of Claim 14. Thus, $p_{6^*}(f) \leq 6$ and $\omega^*(f) \geq 1 - \frac{1}{6} \times 6 = 0$ by (R4b).

• Now we may assume $m_3(f) = 1$, which implies that $f$ is a 7*-face and it is adjacent to exactly one 3-face. Without loss of generality, let $f_1 = [vv_1v_2]$ be such a 3-face that $f$ sends 1 to $f_1$. By Claim 1, we notice that $v$ is not lied on $b(f)$. Moreover, for each $j \in \{2, \cdots, 7\}$, we deduce that $f_j$ is neither a 5-face nor a 6*-face by (O5). It implies that $f$ sends nothing to each $f_j$ with $j \in \{2, \cdots, 7\}$. Applying (R1), we deduce that $\omega^*(f) \geq 1 - 1 = 0$.

**Case 9** $d(f) = 8$.

Clearly, $\omega(f) = 2$ and $f$ can not be adjacent to any 3-face by Claim 3. So we only need to consider the size of $p_5(f)$ and $p_{6^*}(f)$ since they may take charge from $f$. It is easy to calculate that $p_5(f) \leq 6$ by the fact that there is no sunflower in $G$. We have to consider the following possibilities by the value of $p_5(f)$.

**Subcase 9.1** $p_5(f) = 6$.

It implies that there are at least seven vertices in $V(f)$ are 3-vertices. Thus, the remaining two faces adjacent to $f$, which are not nontriangular 5-faces, can not be any 6*-faces by Claim 11. So $\omega^*(f) \geq 2 - 6 \times \frac{1}{3} = 0$ by (R4).

**Subcase 9.2** $p_5(f) = 5$.

One can easily notice that there is at most one of $f_i$ with $i \in \{1, \cdots, 8\}$ which is a 6*-face because no 5-face can be adjacent to a 6*-face by Claim 11 again. Therefore, $\omega^*(f) \geq 2 - 5 \times \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$.

**Subcase 9.3** $0 \leq p_5(f) \leq 4$.

By (R4), we derive that

$$\begin{aligned}
\omega^*(f) &\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}p_{6^*}(f) \\
&\geq 2 - \frac{1}{3}p_5(f) - \frac{1}{6}(8 - p_5(f)) \\
&= \frac{2}{3} - \frac{1}{6}p_5(f) \\
&\geq \frac{2}{3} - \frac{1}{6} \cdot 4 \\
&= 0.
\end{aligned}$$

Next, we will discuss several cases where $d(f) \geq 10$. Let $f$ be such a $10^+$-face that $f'$ is adjacent to $f$. We call $f'$ is *special* if it takes charge 1 from $f$. Let $|F_1(f)|$ denote the number of adjacent special faces. Let $S_i$ be a face adjacent to $f$ by an edge $e_i$ for $i = 1, 2$. If $e_1$ is not incident to $e_2$, then we call $S_1$ and $S_2$ are *mutually disjoint*. According to (R1) and (R5), we see that only 3-face and $5^*$-face may take charge 1 from $f$, respectively. It implies that each special face is either a 3-face or a $5^*$-face. We first observe that:

**Observation 1** *If $f$ is adjacent to two special faces which share at least one common vertex $v$ that is lied on $b(f)$, then $\tau(v \to f) \geq 1$.*

**Proof.** Without loss of generality, assume $f_1$ and $f_2$ are both such two special faces that $v_2 \in V(f_1) \cap V(f_2)$ and $v_2 \in V(f)$. Since each $5^*$-face taking charge 1 from a $10^+$-face must be adjacent to $f$ by a good common $(3^+, 3^+)$-edge, we see that $f_1$ and $f_2$ are either both 3-faces or both $5^*$-faces. By the absence of two adjacent 3-faces and two adjacent $5^*$-faces, we confirm that $d(v_2) \geq 4$. If $d(f_1) = d(f_2) = 3$, then by (R2a), (R3a) and (R6), it is easy to deduce that $\tau(v_2 \to f) \geq 1$. Otherwise, we may suppose $f_1$ and $f_2$ are both $5^*$-faces. According to (R2c3.3), (R3c) and (R6), we derive that $\tau(v_2 \to f) = 1$. This complete the proof of Observation 1. □

If there exist two special faces which share at least one common vertex $v$ that is lied on $b(f)$, i.e., let $f_i$ and $f_{i+1}$ be such two special faces that $v_{i+1} \in V(f_i) \cap V(f_{i+1})$ and $v_{i+1} \in V(f)$, then we see that $\tau(v_{i+1} \to f) \geq 1$ by Observation 1 and $f$ sends at most $2 \times 1$ to $f_i$ and $f_{i+1}$. It means that $f$ takes charge 1 from $v_{i+1}$ and then sends it to $f_{i+1}$. Thus, we can consider that $f_{i+1}$ takes nothing from $f$. Therefore, we may suppose that all of the special faces adjacent to $f$ are mutually disjoint, which implies $|F_1(f)| \leq \left\lfloor \frac{d(f)}{2} \right\rfloor$.

**Observation 2** *If $f_i$ is a special face with a (good) common $(3^+, 3^+)$-edge $v_iv_{i+1}$, then $f$ sends nothing to $f_{i-1}$ and $f_{i+1}$.*

**Proof.** If $f_i$ is a 3-face, then $f_{i-1}$ and $f_{i+1}$ can not be any special faces by the assumption that special faces adjacent to $f$ are mutually disjoint. Since $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ are not good common edges, we conclude that $f$ sends nothing to $f_{i-1}$ and $f_{i+1}$ by (R4) and (R5).

Now we may assume that $f_i$ is a $5^*$-face. By symmetry, we only need to consider $v_i$, depending on $d(v_i)$.

- If $d(v_i) = 3$, then $f_{i-1}$ can not be a 3-face since $v_iv_{i+1}$ is a good common edge. Moreover, $f_{i-1}$ can not be any nontriangular 5-face, $5^*$-face or $6^*$-face by Claim 5, Claim 6 and Claim 12. So $\tau(f \to f_{i-1}) = 0$.
- Next, we may suppose $d(v_i) \geq 4$. Clearly, $f_{i-1}$ can not be any special faces by the assumption that special faces adjacent to $f$ are mutually disjoint. Furthermore, if $f_{i-1}$ is a nontriangular 5-face, then $f$ sends nothing to it because $v_{i-1}v_i$ is not a $(3,3)$-edge. If $f_{i-1}$ is a $6^*$-face, then we discuss as follows: when $v_i$ is a $5^+$-vertex, then $\tau(f \to f_{i-1}) = 0$ since $v_{i-1}v_i$ is neither a $(3,3)$-edge nor a $(3,4)$-edge; when $v_i$ is a 4-vertex, then $f_i$ is the opposite

face of $f_{i-1}$ by a 4-vertex, which contradicts (O3). Thus, we prove that $\tau(f \to f_{i-1}) = 0$ and $\tau(f \to f_{i+1}) = 0$. This completes the proof of Observation 2. $\square$

By using Observation 2, one can easily deduce Observation 3:

**Observation 3** $p_5(f) + p_{6^*}(f) \leq d(f) - 2|F_1(f)| - 1$.

**Case 10** $d(f) = 10$.

Then $\omega(f) = 4$ and $|F_1(f)| \leq 5$. We divide the argument into the following three subcases in light of $|F_1(f)|$.

**Case 10.1** $|F_1(f)| = 5$.

It implies that there exist five mutually disjoint special faces that are adjacent to $f$. Since $G$ does not contain lotus, there exists at least one $4^+$-vertex on $b(f)$. Without loss of generality, suppose $v_1$ is such a vertex that $f_1, f_3, f_5, f_7, f_9$ are all special faces. If $v_1$ is a $5^+$-vertex, then $v_1$ sends at least 1 to $f$ by (R3) and (R6). Now we assume that $v_1$ is a 4-vertex. If $d(v_{10}) = 3$, then $f_{10}$ is not a nontriangular 5-face since $f_9$ is a special face. So $\tau(v_1 \to f) = 1$ by (R2b2), (R2c2) and (R2c3.3). Otherwise, $d(v_{10}) \geq 4$ and $f$ receives at least $\frac{2}{3} \times 2 = \frac{4}{3}$ from $v_1$ and $v_{10}$ totally by (R2b1), (R2b2), (R2c2), (R2c3.2) and (R2c3.3). Thus, $\omega^*(f) \geq 4 - 1 \times 5 + 1 = 0$.

**Case 10.2** $|F_1(f)| = 4$.

It implies that $f$ is adjacent to exactly four special faces by four (good) common edges which are disjoint each other. Denote $S_i$ be such a special face adjacent to $f$ by a common edge $e_i$, where $i = 1, 2, 3, 4$. Noting that $e_i$ can not be incident to $e_j$ for each pair $(i, j) \subset \{1, \cdots, 4\}$. Thus, it follows that there are exist two vertices lied on $b(f)$ which are not incident to any common edge $e_i$ with $i \in \{1, \cdots, 4\}$. W.l.o.g., assume $i < j$. If $j = i + 1$, then $v_i v_j$ is an edge of $b(f)$. Notice that $f_i$ can not be a special face, i.e., $f_i$ is neither a 3-face nor a $5^*$-face. Furthermore, if $f_i$ is nontriangular 5-face or a $6^*$-face, then there exists at least one vertex in $V(f)$ whose degree is at least 4 by the absence of lotus. Let $v^*$ be such $4^+$-vertex. If $d(v^*) \geq 5$, then $\tau(v^* \to f) \geq \frac{4}{5}$ by (R3) and (R6). Now we may suppose $d(v^*) = 4$. According to (R2c2), (R2c3.2) and (R2c3.3), it is obvious that each 4-vertex sends at least $\frac{2}{3}$ to its incident face which is adjacent to a special face. Thus, we have that $\omega^*(f) \geq 4 - 1 \times 4 - \frac{1}{3} + \frac{2}{3} = \frac{1}{3} > 0$.

**Case 10.3** $0 \leq |F_1(f)| \leq 3$.

By Observation 3, $p_5(f) + p_{6^*}(f) \leq 9 - 2|F_1(f)|$. Therefore, $\omega^*(f) \geq 4 - |F_1(f)| - \frac{1}{3}(9 - 2|F_1(f)|) = 1 - \frac{1}{3}|F_1(f)| \geq 1 - \frac{1}{3} \times 3 = 0$.

**Case 11** $d(f) = 11$.

Clearly, $\omega(f) = 5$ and $|F_1(f)| \leq 5$. By Observation 3, $p_5(f) + p_{6^*}(f) \leq 10 - 2|F_1(f)|$. Then $\omega^*(f) \geq 5 - |F_1(f)| - \frac{1}{3}(10 - 2|F_1(f)|) = \frac{5}{3} - \frac{1}{3}|F_1(f)| \geq \frac{5}{3} - \frac{1}{3} \times 5 = 0$.

**Case 12** $d(f) \geq 12$.

By Observation 3, $p_5(f) + p_{6^*}(f) \leq d(f) - 2|F_1(f)| - 1$. Moreover, $|F_1(f)| \leq \lfloor \frac{1}{2} d(f) \rfloor$. Thus, we have that

$$
\begin{aligned}
\omega^*(f) &\geq (d(f)-6) - |F_1(f)| - \frac{1}{3}(d(f) - 2|F_1(f)| - 1) \\
&= \frac{2}{3}d(f) - \frac{17}{3} - \frac{1}{3}|F_1(f)| \\
&\geq \frac{2}{3}d(f) - \frac{17}{3} - \frac{1}{3}\cdot\frac{d(f)}{2} \\
&= \frac{1}{2}d(f) - \frac{17}{3} \\
&\geq \frac{1}{2}\cdot 12 - \frac{17}{3} \\
&= \frac{1}{3}.
\end{aligned}
$$

Up to now, we proved Theorem 1 for 2-connected graphs.

Suppose now that $G$ is not a 2-connected planar graph and we will construct a 2-connected plane graph $G^*$ with $\delta(G^*) \geq 3$ having neither 4-cycles nor 9-cycles and satisfying structural properties (C1) to (C5). This obviously contradicts the result just established before.

We remark that the following proof is stimulated by the technique used in [3].

Let $B$ be an end block of $G$ with the unique cut-vertex $x$. Let $f$ be the outside face of $G$. Notice that $d_B(x) \geq 2$ and $d_B(v) \geq 3$ for each $v \in V(B)\backslash\{x\}$. Choosing another vertex $y$ of $B$ such that $y \neq x$ and $y$ lies on the boundary of $B$. Obviously, $x$ and $y$ are both belonging to $b(f)$. Then we take ten copies of $B$, i.e., $B_k$ with $k = 1, \cdots, 10$. In each copy $B_k$, the vertices corresponding to $x$ and $y$ are denoted by $x_k$ and $y_k$, respectively. Then one can embed $B_k$, $k = 1, \cdots, 10$, into $f$ in the following way: first, let $B = B_1$. Next, for each $k = 2, \cdots, 10$, consecutively embed $B_k$ into $f$ by identifying $x_k$ with $y_{k-1}$. Finally, identify $y_{10}$ with a vertex $u \in V(f)\backslash V(B)$. Then the first resulting graph, denoted by $G_1$.

Obviously, in the processing of constructing $G_1$, we confirm that there are no new adjacent cycles established. Furthermore, no 4-cycles and 9-cycles are formed. Thus, it is easy to deduce that $G_1$ satisfies the following structural properties.

(A1) Fewer end blocks than $G$;

(A2) The minimum degree is at least 3;

(A3) Neither 4-cycles nor 9-cycles;

(A4) A 5-cycle or a 6-cycle is adjacent to at most one 3-cycle;

(A5) A $5^*$-cycle is neither adjacent to a $5^*$-cycle normally, nor adjacent to an $i$-cycle with $i \in \{7, 8\}$;

(A6) A $6^*$-cycle is not adjacent to a 6-cycle;

(A7) A nontriangular 7-cycle is not adjacent to two 5-cycles which are normally adjacent;

(A8) A $7^*$-cycle is neither adjacent to a 5-cycle nor a $6^*$-cycle.

Furthermore, we confirm that $G_1$ also satisfies the following two structural properties:

(P1) $G_1$ has neither orchid, nor sunflower, nor lotus;

(P2) A $6^*$-cycle is not incident to an $i$-cycle $C$ with $i \in \{3, 5\}$, where $C$ is opposite to such $6^*$-cycle by a 4-vertex.

(P1) For some $k \in \{2, \cdots, 10\}$, notice that we just identify some vertex $x_k$ with $y_{k-1}$. It implies that any new cycle, which is not completely belong to some $B_k$, must be an $11^+$-cycles, i.e., $C^* = x_1 \cdots x_{10} u \cdots x_1$. Thus, any orchid, sunflower, or lotus can not be established.

(P2) Assume to the contrary that $G_1$ contains a $6^*$-cycle, denoted by $C_6^*$, which is incident to a 3-cycle $C_3$ or a 5-cycle $C_5$ by a 4-vertex $v^*$. Clearly, $v^*$ must be equal to $u$ or some vertex $x_k$ with $k \in \{2, \cdots, 10\}$. However, $d_{G_1}(u) = d_{B_{10}}(u) + d_{G \setminus B_1}(u) \geq 2 + 3 = 5$ or $d_{G_1}(x_k) = d_{B_{k-1}}(x_k) + d_{B_k}(x_k) \geq 3 + 2 = 5$ for all $k \in \{2, \cdots, 10\}$. We always get a contradiction to $d_{G_1}(v^*) = 4$.

Now, if $G_1$ is 2-connected, then we well done. Otherwise, we may repeat the process described above and finally obtain a desired $G^*$.

Thus, we complete the proof of Theorem 1. □

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