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A multiple stochastic integral criterion for almost sure limit theorems

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Abstract: In this paper, we study almost sure central limit theorems for multiple stochastic integrals and provide a criterion based on the kernel of these multiple integrals. We apply our result to normalized partial sums of Hermite polynomials of increments of fractional Brownian motion. We obtain almost sure central limit theorems for these normalized sums when they converge in law to a normal distribution.

Key words: Almost sure limit theorem; multiple stochastic integrals; fractional Brownian motion; Hermite power variation.

2000 Mathematics Subject Classification: 60F05; 60G15; 60H05; 60H07.

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1 Introduction

Let $\{X_n\}_{n \geq 1}$ be a sequence of real-valued independent identically distributed random variables with $E[X_n] = 0$ and $E[X_n^2] = 1$, and denote

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k.$$

The celebrated almost sure central limit theorem (ASCLT) states that the sequence of random empirical measures, given by

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{S_k},$$

converges almost surely to the $\mathcal{N}(0, 1)$ distribution as $n \rightarrow \infty$. In other words, if N is a $\mathcal{N}(0, 1)$ random variable, then, almost surely, for all $x \in \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \mathbf{1}_{\{S_k \leq x\}} \longrightarrow P(N \leq x), \quad \text{as } n \rightarrow \infty.$$

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or, equivalently, almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(S_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \rightarrow \infty. \quad (1.1)$$

The ASCLT was stated first by Lévy [14] without proof. It was then forgotten for half century. It was rediscovered by Brosamler [8] and Schatte [19] and proven, in its present form, by Lacey and Philipp [13]. We refer the reader to Berkes and Csáki [1] for a universal ASCLT covering a large class of limit theorems for partial sums, extremes, empirical distribution function and local times associated with independent random variables $\{X_n\}$, as well as the PhD thesis by Gonchigdanzan [9], where extensions of the ASCLT to weakly dependent random variables are studied, for example in the context of strong mixing or ρ -mixing. Ibragimov and Lifshits [10, 11] have provided a criterion for (1.1) which does not require the sequence $\{X_n\}$ of random variables to be necessarily independent nor that the sequence $\{S_n\}$ have the specific form of partial sums. This criterion is stated in Proposition 3.1 below.

Our goal is to investigate the ASCLT for a sequence of multiple stochastic integrals. Conditions ensuring the convergence in law of this sequence to the standard $\mathcal{N}(0, 1)$ distribution are now well-known, see Nualart and Peccati [16]. We will derive a criterion for this sequence of multiple integrals to satisfy also the ASCLT.

As an application, we consider some non-linear functions of strongly dependent Gaussian random variables. We will generate strong dependence by using increments of a standard fractional Brownian motion B^H . Recall that $B^H = (B_t^H)_{t \geq 0}$ is a centered Gaussian process with continuous paths such that

$$E[B_t^H B_s^H] = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad s, t \geq 0.$$

The process B^H is self-similar with stationary increments and we refer the reader to Nualart [17] and Samorodnitsky and Taqqu [18] for its main properties. The increments

$$Y_k = B_{k+1}^H - B_k^H, \quad k \geq 0,$$

called “fractional Gaussian noise”, are centered stationary Gaussian random variables with covariance

$$\rho(r) = E[Y_k Y_{k+r}] = \frac{1}{2} \left(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H} \right), \quad r \in \mathbb{Z}. \quad (1.2)$$

This covariance behaves asymptotically as

$$\rho(r) \sim H(2H-1)|r|^{2H-2} \quad \text{as } |r| \rightarrow \infty.$$

Observe that $\rho(0) = 1$ and (i) for $0 < H < 1/2$: $\rho(r) < 0$ for $r \neq 0$, $\sum_{r \in \mathbb{Z}} |\rho(r)| < \infty$ and $\sum_{r \in \mathbb{Z}} \rho(r) = 0$; (ii) for $H = 1/2$: $\rho(r) = 0$ if $r \neq 0$; and (iii) for $1/2 < H < 1$: $\sum_{r \in \mathbb{Z}} |\rho(r)| = \infty$. The Hurst index measures the strenght of the dependence when $H \geq 1/2$: the larger H , the stronger the dependence.

We shall consider random variables

$$H_q(Y_k) = H_q(B_{k+1}^H - B_k^H), \quad k \geq 0, \quad (1.3)$$

where H_q is a Hermite polynomial of order $q \geq 1$. The first few Hermite polynomials are $H_1(x) = x$, $H_2(x) = x^2 - 1$ and $H_3(x) = x^3 - 3x$. Why Hermite polynomials? This is because they can be expressed as multiple stochastic integrals. and because the limit distribution, as $n \rightarrow \infty$, of the partial sums $\sum_{k=0}^{n-1} H_q(Y_k)$, adequately normalized, is known (see Breuer and Major [4], Dobrushin and Major [6], Giraitis and Surgailis [7] and Taqqu [20]). This limit can be Gaussian or not, depending on the order q of the polynomial and on the Hurst index H .

Using our multiple stochastic integral criterion for ASCLT and results from Malliavin calculus, we will show that the normalized sums of $\sum_{k=0}^{n-1} H_q(Y_k)$, when $q \geq 2$, satisfy an ASCLT if their limit is Gaussian, that is, when the Hurst index H satisfies $0 < H \leq 1 - \frac{1}{2q}$.

Our criterion, when applied to the simple linear case $q = 1$, yields the following ASCLT for fractional Brownian motion. We show that, almost surely, for any bounded continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and any $0 < H < 1$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi \left(\frac{B_k^H}{k^H} \right) \longrightarrow E[\varphi(N)] \quad \text{as } n \rightarrow \infty, \quad (1.4)$$

where $N \sim \mathcal{N}(0, 1)$. Berkes and Horváth [2], using other techniques, have obtained a continuous-time version of (1.4).

The paper is organized as follows. In Section 2, we present the basic elements of Gaussian analysis and Malliavin calculus used in this paper. The ASCLT criterion is stated and proved in Section 3. Its application to partial sums of Hermite polynomials of increments of fractional Brownian motion is presented in Section 4, when the limit in distribution is Gaussian. In section 5, we discuss the case where the limit in distribution is non-Gaussian.

2 Multiple stochastic integrals and Malliavin calculus

We shall now present the basic elements of Gaussian analysis and Malliavin calculus that are used in this paper. The reader is referred to the monograph by Nualart [17] for any unexplained definition or result.

Let \mathfrak{H} be a real separable Hilbert space. For any $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ be the q th tensor product of \mathfrak{H} and denote by $\mathfrak{H}^{\odot q}$ the associated q th symmetric tensor product. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate an isonormal Gaussian process over \mathfrak{H} , defined on some probability space (Ω, \mathcal{F}, P) . This means that X is a centered Gaussian family, whose covariance is given in terms of the inner product of \mathfrak{H} by $E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$.

For every $q \geq 1$, let \mathcal{H}_q be the q th Wiener chaos of X , that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables of the type $\{H_q(X(h)), h \in$

$\mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the q th Hermite polynomial defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} \left(e^{-\frac{x^2}{2}} \right). \quad (2.5)$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$, $x \in \mathbb{R}$. For any $q \geq 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\|\cdot\|_{\mathfrak{H}^{\odot q}} = \sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the q th Wiener chaos \mathcal{H}_q . Then

$$E[I_p(f)I_q(g)] = \delta_{p,q} \times p! \langle f, g \rangle_{\mathfrak{H}^{\otimes p}} \quad (\delta_{p,q} \text{ stands for the Kronecker symbol})$$

for $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $p, q \geq 1$. Moreover, if $f \in \mathfrak{H}^{\odot q}$, we have

$$I_q(f) = I_q(\tilde{f}), \quad (2.6)$$

where $\tilde{f} \in \mathfrak{H}^{\odot q}$ is the symmetrization of f .

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in \mathfrak{H} . Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \dots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \quad (2.7)$$

Since $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \tilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Observe that $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$, namely the scalar product of f and g . In the particular case $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where (A, \mathcal{A}) is a measurable space and μ is a σ -finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L_s^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q . In this case, (2.7) can be written as

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) = \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) \\ \times g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r) d\mu(s_1) \dots d\mu(s_r),$$

that is, we identify r variables in f and g and integrate them out. The following useful *multiplication formula* holds: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g). \quad (2.8)$$

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X . Let \mathcal{S} be the set of all cylindrical random variables of the form

$$F = \varphi(X(h_1), \dots, X(h_n)), \quad (2.9)$$

where $n \geq 1$, $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an infinitely differentiable function with compact support and $h_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} (X(h_1), \dots, X(h_n)) h_i.$$

By iteration, one can define the m th derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geq 2$. For $m \geq 1$ and $p \geq 1$, $\mathbb{D}^{m,p}$ denotes the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p = E[|F|^p] + \sum_{i=1}^m E(\|D^i F\|_{\mathfrak{H}^{\odot i}}^p).$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. The Malliavin derivative D verifies moreover the following *chain rule*. If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} (F) DF_i.$$

Let now $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ with μ non-atomic. Then an element $u \in \mathfrak{H}$ can be expressed as $u = \{u_t, t \in A\}$ and the Malliavin derivative of a multiple integral F of the form $I_q(f)$ (with $f \in \mathfrak{H}^{\odot q}$) is the element $DF = \{D_t F, t \in A\}$ of $L^2(A \times \Omega)$ given by

$$D_t F = D_t [I_q(f)] = q I_{q-1}(f(\cdot, t)). \quad (2.10)$$

Thus the derivative of the random variable $I_q(f)$ is the stochastic process $q I_{q-1}(f(\cdot, t))$, $t \in A$. Moreover,

$$\|D[I_q(f)]\|_{\mathfrak{H}}^2 = q^2 \int_A I_{q-1}(f(\cdot, t))^2 \mu(dt).$$

We shall also use the following bound, proved by Nourdin and Peccati in [15], for the difference between the law of multiple integrals of order $q \geq 2$ with unit variance and the law of a standard Gaussian random variable.

Proposition 2.1 *Let $q \geq 2$ be an integer, $f \in \mathfrak{H}^{\odot q}$ with $q! \|f\|_{\mathfrak{H}^{\otimes q}}^2 = E[I_q(f)^2] = 1$ and $N \sim \mathcal{N}(0, 1)$. Then, for all $h : \mathbb{C} \rightarrow \mathbb{R}$ such that*

$$|h(x) - h(y)| \leq |x - y|, \quad x, y \in \mathbb{C}, \quad (2.11)$$

we have

$$|E[h(I_q(f))] - E[h(N)]| \leq \sqrt{E \left[\left(1 - \frac{1}{q} \|D[I_q(f)]\|_{\mathfrak{H}}^2 \right)^2 \right]}. \quad (2.12)$$

3 A multiple stochastic integral criterion for ASCLT

The following result, due to Ibragimov and Lifshits [10] (Theorem 1.1 therein), gives a sufficient condition for extending convergence in law to an almost sure limit theorem.

Proposition 3.1 *Let $\{G_n\}$ be a sequence of random variables converging in distribution towards a random variable G_∞ , and set*

$$\Delta_n(t) = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} (e^{itG_k} - E(e^{itG_\infty})).$$

If

$$\sup_{|t| \leq r} \sum_n \frac{E|\Delta_n(t)|^2}{n \log n} < \infty \quad \text{for all } r > 0, \quad (3.13)$$

then, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(G_\infty)] \quad \text{as } n \rightarrow \infty.$$

The following theorem provides a criterion for an ASCLT for multiple stochastic integrals. It is expressed in terms of the kernels of these integrals.

Theorem 3.2 *Let the notation of Section 2 prevail. Fix $q \geq 2$, and let $\{G_n\}$ be a sequence of the form $G_n = I_q(f_n)$, with $f_n \in \mathfrak{H}^{\odot q}$. Assume that $E[G_n^2] = q! \|f_n\|_{\mathfrak{H}^{\otimes q}}^2 = 1$ for all n , and that $G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0, 1)$ as $n \rightarrow \infty$. If the following two conditions are satisfied:*

$$(A_1) \quad \sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(q-r)}} < \infty \quad \text{for every } r = 1, \dots, q-1;$$

$$(A_2) \quad \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} < \infty,$$

then $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \rightarrow \infty.$$

Remark 3.3 We have $E(G_k G_l) = q! \langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}$. Consequently, condition (A_2) can be replaced by the following equivalent condition:

$$(A'_2) \quad \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{|E(G_k G_l)|}{kl} < \infty.$$

Proof of Theorem 3.2. We shall verify the sufficient condition (3.13), that is the Ibragimov-Lifshits criterion. For simplicity, let $g(t) = E(e^{itN}) = e^{-t^2/2}$. Then

$$\begin{aligned}
& E|\Delta_n(t)|^2 \\
&= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{1}{kl} E[(e^{itG_k} - g(t))(e^{-itG_l} - g(t))] \\
&= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{1}{kl} [E(e^{it(G_k-G_l)}) - g(t)(E(e^{itG_k}) + E(e^{-itG_l})) + g^2(t)] \\
&= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{1}{kl} [(E(e^{it(G_k-G_l)}) - g^2(t)) - g(t)(E(e^{itG_k}) - g(t)) - g(t)(E(e^{-itG_l}) - g(t))].
\end{aligned}$$

Let $t \in \mathbb{R}$ and $r > 0$ be such that $|t| \leq r$. The function $\varphi(x) = \frac{1}{r}e^{itx}$ satisfies $|\varphi'(x)| \leq 1$ and hence (2.11). Therefore, by (2.12), we have, for $j \in \{k, l\}$,

$$|E(e^{\pm itG_j}) - g(t)| \leq r \sqrt{E \left[\left(1 - \frac{1}{q} \|DG_j\|_{\mathfrak{H}}^2 \right)^2 \right]}.$$

Similarly,

$$\begin{aligned}
|E(e^{it(G_k-G_l)}) - g^2(t)| &= \left| E(e^{it\sqrt{2}\frac{G_k-G_l}{\sqrt{2}}}) - g(\sqrt{2}t) \right| \\
&\leq \sqrt{2}r \sqrt{E \left[\left(1 - \frac{1}{q} \left\| D\left(\frac{G_k-G_l}{\sqrt{2}}\right) \right\|_{\mathfrak{H}}^2 \right)^2 \right]} \\
&= \sqrt{2}r \sqrt{E \left[\left(1 - \frac{1}{2q} \|DG_k - DG_l\|_{\mathfrak{H}}^2 \right)^2 \right]}.
\end{aligned}$$

But

$$1 - \frac{1}{2q} \|DG_k - DG_l\|_{\mathfrak{H}}^2 = \frac{1}{2} \left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right) + \frac{1}{2} \left(1 - \frac{1}{q} \|DG_l\|_{\mathfrak{H}}^2 \right) + \frac{1}{q} \langle DG_k, DG_l \rangle_{\mathfrak{H}}$$

so that, since $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and $\sqrt{u+v+w} \leq \sqrt{u} + \sqrt{v} + \sqrt{w}$,

$$\begin{aligned}
\sqrt{E \left[\left(1 - \frac{1}{q} \|DG_k - DG_l\|_{\mathfrak{H}}^2 \right)^2 \right]} &\leq \frac{\sqrt{3}}{2} \sqrt{E \left[\left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2 \right]} \\
&\quad + \frac{\sqrt{3}}{2} \sqrt{E \left[\left(1 - \frac{1}{q} \|DG_l\|_{\mathfrak{H}}^2 \right)^2 \right]} + \frac{\sqrt{3}}{q} \sqrt{E[\langle DG_k, DG_l \rangle_{\mathfrak{H}}^2]}.
\end{aligned}$$

Consequently, to get (3.13), it suffices to prove that the two following conditions hold:

$$\sum_n \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{1}{kl} \sqrt{E \left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2} < \infty \quad (3.14)$$

and

$$\sum_n \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{1}{kl} \sqrt{E \langle DG_k, DG_l \rangle_{\mathfrak{H}}^2} < \infty. \quad (3.15)$$

Since $\sum_{l=1}^n \frac{1}{l} \leq 1 + \log n$, one can observe that (3.14) is a consequence of

$$\sum_n \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \sqrt{E \left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2} < \infty. \quad (3.16)$$

We shall now prove that (3.16) and (3.15) are satisfied.

Proof of (3.16). In view of the normalization $q \|f_k\|_{\mathfrak{H}}^2 = 1$ for all k , we have, by Lemma 3.4 below:

$$\frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 - 1 = q \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f_k \tilde{\otimes}_r f_k).$$

Hence, taking into account the orthogonality between multiple stochastic integrals, we have

$$E \left[\left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2 \right] = q^2 \sum_{r=1}^{q-1} (r-1)!^2 \binom{q-1}{r-1}^4 (2q-2r)! \|f_k \tilde{\otimes}_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2.$$

Via the straightforward inequality

$$\sqrt{x_1^2 + \dots + x_q^2} \leq |x_1| + \dots + |x_q|, \quad (3.17)$$

and since $\|f_k \tilde{\otimes}_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leq \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}}$, we get

$$\sqrt{E \left[\left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2 \right]} \leq q \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 \sqrt{(2q-2r)!} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}}.$$

Combining all these bounds, we obtain

$$\begin{aligned} & \sum_n \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \sqrt{E \left(1 - \frac{1}{q} \|DG_k\|_{\mathfrak{H}}^2 \right)^2} \\ & \leq q \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 \sqrt{(2q-2r)!} \times \sum_n \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}}, \end{aligned}$$

so that assumption (A_1) immediately implies (3.16).

Proof of (3.15). By Lemma 3.4 below and the orthogonality between multiple stochastic integrals, we have

$$E\langle DG_k, DG_l \rangle_{\mathfrak{H}}^2 = q^4 \sum_{r=1}^{q-1} (r-1)!^2 \binom{q-1}{r-1}^4 (2q-2r)! \|f_k \otimes_r f_l\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2 + (qq!)^2 |\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|^2.$$

By Lemma 3.5 below, we also have

$$\|f_k \otimes_r f_l\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2 = \langle f_k \otimes_{q-r} f_k, f_l \otimes_{q-r} f_l \rangle_{\mathfrak{H}^{\otimes 2r}},$$

so that, by Cauchy-Schwarz inequality:

$$\begin{aligned} \|f_k \otimes_r f_l\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2 &\leq \|f_k \otimes_{q-r} f_k\|_{\mathfrak{H}^{\otimes 2r}} \|f_l \otimes_{q-r} f_l\|_{\mathfrak{H}^{\otimes 2r}} \\ &\leq \frac{1}{2} (\|f_k \otimes_{q-r} f_k\|_{\mathfrak{H}^{\otimes 2r}}^2 + \|f_l \otimes_{q-r} f_l\|_{\mathfrak{H}^{\otimes 2r}}^2). \end{aligned}$$

Consequently, using again (3.17), we obtain that

$$\begin{aligned} \sqrt{E\langle DG_k, DG_l \rangle_{\mathfrak{H}}^2} &\leq q^2 \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 \sqrt{(2q-2r)!} \\ &\quad \times \frac{1}{\sqrt{2}} (\|f_k \otimes_{q-r} f_k\|_{\mathfrak{H}^{\otimes 2r}} + \|f_l \otimes_{q-r} f_l\|_{\mathfrak{H}^{\otimes 2r}}) + qq! |\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|. \end{aligned}$$

Finally, we obtain (3.15) from the conjunction of (A_1) and (A_2) , which completes the proof of Theorem 3.2. \square

In the previous proof, we used the two following lemmas:

Lemma 3.4 *Consider two random variables $F = I_q(f)$, $G = I_q(g)$, with $f, g \in \mathfrak{H}^{\odot q}$. Then*

$$\langle DF, DG \rangle_{\mathfrak{H}} = q^2 \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f \tilde{\otimes}_r g) + qq! \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}.$$

Proof. Without loss of generality, we can assume that $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ where (A, \mathcal{A}) is a measurable space, and μ is a σ -finite and non-atomic measure. Thus, we can write

$$\begin{aligned} \langle DF, DG \rangle_{\mathfrak{H}} &= \int_A D_t F D_t G \mu(dt) \\ &= q^2 \int_A I_{q-1}(f(\cdot, t)) I_{q-1}(g(\cdot, t)) \mu(dt) \quad \text{by (2.10)} \\ &= q^2 \int_A \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2-2r}(f(\cdot, t) \tilde{\otimes}_r g(\cdot, t)) \mu(dt) \quad \text{by (2.8)} \\ &= q^2 \int_A \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2-2r}(f(\cdot, t) \otimes_r g(\cdot, t)) \mu(dt) \quad \text{by (2.6)} \end{aligned}$$

$$\begin{aligned}
&= q^2 \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2-2r}(f \otimes_{r+1} g) \quad \text{by linearity} \\
&= q^2 \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2-2r}(f \tilde{\otimes}_{r+1} g) \\
&= q^2 \sum_{r=1}^q (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f \tilde{\otimes}_r g) \\
&= q^2 \sum_{r=1}^{q-1} (r-1)! \binom{q-1}{r-1}^2 I_{2q-2r}(f \tilde{\otimes}_r g) + q q! \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}.
\end{aligned}$$

□

Lemma 3.5 *Let $f, g \in \mathfrak{H}^{\otimes q}$. Then $\|f \otimes_r g\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2 = \langle f \otimes_{q-r} f, g \otimes_{q-r} g \rangle_{\mathfrak{H}^{\otimes 2r}}$.*

Proof. Without loss of generality, we can assume that $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ where (A, \mathcal{A}) is a measurable space, and μ is a σ -finite and non-atomic measure. Using the definition of contractions and Fubini theorem, we can write

$$\begin{aligned}
&\|f \otimes_r g\|_{\mathfrak{H}^{\otimes(2q-2r)}}^2 \\
&= \int_{A^{2q-2r}} \mu(dx_1) \dots \mu(dx_{q-r}) \mu(dy_1) \dots \mu(dy_{q-r}) \\
&\quad \times \left(\int_{A^r} \mu(dz_1) \dots \mu(dz_r) f(x_1, \dots, x_{q-r}, z_1, \dots, z_r) g(y_1, \dots, y_{q-r}, z_1, \dots, z_r) \right)^2 \\
&= \int_{A^{2q-2r}} \mu(dx_1) \dots \mu(dx_{q-r}) \mu(dy_1) \dots \mu(dy_{q-r}) \\
&\quad \times \int_{A^r} \mu(dz_1) \dots \mu(dz_r) f(x_1, \dots, x_{q-r}, z_1, \dots, z_r) g(y_1, \dots, y_{q-r}, z_1, \dots, z_r) \\
&\quad \times \int_{A^r} \mu(dt_1) \dots \mu(dt_r) f(x_1, \dots, x_{q-r}, t_1, \dots, t_r) g(y_1, \dots, y_{q-r}, t_1, \dots, t_r) \\
&= \int_{A^{2r}} \mu(dz_1) \dots \mu(dz_r) \mu(dt_1) \dots \mu(dt_r) \\
&\quad \times \int_{A^{q-r}} \mu(dx_1) \dots \mu(dx_{q-r}) f(x_1, \dots, x_{q-r}, z_1, \dots, z_r) f(x_1, \dots, x_{q-r}, t_1, \dots, t_r) \\
&\quad \times \int_{A^{q-r}} \mu(dy_1) \dots \mu(dy_{q-r}) g(y_1, \dots, y_{q-r}, z_1, \dots, z_r) g(y_1, \dots, y_{q-r}, t_1, \dots, t_r) \\
&= \int_{A^{2r}} \mu(dz_1) \dots \mu(dz_r) \mu(dt_1) \dots \mu(dt_r) \\
&\quad \times f \otimes_{q-r} f(z_1, \dots, z_r, t_1, \dots, t_r) g \otimes_{q-r} g(z_1, \dots, z_r, t_1, \dots, t_r) = \langle f \otimes_{q-r} f, g \otimes_{q-r} g \rangle_{\mathfrak{H}^{\otimes 2r}}.
\end{aligned}$$

□

When $q = 1$, we have the following result:

Proposition 3.6 *Let $\{G_n\}$ be a centered Gaussian sequence with unit variance. If the following condition is satisfied,*

$$(A'_2) \quad \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{|E(G_k G_l)|}{kl} < \infty,$$

then $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \rightarrow \infty.$$

Proof. Let $t \in \mathbb{R}$ and $r > 0$ be such that $|t| \leq r$. Since $E[e^{itG_k}]$ equals $g(t) = e^{-t^2/2}$, we have

$$\begin{aligned} E|\Delta_n(t)|^2 &= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{1}{kl} E \left[(e^{itG_k} - e^{-t^2/2})(e^{-itG_l} - e^{-t^2/2}) \right] \\ &= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{1}{kl} \left[E(e^{it(G_k - G_l)}) - e^{-t^2} \right] \\ &= \frac{1}{\log^2 n} \sum_{k,l=1}^n \frac{e^{-t^2}}{kl} (e^{E(G_k G_l)t^2} - 1) \\ &\leq \frac{r^2 e^{r^2}}{\log^2 n} \sum_{k,l=1}^n \frac{|E(G_k G_l)|}{kl} \quad \text{since } |e^x - 1| \leq e^{|x|}|x| \text{ and } |E(G_k G_l)| \leq 1. \end{aligned}$$

Therefore, assumption (A'_2) implies (3.13), and the proof of the proposition is done. \square

4 Partial sums of Hermite polynomials of increments of fBm: the Gaussian case

Let B^H be a fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. We are interested in an ASCLT for the q -Hermite power variations of B^H , defined as

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H), \quad n \geq 1, \quad (4.18)$$

in cases where V_n , adequately normalized, converges to a normal distribution. Here, H_q stands for the Hermite polynomial of degree q , given by (2.5).

We first treat the case $q = 1$ and $0 < H < 1$. Convergence in distribution of

$$G_n = \frac{V_n}{n^H} = \frac{B_n^H}{n^H}$$

to a normal law is trivial because, by self-similarity, $G_n \stackrel{\text{law}}{=} B_1^H \sim \mathcal{N}(0, 1)$. The following theorem provides the corresponding ASCLT. A continuous time version of the result was obtained by Berkes and Horváth [2] using different methods.

Theorem 4.1 *For all $H \in (0, 1)$, the sequence $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,*

$$\frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{k} \varphi(B_k^H/k^H) \longrightarrow E[\varphi(N)] \quad \text{as } n \rightarrow \infty.$$

Proof. We shall apply Proposition 3.6. The cases $H < 1/2$ and $H \geq 1/2$ are treated separately. From now on, the value of a constant $C > 0$ may change from line to line.

Case $H < 1/2$. For any $b \geq a \geq 0$, we have

$$b^{2H} - a^{2H} = 2H \int_0^{b-a} \frac{dx}{(x+a)^{1-2H}} \leq 2H \int_0^{b-a} \frac{dx}{x^{1-2H}} = (b-a)^{2H}.$$

Hence, for $l \geq k \geq 1$, we have $l^{2H} - (l-k)^{2H} \leq k^{2H}$ so that

$$|E[B_k^H B_l^H]| = \frac{1}{2} (k^{2H} + l^{2H} - (l-k)^{2H}) \leq k^{2H}.$$

Thus

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \sum_{k=1}^l \frac{|E[G_k G_l]|}{k} &= \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{1+H}} \sum_{k=1}^l \frac{|E[B_k^H B_l^H]|}{k^{1+H}} \\ &\leq \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{1+H}} \sum_{k=1}^l \frac{1}{k^{1-H}} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \leq C \sum_{n \geq 2} \frac{1}{n \log^2 n} < \infty. \end{aligned}$$

Consequently, condition (A'_2) in Proposition 3.6 is fulfilled.

Case $H \geq 1/2$. For $l \geq k \geq 1$, it follows from (1.2) that

$$\begin{aligned} |E[B_k^H B_l^H]| &= \left| \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} E[(B_{i+1}^H - B_i^H)(B_{j+1}^H - B_j^H)] \right| \leq \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} |\rho(i-j)| \\ &\leq k \sum_{r=-l+1}^{l-1} |\rho(r)| \leq C k l^{2H-1}. \end{aligned}$$

The last inequality comes from the fact that $\rho(0) = 1$, $\rho(1) = \rho(-1) = (2^{2H} - 1)/2$ and, if $r \geq 2$,

$$\begin{aligned} |\rho(-r)| &= |\rho(r)| = |E[(B_{r+1}^H - B_r^H)B_1^H]| = H(2H - 1) \int_0^1 du \int_r^{r+1} dv (v - u)^{2H-2} \\ &\leq H(2H - 1) \int_0^1 (r - u)^{2H-2} du \leq H(2H - 1)(r - 1)^{2H-2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \sum_{k=1}^l \frac{|E[G_k G_l]|}{k} &= \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{1+H}} \sum_{k=1}^l \frac{|E[B_k^H B_l^H]|}{k^{1+H}} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{2-H}} \sum_{k=1}^l \frac{1}{k^H} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \leq C \sum_{n \geq 2} \frac{1}{n \log^2 n} < \infty. \end{aligned}$$

Finally, condition (A'_2) in Proposition 3.6 is satisfied, which completes the proof of Theorem 4.1. \square

In the remaining part of this section, we assume that $q \geq 2$ and $0 < H \leq 1 - 1/(2q)$. When $H \neq 1/2$, since the increments of B^H are not independent and V_n is not linear, the asymptotic behavior of (4.18) is more difficult to investigate. In fact, thanks to the seminal works of Breuer and Major [4], Giraitis and Surgailis [7] and Taqqu [20], it is known that, as $n \rightarrow \infty$:

- If $q \geq 2$ and $0 < H < 1 - 1/(2q)$ then

$$G_n := \frac{V_n}{\sigma_n \sqrt{n}} \xrightarrow{\text{Law}} \mathcal{N}(0, 1). \quad (4.19)$$

- If $q \geq 2$ and $H = 1 - 1/(2q)$ then

$$G_n := \frac{V_n}{\sigma_n \sqrt{n \log n}} \xrightarrow{\text{Law}} \mathcal{N}(0, 1). \quad (4.20)$$

Here, σ_n denotes the positive normalizing constant which ensures that $E[G_n^2] = 1$. The case $H > 1 - \frac{1}{2q}$ will be considered in Section 5. Proofs of (4.19) and (4.20), together with rates of convergence, can be found in [15] and [3], respectively.

We want to see if one can associate almost sure central limit theorems to the convergences (4.19) and (4.20). To do so, we need a few lemmas.

Lemma 4.2 *Let $q \geq 2$. As $n \rightarrow \infty$,*

1. *if $H < 1 - \frac{1}{2q}$, then $\sigma_n \rightarrow q! \sum_{r \in \mathbb{Z}} \rho(r)^q > 0$;*
2. *if $H = 1 - \frac{1}{2q}$, then $\sigma_n \rightarrow 2q! \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q > 0$.*

Proof. We have $E[(B_{k+1}^H - B_k^H)(B_{l+1}^H - B_l^H)] = \rho(k-l)$ where ρ is given in (1.2). Recall that ρ is an even function and that

$$\rho(r) = H(2H-1)r^{2H-2} + o(r^{2H-2}), \quad \text{as } |r| \rightarrow \infty.$$

We deduce that $\sum_{r \in \mathbb{Z}} |\rho(r)|^q < \infty$ if and only if $H < 1 - 1/(2q)$. On the other hand,

$$\begin{aligned} E[V_n^2] &= \sum_{k,l=0}^{n-1} E(H_q(B_{k+1}^H - B_k^H)H_q(B_{l+1}^H - B_l^H)) = q! \sum_{k,l=0}^{n-1} \rho(k-l)^q \\ &= q! \sum_{l=0}^{n-1} \sum_{r=-l}^{n-1-l} \rho(r)^q = q! \sum_{|r|<n} (n-1-|r|)\rho(r)^q \\ &= q! \left(n \sum_{|r|<n} \rho(r)^q - \sum_{|r|<n} (|r|+1)\rho(r)^q \right). \end{aligned}$$

Assume first that $H < 1 - 1/(2q)$. In this case,

$$\sigma_n^2 = q! \sum_{r \in \mathbb{Z}} \rho(r)^q \left(1 - \frac{|r|+1}{n}\right) \mathbf{1}_{\{|r|<n\}}.$$

In addition, we also have $\sum_{r \in \mathbb{Z}} |\rho(r)|^q < \infty$. Hence, we deduce by bounded Lebesgue convergence that

$$\sigma_n^2 \rightarrow q! \sum_{r \in \mathbb{Z}} \rho(r)^q, \quad \text{as } n \rightarrow \infty.$$

Assume now that $H = 1 - \frac{1}{2q}$. In that case, as $|r| \rightarrow \infty$,

$$\rho(r)^q \sim H^q (2H-1)^q |r|^{(2H-2)q} = \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q \frac{1}{|r|}.$$

Therefore, as $n \rightarrow \infty$,

$$\sum_{|r|<n} \rho(r)^q \sim \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q \sum_{0<|r|<n} \frac{1}{|r|} \sim 2 \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q \log n$$

and

$$\sum_{|r|<n} (|r|+1)\rho(r)^q \sim \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q \sum_{|r|<n} 1 \sim 2n \left(1 - \frac{1}{2q}\right)^q \left(1 - \frac{1}{q}\right)^q.$$

The desired conclusion follows.

□

The next lemma follows from Nourdin and Peccati [15] and Breton and Nourdin [3]. It will play a crucial role in the proof of Theorem 4.4.

Lemma 4.3 *Fix $q \geq 2$, and let \mathfrak{H} be the real and separable Hilbert space defined as follows: (i) denote by \mathcal{E} the set of all \mathbb{R} -valued step functions on $[0, \infty)$, (ii) define \mathfrak{H} as the Hilbert space obtained by closing \mathcal{E} with respect to the scalar product*

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = E[B_t^H B_s^H].$$

For any $n \geq 2$, let f_n be the element of $\mathfrak{H}^{\otimes q}$ defined by

$$f_n = \begin{cases} \frac{1}{\sigma_n \sqrt{n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k, k+1]}^{\otimes q} & \text{if } H < 1 - \frac{1}{2q} \\ \frac{1}{\sigma_n \sqrt{n \log n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k, k+1]}^{\otimes q} & \text{if } H = 1 - \frac{1}{2q} \end{cases} \quad (4.21)$$

with σ_n the positive normalizing constant which ensures that $G_n = I_q(f_n)$ has variance one. Then there exists a constant $C > 0$, depending only on q and H (but not on n), such that, for all $n \geq 1$ and $r = 1, \dots, q-1$:

$$\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes (2q-2r)}} \leq C \times \begin{cases} n^{-1/2} & \text{if } H \leq \frac{1}{2} \\ n^{H-1} & \text{if } \frac{1}{2} \leq H \leq \frac{2q-3}{2q-2} \\ n^{qH-q+1/2} & \text{if } \frac{2q-3}{2q-2} \leq H < 1 - \frac{1}{2q} \\ (\log n)^{-1/2} & \text{if } H = 1 - \frac{1}{2q} \end{cases}.$$

We can now state and prove the main result of this section.

Theorem 4.4 *Let $q \geq 2$ and $H \leq 1 - 1/(2q)$. For $n \geq 1$, set*

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H),$$

and define

$$G_n = \begin{cases} V_n / (\sigma_n \sqrt{n}) & \text{if } H < 1 - \frac{1}{2q} \\ V_n / (\sigma_n \sqrt{n \log n}) & \text{if } H = 1 - \frac{1}{2q} \end{cases}.$$

Here, σ_n denotes the positive normalizing constant which ensures that $E[G_n^2] = 1$. Then $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)] \quad \text{as } n \rightarrow \infty.$$

Proof. We shall apply Theorem 3.2, and let C be a positive constant, depending only on q and H , whose value changes from line to line. We consider the real and separable Hilbert space \mathfrak{H} as defined in Lemma 4.3.

First, we focus on the case $H < 1 - 1/(2q)$. We have $G_n = I_q(f_n)$ with f_n given by (4.21). Let us verify assumptions (A_1) and (A_2) in Theorem 3.2. According to Lemma 4.3, there exists $\alpha > 0$, depending only on q and H (but not on k and r), such that, for all $k \geq 1$ and $r = 1, \dots, q-1$, $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leq Ck^{-\alpha}$. Hence

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leq C \sum_{k \geq 1} \frac{1}{k^{1+\alpha}} \times \sum_{n \geq 2} \frac{1}{n \log^2 n} < \infty,$$

that is assumption (A_1) is verified. On the other hand, we have

$$\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}} = \frac{1}{\sigma_k \sigma_l \sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho(j-i)^q$$

with ρ given by (1.2). Since $\sigma_k \rightarrow \sigma_\infty > 0$ as $k \rightarrow \infty$ (see Lemma 4.2), we have, for $l \geq k \geq 1$,

$$\begin{aligned} |\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}| &\leq \frac{C}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} |\rho(j-i)|^q = \frac{C}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{r=-i}^{l-1-i} |\rho(r)|^q \\ &\leq C \sqrt{\frac{k}{l}} \sum_{r \in \mathbb{Z}} |\rho(r)|^q \leq C \sqrt{\frac{k}{l}}, \end{aligned}$$

where the last inequality follows from the fact that $\sum_{r \in \mathbb{Z}} |\rho(r)|^q < \infty$. Consequently, assumption (A_2) is verified as well, because

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} &\leq 2 \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \sum_{k=1}^l \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{3/2}} \sum_{k=1}^l \frac{1}{\sqrt{k}} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \leq C \sum_{n \geq 2} \frac{1}{n \log^2 n} < \infty. \end{aligned}$$

It remains now to consider the critical case $H = 1 - 1/(2q)$. We have $G_n = I_q(f_n)$ with $f_n = \frac{1}{\sigma_n \sqrt{n \log n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k, k+1]}^{\otimes q} \in \mathfrak{H}^{\otimes q}$. According to Lemma 4.3, we have, for all $k \geq 1$ and $r = 1, \dots, q-1$, $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leq C/\sqrt{\log k}$. Hence

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} &\leq C \sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k \sqrt{\log k}} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^{3/2} n} < \infty, \end{aligned}$$

that is assumption (A_1) is verified. Concerning (A_2) , note that

$$\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}} = \frac{1}{\sigma_k \sigma_l \sqrt{k \log k} \sqrt{l \log l}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho(j-i)^q.$$

Since $\sigma_k \rightarrow \sigma_\infty > 0$ as $k \rightarrow \infty$ (see Lemma 4.2), we have, for $l \geq k \geq 1$,

$$\begin{aligned} |\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}| &\leq \frac{C}{\sqrt{k \log k} \sqrt{l \log l}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} |\rho(j-i)|^q \\ &= \frac{C}{\sqrt{k \log k} \sqrt{l \log l}} \sum_{i=0}^{k-1} \sum_{r=-i}^{l-1-i} |\rho(r)|^q \\ &\leq C \frac{\sqrt{k}}{\sqrt{\log k} \sqrt{l \log l}} \sum_{r=-l}^l |\rho(r)|^q \leq C \sqrt{\frac{k \log l}{l \log k}}. \end{aligned}$$

The last inequality follows from the fact that $\sum_{r=-l}^l |\rho(r)|^q \leq C \log l$ since, as $|r| \rightarrow \infty$,

$$\rho(r) \sim \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{2q}\right) |r|^{-1/q}.$$

Consequently, assumption (A_2) is verified because

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=2}^n \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} &\leq 2 \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=2}^n \sum_{k=2}^l \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=2}^n \frac{\sqrt{\log l}}{l^{3/2}} \sum_{k=2}^l \frac{1}{\sqrt{k \log k}} \\ &\leq C \sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{l=2}^n \frac{1}{l} \leq C \sum_{n \geq 2} \frac{1}{n \log^2 n} < \infty. \end{aligned}$$

□

5 Partial sums of Hermite polynomials of increments of fBm: the non-Gaussian case

Fix $q \geq 2$. In the previous section, we saw that the limit distribution of $V_n = \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H)$, adequately normalized, is Gaussian when $H \leq 1 - 1/(2q)$. We consider here the case $H > 1 - 1/(2q)$. In contrast to (4.19)-(4.20), we have

$$G_n := n^{q(1-H)-1} V_n \xrightarrow{\text{Law}} G_\infty. \quad (5.22)$$

The law of G_∞ is called the ‘‘Hermite distribution’’. A short proof of (5.22) is given in Proposition 5.1 below. It is based on the fact that, for *fixed* n , Z_n defined in (5.23) below and G_n share the same law, because of the self-similarity property of fractional Brownian motion.

Proposition 5.1 *Fix $q \geq 2$ and $H > 1 - 1/(2q)$, and define Z_n by*

$$Z_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q(n^H(B_{(k+1)/n}^H - B_{k/n}^H)), \quad n \geq 1. \quad (5.23)$$

Then, as $n \rightarrow \infty$, $\{Z_n\}$ converges almost surely and in $L^2(\Omega)$ to a limit denoted by Z_∞ .

Proof. Let us first prove the convergence in $L^2(\Omega)$. For $n, m \geq 1$, we have

$$E[Z_n Z_m] = q!(nm)^{q-1} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} (E[(B_{(k+1)/n}^H - B_{k/n}^H)(B_{(l+1)/m}^H - B_{l/m}^H)])^q.$$

On the other hand, since $H > 1/2$, we have, for all $s, t \geq 0$,

$$E[B_s^H B_t^H] = H(2H - 1) \int_0^t du \int_0^s dv |u - v|^{2H-2}.$$

Hence

$$E[Z_n Z_m] = q! H^q (2H - 1)^q \times \frac{1}{nm} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left(nm \int_{k/n}^{(k+1)/n} du \int_{l/m}^{(l+1)/m} dv |v - u|^{2H-2} \right)^q.$$

Therefore, as $n, m \rightarrow \infty$, we have,

$$E[Z_n Z_m] \rightarrow q! H^q (2H - 1)^q \int_{[0,1]^2} |u - v|^{(2H-2)q} dudv,$$

and the limit is finite since $H > 1 - 1/(2q)$. In other words, the sequence $\{Z_n\}$ is Cauchy in $L^2(\Omega)$, and hence converges in $L^2(\Omega)$ to some Z_∞ .

Let us now prove that $\{Z_n\}$ converges also almost surely. Observe first that, since Z_n belongs to the q th chaos of B^H for all n and because $\{Z_n\}$ converges in $L^2(\Omega)$ to Z_∞ , we have that Z_∞ also belongs to the q th chaos of B^H . In [3, Proposition 3.1], it is shown that $E[|Z_n - Z_\infty|^2] \leq C n^{2q-1-2qH}$, for some positive constant C not depending on n . Inside a fixed chaos, all the $L^p(\Omega)$ -norms are equivalent (see e.g. [12, Theorem 5.10]). Hence, for any $p > 2$, we have $E[|Z_n - Z_\infty|^p] \leq C n^{p(q-1/2-qH)}$. Since $H > 1 - 1/(2q)$, there exists $p > 2$ large enough such that $(q - 1/2 - qH)p < -1$. Consequently

$$\sum_{n \geq 1} E[|Z_n - Z_\infty|^p] < \infty$$

leading, for all $\varepsilon > 0$, to

$$\sum_{n \geq 1} P[|Z_n - Z_\infty| > \varepsilon] < \infty.$$

Therefore, we deduce from the Borel-Cantelli lemma that $\{Z_n\}$ converges almost surely to Z_∞ .

□

We now face some difficulties. First, since the limit of G_n in (5.22) is not Gaussian, we cannot apply our general criterion Theorem 3.2 to obtain an ASCLT. To modify adequately the criterion, we would need a version of Proposition 2.1 for random variables with a Hermite distribution, a result which is not presently available. Thus an ASCLT associated to the convergence in law (5.22) falls outside the scope of this paper. We can nevertheless make a number of observations. First, changing the nature of the random variables without changing their law has no impact on CLTs as in (5.22), but may have a great impact on an ASCLT. To see this, observe that for each fixed n , the ASCLT involves not only the distribution of G_n , but also that of (G_1, \dots, G_n) . Consider, moreover, the following example. Let $\{G_n\}$ be a sequence of random variables converging in law to a limit G_∞ . According to a theorem of Skorohod, there is a sequence $\{G_n^*\}$ such that for any fixed n , $G_n^* \stackrel{\text{Law}}{=} G_n$ and such that $\{G_n^*\}$ converges almost surely as $n \rightarrow \infty$ to a random variable G_∞^* with $G_\infty^* \stackrel{\text{Law}}{=} G_\infty$. Then, for any bounded continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we have $\varphi(G_n^*) \xrightarrow{\text{a.s.}} \varphi(G_\infty^*)$ and by a classical theorem of Hardy (see [5] p.35), as $n \rightarrow \infty$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k^*) \xrightarrow{\text{a.s.}} \varphi(G_\infty^*).$$

This limit is different from $E[\varphi(G_\infty^*)]$ (or equivalently $E[\varphi(G_\infty)]$), that is, different from the limit if one had an ASCLT. Thus, knowing the law of G_n , for a fixed n , does not allow to determine whether an ASCLT holds or not.

Remark 5.2 In view of Proposition 5.1, the Skorohod version of

$$G_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H)$$

is

$$G_n^* = Z_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q(n^H (B_{(k+1)/n}^H - B_{k/n}^H)),$$

since $G_n^* \stackrel{\text{Law}}{=} G_n$ and G_n^* converges almost surely.

Hence, in the case of Hermite distributions, by suitably modifying the argument of the Hermite polynomial H_q in a way which does not change the limit in law, namely by considering Z_n in (5.23) instead of G_n in (5.22), we obtain the almost sure convergence

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(Z_k) \rightarrow \varphi(Z_\infty).$$

Note that the limit is different from the limit expected under an ASCLT, namely $E[\varphi(Z_\infty)]$. Recall indeed that Z_∞ is a non-constant random variable with a Hermite distribution (Dobrushin and Major [6], Taqqu [20]) and therefore one has $E[\varphi(Z_\infty)] \neq \varphi(Z_\infty)$ in general.

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