



HAL
open science

The index of centralizers of elements of reductive Lie algebras

Jean-Yves Charbonnel, Anne Moreau

► **To cite this version:**

Jean-Yves Charbonnel, Anne Moreau. The index of centralizers of elements of reductive Lie algebras. 2009. hal-00375022v1

HAL Id: hal-00375022

<https://hal.science/hal-00375022v1>

Preprint submitted on 11 Apr 2009 (v1), last revised 16 Oct 2015 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE INDEX OF CENTRALIZERS OF ELEMENTS OF REDUCTIVE LIE ALGEBRAS

JEAN-YVES CHARBONNEL AND ANNE MOREAU

ABSTRACT. For a finite dimensional complex Lie algebra, its index is the minimal dimension of stabilizers for the coadjoint action. A famous conjecture due to A.G. Elashvili says that the index of the centralizer of an element of a reductive Lie algebra is equal to the rank. That conjecture caught attention of several Lie theorists for years. It reduces to the case of nilpotent elements. In [Pa03a] and [Pa03b], D.I. Panyushev proved the conjecture for some classes of nilpotent elements (e.g. regular, subregular and spherical nilpotent elements). Then the conjecture has been proven for the classical Lie algebras in [Y06a] and checked with a computer programme for the exceptional ones (cf. [De08]). In this paper we give an almost general proof of that conjecture.

1. INTRODUCTION

In this note \mathbb{k} is an algebraically closed field of characteristic 0.

1.1. Let \mathfrak{g} be a finite dimensional Lie algebra over \mathbb{k} and consider the coadjoint representation of \mathfrak{g} . By definition, the *index* of \mathfrak{g} is the minimal dimension of stabilizers \mathfrak{g}^x , $x \in \mathfrak{g}^*$, for the coadjoint representation:

$$\text{ind } \mathfrak{g} := \min\{\dim \mathfrak{g}^x; x \in \mathfrak{g}^*\} .$$

The definition of the index goes back to Dixmier [Di74]. It is a very important notion in representation theory and in invariant theory. By Rosenlicht's theorem [Ro63], generic orbits of an arbitrary action of a linear algebraic group on an irreducible algebraic variety are separated by rational invariants; in particular, if \mathfrak{g} is an algebraic Lie algebra,

$$\text{ind } \mathfrak{g} = \deg \text{tr } \mathbb{k}(\mathfrak{g}^*)^{\mathfrak{g}},$$

where $\mathbb{k}(\mathfrak{g}^*)^{\mathfrak{g}}$ is the field of \mathfrak{g} -invariant rational functions on \mathfrak{g}^* . The index of a reductive algebra equals its rank. For an arbitrary Lie algebra, computing its index seems to be a wild problem. However, there is a large number of interesting results for several classes of nonreductive subalgebras of reductive Lie algebras. For instance, parabolic subalgebras and their relatives as nilpotent radicals, seaweeds, are considered in [Pa03a], [TY04], [J07]. The centralizers, or normalizers of centralizers, of elements form another interesting class of such subalgebras, [E85a], [Pa03a], [Mo06b]. The last topic is closely related to the theory of integrable Hamiltonian systems [Bol91]. Let us precise this link.

Date: April 11, 2009.

1991 *Mathematics Subject Classification.* 22E46, 17B80, 17B20, 14L24.

Key words and phrases. reductive Lie algebra; index; centralizer; argument shift method; Poisson-commutative family of polynomials; rigid nilpotent orbit; Slodowy slice.

From now on, \mathfrak{g} is supposed to be reductive. Denote by G the adjoint group of \mathfrak{g} . The symmetric algebra $S(\mathfrak{g})$ carries a natural Poisson structure. By the so-called *argument shift method*, for x in \mathfrak{g}^* , we can construct a Poisson-commutative family \mathcal{F}_x in $S(\mathfrak{g}) = \mathbb{k}[\mathfrak{g}^*]$; see [MF78] or Remark 1.3. It is generated by the derivatives of all orders in the direction $x \in \mathfrak{g}^*$ of all elements of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ of \mathfrak{g} -invariants of $S(\mathfrak{g})$. Moreover, if $G.x$ denotes the coadjoint orbit of $x \in \mathfrak{g}^*$:

Theorem 1.1 ([Bol91], Theorems 2.1 and 3.2). *There is a Poisson-commutative family of polynomial functions on \mathfrak{g}^* , constructed by the argument shift method, such that its restriction to $G.x$ contains $\frac{1}{2}\dim(G.x)$ algebraically independent functions if and only if $\text{ind } \mathfrak{g}^x = \text{rk } \mathfrak{g}$.*

Motivated by the preceding result of Bolsinov, A.G. Elashvili formulated the conjecture: $\text{ind } \mathfrak{g}^x = \text{rk } \mathfrak{g}$ for all $x \in \mathfrak{g}^*$, where $\text{rk } \mathfrak{g}$ is the rank of \mathfrak{g} . Elashvili's conjecture also appears in the following problem: Is the algebra $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ of invariants in $S(\mathfrak{g}^x)$ under the adjoint action a polynomial algebra? This question was formulated by A. Premet in [PPY07, Conjecture 0.1]. After that, O. Yakomiva discovered a counterexample [Y07], but the question remains very interesting. As an example, under certain hypothesis, and under the condition that Elashvili's conjecture holds, the algebra of invariants $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in $\text{rk } \mathfrak{g}$ variables, [PPY07, Theorem 0.3].

In this paper, we give a proof of Elashvili's conjecture. Hence we claim:

Theorem 1.2. *Let \mathfrak{g} be a reductive Lie algebra. Then $\text{ind } \mathfrak{g}^x = \text{rk } \mathfrak{g}$ for all $x \in \mathfrak{g}^*$.*

During the last decade, Elashvili's conjecture caught attention of many invariant theorists [Pa03a], [Ch04], [Y06a], [De08]. To begin with, describe some easy but useful reductions. Since the \mathfrak{g} -modules \mathfrak{g} and \mathfrak{g}^* are isomorphic, it is equivalent to prove Theorem 1.2 for centralizers of elements of \mathfrak{g} . On the other hand, by a result due to E.B. Vinberg, pointed out in [Pa03a], the inequality $\text{ind } \mathfrak{g}^x \geq \text{rk } \mathfrak{g}$ holds for all $x \in \mathfrak{g}$. So only remains to prove the opposite one. Given $x \in \mathfrak{g}$, let $x = x_s + x_n$ be its Jordan decomposition. Then $\mathfrak{g}^x = (\mathfrak{g}^{x_s})^{x_n}$. The subalgebra \mathfrak{g}^{x_s} is reductive of rank $\text{rk } \mathfrak{g}$. Thus, the verification of the Elashvili's conjecture reduces to the case of nilpotent elements. At last, one can clearly restrict itself to the case of simple \mathfrak{g} .

Review now the main results obtained so far on Elashvili's conjecture. If x is regular, then \mathfrak{g}^x is a commutative Lie algebra of dimension $\text{rk } \mathfrak{g}$. So, Elashvili's conjecture is obviously true in that case. Further, the conjecture is known for subregular nilpotent elements and nilpotent elements of height 2 and 3, [Pa03a], [Pa03b]. Remind that the *height* of a nilpotent element e is the maximal integer m such that $(\text{ad } e)^m \neq 0$. More recently, O. Yakimova proved the conjecture in the classical case [Y06a]. To valid the conjecture in the exceptional type, W.A. DeGraaf used the computer programme **GAP** (cf. [De08]). Since there are many nilpotent orbits in the Lie algebras of exceptional type, it is difficult to present the results of such computations in a concise way. In 2004, the first author published a case-free proof of Elashvili's conjecture applicable to all simple Lie algebras; see [Ch04]. Unfortunately, the argument in [Ch04] has a gap in the final part of the proof, which was pointed out by L. Rybnikov.

Because of the importance of Elashvili's conjecture in invariant theory, it would be very appreciated to find a conceptual proof of Elashvili's conjecture applicable to all finite-dimensional

simple Lie algebras. The proof we propose in this paper is fresh and almost general. More precisely, it remains 7 isolated cases; one nilpotent orbit in type E_7 and six nilpotent orbits in type E_8 have to be considered separately. For these 7 orbits, the use of GAP is unfortunately necessary.

1.2. Description of the paper. Let us briefly explain our approach. Denote by $\mathcal{N}(\mathfrak{g})$ the nilpotent cone of \mathfrak{g} . As noticed previously, it suffices to prove $\text{ind } \mathfrak{g}^e = \text{rk } \mathfrak{g}$ for all e in $\mathcal{N}(\mathfrak{g})$. If the equality holds for e , it does for all elements of $G.e$; we shortly say that $G.e$ satisfies the Elashvili's conjecture.

From a nilpotent orbit $\mathcal{O}_\mathfrak{l}$ of a reductive factor \mathfrak{l} of a parabolic subalgebra of \mathfrak{g} , we can construct a nilpotent orbit of \mathfrak{g} having the same codimension in \mathfrak{g} as $\mathcal{O}_\mathfrak{l}$ in \mathfrak{l} and having other remarkable properties. The nilpotent orbits obtained in such a way are called *induced*; the other ones are called *rigid*. *Richardson* orbits are induced nilpotent orbits from a zero orbit. We refer the reader to Subsection 2.3 for more precisions about this topic. Using Bolsinov's criterion of Theorem 1.1, we first prove Theorem 1.2 for all Richardson nilpotent orbits and then, by an induction process, we show that the conjecture reduces to the case of rigid nilpotent orbits. To deal with rigid nilpotent orbits, we use methods developed in [Ch04] by the first author, and resumed in [Mo06a] by the second author, based on nice properties of Slodowy slices of nilpotent orbits.

In more details, the paper is organized as follows:

We state in Section 2 the necessary preliminary results. In particular, we investigate in Subsection 2.2 extensions of Bolsinov's criterion and we establish an important result (Theorem 2.7) which will be used repeatedly in the sequel. We prove in Section 3 the conjecture for all Richardson nilpotent orbits. Starting with Section 4, we develop a reduction method: we introduce in Section 4 a property (P_1) given by Definition 4.3 which turns out to be equivalent to Elashvili's conjecture. By using this equivalence and results of Section 2, we show in Section 4 that Elashvili's conjecture reduces to the case of rigid nilpotent orbits. From Section 5, we handle the rigid nilpotent orbits: we introduce and study in Section 5 a property (P_2) given by Definition 5.2. Then, in Section 6, we are able to deal with almost all rigid nilpotent orbits. The remaining cases are dealt with set-apart by using a different approach, still in Section 6.

1.3. Notations. • If E is a subset of a vector space V , we denote by $\text{span}(E)$ the vector subspace of V generated by E . The grassmanian of all d -dimensional subspaces of V is denoted by $\text{Gr}_d(V)$. By *cone* of V , we mean a subset of V invariant under the natural action of $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$ and by a *bicone* of $V \times V$ we mean a subset of $V \times V$ invariant under the natural action of $\mathbb{k}^* \times \mathbb{k}^*$ on $V \times V$.

- From now on, we assume that \mathfrak{g} is semisimple of rank ℓ and we denote by $\langle \cdot, \cdot \rangle$ the Killing form of \mathfrak{g} . We identify \mathfrak{g} to \mathfrak{g}^* through $\langle \cdot, \cdot \rangle$. Unless otherwise specified, the notion of orthogonality refers to the bilinear form $\langle \cdot, \cdot \rangle$.

- Denote by $S(\mathfrak{g})^\mathfrak{g}$ the algebra of \mathfrak{g} -invariant elements of $S(\mathfrak{g})$. Let f_1, \dots, f_ℓ be homogeneous generators of $S(\mathfrak{g})^\mathfrak{g}$ of degree d_1, \dots, d_ℓ respectively. We choose the polynomials f_1, \dots, f_ℓ so that $d_1 \leq \dots \leq d_\ell$. For $i = 1, \dots, d_i$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, we may consider a shift of f_i in direction y :

$f_i(x + ty)$ where $t \in \mathbb{k}$. Expanding $f_i(x + ty)$ as a polynomial in t , we obtain

$$(1) \quad f_i(x + ty) = \sum_{m=0}^{d_i} f_i^{(m)}(x, y)t^m; \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

where $y \mapsto (m!)f_i^{(m)}(x, y)$ is the differential at x of f_i at the order m in the direction y . The elements $f_i^{(m)}$ as defined by (1) are invariant elements of $S(\mathfrak{g}) \otimes_{\mathbb{k}} S(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Note that $f_i^{(0)}(x, y) = f_i(x)$ while $f_i^{(d_i)}(x, y) = f_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.3. The family $\mathcal{F}_x := \{f_i^{(m)}(x, \cdot); 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$, for $x \in \mathfrak{g}$, is a Poisson-commutative family of $S(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family \mathcal{F}_x is constructed by the *argument shift method*.

- Let $i \in \{1, \dots, \ell\}$. For x in \mathfrak{g} , we denote by $\varphi_i(x)$ the element of \mathfrak{g} satisfying $(df_i)_x(y) = f_i^{(1)}(x, y) = \langle \varphi_i(x), y \rangle$, for all y in \mathfrak{g} . Thereby, φ_i is an invariant element of $S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ under the canonical action of G . We denote by $\varphi_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $S(\mathfrak{g}) \otimes_{\mathbb{k}} S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ defined by the equality:

$$(2) \quad \varphi_i(x + ty) = \sum_{m=0}^{d_i-1} \varphi_i^{(m)}(x, y)t^m, \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}.$$

- For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [y, x] = 0\}$ the centralizer of x in \mathfrak{g} and by $\mathfrak{z}(\mathfrak{g}^x)$ the center of \mathfrak{g}^x . The set of regular elements of \mathfrak{g} is

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and we denote by $\mathfrak{g}_{\text{reg,ss}}$ the set of regular semisimple elements of \mathfrak{g} . Both $\mathfrak{g}_{\text{reg}}$ and $\mathfrak{g}_{\text{reg,ss}}$ are G -invariant open dense subsets of \mathfrak{g} .

- The nilpotent cone of \mathfrak{g} is $\mathcal{N}(\mathfrak{g})$. As a rule, for $e \in \mathcal{N}(\mathfrak{g})$, we choose an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} given by the Jacobson-Morozov theorem [CMA93, Theorem 3.3.1]. In particular, it satisfies the equalities:

$$[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f.$$

The action of $\text{ad } h$ on \mathfrak{g} induces a \mathbb{Z} -grading:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

Recall that e , or $G.e$, is said to be *even* if $\mathfrak{g}(i) = 0$ for odd i . Note that $e \in \mathfrak{g}(2)$, $f \in \mathfrak{g}(-2)$ and that \mathfrak{g}^e , $\mathfrak{z}(\mathfrak{g}^e)$ and \mathfrak{g}^f are all $\text{ad } h$ -stable.

- All topological terms refer to the Zariski topology. If Y is a subset of a topological space X , we denote by \overline{Y} the closure of Y in X .

1.4. Acknowledgments. We would like to thank O. Yakimova for her interest and useful discussions; and more particularly for bringing Bolsinov's paper to our attention. We also thank A.G. Elashvili for suggesting Lawther-Testerman's paper [LT08] about the centers of centralizers of nilpotent elements.

CONTENTS

1. Introduction	1
2. Preliminary results	5
3. Proof of Theorem 1.2 for Richardson nilpotent orbits	10
4. The property (P ₁) and reduction to rigid nilpotent orbits	12
5. The Slodowy slice and the property (P ₂)	17
6. Proof of Theorem 1.2 for rigid nilpotent orbits	23
Appendix A. Proof of Proposition 6.9: explicit computations.	27
References	31

2. PRELIMINARY RESULTS

We start in this section by reviewing some facts about the differentials of generators of $S(\mathfrak{g})^{\mathfrak{g}}$. Then, the goal of Subsection 2.2 is Theorem 2.7. We collect in Subsection 2.3 basics facts about induced nilpotent orbits.

2.1. Differentials of generators of $S(\mathfrak{g})^{\mathfrak{g}}$. Recall that the elements $\varphi_1, \dots, \varphi_\ell$ of $S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ are identified, through the Killing form $\langle \cdot, \cdot \rangle$, to the differentials of generators f_1, \dots, f_ℓ of $S(\mathfrak{g})^{\mathfrak{g}}$, see Subsection 1.3. Since $f_i(g(x)) = f_i(x)$ for all $(x, g) \in \mathfrak{g} \times G$, the element $\varphi_i(x)$ centralizes x for all $x \in \mathfrak{g}$. Moreover:

Lemma 2.1. (i)[Ri87, Lemma 2.1] *The elements $\varphi_1(x), \dots, \varphi_\ell(x)$ belong to $\mathfrak{z}(\mathfrak{g}^e)$.*

(ii)[Ko63, Theorem 9] *The elements $\varphi_1(x), \dots, \varphi_\ell(x)$ are linearly independent elements of \mathfrak{g} if and only if x is regular. Moreover, if so, $\varphi_1(x), \dots, \varphi_\ell(x)$ is a basis of \mathfrak{g}^x .*

We turn now to the elements $\varphi_i^{(m)}$, for $i = 1, \dots, \ell$ and $0 \leq m \leq d_i - 1$, defined in Subsection 1.3 by (2). Recall that d_i is the degree of the homogeneous polynomial f_i , for $i = 1, \dots, \ell$. The integers $d_1 - 1, \dots, d_\ell - 1$ are thus the exponents of \mathfrak{g} . By a classical result [Bou02, Ch. V, §5, Proposition 3], we have $\sum d_i = b_{\mathfrak{g}}$ where $b_{\mathfrak{g}}$ is the dimension of Borel subalgebras of \mathfrak{g} . For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, we set:

$$(3) \quad V_{x,y} := \text{span}\{\varphi_i^{(m)}(x, y) ; 1 \leq i \leq \ell, 0 \leq m \leq d_i - 1\}.$$

The subspaces $V_{x,y}$ will play a central role throughout the note.

Remark 2.2. (1) For $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, the dimension of $V_{x,y}$ is at most $b_{\mathfrak{g}}$ since $\sum d_i = b_{\mathfrak{g}}$. Moreover, for all (x, y) in a nonempty open subset of $\mathfrak{g} \times \mathfrak{g}$, the equality holds [Bol91]. Actually, in this note, we do not need this observation.

(2) By Lemma 2.1(ii), if x is regular, then \mathfrak{g}^x is contained in $V_{x,y}$ for all $y \in \mathfrak{g}$. In particular, if so, $\dim[x, V_{x,y}] = \dim V_{x,y} - \ell$.

The subspaces $V_{x,y}$ were introduced and studied by Bolsinov in [Bol91], motivated by the maximality of Poisson-commutative families in $S(\mathfrak{g})$. These subspaces have been recently exploited

in [PY08] and [CMo08]. The following results are mostly due to Bosinov, [Bol91]. We refer to [PY08] for a more recent account about this topic.

Lemma 2.3. *Let (x, y) be in $\mathfrak{g}_{\text{reg}} \times \mathfrak{g}$.*

(i)[PY08, Lemma A1] *The subspace $V_{x,y}$ of \mathfrak{g} is the sum of the subspaces \mathfrak{g}^{x+ty} where t runs through any nonempty open subset of \mathbb{k} .*

(ii)[PY08, Lemma A4] *The subspace $\mathfrak{g}^y + V_{x,y}$ is a totally isotropic subspace of \mathfrak{g} with respect to the Kirillov form K_y on $\mathfrak{g} \times \mathfrak{g}$, $(v, w) \mapsto \langle y, [v, w] \rangle$. Furthermore, $\dim(\mathfrak{g}^y + V_{x,y})^\perp \geq \frac{1}{2} \dim G.y$.*

(iii)[PY08, Lemma A2] *The subspaces $[x, V_{x,y}]$ and $[y, V_{x,y}]$ are equal.*

Let σ and σ_i , for $i = 1, \dots, \ell$, be the maps

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\sigma} & \mathbb{k}^{\text{b}_{\mathfrak{g}} + \ell} \\ (x, y) & \longmapsto & (f_i^{(m)}(x, y))_{\substack{1 \leq i \leq \ell, \\ 0 \leq m \leq d_i}} \end{array}, \quad \begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\sigma_i} & \mathbb{k}^{d_i + 1} \\ (x, y) & \longmapsto & (f_i^{(m)}(x, y))_{0 \leq m \leq d_i} \end{array}$$

respectively, and denote by $\sigma'(x, y)$ and $\sigma'_i(x, y)$ the tangent map at (x, y) of σ and σ_i respectively. Then $\sigma'_i(x, y)$ is given by the differentials of the $f_i^{(m)}$'s at (x, y) and $\sigma'(x, y)$ is given by the elements $\sigma'_i(x, y)$.

Lemma 2.4. *Let (x, y) and (v, w) be in $\mathfrak{g} \times \mathfrak{g}$.*

(i) *For $i = 1, \dots, \ell$, $\sigma'_i(x, y)$ maps (v, w) to*

$$\begin{aligned} & (\langle \varphi_i(x), v \rangle, \langle \varphi_i^{(1)}(x, y), v \rangle + \langle \varphi_i^{(0)}(x, y), w \rangle, \\ & \dots, \langle \varphi_i^{(d_i-1)}(x, y), v \rangle + \langle \varphi_i^{(d_i-2)}(x, y), w \rangle, \langle \varphi_i(y), w \rangle). \end{aligned}$$

(ii) *Suppose that $\sigma'(x, y)(v, w) = 0$. Then, for w' in \mathfrak{g} , $\sigma'(x, y)(v, w') = 0$ if and only if $w - w'$ is orthogonal to $V_{x,y}$.*

(iii) *For $x \in \mathfrak{g}_{\text{reg}}$, $\sigma'(x, y)(v, w') = 0$ for some $w' \in \mathfrak{g}$ if and only if $v \in [x, \mathfrak{g}]$.*

Proof. (i) The verifications are easy and left to the reader.

(ii) Since $\sigma'(x, y)(v, w) = 0$, $\sigma'(x, y)(v, w') = 0$ if and only if $\sigma'(x, y)(v, w - w') = 0$ whence the statement by (i).

(iii) Suppose that x is regular and suppose that $\sigma'(x, y)(v, w') = 0$ for some $w' \in \mathfrak{g}$. Then by (i), v is orthogonal to the elements $\varphi_1(x), \dots, \varphi_\ell(x)$. So by Lemma 2.1(ii), v is orthogonal to \mathfrak{g}^x . Since \mathfrak{g}^x is the orthogonal complement of $[x, \mathfrak{g}]$ in \mathfrak{g} , we deduce that v lies in $[x, \mathfrak{g}]$. Conversely, since $\sigma(x, y) = \sigma(g(x), g(y))$ for all g in G , the element $([u, x], [u, y])$ belongs to the kernel of $\sigma'(x, y)$ for all $u \in \mathfrak{g}$. So, the converse implication follows. \square

2.2. On Bolsinov's criterion. Let a be in \mathfrak{g} and denote by π the map

$$\begin{array}{ccc} \mathfrak{g} \times G.a & \xrightarrow{\pi} & \mathfrak{g} \times \mathbb{k}^{\text{b}_{\mathfrak{g}} + \ell} \\ (x, y) & \longmapsto & (x, \sigma(x, y)). \end{array}$$

Remark 2.5. Recall that the family $(\mathcal{F}_x)_{x \in \mathfrak{g}}$ constructed by the argument shift method consists of all elements $f_i^{(m)}(x, \cdot)$ for $i = 1, \dots, \ell$, $1 \leq m \leq d_i$, see Remark 1.3. By definition of the morphism π , there is a family constructed by the argument shift method whose restriction to $G.a$

contains $\frac{1}{2}\dim G.a$ algebraically independent functions if and only if π has a fiber of dimension $\frac{1}{2}\dim G.a$.

In view of Theorem 1.1 and the above remark, we now concentrate on the fibers of π . For $(x, y) \in \mathfrak{g} \times G.a$, denote by $F_{x,y}$ the fiber of π at $\pi(x, y)$:

$$F_{x,y} := \{x\} \times \{y' \in G.a \mid \sigma(x, y') = \sigma(x, y)\}.$$

Lemma 2.6. *Let (x, y) be in $\mathfrak{g} \times G.a$.*

(i) *The irreducible components of $F_{x,y}$ have dimension at least $\frac{1}{2}\dim G.a$.*

(ii) *The fiber $F_{x,y}$ has dimension $\frac{1}{2}\dim G.a$ if and only if any irreducible component of $F_{x,y}$ contains an element (x, y') such that $(\mathfrak{g}^{y'} + V_{x,y'})^\perp$ has dimension $\frac{1}{2}\dim G.a$.*

Proof. We prove (i) and (ii) all together. The tangent space $T_{x,y'}(F_{x,y})$ of $F_{x,y}$ at (x, y') in $F_{x,y}$ identifies to the subspace of elements w of $[y', \mathfrak{g}]$ such that $\sigma'(x, y')(0, w) = 0$. Hence, by Lemma 2.4(ii),

$$T_{x,y'}(F_{x,y}) = [y', \mathfrak{g}] \cap V_{x,y'}^\perp = (\mathfrak{g}^{y'} + V_{x,y'})^\perp,$$

since $[y', \mathfrak{g}] = (\mathfrak{g}^{y'})^\perp$. But by Lemma 2.3(ii), $(\mathfrak{g}^{y'} + V_{x,y'})^\perp$ has dimension at least $\frac{1}{2}\dim G.a$; so does $\dim T_{x,y'}(F_{x,y})$. This proves (i). Moreover, the equality holds if and only if $(\mathfrak{g}^{y'} + V_{x,y'})^\perp$ has dimension $\frac{1}{2}\dim G.a$, whence the statement (ii). \square

Theorem 2.7. *The following conditions are equivalent:*

- (1) $\text{ind } \mathfrak{g}^a = \ell$;
- (2) π has a fiber of dimension $\frac{1}{2}\dim G.a$;
- (3) there exists $(x, y) \in \mathfrak{g} \times G.a$ such that $(\mathfrak{g}^y + V_{x,y})^\perp$ has dimension $\frac{1}{2}\dim G.a$;
- (4) there exists x in $\mathfrak{g}_{\text{reg}}$ such that $\dim(\mathfrak{g}^a + V_{x,a}) = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^a)$;
- (5) there exists x in $\mathfrak{g}_{\text{reg}}$ such that $\dim V_{x,a} = \frac{1}{2}\dim G.a + \ell$;
- (6) $\sigma(\mathfrak{g} \times \{a\})$ has dimension $\frac{1}{2}\dim G.a + \ell$.

Proof. By Theorem 1.1 and Remark 2.5, we have (1) \Leftrightarrow (2). Moreover, by Lemma 2.6(ii), we have (2) \Leftrightarrow (3).

(3) \Leftrightarrow (4): If (4) holds, so does (3). Indeed, if so,

$$\dim \mathfrak{g} - \frac{1}{2}\dim G.a = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^a) = \dim(\mathfrak{g}^a + V_{x,a}).$$

Conversely, suppose that (3) holds. By Lemma 2.3(ii), $\mathfrak{g}^y + V_{x,y}$ has maximal dimension $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^y)$. So the same goes for all (x, y) in a G -invariant nonempty open subset of $\mathfrak{g} \times G.a$. Hence, since the map $(x, y) \mapsto V_{x,y}$ is G -equivariant, there exists x in $\mathfrak{g}_{\text{reg}}$ such that

$$\dim(V_{x,a} + \mathfrak{g}^a) = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^a).$$

(4) \Leftrightarrow (5): Let x be in $\mathfrak{g}_{\text{reg}}$. By Lemma 2.3(iii), $[x, V_{x,a}] = [a, V_{x,a}]$. Hence $\mathfrak{g}^a \cap V_{x,a}$ has dimension ℓ by Remark 2.2(2). As a consequence,

$$\dim(\mathfrak{g}^a + V_{x,a}) = \dim \mathfrak{g}^a + \dim V_{x,a} - \ell,$$

whence the equivalence.

(2) \Leftrightarrow (6): Suppose that (2) holds. By Lemma 2.6, $\frac{1}{2}\dim G.a$ is the minimal dimension of the fibers of π . So, $\pi(\mathfrak{g} \times G.a)$ has dimension

$$\dim \mathfrak{g} + \dim G.a - \frac{1}{2}\dim G.a = \dim \mathfrak{g} + \frac{1}{2}\dim G.a.$$

Denote by τ the restriction to $\pi(\mathfrak{g} \times G.a)$ of the projection map $\mathfrak{g} \times \mathbb{k}^{\mathfrak{b}_{\mathfrak{g}}+\ell} \rightarrow \mathbb{k}^{\mathfrak{b}_{\mathfrak{g}}+\ell}$. Then $\tau \circ \pi$ is the restriction of σ to $\mathfrak{g} \times G.a$. Since σ is a G -invariant map, $\sigma(\mathfrak{g} \times \{a\}) = \sigma(\mathfrak{g} \times G.a)$. Let $(x, y) \in \mathfrak{g}_{\text{reg,ss}} \times G.a$. The fiber of τ at $z = \sigma(x, y)$ is $G.x$ since x is a regular semisimple element of \mathfrak{g} . Hence,

$$\dim \sigma(\mathfrak{g} \times \{a\}) = \dim \pi(\mathfrak{g} \times G.a) - (\dim \mathfrak{g} - \ell) = \frac{1}{2}\dim G.a + \ell$$

and we obtain (6).

Conversely, suppose that (6) holds. Then $\pi(\mathfrak{g} \times G.a)$ has dimension $\dim \mathfrak{g} + \frac{1}{2}\dim G.a$ by the above equality. So the minimal dimension of the fibers of π is equal to

$$\dim \mathfrak{g} + \dim G.a - (\dim \mathfrak{g} + \frac{1}{2}\dim G.a) = \frac{1}{2}\dim G.a$$

and (2) holds. \square

2.3. Induced, rigid and Richardson nilpotent orbits. The definitions and results of this subsection are mostly extracted from [Di74], [Di75], [LS79] and [BoK79]. We refer to [CMa93] and [TY05] for recent surveys.

Let \mathfrak{p} be a proper parabolic subalgebra of \mathfrak{g} and let \mathfrak{l} be a reductive factor of \mathfrak{p} . We denote by $\mathfrak{p}_{\mathfrak{u}}$ the nilpotent radical of \mathfrak{p} . Denote by L the connected closed subgroup of G whose Lie algebra is \mathfrak{l} and denote by P the normalizer of \mathfrak{p} in G .

Theorem 2.8 ([CMa93], Theorem 7.1.1). *Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit of \mathfrak{l} . There exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in \mathfrak{g} whose intersection with $\mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_{\mathfrak{u}}$ is a dense open subset of $\mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_{\mathfrak{u}}$. Moreover, the intersection of $\mathcal{O}_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{l}} + \mathfrak{p}_{\mathfrak{u}}$ consists of a single P -orbit and $\text{codim}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \text{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}})$.*

The orbit $\mathcal{O}_{\mathfrak{g}}$ only depends on \mathfrak{l} and not on the choice of a parabolic subalgebra \mathfrak{p} containing it [CMa93, Theorem 7.1.3]. By definition, the orbit $\mathcal{O}_{\mathfrak{g}}$ is called the *induced orbit from $\mathcal{O}_{\mathfrak{l}}$* ; it is denoted by $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$. If $\mathcal{O}_{\mathfrak{l}} = 0$, then we call $\mathcal{O}_{\mathfrak{g}}$ a *Richardson orbit*. For example all even nilpotent orbits are Richardson [CMa93, Corollary 7.1.7]. In turn, not all nilpotent orbits are induced from another one. A nilpotent orbit which is not induced in a proper way from another one is called *rigid*.

We shall say that $e \in \mathcal{N}(\mathfrak{g})$ is an induced (respectively rigid, Richardson) nilpotent element of \mathfrak{g} if the G -orbit of e is an induced (respectively rigid, Richardson) nilpotent orbit of \mathfrak{g} .

Proposition 2.9 ([CMa93], Proposition 7.1.4). *If \mathfrak{l}_1 and \mathfrak{l}_2 are two Levi subalgebras of \mathfrak{g} with $\mathfrak{l}_1 \subset \mathfrak{l}_2$. Then $\text{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\text{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = \text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1})$, for all nilpotent orbit $\mathcal{O}_{\mathfrak{l}_1}$ of \mathfrak{l}_1 .*

In other words, Proposition 2.9 says that the induction of orbits is transitive.

Remark 2.10. As a consequence of Proposition 2.9, a nilpotent orbit is always induced, not necessarily in a unique way, from a rigid nilpotent orbit.

We use now classical results about polarizable elements and sheets to obtain interesting properties of Richardson elements useful for Section 4. All results we need to state Theorem 2.11 are contained in [TY05, §33 and §39]. For any x in \mathfrak{g} , we denote by $C(x)$ the G -invariant cone generated by x and we denote by $\overline{C(x)}$ the closure of $C(x)$ in \mathfrak{g} .

Theorem 2.11. *Let $e \in \mathcal{N}(\mathfrak{g})$ be a Richardson element. There is a semisimple element s of \mathfrak{g} satisfying the following two conditions:*

- (1) $\dim G.e = \dim G.s$;
- (2) *the subset of nilpotent elements contained in $\overline{C(s)}$ is the closure of $G.e$.*

Proof. Remind that a sheet of \mathfrak{g} is by definition an irreducible component of a set

$$\{x \in \mathfrak{g} \mid \dim G.x = m\},$$

for some $m \geq 0$. Dixmier sheets are those which contain semisimple elements. By [TY05, Theorem 39.4.8], the set of polarizable elements of \mathfrak{g} are the union of the Dixmier sheets of \mathfrak{g} . As e is Richardson, it is polarizable [TY05, Theorem 33.5.6]. Hence, there is a Dixmier sheet \mathcal{S} of \mathfrak{g} containing e . Let s be a semisimple element of \mathfrak{g} belonging to \mathcal{S} . Then $C(s)$ is contained in \mathcal{S} and by [TY05, Proposition 33.5.4], there is a nilpotent element e' in $\overline{C(s)}$ satisfying $\dim G.e' = \dim G.s$. Moreover, e' belongs to \mathcal{S} and by [TY05, Proposition 35.3.5], the set of nilpotent elements belonging to \mathcal{S} is precisely $G.e'$, whence $G.e = G.e'$. So, $G.e$ is the subset of nilpotent elements of $\overline{C(s)}$ whose orbit has dimension $\dim G.e$.

Since s is a nonzero semisimple element, there exists i in $\{1, \dots, \ell\}$ such that $f_i(s) \neq 0$. For any x in $G.s$,

$$\frac{f_j(x)^{d_i}}{f_i(x)^{d_j}} = \frac{f_j(s)^{d_i}}{f_i(s)^{d_j}},$$

for $j = 1, \dots, \ell$. Hence $\overline{C(s)}$ is contained in the nullvariety in \mathfrak{g} of the functions:

$$f_i(s)^{d_j} f_j^{d_i} - f_j(s)^{d_i} f_i^{d_j}; \quad j = 1, \dots, \ell \quad (j \neq i).$$

Recall that $\mathcal{N}(\mathfrak{g})$ is the nullvariety in \mathfrak{g} of the f_j 's. As a consequence, the set $\mathcal{N}(\mathfrak{g}) \cap \overline{C(s)}$ is the nullvariety in $\overline{C(s)}$ of f_i . Hence, it is an equidimensional variety of dimension $\dim G.e$ since $\overline{C(s)}$ has dimension $\dim G.s + 1 = \dim G.e + 1$. In addition, it is a finite union of nilpotent orbits. Hence the subset of nilpotent elements belonging to $\overline{C(s)}$ is exactly the closure of $G.e$. Indeed, a nilpotent orbit contained in $\overline{C(s)}$ different from $G.e$ has dimension strictly smaller than $\dim G.e$ as we saw before. In conclusion, s satisfies the desired conditions (1) and (2). \square

Remark 2.12. The above proof shows that, for semisimple s as in Theorem 2.11, $\mathcal{N}(\mathfrak{g}) \cap \overline{C(s)}$ is the nullvariety in $\overline{C(s)}$ of f_i if $f_i(s) \neq 0$.

3. PROOF OF THEOREM 1.2 FOR RICHARDSON NILPOTENT ORBITS

Let e be a Richardson nilpotent element of \mathfrak{g} . Our goal is to prove: $\text{ind } \mathfrak{g}^e = \ell$. Let s be a semisimple element of \mathfrak{g} verifying the two conditions of Theorem 2.11 and let x_* be a regular element of \mathfrak{g} such that $f_j(s) = f_j(x_*)$ for all $j = 1, \dots, \ell$. Let $i \in \{1, \dots, \ell\}$ such that $f_i(s) \neq 0$.

Lemma 3.1. (i) *The subset $\overline{C(x_*)}$ is an irreducible component of the nullvariety in \mathfrak{g} of the functions $f_i(s)^{d_j} f_j^{d_i} - f_j(s)^{d_i} f_i^{d_j}$, $1 \leq j \leq \ell$, $j \neq i$. Furthermore, $C(s)$ is contained in $\overline{C(x_*)}$.*

(ii) *The set $\overline{C(x_*)}$ is the union of the nilpotent cone and of the cone generated by the closure of $G.x_*$.*

Proof. (i) As $f_i(x_*) \neq 0$, x_* is not nilpotent. Hence $\overline{C(x_*)}$ has dimension $\dim \mathfrak{g} - \ell + 1$ since x_* is regular. Since the morphism $\mathfrak{g} \rightarrow \mathbb{k}^\ell, x \in \mathfrak{g} \mapsto (f_i)_{1 \leq i \leq \ell}$ is flat, the nullvariety in \mathfrak{g} of the functions $f_i(s)^{d_j} f_j^{d_i} - f_j(s)^{d_i} f_i^{d_j}$, $1 \leq j \leq \ell$, $j \neq i$ is equidimensional of dimension $\dim \mathfrak{g} - \ell + 1$. As a consequence, $\overline{C(x_*)}$ is an irreducible component of this nullvariety. The orbit $G.s$ is the unique closed orbit contained in $\overline{G.x_*}$. Hence $C(s)$ is contained in $\overline{C(x_*)}$.

(ii) By (i), the nullvariety in $\overline{C(x_*)}$ of f_i is contained in $\mathcal{N}(\mathfrak{g})$. Furthermore, the former variety has dimension $\dim \mathfrak{g} - \ell$ since $\overline{C(x_*)}$ has dimension $\dim \mathfrak{g} - \ell + 1$ since f_i is not identically zero on $\overline{C(x_*)}$. Hence the nilpotent cone is contained in $\overline{C(x_*)}$ since it is irreducible and of dimension $\dim \mathfrak{g} - \ell$. Let $D(f_i)_*$ be the complement subset of the nilpotent cone in $\overline{C(x_*)}$; it is a principal open subset of $\overline{C(x_*)}$ which is irreducible. Let τ be the morphism:

$$\begin{array}{ccc} \mathbb{k}^* \times \overline{G.x_*} & \xrightarrow{\tau} & D(f_i)_* \\ (t, x) & \longmapsto & tx. \end{array}$$

Then τ is a proper morphism and its fiber are finite. So τ is surjective since $D(f_i)_*$ and $\mathbb{k}^* \times \overline{G.x_*}$ have the same dimension, whence the statement. \square

As an abbreviation, we set:

$$\begin{aligned} \mathbb{k}^{\underline{d}} &:= \mathbb{k}^{d_1+1} \times \dots \times \mathbb{k}^{d_\ell+1} \simeq \mathbb{k}^{b_{\mathfrak{g}}+\ell}, \\ \mathbb{k}^{\underline{d}^\times} &:= (\mathbb{k}^{d_1+1} \setminus \{0\}) \times \dots \times (\mathbb{k}^{d_\ell+1} \setminus \{0\}), \\ \mathbb{P}^{\underline{d}} &:= \mathbb{P}(\mathbb{k}^{d_1+1}) \times \dots \times \mathbb{P}(\mathbb{k}^{d_\ell+1}) = \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_\ell}. \end{aligned}$$

For $j = 1, \dots, \ell$, recall that σ_j is the map:

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\sigma_j} \mathbb{k}^{d_j+1}, (x, y) \mapsto (f_j^{(m)}(x, y))_{0 \leq m \leq d_j}.$$

Let \mathcal{B}_j be the nullvariety of σ_j in $\mathfrak{g} \times \mathfrak{g}$ and let \mathcal{B} be the union of $\mathcal{B}_1, \dots, \mathcal{B}_\ell$; it is a bicone of $\mathfrak{g} \times \mathfrak{g}$. Denote by ρ and τ the canonical maps:

$$\begin{array}{ccc} (\mathfrak{g} \times \mathfrak{g}) \setminus \{0\} & \xrightarrow{\rho} & \mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \\ \mathbb{k}^{\underline{d}^\times} & \xrightarrow{\tau} & \mathbb{P}^{\underline{d}}. \end{array}$$

Let σ_0 be the restriction to $(\mathfrak{g} \times \mathfrak{g}) \setminus \mathcal{B}$ of σ ; it has values in $\mathbb{k}^{\underline{d}^\times}$. As $\sigma_j(sx, sy) = s^{d_j} \sigma_j(x, y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ and $j = 1, \dots, \ell$, the map $\tau \circ \sigma_0$ factors through ρ . Denote by $\overline{\sigma_0}$ the map from $\rho(\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B})$ to $\mathbb{P}^{\underline{d}}$ making the following diagram commutative:

$$\begin{array}{ccc}
\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B} & \xrightarrow{\sigma_0 = \sigma|_{\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B}}} & \mathbb{k}^{d \times} \\
\rho \downarrow & & \tau \downarrow \\
\rho(\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{B}) & \xrightarrow{\overline{\sigma_0}} & \mathbb{P}^d
\end{array}$$

and let Γ be the graph of the restriction to $\rho(\mathfrak{g} \times \overline{C(s)} \setminus \mathcal{B})$ of $\overline{\sigma_0}$.

Lemma 3.2. *The set Γ is a closed subset of $\mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \times \mathbb{P}^d$.*

Proof. Let $\tilde{\Gamma}$ be the inverse image of Γ by the map $\rho \times \tau$. Then $\tilde{\Gamma}$ is the intersection of the graph of σ and $(\mathfrak{g} \times \overline{C(s)} \setminus \mathcal{B}) \times \mathbb{k}^{d \times}$. Since $\sigma^{-1}(\mathbb{k}^d \setminus \mathbb{k}^{d \times})$ is contained in \mathcal{B} , $\tilde{\Gamma}$ is the intersection of the graph of σ and $(\mathfrak{g} \times \overline{C(s)} \setminus \{0\}) \times \mathbb{k}^{d \times}$; so $\tilde{\Gamma}$ is closed in $(\mathfrak{g} \times \overline{C(s)} \setminus \{0\}) \times \mathbb{k}^{d \times}$. As a consequence, Γ is closed in $\mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \times \mathbb{P}^d$, since $\mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \times \mathbb{P}^d$ is endowed with the quotient topology. \square

Denote by Z the closure of $\sigma(\mathfrak{g} \times \overline{C(s)})$ in \mathbb{k}^d .

Proposition 3.3. *There exists an open subset U of Z such that $U \cap \sigma(\mathfrak{g} \times G.e)$ is not empty.*

Proof. Let Γ_2 be the projection of $\Gamma \subset \mathbb{P}(\mathfrak{g} \times \mathfrak{g}) \times \mathbb{P}^d$ to \mathbb{P}^d . By Lemma 3.2, Γ_2 is a closed subset of \mathbb{P}^d since $\mathbb{P}(\mathfrak{g} \times \mathfrak{g})$ is complete. So $\tau^{-1}(\Gamma_2)$ is a closed subset of $\mathbb{k}^{d \times}$. Moreover,

$$\tau^{-1}(\Gamma_2) = \sigma(\mathfrak{g} \times \overline{C(s)} \setminus \mathcal{B})$$

since $\sigma(\mathfrak{g} \times \overline{C(s)})$ is stable under the action of $\mathbb{k}^* \times \cdots \times \mathbb{k}^*$ on \mathbb{k}^d . Hence the open subset $Z \cap \mathbb{k}^{d \times}$ of Z is contained in $\sigma(\mathfrak{g} \times \overline{C(s)})$. But for all x in \mathfrak{g} such that $f_j(x) \neq 0$ for any j , $\sigma(x, e)$ belongs to $\mathbb{k}^{d \times}$. Thus, the open subset $U = Z \cap \mathbb{k}^{d \times}$ of Z is convenient and the proposition follows. \square

Theorem 3.4. *Let e be a Richardson nilpotent element of \mathfrak{g} . Then the index of \mathfrak{g}^e is equal to ℓ .*

Proof. Let s be as in Theorem 2.11. Since s is semisimple, \mathfrak{g}^s is a reductive Lie algebra of rank ℓ . So the index of \mathfrak{g}^s is equal to ℓ . Besides, by Theorem 2.7, (1) \Rightarrow (6), applied to s ,

$$\dim \sigma(\mathfrak{g} \times \{s\}) = \frac{1}{2} \dim G.s + \ell.$$

Since σ is G -invariant, $\sigma(\mathfrak{g} \times \{s\}) = \sigma(\mathfrak{g} \times G.s)$. Hence for all z in a dense subset of $\sigma(\mathfrak{g} \times G.s)$, the fiber of the restriction of σ to $\mathfrak{g} \times G.s$ at z has minimal dimension

$$\dim \mathfrak{g} + \dim G.s - \left(\frac{1}{2} \dim G.s + \ell \right) = \dim \mathfrak{g} + \frac{1}{2} \dim G.s - \ell.$$

Recall that Z is the closure of $\sigma(\mathfrak{g} \times \overline{C(s)})$ in \mathbb{k}^d . We deduce from the above equality that Z has dimension

$$\begin{aligned}
\dim \mathfrak{g} + \dim C(s) - \left(\dim \mathfrak{g} + \frac{1}{2} \dim G.s - \ell \right) &= \dim C(s) - \frac{1}{2} \dim G.s + \ell \\
&= \frac{1}{2} \dim G.e + \ell + 1,
\end{aligned}$$

since $\dim C(s) = \dim G.s + 1 = \dim G.e + 1$.

By Proposition 3.3, there exists an open subset U of Z contained in $\sigma(\mathfrak{g} \times \overline{C(s)})$ having a nonempty intersection with $\sigma(\mathfrak{g} \times G.e)$. Let i be in $\{1, \dots, \ell\}$ such that $f_i(s) \neq 0$. For $z \in \mathbb{k}^d$,

we write $z = (z_{i,j})_{\substack{1 \leq i \leq \ell \\ 0 \leq j \leq d_i}}$ its coordinates. Let \mathcal{V}_i be the nullvariety in U of the coordinate z_{i,d_i} . Then \mathcal{V}_i is not empty by the choice of U . Since U is irreducible and since z_{i,d_i} is not identically zero on U , \mathcal{V}_i is equidimensional of dimension $\frac{1}{2}\dim G.e + \ell$. By Theorem 2.11 and Remark 2.12, the nullvariety of f_i in $\overline{C(s)}$ is equal to $\overline{G.e}$. Hence $\sigma^{-1}(\mathcal{V}_i) \cap (\mathfrak{g} \times \overline{C(s)}) = \sigma^{-1}(U) \cap (\mathfrak{g} \times \overline{G.e})$ is an open subset of $\mathfrak{g} \times \overline{G.e}$. So $\sigma(\mathfrak{g} \times G.e)$ has dimension $\frac{1}{2}\dim G.e + \ell$. Then by Theorem 2.7, (6) \Rightarrow (1), the index of \mathfrak{g}^e is equal to ℓ . \square

4. THE PROPERTY (P₁) AND REDUCTION TO RIGID NILPOTENT ORBITS

Let \mathfrak{p} be a proper parabolic subalgebra of \mathfrak{g} , let \mathfrak{l} be a Levi factor of \mathfrak{p} and denote by \mathfrak{p}_u the nilpotent radical of \mathfrak{p} . We denote by L the connected subgroup of G whose Lie algebra is $\text{ad } \mathfrak{l}$. We consider a nilpotent orbit $L.\tilde{e}$ of \mathfrak{l} and we choose an element e of $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(L.\tilde{e})$ which belongs to $\tilde{e} + \mathfrak{p}_u$. Note that $\text{rk } \mathfrak{l} = \ell$ and that $\dim \mathfrak{g}^e = \dim \mathfrak{l}^{\tilde{e}}$ by Theorem 2.8. It will be shown in this Section that under the assumption $\text{ind } \mathfrak{l}^{\tilde{e}} = \ell$, the index of \mathfrak{g}^e is equal to ℓ (Theorem 4.10). This will enable us to reduce the proof of Elashvili's conjecture to the case of rigid nilpotent orbits.

4.1. A preliminary result. Let $\tilde{f}_1, \dots, \tilde{f}_\ell$ be homogeneous generators of $S(\mathfrak{l})^{\mathfrak{l}}$ of degrees $\tilde{d}_1, \dots, \tilde{d}_\ell$ respectively, with $\tilde{d}_1 \leq \dots \leq \tilde{d}_\ell$. As a rule, we will write down with a tilde all relative notions to the \tilde{f}_i 's. For example, we denote by $\tilde{\sigma}$ the map

$$\begin{aligned} \mathfrak{l} \times \mathfrak{l} &\xrightarrow{\tilde{\sigma}} \mathbb{k}^{\text{b}_\mathfrak{l} + \ell} \\ (x, y) &\longmapsto (\tilde{f}_i^{(m)})_{\substack{0 \leq i \leq \ell, \\ 0 \leq m \leq \tilde{d}_i}} \end{aligned}$$

where $\text{b}_\mathfrak{l}$ is the dimension of Borel subalgebras of \mathfrak{l} . Recall that the morphism σ was defined in Subsection 2.1.

For $(x, y) \in \mathfrak{l} \times \mathfrak{l}$, denote by $\tilde{\Gamma}_{x,y}$ the fiber at $\tilde{\sigma}(x, y)$ of $\tilde{\sigma}$ and denote by $\Gamma_{x,y}$ the fiber at $\sigma(x, y)$ of the restriction of σ to $\mathfrak{l} \times \mathfrak{l}$.

Lemma 4.1. *Let $(x, y) \in \mathfrak{l} \times \mathfrak{l}$. The fiber $\tilde{\Gamma}_{x,y}$ is a union of irreducible components of $\Gamma_{x,y}$ and $\tilde{\sigma}$ is constant on each irreducible component of $\Gamma_{x,y}$.*

Proof. Let $d \geq \max(d_\ell, \tilde{d}_\ell)$ and let us choose t_1, \dots, t_d pairwise different elements in \mathbb{k} . Denote by κ and $\tilde{\kappa}$ the maps from \mathfrak{l} to \mathbb{k}^ℓ ,

$$\begin{aligned} x' &\xrightarrow{\kappa} (f_1(x'), \dots, f_\ell(x')) \\ x' &\xrightarrow{\tilde{\kappa}} (\tilde{f}_1(x'), \dots, \tilde{f}_\ell(x')), \end{aligned}$$

respectively. For $j = 1, \dots, d$, we denote by τ_j the map $\mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}, (x', y') \mapsto x' + t_j y'$ and we denote by χ and $\tilde{\chi}$ the maps from $\mathfrak{l} \times \mathfrak{l}$ to $\mathbb{k}^{d\ell}$,

$$\begin{aligned} (x', y') &\xrightarrow{\chi} (\kappa \circ \tau_1(x', y'), \dots, \kappa \circ \tau_d(x', y')) \\ (x', y') &\xrightarrow{\tilde{\chi}} (\tilde{\kappa} \circ \tau_1(x', y'), \dots, \tilde{\kappa} \circ \tau_d(x', y')), \end{aligned}$$

respectively. Then $\tilde{\chi}$ is constant on $\tilde{\Gamma}_{x,y}$ since $\tilde{f}_i(x' + t_j y') = \sum_{m=0}^{\tilde{d}_i} \tilde{f}_i^{(m)}(x', y') t_j^m$ for all (x', y') in $\mathfrak{l} \times \mathfrak{l}$ and all $j = 1, \dots, d$. Conversely, if F is the fiber of $\tilde{\chi}$ at $\tilde{\chi}(x, y)$, then for all $(x', y') \in F$, all $j = 1, \dots, d$ and all $i = 1, \dots, \ell$, we have:

$$\sum_{m=0}^{\tilde{d}_i} \tilde{f}_i^{(m)}(x', y') t_j^m = \sum_{m=0}^{\tilde{d}_i} \tilde{f}_i^{(m)}(x, y) t_j^m.$$

So, for $i = 1, \dots, \ell$, the elements $\tilde{f}_i^{(m)}(x', y')$ are solutions of a linear system whose determinant is a Van der Monde determinant defined by some elements among the t_j 's. As these elements are pairwise different, this system has a unique solution. As a consequence, $\tilde{\sigma}$ is constant on F . In conclusion, $F = \tilde{\Gamma}_{x,y}$. The same argument shows that $\Gamma_{x,y}$ is the fiber of χ at $\chi(x, y)$.

The subalgebra $S(\mathfrak{l})^{\mathfrak{l}}$ is a finite extension of the subalgebra generated by the restrictions to \mathfrak{l} of f_1, \dots, f_ℓ since \mathfrak{l} contains a Cartan subalgebra of \mathfrak{g} . So there exists a finite morphism $\rho : \mathbb{k}^\ell \rightarrow \mathbb{k}^\ell$ such that $\kappa = \rho \circ \tilde{\kappa}$. Let ρ_d be the map

$$\begin{aligned} \mathbb{k}^{d\ell} &\longrightarrow \mathbb{k}^{d\ell} \\ (x_{i,j})_{\substack{1 \leq i \leq \ell, \\ 1 \leq j \leq d}} &\longmapsto (\rho(x_{i,j}))_{\substack{1 \leq i \leq \ell, \\ 1 \leq j \leq d}}. \end{aligned}$$

Then ρ_d is a finite morphism and $\chi = \rho_d \circ \tilde{\chi}$. As a result, $\tilde{\Gamma}_{x,y}$ is contained in $\Gamma_{x,y}$. Moreover, $\tilde{\chi}$ is constant on each irreducible component of $\Gamma_{x,y}$ since $\tilde{\chi}(Y)$ is finite for any irreducible component Y of $\Gamma_{x,y}$. This completes the proof of the lemma. \square

Proposition 4.2. *Assume that $\text{ind } \mathfrak{l}^{\tilde{e}} = \ell$. Then $\sigma(\mathfrak{p} \times \{e\})$ has dimension $\frac{1}{2} \dim G \cdot e + \ell - \dim \mathfrak{p}_u$.*

Proof. By Theorem 2.7, (1) \Rightarrow (6), applied to \mathfrak{l} and \tilde{e} , we deduce from our hypothesis that $\tilde{\sigma}(\mathfrak{l} \times \{\tilde{e}\})$ has dimension $\frac{1}{2} \dim L \cdot \tilde{e} + \ell$. On the other hand, Lemma 4.1 tells us that the minimal dimension of fibers of the restriction of σ to $\mathfrak{l} \times \{\tilde{e}\}$ is the minimal dimension of fibers of the restriction of $\tilde{\sigma}$ to $\mathfrak{l} \times \{\tilde{e}\}$. Hence $\sigma(\mathfrak{l} \times \{\tilde{e}\})$ has dimension $\frac{1}{2} \dim L \cdot \tilde{e} + \ell$ as well.

For $(x, y) \in \mathfrak{l} \times \mathfrak{l}$ and $(v, w) \in \mathfrak{p}_u \times \mathfrak{p}_u$, we have $\sigma(x+v, y+w) = \sigma(x, y)$; so $\sigma(\mathfrak{p} \times \{e\}) = \sigma(\mathfrak{l} \times \{\tilde{e}\})$. In conclusion $\sigma(\mathfrak{p} \times \{e\})$ has dimension $\frac{1}{2} \dim L \cdot \tilde{e} + \ell$. Now, since $\dim \mathfrak{l}^{\tilde{e}} = \dim \mathfrak{g}^e$, we have

$$\frac{1}{2} \dim L \cdot \tilde{e} = \frac{1}{2} (\dim \mathfrak{l} - \dim \mathfrak{g}^e) = \frac{1}{2} \dim G \cdot e - \dim \mathfrak{p}_u,$$

and the proposition follows. \square

4.2. The property (P₁). Let \mathfrak{p}_- be the opposite parabolic subalgebra to \mathfrak{p} . Denote by $\mathfrak{p}_{u,-}$ the nilpotent radical of \mathfrak{p}_- , by P_- the normalizer of \mathfrak{p}_- in G and by $P_{u,-}$ the unipotent radical of P_- . We introduce now a property (P₁). It will turn out to be equivalent to the equality $\text{ind } \mathfrak{g}^e = \ell$ under the assumption $\text{ind } \mathfrak{l}^{\tilde{e}} = \ell$ (Corollary 4.7).

Definition 4.3. We say that e has Property (P₁) if for all (g, x) in a nonempty open subset of $P_{u,-} \times \mathfrak{p}$, we have $[g(x), V_{g(x),e}]^\perp \cap \mathfrak{p}_{u,-} = 0$.

Remark 4.4. The image of $P_{u,-} \times \mathfrak{p}$ by the map $(g, x) \mapsto g(x)$ is dense in \mathfrak{g} . Hence, e has Property (P₁) if and only if for all y in a nonempty open subset of \mathfrak{g} , we have $[y, V_{y,e}]^\perp \cap \mathfrak{p}_{u,-} = 0$.

We start with a preliminary result. For $(g, x) \in P_{\mathfrak{u},-} \times \mathfrak{p}$, set:

$$\Sigma_{g,x} := \{k \in P_{\mathfrak{u},-} \mid \sigma(k(x), e) = \sigma(g(x), e)\}.$$

Note that $\Sigma_{g,x}$ is a closed subset of $P_{\mathfrak{u},-}$ containing g . We denote by $T_g(\Sigma_{g,x})$ the tangent space of $\Sigma_{g,x}$ at g .

Lemma 4.5. *Let $(g, x) \in P_{\mathfrak{u},-} \times \mathfrak{p}$. For $v \in \mathfrak{p}_{\mathfrak{u},-}$, the element $(\text{ad } v)g$ belongs to $T_g(\Sigma_x)$ if and only if $v \in [g(x), V_{g(x),e}^\perp]$.*

Proof. Let v be in $\mathfrak{p}_{\mathfrak{u},-}$. By Lemma 2.4, $(\text{ad } v)g$ belongs to $T_g(\Sigma_x)$ if and only if $[v, g(x)]$ is orthogonal to $V_{g(x),e}$ whence the lemma. \square

Proposition 4.6. *If e has Property (P_1) then*

$$\dim \sigma(\mathfrak{g} \times \{e\}) \geq \dim \sigma(\mathfrak{p} \times \{e\}) + \dim \mathfrak{p}_{\mathfrak{u}}.$$

Moreover, if the following equality holds:

$$\dim \sigma(\mathfrak{g} \times \{e\}) = \dim \sigma(\mathfrak{p} \times \{e\}) + \dim \mathfrak{p}_{\mathfrak{u}},$$

then e has Property (P_1) .

Proof. Since the image of $P_{\mathfrak{u},-} \times \mathfrak{p}$ by the map $(g, x) \mapsto g(x)$ is dense in \mathfrak{g} , the image of the map ψ ,

$$\begin{aligned} P_{\mathfrak{u},-} \times \mathfrak{p} &\xrightarrow{\psi} \mathbb{k}^{\text{b}_{\mathfrak{g}} + \ell} \\ (g, x) &\longmapsto \sigma(g(x), e), \end{aligned}$$

has dimension $\dim \sigma(\mathfrak{g} \times \{e\})$. For (g, x) in $P_{\mathfrak{u},-} \times \mathfrak{p}$, we set:

$$S_{g,x} := \{y \in \mathfrak{p} \mid \sigma(g(y), e) = \sigma(g(x), e)\}.$$

By Lemma 2.4, for v in \mathfrak{g} , v belongs to the tangent space at x of $S_{g,x}$ if and only if $g(v)$ is orthogonal to $V_{g(x),e}$ or, equivalently, if and only if v is orthogonal to $V_{x,g^{-1}(e)}$. Since $S_{1,x}$ is the fiber at $\sigma(x, e)$ of the map $x \mapsto \sigma(x, e)$ from \mathfrak{p} to $\sigma(\mathfrak{p} \times \{e\})$ we deduce that

$$\dim(V_{x,e}^\perp \cap \mathfrak{p}) = \dim \mathfrak{p} - \dim \sigma(\mathfrak{p} \times \{e\})$$

for all x in a nonempty open subset of \mathfrak{p} . So, for all (g, x) in a dense open subset of $P_{\mathfrak{u},-} \times \mathfrak{p}$, we have

$$\dim(V_{x,g^{-1}(e)}^\perp \cap \mathfrak{p}) \leq \dim \mathfrak{p} - \dim \sigma(\mathfrak{p} \times \{e\}).$$

Hence, there exists a dense open subset U of $P_{\mathfrak{u},-} \times \mathfrak{p}$ satisfying the following condition:

$$(C_1) \text{ for all } (g, x) \in U, \dim S_{g,x} \leq \dim \mathfrak{p} - \dim \sigma(\mathfrak{p} \times \{e\}).$$

Suppose now that e has Property (P_1) and show $\dim \psi(P_{\mathfrak{u},-} \times \mathfrak{p}) \geq \dim \sigma(\mathfrak{p} \times \{e\}) + \dim \mathfrak{p}_{\mathfrak{u}}$. Since e has Property (P_1) , we can assume furthermore that U satisfies the following condition:

$$(C_2) \text{ for all } (g, x) \in U, \text{ we have } [g(x), V_{g(x),e}^\perp]^\perp \cap \mathfrak{p}_{\mathfrak{u},-} = 0.$$

Let F be an irreducible component of a fiber of the restriction of ψ to U and let τ be the restriction to F of the projection map $P_{\mathfrak{u},-} \times \mathfrak{p} \rightarrow \mathfrak{p}$. By (C₂) and Lemma 4.5, the fibers of τ are finite. So there exists a dense open subset O of $\tau(F)$ such that the restriction of τ to $\tau^{-1}(O)$ is an étale covering. Therefore, for $x \in O$ and for $g \in P_{\mathfrak{u},-}$ such that $(g, x) \in \tau^{-1}(O)$, $\{g\} \times S_{g,x}$ is a neighborhood of (g, x) in F . Hence for all $(g, x) \in F$, F has dimension $\dim S_{g,x}$. Then, by (C₁), we get $\dim \psi(U) = \psi(P_{\mathfrak{u},-} \times \mathfrak{p}) \geq \dim \sigma(\mathfrak{p} \times \{e\}) + \dim \mathfrak{p}_{\mathfrak{u}}$.

Suppose now that the image of ψ has dimension $\dim \sigma(\mathfrak{p} \times \{e\}) + \mathfrak{p}_{\mathfrak{u}}$ and prove that e has Property (P₁). In this case, we can assume that the dense open subset U of $P_{\mathfrak{u},-} \times \mathfrak{p}$ satisfies (C₁) and also the following condition:

(C₃) for all (g, x) in U , the fiber at $\psi(g, x)$ of the restriction ψ_0 of ψ to U has dimension $\dim \mathfrak{p} - \dim \sigma(\mathfrak{p} \times \{e\})$.

By (C₃), for all $(g, x) \in U$, $(\{g\} \times S_{g,x}) \cap U$ is a union of irreducible components of $(\psi_0)^{-1}(\psi(g, x))$. As a result, for all $(g, x) \in U$, the subset $\Sigma_{g,x}$ is finite. Let (g, x) be in U and let v be in $\mathfrak{p}_{\mathfrak{u},-}$ such that v is orthogonal to $[g(x), V_{g(x),e}]$. Then $[v, g(x)]$ is orthogonal to $V_{g(x),e}$. So by Lemma 4.5, $(\text{ad } v)g$ belongs to the tangent space at g of $\Sigma_{g,x}$. Hence $v = 0$. In other words, e has Property (P₁). \square

The following corollary yields the expected equivalence:

Corollary 4.7. *Assume that $\text{ind } \tilde{e} = \ell$. Then, e has property (P₁) if and only if $\text{ind } \mathfrak{g}^e = \ell$.*

Proof. By Proposition 4.2, $\sigma(\mathfrak{p} \times \{e\})$ has dimension $\frac{1}{2} \dim G.e + \ell - \dim \mathfrak{p}_{\mathfrak{u}}$ and by Lemma 2.6(i), $\dim \sigma(\mathfrak{g} \times \{e\}) \leq \frac{1}{2} \dim G.e + \ell$. Hence by Proposition 4.6, $\sigma(\mathfrak{g} \times \{e\})$ has dimension $\frac{1}{2} \dim G.e + \ell$ if and only if e has property (P₁). But Theorem 2.7, (1) \Leftrightarrow (6), says that $\text{ind } \mathfrak{g}^e = \ell$ if and only if e has property (P₁). \square

4.3. Reduction to rigid nilpotent orbits. The aim of this section is Theorem 4.10. Let \mathfrak{p} , $\mathfrak{p}_{\mathfrak{u}}$, \mathfrak{l} , e , \tilde{e} be as in Subsections 4.1 and 4.2. Recall that e is a nilpotent element which belongs to $\tilde{e} + \mathfrak{p}_{\mathfrak{u}}$. For x in \mathfrak{p} and for y in $\mathcal{N}(\mathfrak{l})$, we set:

$$s_x := \inf \{ \dim [z, V_{z,x}]^{\perp} \mid z \in \mathfrak{g}_{\text{reg}} \}.$$

Since the map $(x, y) \mapsto V_{x,y}$ is G -equivariant, s_x only depends on $G.x$. As an abbreviation, we denote by Ω_y the intersection $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(L.y) \cap (L.y + \mathfrak{p}_{\mathfrak{u}})$. By Theorem 2.8, remind that Ω_y is a P -orbit.

Lemma 4.8. *Let y be in $\mathcal{N}(\mathfrak{l})$.*

- (i) *If $x \in \Omega_y$ has Property (P₁), then so does any element of Ω_y .*
- (ii) *For all (z, x) in a nonempty open subset of $\mathfrak{g}_{\text{reg}} \times \Omega_y$, $[z, V_{z,x}]^{\perp}$ has dimension s_x .*

Proof. (i) By Proposition 4.6, x has Property (P₁) if and only if

$$\dim \sigma(\mathfrak{g} \times \{x\}) = \sigma(\mathfrak{p} \times \{x\}) + \dim \mathfrak{p}_{\mathfrak{u}}.$$

Since σ is a P -equivariant map,

$$\sigma(\mathfrak{g} \times \{x\}) = \sigma(\mathfrak{g} \times P.x) \text{ and } \sigma(\mathfrak{p} \times \{x\}) = \sigma(\mathfrak{p} \times P.x),$$

whence the statement since $\Omega_y = P.x$.

(ii) For x in Ω_y , set:

$$s'_x := \max\{\dim V_{z,x} \mid z \in \mathfrak{g}_{\text{reg}}\}.$$

Let (z, x) be in $\mathfrak{g}_{\text{reg}} \times \Omega_y$ such that $s'_x = \dim V_{z,x}$. Then there are s'_x elements $\eta_1, \dots, \eta_{s'_x}$ among the elements $\varphi_i^{(m)}$'s, such that $\eta_1(z, x), \dots, \eta_{s'_x}(z, x)$ is a basis of $V_{z,x}$. By continuity, for all (z', x') in an open subset U of $\mathfrak{g}_{\text{reg}} \times \Omega_y$ containing (z, x) , the elements $\eta_1(z', x'), \dots, \eta_{s'_x}(z', x')$ are linearly independent. Hence by maximality of s'_x , $V_{z',x'}$ has dimension s'_x for all (z', x') in U . Then for all (z', x') in U , $[z', V_{z',x'}]$ has dimension $s'_x - \ell$ by Remark 2.2(2). Furthermore, for (z', x') running through $\mathfrak{g}_{\text{reg}} \times \Omega_y$, $\dim V_{z',x'}$ is maximal whenever $[z', V_{z',x'}]^\perp$ is minimal. Hence for all (z', x') in U , $[z', V_{z',x'}]^\perp$ has dimension s_x . \square

Let y be in $\mathcal{N}(\mathfrak{l})$. The integer s_x does not depend on the choice of an element x in Ω_y . So one is justified to set $s_y := s_x$ for any x in Ω_y .

Lemma 4.9. *Let $y, y' \in \mathcal{N}(\mathfrak{l})$ such that $L.y$ is contained in the closure of $L.y'$ in \mathfrak{l} . If $x \in \Omega_y$ has Property (P_1) then any element of $\Omega_{y'}$ has Property (P_1) too.*

Proof. Let x be in Ω_y and assume that x has Property (P_1) . Suppose that there exists $x' \in \Omega_{y'}$ such that x' has not Property (P_1) . We are going to prove that, for all z in a nonempty open subset of $\mathfrak{g}_{\text{reg}}$, there exists a subspace E of \mathfrak{g} of dimension $s_{x'}$, contained in $[z, V_{z,x}]^\perp$ and such that $E \cap \mathfrak{p}_{u,-} \neq 0$. Once proved, this will lead us to a contradiction. Indeed, by Remark 4.4, Property (P_1) for x means that, for any z in a nonempty open subset of $\mathfrak{g}_{\text{reg}}$, we have $[z, V_{y,z}]^\perp \cap \mathfrak{p}_{u,-} = 0$.

So, let us prove the above statement. From our assumption and by Lemma 4.8(i), no element of $\Omega_{y'}$ has Property (P_1) . Remind that $s_{y'} = s_{x'}$ for all $x' \in \Omega_{y'}$ by definition of $s_{y'}$. By Lemma 4.8(ii), for all (z, x') in a nonempty open subset U of $\mathfrak{g}_{\text{reg}} \times \Omega_{y'}$,

$$\dim [z, V_{z,x'}]^\perp = s_{y'} \text{ and } [z, V_{z,x'}]^\perp \cap \mathfrak{p}_{u,-} \neq 0.$$

Let z be in the projection of U to $\mathfrak{g}_{\text{reg}}$ and set:

$$U_z := \{x' \in \Omega_{y'} \mid (z, x') \in U\}.$$

Then U_z is a dense open subset of $\Omega_{y'}$. Denote by T the subset of elements (E, D) of $\text{Gr}_{s_{y'}}(\mathfrak{g}) \times \mathbb{P}(\mathfrak{p}_{u,-})$ such that D is contained in E . Then T is a projective variety. Set:

$$T_0 := \{(x', [z, V_{z,x'}]^\perp, D) \mid x' \in U_z, ([z, V_{z,x'}]^\perp, D) \in T\}.$$

Denote by $\overline{T_0}$ the closure of T_0 in $\mathfrak{g} \times T$. Since T is a projective variety, the projection of $\overline{T_0}$ to \mathfrak{g} is closed. In particular, it contains the closure of $\Omega_{y'}$ in \mathfrak{g} . Since $L.y$ is contained in the closure of $L.y'$ in \mathfrak{l} , there exists (E, D) in T such that (x, E, D) belongs to $\overline{T_0}$. For any $i = 1, \dots, \ell$, $0 \leq m \leq d_i - 1$, and for all x' in U_z , $\varphi_i^{(m)}(z, x')$ is orthogonal to $[z, V_{z,x'}]$ by Lemma 2.3(ii). Hence $\varphi_i^{(m)}(z, x)$ is orthogonal to E for any (i, m) . As a consequence, E is contained in $[z, V_{z,x}]^\perp$ since $V_{z,x}$ is the subspace generated by the $\varphi_i^{(m)}(z, x)$'s. So, E is convenient since its intersection with $\mathfrak{p}_{u,-}$ contains the line D and since the projection of U to $\mathfrak{g}_{\text{reg}}$ is an open subset of $\mathfrak{g}_{\text{reg}}$. \square

We are now ready to prove the main result of this section:

Theorem 4.10. *Assume that $\text{ind} \mathfrak{a}^x = \text{rk} \mathfrak{a}$ for all reductive subalgebra \mathfrak{a} strictly contained in \mathfrak{g} and for all x in \mathfrak{a} . Then for all induced nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in \mathfrak{g} and for all e in $\mathcal{O}_{\mathfrak{g}}$, $\text{ind} \mathfrak{g}^e = \ell$.*

Proof. Let \mathfrak{l} be a Levi subalgebra of \mathfrak{g} and let $\mathcal{O}_{\mathfrak{g}}$ the induced orbit from a nilpotent orbit $L.\tilde{e}$ of \mathfrak{l} . From our assumption, $\text{ind} \mathfrak{l}^{\tilde{e}} = \ell$. Choose now e in $\mathcal{O}_{\mathfrak{g}} \cap \mathfrak{p}$; we wish to prove $\text{ind} \mathfrak{g}^e = \ell$. Let \mathcal{O}_0 be the induced nilpotent orbit of \mathfrak{g} from the zero orbit of \mathfrak{l} . It is a Richardson orbit by definition. Therefore by Theorem 3.4, for all x in \mathcal{O}_0 , $\text{ind} \mathfrak{g}^x = \ell$. So by Corollary 4.7, all elements of \mathcal{O}_0 have Property (P₁). Since 0 belongs to the closure of $L.\tilde{e}$ in \mathfrak{l} , any element of $\Omega_{\tilde{e}}$ has Property (P₁) by Lemma 4.9. So by Corollary 4.7, $\text{ind} \mathfrak{g}^e = \ell$ since e belongs to $\Omega_{\tilde{e}}$ by the choice of e . \square

From that point, our goal is to prove Theorem 1.2 for rigid nilpotent element; Theorem 4.10 tells us that this is enough to complete the proof.

5. THE SLOWOWY SLICE AND THE PROPERTY (P₂)

In this section, we introduce a property (P₂) in Definition 5.2 and we prove that $e \in \mathcal{N}(\mathfrak{g})$ has Property (P₂) if and only if $\text{ind} \mathfrak{g}^e = \ell$. Then, for most rigid nilpotent orbits of \mathfrak{g} , we will show next section that they do have Property (P₂).

5.1. Blowing up of \mathcal{S} . Let e be a nilpotent element of \mathfrak{g} and consider an \mathfrak{sl}_2 -triple (e, h, f) containing e as in Subsection 1.3. The *Slowowy slice* is the affine subspace $\mathcal{S} := e + \mathfrak{g}^f$ of \mathfrak{g} which is a transverse variety to the adjoint orbit $G.e$. Denote by $B_e(\mathcal{S})$ the blowing up of \mathcal{S} centered at e and let $p : B_e(\mathcal{S}) \rightarrow \mathcal{S}$ be the canonical morphism. The variety \mathcal{S} is smooth and $p^{-1}(e)$ is a smooth, irreducible, hypersurface of $B_e(\mathcal{S})$. The use of the blowing-up $B_e(\mathcal{S})$ for the computation of the index was initiated by the first author in [Ch04] and resumed by the second author in [Mo06a]. Here, we use again this technique to study the index of \mathfrak{g}^e . Describe first the main tools extracted from [Ch04] we need.

For Y an open subset of $B_e(\mathcal{S})$, we denote by $\mathbb{k}[Y]$ the algebra of regular functions of Y . By [Ch04, Théorème 3.3], we have:

Theorem 5.1. *The following two assertions are equivalent:*

- (A) *The equality $\text{ind} \mathfrak{g}^e = \ell$ holds,*
- (B) *there exists an affine open subset $Y \subset B_e(\mathcal{S})$ such that $Y \cap p^{-1}(e) \neq \emptyset$ and satisfying the following property:*

for any regular map $\varphi \in \mathbb{k}[Y] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y$, there exists $\psi \in \mathbb{k}[Y] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), p(x)]$ for all $x \in Y$.

An open subset $\Omega \subset B_e(\mathcal{S})$ is called a *big open subset* if $B_e(\mathcal{S}) \setminus \Omega$ has codimension at least 2 in $B_e(\mathcal{S})$. As explained in [Ch04, Section 2], there exists a big open subset Ω of $B_e(\mathcal{S})$ and a regular map

$$\alpha : \Omega \rightarrow \text{Gr}_{\ell}(\mathfrak{g})$$

such that $\alpha(x) = \mathfrak{g}^{p(x)}$ if $p(x)$ is regular. In fact, this result is a consequence of [Sh94, Ch. VI, Theorem 1]. As $p^{-1}(e)$ is an hypersurface and Ω a big open subset of $B_e(\mathcal{S})$, note that $\Omega \cap p^{-1}(e)$

is a nonempty set. In addition, $\alpha(x) \subset \mathfrak{g}^{p(x)}$ for all $x \in \Omega$.

Definition 5.2. We say that e has Property (P₂) if $\mathfrak{z}(\mathfrak{g}^e) \subset \alpha(x)$ for all x in $\Omega \cap p^{-1}(e)$.

Remark 5.3. Suppose that e is regular. Then \mathfrak{g}^e is a commutative algebra, i.e. $\mathfrak{z}(\mathfrak{g}^e) = \mathfrak{g}^e$. If $x \in \Omega \cap p^{-1}(e)$, then $\alpha(x) = \mathfrak{g}^e$ since $p(x) = e$ is regular in this case. On the other hand, $\text{ind } \mathfrak{g}^e = \dim \mathfrak{g}^e = \ell$ since e is regular. So e has Property (P₂) and $\text{ind } \mathfrak{g}^e = \ell$.

We aim to prove that e has Property (P₂) if and only if $\text{ind } \mathfrak{g}^e = \ell$. As a consequence of Remark 5.3, we will assume that e is a nonregular nilpotent element of \mathfrak{g} .

5.2. On the property (P₂). This subsection aims to show: Property (P₂) holds for e if and only if $\text{ind } \mathfrak{g}^e = \ell$. We start by stating that if (P₂) holds, then so does the assertion (B) of Theorem 5.1.

Let $L_{\mathfrak{g}}$ be the $S(\mathfrak{g})$ -submodule $\varphi \in S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ satisfying $[\varphi(x), x] = 0$ for all x in \mathfrak{g} . It is known that $L_{\mathfrak{g}}$ is a free module of basis $\varphi_1, \dots, \varphi_{\ell}$, cf. [Di79]. We investigate an analogous property for the Slodowy slice $\mathcal{S} = e + \mathfrak{g}^f$. We denote by \mathcal{S}_{reg} the intersection of \mathcal{S} and $\mathfrak{g}_{\text{reg}}$. As e is nonregular, the set $(\mathcal{S} \setminus \mathcal{S}_{\text{reg}})$ contains e .

Lemma 5.4. *The set $\mathcal{S} \setminus \mathcal{S}_{\text{reg}}$ has codimension 3 in \mathcal{S} and each irreducible component of $\mathcal{S} \setminus \mathcal{S}_{\text{reg}}$ contains e .*

Proof. Let us consider the morphism

$$\begin{aligned} G \times \mathcal{S} &\longrightarrow \mathfrak{g} \\ (g, x) &\longmapsto g(x). \end{aligned}$$

By a Slodowy's result [Sl80], this morphism is a smooth morphism. So its fibers are equidimensional of dimension $\dim \mathfrak{g}^f$. In addition, by [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$ is a G -invariant equidimensional closed subset of \mathfrak{g} of codimension 3. Hence $\mathcal{S} \setminus \mathcal{S}_{\text{reg}}$ is an equidimensional closed subset of \mathcal{S} of codimension 3.

Denoting by $t \mapsto g(t)$ the one parameter subgroup of G generated by $\text{ad } h$, \mathcal{S} and $\mathcal{S} \setminus \mathcal{S}_{\text{reg}}$ are stable under the action of $t^{-2}g(t)$ for all t in \mathbb{k}^* . Furthermore, for all x in \mathcal{S} , $t^{-2}g(t)(x)$ goes to e when t goes to ∞ , whence the lemma. \square

Denote by $\mathbb{k}[\mathcal{S}]$ the algebra of regular functions on \mathcal{S} and denote by $L_{\mathcal{S}}$ the $\mathbb{k}[\mathcal{S}]$ -submodule of $\varphi \in \mathbb{k}[\mathcal{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ satisfying $[\varphi(x), x] = 0$ for all x in \mathcal{S} .

Lemma 5.5. *The module $L_{\mathcal{S}}$ is a free module of basis $\varphi_1|_{\mathcal{S}}, \dots, \varphi_{\ell}|_{\mathcal{S}}$ where $\varphi_i|_{\mathcal{S}}$ is the restriction to \mathcal{S} of φ_i for $i = 1, \dots, \ell$.*

Proof. Let φ be in $L_{\mathcal{S}}$. There are regular functions a_1, \dots, a_{ℓ} on \mathcal{S}_{reg} satisfying

$$\varphi(x) = a_1(x)\varphi_1|_{\mathcal{S}}(x) + \dots + a_{\ell}(x)\varphi_{\ell}|_{\mathcal{S}}(x)$$

for all $x \in \mathcal{S}_{\text{reg}}$, by Lemma 2.1(ii). By Lemma 5.4, $\mathcal{S} \setminus \mathcal{S}_{\text{reg}}$ has codimension 3 in \mathcal{S} . Hence a_1, \dots, a_{ℓ} have polynomial extensions to \mathcal{S} since \mathcal{S} is normal. So the maps $\varphi_1|_{\mathcal{S}}, \dots, \varphi_{\ell}|_{\mathcal{S}}$ generate

$L_{\mathfrak{S}}$. Moreover, by Lemma 2.1(ii) for all $x \in \mathfrak{S}_{\text{reg}}$, $\varphi_1(x), \dots, \varphi_\ell(x)$ are linearly independent, whence the statement. \square

The following proposition accounts for an important step to interpret Assertion (B) of Theorem 5.1:

Proposition 5.6. *Let φ be in $\mathbb{k}[\mathfrak{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) \in [\mathfrak{g}, x]$ for all x in a nonempty open subset of \mathfrak{g} . Then there exists a polynomial map $\psi \in \mathbb{k}[\mathfrak{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), x]$ for all $x \in \mathfrak{S}$.*

Proof. Since \mathfrak{g}^x is the orthogonal complement of $[x, \mathfrak{g}]$ in \mathfrak{g} , our hypothesis says that $\varphi(x)$ is orthogonal to \mathfrak{g}^x for all x in a nonempty open subset \mathfrak{S}' of \mathfrak{S} . The intersection $\mathfrak{S}' \cap \mathfrak{S}_{\text{reg}}$ is not empty; so by Lemma 2.1(ii), $\langle \varphi(x), \varphi_i|_{\mathfrak{S}}(x) \rangle = 0$ for all $i = 1, \dots, \ell$ and for all $x \in \mathfrak{S}' \cap \mathfrak{S}_{\text{reg}}$. Therefore, by continuity, $\langle \varphi(x), \varphi_i|_{\mathfrak{S}}(x) \rangle = 0$ for all $i = 1, \dots, \ell$ and all $x \in \mathfrak{S}$. Hence $\varphi(x) \in [x, \mathfrak{g}]$ for all $x \in \mathfrak{S}_{\text{reg}}$ by Lemma 2.1(ii) again. Consequently by Lemma 5.4, Lemma 5.5 and the proof of the main theorem of [Di79], there exists an element $\psi \in \mathbb{k}[\mathfrak{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ which satisfies the condition of the proposition. \square

Let u_1, \dots, u_m be a basis of \mathfrak{g}^f and let u_1^*, \dots, u_m^* be the corresponding coordinate system of $\mathfrak{S} = e + \mathfrak{g}^f$. There is an affine open subset $Y \subset B_e(\mathfrak{S})$ with $Y \cap p^{-1}(e) \neq \emptyset$ such that $\mathbb{k}[Y]$ is the set of linear combinations of monomials in $(u_1^*)^{-1}, u_1^*, \dots, u_m^*$ whose total degree is nonnegative. In particular, we have a global coordinates system $u_1^*, v_2^*, \dots, v_m^*$ on Y satisfying the relations:

$$(4) \quad u_2^* = u_1^* v_2^* \quad , \dots, \quad u_m^* = u_1^* v_m^*.$$

Note that, for $x \in Y$, we so have: $p(x) = e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \dots + v_m^*(x)u_m)$. So, the image of Y by p is the union of $\{e\}$ and the complementary in \mathfrak{S} of the nullvariety of u_1^* . Let Y' be an affine open subset of Y contained in Ω and having a nonempty intersection with $p^{-1}(e)$. Denote by $L_{Y'}$ the set of regular maps φ from Y' to \mathfrak{g} satisfying $[\varphi(x), p(x)] = 0$ for all $x \in Y'$.

Lemma 5.7. *Suppose that e has Property (P₂). For each $z \in \mathfrak{z}(\mathfrak{g}^e)$, there exists $\psi_z \in \mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $z - u_1^* \psi_z$ belongs to $L_{Y'}$.*

Proof. Let z be in $\mathfrak{z}(\mathfrak{g}^e)$. Since $Y' \subset \Omega$, for each $y \in Y'$, there exists an affine open subset U_y of Y' containing y and regular maps ν_1, \dots, ν_ℓ from U_y to \mathfrak{g} such that $\nu_1(x), \dots, \nu_\ell(x)$ is a basis of $\alpha(x)$ for all $x \in U_y$. Let y be in Y' . We consider two cases:

(1) Suppose $p(y) = e$.

Since e has Property (P₂), there exist regular functions a_1, \dots, a_ℓ on U_y satisfying

$$z = a_1(x)\nu_1(x) + \dots + a_\ell(x)\nu_\ell(x),$$

for all $x \in U_y \cap p^{-1}(e)$. The intersection $U_y \cap p^{-1}(e)$ is the set of zeroes of u_1^* in U_y . So there exists a regular map ψ from U_y to \mathfrak{g} which satisfies the equality:

$$z - u_1^* \psi = a_1 \nu_1 + \dots + a_\ell \nu_\ell.$$

Hence $[z - u_1^*(x)\psi(x), p(x)] = 0$ for all $x \in U_y$ since $\alpha(x)$ is contained in $\mathfrak{g}^{p(x)}$ for all $x \in \Omega$.

(2) Suppose $p(y) \neq e$.

Then we can assume that $U_y \cap p^{-1}(e) = \emptyset$ and the map $\psi = (u_1^*)^{-1}z$ satisfies the condition: $[z - u_1^*(x)\psi(x), p(x)] = 0$ for all $x \in U_y$.

In both cases (1) or (2), we have found a regular map ψ_y from U_y to \mathfrak{g} satisfying: $[z - (u_1^*\psi_y)(x), p(x)] = 0$ for all $x \in U_y$.

Let y_1, \dots, y_k be in Y' such that the open subsets U_{y_1}, \dots, U_{y_k} cover Y' . For $i = 1, \dots, k$, we denote by ψ_i a regular map from U_{y_i} to \mathfrak{g} such that $z - u_1^*\psi_i$ is in $\Gamma(U_{y_i}, \mathcal{L})$ where \mathcal{L} is the localization of $L_{Y'}$ on Y' . Then for $i, j = 1, \dots, k$, $\psi_i - \psi_j$ is in $\Gamma(U_{y_i} \cap U_{y_j}, \mathcal{L})$. Since Y' is affine, $H^1(Y', \mathcal{L}) = 0$. So, for $i = 1, \dots, k$, there exists $\tilde{\psi}_i$ in $\Gamma(U_{y_i}, \mathcal{L})$ such that $\tilde{\psi}_i - \tilde{\psi}_j$ is equal to $\psi_i - \psi_j$ on $U_{y_i} \cap U_{y_j}$ for all i, j . Then there exists a well-defined map ψ_z from Y' to \mathfrak{g} whose restriction to U_{y_i} is equal to $\psi_i - \tilde{\psi}_i$ for all i , and such that $z - u_1^*\psi_z$ belongs to $L_{Y'}$. Finally, the map ψ_z verifies the required property. \square

Let z be in $\mathfrak{z}(\mathfrak{g}^e)$. We denote by φ_z the regular map from Y to \mathfrak{g} defined by:

$$(5) \quad \varphi_z(x) = [z, u_1] + v_2^*(x)[z, u_2] + \dots + v_m^*(x)[z, u_m], \quad \text{for all } x \in Y.$$

Corollary 5.8. *Suppose that e has Property (P₂) and let z be in $\mathfrak{z}(\mathfrak{g}^e)$. There exists ψ_z in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi_z(x) = [\psi_z(x), p(x)]$ for all $x \in Y'$.*

Proof. By Lemma 5.7, there exists ψ_z in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $z - u_1^*\psi_z$ is in $L_{Y'}$. Then

$$u_1^*\varphi_z(x) = [z, p(x)] = [z - u_1^*\psi_z(x), p(x)] + u_1^*[\psi_z(x), p(x)],$$

for all $x \in Y'$. So the map ψ_z is convenient, since u_1^* is not identically zero on Y' . \square

The following lemma is easy but helpful for Proposition 5.10:

Lemma 5.9. *Let v be in \mathfrak{g}^e . Then, v belongs to $\mathfrak{z}(\mathfrak{g}^e)$ if and only if $[v, \mathfrak{g}^f] \subset [e, \mathfrak{g}]$.*

Proof. As $[x, \mathfrak{g}]$ is the orthogonal complement of \mathfrak{g}^x in \mathfrak{g} for all $x \in \mathfrak{g}$, we have:

$$[v, \mathfrak{g}^f] \subset [e, \mathfrak{g}] \iff \langle [v, \mathfrak{g}^f], \mathfrak{g}^e \rangle = 0 \iff \langle [v, \mathfrak{g}^e], \mathfrak{g}^f \rangle = 0 \iff [v, \mathfrak{g}^e] \subset [f, \mathfrak{g}].$$

But \mathfrak{g} is the direct sum of \mathfrak{g}^e and $[f, \mathfrak{g}]$ and $[v, \mathfrak{g}^e]$ is contained in \mathfrak{g}^e since $v \in \mathfrak{g}^e$. Hence $[v, \mathfrak{g}^f]$ is contained in $[e, \mathfrak{g}]$ if and only if v is in $\mathfrak{z}(\mathfrak{g}^e)$. \square

Proposition 5.10. *Suppose that e has Property (P₂) and let φ be in $\mathbb{k}[Y] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) \in [e, p(x)]$ for all $x \in Y$. Then there exists ψ in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), p(x)]$ for all $x \in Y'$.*

Proof. Since φ is a regular map from Y to \mathfrak{g} , there is a nonnegative integer d and $\tilde{\varphi} \in \mathbb{k}[\mathcal{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ such that

$$(6) \quad (u_1^*)^d(x)\varphi(x) = (\tilde{\varphi} \circ p)(x), \quad \forall x \in Y$$

and $\tilde{\varphi}$ is a linear combination of monomials in u_1^*, \dots, u_m^* whose total degree is at least d . By hypothesis on φ , we deduce that for all $x \in \mathcal{S}$ such that $u_1^*(x) \neq 0$, $\tilde{\varphi}(x)$ is in $[e, p(x)]$. Hence by Proposition 5.6, there exists $\tilde{\psi}$ in $\mathbb{k}[\mathcal{S}] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\tilde{\varphi}(x) = [\tilde{\psi}(x), p(x)]$ for all $x \in \mathcal{S}$.

Denote by $\tilde{\psi}'$ the sum of monomials of degree at least d in $\tilde{\psi}$ and denote by ψ' the element of $\mathbb{k}[Y] \otimes_{\mathbb{k}} \mathfrak{g}$ satisfying

$$(7) \quad (u_1^*)^d(x)\psi'(x) = (\tilde{\psi}' \circ p)(x), \quad \forall x \in Y.$$

Then we set, for $x \in Y$, $\varphi'(x) := \varphi(x) - [\psi'(x), p(x)]$. We have to prove the existence of an element ψ'' in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi'(x) = [\psi''(x), p(x)]$ for all $x \in Y'$.

- If $d = 0$, then $\varphi = \tilde{\varphi} \circ p$, $\psi' = \psi$ and $\varphi' = 0$; so ψ' is convenient in that case.
- If $d = 1$, we can write

$$u_1^*(x)\varphi(x) = \tilde{\varphi}(p(x)) = [\tilde{\psi}(p(x)), e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \cdots + v_m^*(x)u_m)],$$

for all $x \in Y$, whence we deduce

$$u_1^*(x)(\varphi(x) - [\psi'(x), p(x)]) = [\tilde{\psi}(e), e + u_1^*(x)(u_1 + v_2^*(x)u_2 + \cdots + v_m^*(x)u_m)]$$

for all $x \in Y$. Hence $\tilde{\psi}(e)$ belongs to \mathfrak{g}^e and $[\tilde{\psi}(e), u_i] \in [e, \mathfrak{g}]$ for all $i = 1, \dots, m$, since $\varphi(x) \in [e, \mathfrak{g}]$ for all $x \in Y \cap p^{-1}(e)$. Then $\tilde{\psi}(e)$ is in $\mathfrak{z}(\mathfrak{g}^e)$ by Lemma 5.9. So by Corollary 5.8, φ' has the desired property.

• Suppose $d > 1$. For $\underline{i} = (i_1, \dots, i_m) \in \mathbb{N}^m$, we set $|\underline{i}| := i_1 + \cdots + i_m$ and we denote by $\psi_{\underline{i}}$ the coefficient of $(u_1^*)^{i_1} \cdots (u_m^*)^{i_m}$ in $\tilde{\psi}$. By Corollary 5.8, it suffices to prove:

$$\begin{cases} \psi_{\underline{i}} = 0 & \text{if } |\underline{i}| < d - 1; \\ \psi_{\underline{i}} \in \mathfrak{z}(\mathfrak{g}^e) & \text{if } |\underline{i}| = d - 1 \end{cases}.$$

For $\underline{i} \in \mathbb{N}^m$ and $j \in \{1, \dots, m\}$, we define the element $\underline{i}(j)$ of \mathbb{N}^m by:

$$\underline{i}(j) := (i_1, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_m).$$

It suffices to prove:

Claim 5.11. For $|\underline{i}| \leq d - 1$, $\psi_{\underline{i}}$ is an element of \mathfrak{g}^e such that $[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0$ for $j = 1, \dots, m$.

Indeed, by Lemma 5.9, if

$$[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0 \text{ and } \psi_{\underline{i}} \in \mathfrak{g}^e$$

for all $j = 1, \dots, m$, then $\psi_{\underline{i}} \in \mathfrak{z}(\mathfrak{g}^e)$. Furthermore, if

$$[\psi_{\underline{i}}, u_j] + [\psi_{\underline{i}(j)}, e] = 0 \text{ and } \psi_{\underline{i}} \in \mathfrak{g}^e \text{ and } \psi_{\underline{i}(j)} \in \mathfrak{g}^e$$

for all $j = 1, \dots, m$, then $\psi_{\underline{i}} = 0$ since $\mathfrak{z}(\mathfrak{g}^e) \cap \mathfrak{g}^f = 0$. So only remains to prove Claim 5.11.

We prove the claim by induction on $|\underline{i}|$. Arguing as in the case $d = 1$, we prove the claim for $|\underline{i}| = 0$. We suppose the claim true for all $|\underline{i}| \leq l - 1$ for some $0 < l \leq d - 2$. We have to prove the statement for all $|\underline{i}| \leq l$. By what foregoes and by induction hypothesis, $\psi_{\underline{i}} = 0$ for $|\underline{i}| \leq l - 2$. For $k = l + 1, l + 2$, we consider the ring $\mathbb{k}[\tau_k]$ where $\tau_k^k = 0$. Since $(u_1^*)^d$ vanishes on the set of $\mathbb{k}[\tau_{l+1}]$ -points $x = x_0 + x_1\tau_{l+1} + \cdots + x_l\tau_{l+1}^l$ of Y whose source x_0 is a zero of u_1^* ,

$$0 = [\tilde{\psi}(e + \tau_{l+1}v), e + \tau_{l+1}v] = \sum_{|\underline{i}|=l} \tau_{l+1}^l [\psi_{\underline{i}}, e] (u_1^*)^{i_1} \cdots (u_m^*)^{i_m}(v),$$

for all $v \in \mathfrak{g}^f$. So $\psi_{\underline{i}} \in \mathfrak{g}^e$ for $|\underline{i}| = l$.

For $|\underline{i}|$ equal to l , the term in

$$\tau_{l+2}^{l+1}(u_1^*)^{i_1} \cdots (u_{i_{j-1}}^*)^{i_{j-1}} (u_{i_j+1}^*)^{i_j+1} (u_{i_{j+1}}^*)^{i_{j+1}} \cdots (u_m^*)^{i_m}(v)$$

of $[\tilde{\psi}(e + \pi_{+2}v), e + \pi_{+2}v]$ is equal to $[\psi_{\underline{i}(j)}, e] + [\psi_{\underline{i}}, u_j]$. Since $(u_1^*)^d$ vanishes on the set of $\mathbb{k}[\tau_{+2}]$ -points of Y whose source is a zero of u_1^* , this term is equal to 0, whence the claim. \square

Recall that Y' is an affine open subset of Y contained in Ω and having a nonempty intersection with $p^{-1}(e)$.

Corollary 5.12. *Suppose that e has Property (P_2) . Let φ be in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y'$. Then there exists ψ in $\mathbb{k}[Y'] \otimes_{\mathbb{k}} \mathfrak{g}$ such that $\varphi(x) = [\psi(x), p(x)]$ for all $x \in Y'$.*

Proof. For $a \in \mathbb{k}[Y]$, denote by $D(a)$ the principal open subset defined by a . Let $D(a_1), \dots, D(a_m)$ be an open covering of Y' by principal open subsets of Y , with a_1, \dots, a_m in $\mathbb{k}[Y]$. Since φ is a regular map from Y' to \mathfrak{g} , there is $m_i \geq 0$ such that $a_i^{m_i} \varphi$ is the restriction to Y' of some regular map φ_i from Y to \mathfrak{g} . For m_i big enough, φ_i vanishes on $Y \setminus D(a_i)$; hence $\varphi_i(x) \in [\mathfrak{g}, p(x)]$ for all $x \in Y$. So, by Proposition 5.6, there is a regular map ψ_i from Y' to \mathfrak{g} such that $\varphi_i(x) = [\psi_i(x), p(x)]$ for all $x \in Y'$. Then for all $x \in D(a_i)$, we have $\varphi(x) = [a_i(x)^{-m_i} \psi_i(x), p(x)]$. Since Y' is an affine open subset of Y , there exists a regular map ψ from Y' to \mathfrak{g} which satisfies the condition of the corollary. \square

We are now in position to prove the main result of this section:

Theorem 5.13. *The equality $\text{ind } \mathfrak{g}^e = \ell$ holds if and only if e has Property (P_2) .*

Proof. By Corollary 5.12, if e has Property (P_2) , then Assertion (B) of Theorem 5.1 is satisfied. Conversely, suppose that $\text{ind } \mathfrak{g}^e = \ell$ and show that e has Property (P_2) . By Theorem 5.1, (A) \Rightarrow (B), Assertion (B) is satisfied. We choose an affine open subset Y' of Y , contained in Ω , such that $Y' \cap p^{-1}(e) \neq \emptyset$ and verifying the condition of the assertion (B). Let $z \in \mathfrak{z}(\mathfrak{g}^e)$. Recall that the map φ_z is defined by (5). Let x be in Y' . If $u_1^*(x) \neq 0$, then $\varphi_z(x)$ belongs to $[\mathfrak{g}, p(x)]$ by (5). If $u_1^*(x) = 0$, then by Lemma 5.9, $\varphi_z(x)$ belongs to $[e, \mathfrak{g}]$. So there exists a regular map ψ from Y' to \mathfrak{g} such that $\varphi_z(x) = [\psi(x), p(x)]$ for all $x \in Y'$ by Assertion (B). Hence we have

$$[z - u_1^* \psi(x), p(x)] = 0,$$

for all $x \in Y'$ since $(u_1^* \varphi_z)(x) = [z, p(x)]$ for all $x \in Y$. So $\alpha(x)$ contains z for all x in $\Omega \cap Y' \cap p^{-1}(e)$. Since $p^{-1}(e)$ is irreducible, we deduce that e has Property (P_2) . \square

5.3. A new formulation of the property (P_2) . Recall that Property (P_2) is introduced in Definition 5.2. As has been noticed in the proof of Lemma 5.4, the morphism $G \times \mathcal{S} \rightarrow \mathfrak{g}, (g, x) \mapsto g(x)$ is smooth. As a consequence, the set \mathcal{S}_{reg} of $v \in \mathcal{S}$ such that v is regular is a nonempty open subset of \mathcal{S} . For x in \mathcal{S}_{reg} , $\mathfrak{g}^{e+t(x-e)}$ has dimension ℓ for all t in a nonempty open subset of \mathbb{k} since $x = e + (x - e)$ is regular. Furthermore, since \mathbb{k} has dimension 1, [Sh94, Ch. VI, Theorem 1] asserts that there is a unique regular map

$$\beta_x : \mathbb{k} \rightarrow \text{Gr}_{\ell}(\mathfrak{g})$$

satisfying $\beta_x(t) = \mathfrak{g}^{e+t(x-e)}$ for all t in a nonempty open subset of \mathbb{k} .

Recall that Y is an affine open subset of $B_e(\mathcal{S})$ with $Y \cap p^{-1}(e) \neq \emptyset$ and that $u_1^*, v_2^*, \dots, v_m^*$ is a global coordinates system of Y , cf. (4). Let $\mathcal{S}'_{\text{reg}}$ be the subset of x in \mathcal{S}_{reg} such that $u_1^*(x) \neq 0$. For x in $\mathcal{S}'_{\text{reg}}$, we denote by \tilde{x} the element of Y whose coordinates are $0, v_2^*(x), \dots, v_m^*(x)$.

Lemma 5.14. *Let x be in $\mathcal{S}'_{\text{reg}}$.*

- (i) *The subspace $\beta_x(0)$ is contained in \mathfrak{g}^e .*
- (ii) *If $\tilde{x} \in \Omega$, then $\alpha(\tilde{x}) = \beta_x(0)$.*

Proof. (i) The map β_x is a regular map and $[\beta_x(t), e + t(x - e)] = 0$ for all t in a nonempty open subset of \mathbb{k} . So, $\beta_x(0)$ is contained in \mathfrak{g}^e .

(ii) Since $\mathcal{S}'_{\text{reg}}$ has an empty intersection with the nullvariety of u_1^* in \mathcal{S} , the restriction of p to $p^{-1}(\mathcal{S}'_{\text{reg}})$ is an isomorphism from $p^{-1}(\mathcal{S}'_{\text{reg}})$ to $\mathcal{S}'_{\text{reg}}$. Furthermore, $\beta_x(t) = \alpha(p^{-1}(e + tx - te))$ for any t in \mathbb{k} such that $e + t(x - e)$ belongs to $\mathcal{S}'_{\text{reg}}$ and $p^{-1}(e + tx - te)$ goes to \tilde{x} when t goes to 0. Hence $\beta_x(0)$ is equal to $\alpha(\tilde{x})$ since α and β are regular maps. \square

Corollary 5.15. *The element e has Property (P₂) if and only if $\mathfrak{z}(\mathfrak{g}^e) \subset \beta_x(0)$ for all x in a nonempty subset of $\mathcal{S} = e + \mathfrak{g}^f$.*

Proof. The map $x \mapsto \tilde{x}$ from $\mathcal{S}'_{\text{reg}}$ to Y is well-defined and its image is an open subset of $Y \cap p^{-1}(e)$. Let $\mathcal{S}''_{\text{reg}}$ be the set of $x \in \mathcal{S}'_{\text{reg}}$ such that $\tilde{x} \in \Omega$ and let Y'' be the image of $\mathcal{S}''_{\text{reg}}$ by the map $x \mapsto \tilde{x}$. Then $\mathcal{S}''_{\text{reg}}$ is open in \mathcal{S}_{reg} and Y'' is dense in $\Omega \cap p^{-1}(e)$ since $p^{-1}(e)$ is irreducible. Since α is regular, e has property (P₂) if and only if $\alpha(x)$ contains $\mathfrak{z}(\mathfrak{g}^e)$ for all x in Y'' . By Lemma 5.14(ii), the latter property is equivalent to the fact that $\beta_x(0)$ contains $\mathfrak{z}(\mathfrak{g}^e)$ for all x in $\mathcal{S}''_{\text{reg}}$. \square

Corollary 5.16. (i) *If $\mathfrak{z}(\mathfrak{g}^e)$ is generated by $\varphi_1(e), \dots, \varphi_\ell(e)$, then e has Property (P₂).*

- (ii) *If $\mathfrak{z}(\mathfrak{g}^e)$ has dimension 1, then e has Property (P₂).*

Proof. Recall that $\varphi_i(e)$ belongs to $\mathfrak{z}(\mathfrak{g}^e)$, for all $i = 1, \dots, \ell$, by Lemma 2.1(i). Moreover, for all x in \mathcal{S}_{reg} and all $i = 1, \dots, \ell$, $\varphi_i(e + t(x - e))$ belongs to $\mathfrak{g}^{e+t(x-e)}$ for any t in \mathbb{k} . So by continuity, $\varphi_i(e)$ belongs to $\beta_x(0)$. As a consequence, whenever $\mathfrak{z}(\mathfrak{g}^e)$ is generated by $\varphi_1(e), \dots, \varphi_\ell(e)$, e has Property (P₂) by Corollary 5.15.

- (ii) is an immediate consequence of (i) since $\varphi_1(e) = e$ by our choice of d_1 . \square

Remark 5.17. When \mathfrak{g} is simple of classical type, $\mathfrak{z}(\mathfrak{g}^e)$ is generated by $\varphi_1(e), \dots, \varphi_\ell(e)$ means that $\mathfrak{z}(\mathfrak{g}^e)$ is generated by powers of e . For example, this situation always occurs when \mathfrak{g} has type A or C and occurs in most cases when \mathfrak{g} has type B or D (cf. [Y06b] and [Mo06c, Théorème 1.1.8]).

6. PROOF OF THEOREM 1.2 FOR RIGID NILPOTENT ORBITS

We intend to prove in this section the following theorem:

Theorem 6.1. *Suppose that \mathfrak{g} is reductive and let e be a rigid nilpotent element of \mathfrak{g} . Then the index of \mathfrak{g}^e is equal to ℓ .*

Theorem 6.1 will complete the proof of Theorem 1.2 by Theorem 4.10. As explained in introduction, we can assume that \mathfrak{g} is simple. We consider two cases, according to \mathfrak{g} has classical type or exceptional type.

6.1. The classical case. We consider here the classical case.

Theorem 6.2. *Assume that \mathfrak{g} is simple of classical type and let e be a rigid nilpotent element. Then $\mathfrak{z}(\mathfrak{g}^e)$ is generated by powers of e . In particular, the index of \mathfrak{g}^e is equal to ℓ .*

Proof. The second assertion results from Remark 5.17, Corollary 5.16(i) and Theorem 5.13. Furthermore, by Remark 5.17, we can assume that \mathfrak{g} has type B or D.

Set $n := 2\ell + 1$ if \mathfrak{g} has type B_ℓ and $n := 2\ell$ if \mathfrak{g} has type D_ℓ . Denote by (n_1, \dots, n_k) the partition of n corresponding to the nilpotent element e . By [Mo06c, Théorème 1.1.8], $\mathfrak{z}(\mathfrak{g}^e)$ is *not* generated by powers of e if and only if n_1 and n_2 are both odd integers and $n_3 < n_2$. On the other hand, as e is rigid, n_k is equal to 1, $n_i \leq n_{i+1} \leq n_i + 1$ and all odd integers of the partition (n_1, \dots, n_k) have a multiplicity different from 2; cf. [CMA93, Corollary 7.3.5]. Hence, the preceding criterion is not satisfied for e ; so $\mathfrak{z}(\mathfrak{g}^e)$ is generated by powers of e . \square

Remark 6.3. Yakimova's proof of Elashvili's conjecture in the classical case is shorter and more elementary [Y06a]. The results of Section 5 will serve the exceptional case in a more relevant way.

6.2. The exceptional case. We let in this subsection \mathfrak{g} be simple of exceptional type and we assume that e is a nonzero rigid nilpotent element of \mathfrak{g} . The dimension of the center of centralizers of nilpotent elements has been recently described in [LT08, Theorem 4]. On the other hand, we have explicit computations for the rigid nilpotent orbits in the exceptional types due to A.G. Elashvili. These computations are collected in [Sp82, Appendix of Chap. II] and a complete version was published later in [E85b]. From all this, we observe that the center of \mathfrak{g}^e has dimension 1 in most cases. In more details, we have:

Proposition 6.4. *Let e be nonzero rigid nilpotent element of \mathfrak{g} .*

- (i) *Suppose that \mathfrak{g} has type G_2, F_4 or E_6 . Then $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*
- (ii) *Suppose that \mathfrak{g} has type E_7 . If \mathfrak{g}^e has dimension 41, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 2$; otherwise $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*
- (iii) *Suppose that \mathfrak{g} has type E_8 . If \mathfrak{g}^e has dimension 112, 84, 76, or 46, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 2$, if \mathfrak{g}^e has dimension 72, then $\dim \mathfrak{z}(\mathfrak{g}^e) = 3$; otherwise $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$.*

By Corollary 5.16(ii), $\text{ind } \mathfrak{g}^e = \ell$ whenever $\dim z(\mathfrak{g}^e) = 1$. So, as an immediate consequence of Proposition 6.4, we obtain:

Corollary 6.5. *Suppose that either \mathfrak{g} has type G_2, F_4, E_6 , or \mathfrak{g} has type E_7 and $\dim \mathfrak{g}^e \neq 41$, or \mathfrak{g} has type E_8 and $\dim \mathfrak{g}^e \notin \{112, 84, 76, 72, 46\}$. Then $\dim \mathfrak{z}(\mathfrak{g}^e) = 1$ and the index of \mathfrak{g}^e is equal to ℓ .*

According to Corollary 6.5, it remains 7 cases; there are indeed two rigid nilpotent orbits of dimension 46 in E_8 . We handle now these remaining cases. We process here in a different way; we

will study technical conditions on \mathfrak{g}^e under which $\text{ind } \mathfrak{g}^e = \ell$. For the moment, we state general results about the index.

Let \mathfrak{a} be an algebraic Lie algebra. Recall that the stabilizer of $\xi \in \mathfrak{a}^*$ for the coadjoint representation is denoted by \mathfrak{a}^ξ and that ξ is regular if $\dim \mathfrak{a}^\xi = \text{ind } \mathfrak{a}$. Choose a commutative subalgebra \mathfrak{t} of \mathfrak{a} consisted of semisimple elements of \mathfrak{a} and denote by $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})$ the centralizer of \mathfrak{t} in \mathfrak{a} . Then $\mathfrak{a} = \mathfrak{z}_\mathfrak{a}(\mathfrak{t}) \oplus [\mathfrak{t}, \mathfrak{a}]$. The dual $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ of $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})$ identifies to the orthogonal complement of $[\mathfrak{t}, \mathfrak{a}]$ in \mathfrak{a}^* . Thus, $\xi \in \mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ if and only if \mathfrak{t} is contained in \mathfrak{a}^ξ .

Lemma 6.6. *Suppose that there exists ξ in $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ such that $\dim(\mathfrak{a}^\xi \cap [\mathfrak{t}, \mathfrak{a}]) \leq 2$. Then*

$$\text{ind } \mathfrak{a} \leq \text{ind } \mathfrak{z}_\mathfrak{a}(\mathfrak{t}) + 1.$$

Proof. Let T be the closure in $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^* \times \text{Gr}_3([\mathfrak{t}, \mathfrak{a}])$ of the subset of elements (η, E) such that η is a regular element of $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ and E is contained in \mathfrak{a}^η . The image T_1 of T by the projection from $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^* \times \text{Gr}_3([\mathfrak{t}, \mathfrak{a}])$ to $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ is closed in $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$. By hypothesis, T_1 is not equal to $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ since for all η in T_1 , $\dim(\mathfrak{a}^\eta \cap [\mathfrak{t}, \mathfrak{a}]) \geq 3$. Hence there exists a regular element ξ_0 in $\mathfrak{z}_\mathfrak{a}(\mathfrak{t})^*$ such that $\dim(\mathfrak{a}^{\xi_0} \cap [\mathfrak{t}, \mathfrak{a}]) \leq 2$. Since \mathfrak{t} is contained in \mathfrak{a}^{ξ_0} ,

$$\mathfrak{a}^{\xi_0} = \mathfrak{z}_\mathfrak{a}(\mathfrak{t})^{\xi_0} \oplus \mathfrak{a}^{\xi_0} \cap [\mathfrak{t}, \mathfrak{a}].$$

If $[\mathfrak{t}, \mathfrak{a}] \cap \mathfrak{a}^{\xi_0} = \{0\}$ then $\text{ind } \mathfrak{a}$ is at most $\text{ind } \mathfrak{z}_\mathfrak{a}(\mathfrak{t})$. Otherwise, \mathfrak{a}^{ξ_0} is not a commutative subalgebra since \mathfrak{t} is contained in \mathfrak{a}^{ξ_0} . Hence ξ_0 is not a regular element of \mathfrak{a}^* , so $\text{ind } \mathfrak{a} < \dim \mathfrak{a}^{\xi_0}$. Since $\dim \mathfrak{a}^{\xi_0} \leq \text{ind } \mathfrak{z}_\mathfrak{a}(\mathfrak{t}) + 2$, the lemma follows. \square

From now on, we assume that $\mathfrak{a} = \mathfrak{g}^e$. As a rigid nilpotent element of \mathfrak{g} , e is a nondistinguished nilpotent element. So we can choose a nonzero commutative subalgebra \mathfrak{t} of \mathfrak{g}^e consisted of semisimple elements. Denote by \mathfrak{l} the centralizer of \mathfrak{t} in \mathfrak{g} . As a Levi subalgebra of \mathfrak{g} , \mathfrak{l} is reductive Lie algebra whose rank is ℓ . Moreover its dimension is strictly smaller than $\dim \mathfrak{g}$. In the preceding notations, we have $\mathfrak{z}_{\mathfrak{g}^e}(\mathfrak{t}) = \mathfrak{z}_\mathfrak{g}(\mathfrak{t})^e = \mathfrak{l}^e$. Let \mathfrak{t}_1 be a commutative subalgebra of \mathfrak{l}^e containing \mathfrak{t} and consisted of semisimple elements of \mathfrak{l} . Then $[\mathfrak{t}, \mathfrak{g}^e]$ is stable under the adjoint action of \mathfrak{t}_1 . For λ in \mathfrak{t}_1^* , denote by \mathfrak{g}_λ^e the λ -weight space of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^e .

Lemma 6.7. *Let $\lambda \in \mathfrak{t}_1^*$ be a nonzero weight of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^e . Then $-\lambda$ is also a weight for this action and λ and $-\lambda$ have same multiplicity. Moreover, \mathfrak{g}_λ^e is contained in $[\mathfrak{t}, \mathfrak{g}^e]$ if and only if the restriction of λ to \mathfrak{t} is not identically zero.*

Proof. By definition, $\mathfrak{g}_\lambda^e \cap \mathfrak{l}^e = \{0\}$ if and only if the restriction of λ to \mathfrak{t} is not identically zero. So \mathfrak{g}_λ^e is contained in $[\mathfrak{t}, \mathfrak{g}^e]$ if and only if the restriction of λ to \mathfrak{t} is not equal to 0 since

$$\mathfrak{g}_\lambda^e = (\mathfrak{g}_\lambda^e \cap \mathfrak{l}^e) \oplus (\mathfrak{g}_\lambda^e \cap [\mathfrak{t}, \mathfrak{g}^e]).$$

The subalgebra \mathfrak{t}_1 is contained in a reductive factor of \mathfrak{g}^e . So we can choose h and f such that \mathfrak{t}_1 is contained in $\mathfrak{g}^e \cap \mathfrak{g}^f$. As a consequence, any weight of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^f is a weight of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^e with the same multiplicity. Furthermore, the \mathfrak{t}_1 -module \mathfrak{g}^f for the adjoint action is isomorphic to the \mathfrak{t}_1 -module $(\mathfrak{g}^e)^*$ for the coadjoint action. So $-\lambda$ is a weight of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^f with the same multiplicity as λ . Hence $-\lambda$ is a weight of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^e with the same multiplicity as λ , whence the lemma. \square

Choose pairwise different elements $\lambda_1, \dots, \lambda_m$ of \mathfrak{t}_1^* so that the weights of the adjoint action of \mathfrak{t}_1 in \mathfrak{g}^e which are not identically zero on \mathfrak{t} are precisely the elements $\pm\lambda_i$. For $i = 1, \dots, m$, let $v_{i,1}, \dots, v_{i,m_i}$ and $w_{i,1}, \dots, w_{i,m_i}$ be basis of $\mathfrak{g}_{\lambda_i}^e$ and $\mathfrak{g}_{-\lambda_i}^e$ respectively. Then we set:

$$q_i := \det([v_{i,k}, w_{i,l}]_{1 \leq k, l \leq m_i}) \in S(\mathfrak{l}^e).$$

Proposition 6.8. *Suppose that $\text{ind } \mathfrak{l}^e = \ell$ and suppose that one of the following two conditions is satisfied:*

- (1) for $i = 1, \dots, m$, $q_i \neq 0$,
- (2) there exists j in $\{1, \dots, m\}$ such that $q_i \neq 0$ for all $i \neq j$ and such that the basis $v_{j,1}, \dots, v_{j,m_j}$ and $w_{j,1}, \dots, w_{j,m_j}$ can be chosen so that

$$\det([v_{j,k}, w_{j,l}]_{1 \leq k, l \leq m_j-1}) \neq 0.$$

Then, $\text{ind } \mathfrak{g}^e = \ell$.

Proof. First, observe that $\text{ind } \mathfrak{g}^e - \text{ind } \mathfrak{g}$ is an even integer. Indeed, we have:

$$\text{ind } \mathfrak{g}^e - \text{ind } \mathfrak{g} = (\text{ind } \mathfrak{g}^e - \dim \mathfrak{g}^e) + (\dim \mathfrak{g}^e - \dim \mathfrak{g}) + (\dim \mathfrak{g} - \text{ind } \mathfrak{g}).$$

But the integers $\text{ind } \mathfrak{g}^e - \dim \mathfrak{g}^e$, $\dim \mathfrak{g}^e - \dim \mathfrak{g}$ and $\dim \mathfrak{g} - \text{ind } \mathfrak{g}$ are all even integers. Thereby if $\text{ind } \mathfrak{g}^e \leq \text{ind } \mathfrak{g} + 1$, then $\text{ind } \mathfrak{g}^e \leq \text{ind } \mathfrak{g}$. In turn, by Vinberg's inequality (cf. Introduction), we have $\text{ind } \mathfrak{g}^e \geq \text{ind } \mathfrak{g}$. Hence, it suffices to prove $\text{ind } \mathfrak{g}^e \leq \text{ind } \mathfrak{l}^e + 1$ since our hypothesis says that $\text{ind } \mathfrak{l}^e = \ell = \text{ind } \mathfrak{g}$. Now, by Lemma 6.6, if there exists ξ in $(\mathfrak{l}^e)^*$ such that $(\mathfrak{g}^e)^\xi \cap [\mathfrak{t}, \mathfrak{g}^e]$ has dimension at most 2, we are done.

Denote by \mathfrak{l}_1 the centralizer of \mathfrak{t}_1 in \mathfrak{g} . Then \mathfrak{l}_1 is contained in \mathfrak{l} and $\mathfrak{l}^e = \mathfrak{l}_1^e \oplus [\mathfrak{t}_1, \mathfrak{l}^e]$. We identify $(\mathfrak{l}_1^e)^*$ to the orthogonal of $[\mathfrak{t}_1, \mathfrak{l}^e]$ in the dual of \mathfrak{l}^e . Moreover, for $i = 1, \dots, m$, q_i belongs to $S(\mathfrak{l}_1^e)$. For ξ in $(\mathfrak{l}_1^e)^*$, denote by B_ξ the bilinear form

$$\begin{aligned} [\mathfrak{t}, \mathfrak{g}^e] \times [\mathfrak{t}, \mathfrak{g}^e] &\longrightarrow \mathbb{k} \\ (v, w) &\longmapsto \xi([v, w]) \end{aligned}$$

and denote by $\ker B_\xi$ its kernel. For $i = 1, \dots, m$, $-q_i(\xi)^2$ is the determinant of the restriction of B_ξ to the subspace

$$(\mathfrak{g}_{\lambda_i}^e \oplus \mathfrak{g}_{-\lambda_i}^e) \times (\mathfrak{g}_{\lambda_i}^e \oplus \mathfrak{g}_{-\lambda_i}^e)$$

in the basis $v_{i,1}, \dots, v_{i,m_i}, w_{i,1}, \dots, w_{i,m_i}$.

If (1) holds, we can find ξ in $(\mathfrak{l}_1^e)^*$ such that $\ker B_\xi$ is zero. If (2) holds, we can find ξ in $(\mathfrak{l}_1^e)^*$ such that $\ker B_\xi$ has dimension 2 since B_ξ is invariant under the adjoint action of \mathfrak{t}_1 . But $\ker B_\xi$ is equal to $(\mathfrak{g}^e)^\xi \cap [\mathfrak{t}, \mathfrak{g}^e]$. Hence such a ξ satisfies the required inequality and the proposition follows. \square

The following proposition is proven in Appendix A since it relies on explicit computations:

Proposition 6.9. (i) *Suppose that either \mathfrak{g} has type E_7 and $\dim \mathfrak{g}^e = 41$ or, \mathfrak{g} has type E_8 and $\dim \mathfrak{g}^e \in \{112, 72\}$. Then, for suitable choices of \mathfrak{t} and \mathfrak{t}_1 , Condition (1) of Proposition 6.8 is satisfied.*

(ii) Suppose that \mathfrak{g} has type E_8 and that \mathfrak{g}^e has dimension 84, 76, or 46. Then, for suitable choices of \mathfrak{t} and \mathfrak{t}_1 , Condition (2) of Proposition 6.4 is satisfied.

6.3. Proof of Theorem 1.2. We are now in position to complete the proof of Theorem 1.2:

Proof of Theorem 1.2. We argue by induction on the dimension of \mathfrak{g} . If \mathfrak{g} has dimension 3, the statement is known. Assume now that $\text{ind} \mathfrak{l}^{e'} = \text{rk} \mathfrak{l}$ for any reductive Lie algebras \mathfrak{l} of dimension at most $\dim \mathfrak{g} - 1$ and any $e' \in \mathcal{N}(\mathfrak{l})$. Let $e \in \mathcal{N}(\mathfrak{g})$ be a nilpotent element of \mathfrak{g} . By Theorem 4.10 and Theorem 6.2, we can assume that e is rigid and that \mathfrak{g} is simple of exceptional type. Furthermore by Corollary 6.5, we can assume that $\dim \mathfrak{z}(\mathfrak{g}^e) > 1$. Then we consider the different cases given by Proposition 6.9.

If, either \mathfrak{g} has type E_7 and $\dim \mathfrak{g}^e = 41$, or \mathfrak{g} has type E_8 and $\dim \mathfrak{g}^e$ equals 112, 72, or 46, then Condition (1) of Proposition 6.8 applies for suitable choices of \mathfrak{t} and \mathfrak{t}_1 by Proposition 6.9. Moreover, if $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$, then \mathfrak{l} is a reductive Lie algebra of rank ℓ and strictly contained in \mathfrak{g} . So, from our induction hypothesis, we deduce that $\text{ind} \mathfrak{g}^e = \ell$ by Proposition 6.8.

If \mathfrak{g} has type E_8 and $\dim \mathfrak{g}^e$ equals 84, 76, or 46, then Condition (2) of Proposition 6.8 applies for suitable choices of \mathfrak{t} and \mathfrak{t}_1 by Proposition 6.9. Arguing as above, we deduce that $\text{ind} \mathfrak{g}^e = \ell$. \square

APPENDIX A. PROOF OF PROPOSITION 6.9: EXPLICIT COMPUTATIONS.

In this appendix, we give the necessary computations to prove Proposition 6.9. These computations are made with the help of **GAP**; we give below our programmes (two cases are detailed; the other ones are similar). Explain now the general approach. To each rigid nilpotent orbit as considered in Proposition 6.9, we provide a representative e of the nilpotent orbit. Then, we exhibit suitable tori \mathfrak{t} and \mathfrak{t}_1 of \mathfrak{g} contained in \mathfrak{g}^e which satisfies conditions (1) or (2) of Proposition 6.8. In each case, our torus \mathfrak{t} is one dimensional; we define it by a generator called \mathfrak{t} in our programmes. Its centralizer in \mathfrak{g}^e is denoted by $\mathfrak{1e}$. The torus \mathfrak{t}_1 has dimension at most 4. It is defined by a basis denoted by $\mathfrak{Bt1}$. The weights of \mathfrak{t}_1 for the adjoint action of \mathfrak{t}_1 on \mathfrak{g}^e are given by their values on the basis $\mathfrak{Bt1}$ of \mathfrak{t}_1 . We list in a matrix \mathfrak{W} almost all weights which have a positive value at $\mathfrak{Bt1}$. The other weights have multiplicity 1. Then in our programmes, by the term \mathfrak{S} we check that no weight is forgotten; this term has to be zero. Then, the matrices corresponding to the weights given by \mathfrak{W} are given by a function \mathfrak{A} . If there is only one other weight, the corresponding matrix is denoted by \mathfrak{a} . At last, we verify that these matrices have the desired property depending on the situations (i) or (ii) of Proposition 6.9.

As examples, we detail below two cases:

(1) the case of E_7 with $\dim \mathfrak{g}^e = 41$ where we intend to check that Condition (1) of Proposition 6.8 is satisfied;

(2) the case of E_8 with $\dim \mathfrak{g}^e = 84$ where we intend to check that Condition (2) of Proposition 6.8 is satisfied.

(1) E_7 , $\dim \mathfrak{g}^e = 41$: In this case, $\dim \mathfrak{l}^e = 23$ and $\dim \mathfrak{t}_1 = 3$. The order of matrices to be considered is at most 2.

```

L := SimpleLieAlgebra("E",7,Rationals);; R := RootSystem(L);;
x := PositiveRootVectors(R);; y := NegativeRootVectors(R);;
e := x[14]+x[26]+x[28]+x[49];;
c := LieCentralizer(L,Subspace(L,[e]));
Bc := BasisVectors(Basis(c));;
z := LieCentre(c);; Bz := BasisVectors(Basis(z));;
t := Bc[Dimension(c)];;
le := LieCentralizer(L,Subspace(L,[t],e));
> <Lie algebra of dimension 23 over Rationals>
n := function(k) if k=2 then return 1;;
    elif k=-2 then return 1;;
    elif k=1 then return 8;;
    elif k=-1 then return 8;;
    fi;; end;;
M := function(k) local m;;
    m := function(j,k) if j=1 then return Position(List([1..Dimension(c)],
    i->t*Bc[i]-k*Bc[i]),0*x[1]);;
    else return m(j-1,k)+Position(List([m(j-1,k)+1..Dimension(c)],
    i->t*Bc[i]-k*Bc[i]),0*x[1]);;
    fi;; end;; return List([1..n(k)],i->m(i,k));; end;;
P := function(k) if k=1 then return [13,16,17,19,20,23,26,28];;
    elif k=-1 then return [10,11,12,14,15,18,22,25];; fi;; end;;
List([1,-1],i->M(i)=P(i));
> [ true, true ]
Bt1 := [Bc[41],Bc[40],Bc[39]];;
N := function(k,p) local n;;
    n := function(j,k,p) if j=1 then return Position(List([1..8],
    i->Bt1[2]*Bc[P(k)[i]]-p*Bc[P(k)[i]],0*x[1]);;
    else return n(j-1,k,p)+Position(List([n(j-1,k,p)+1..8],
    i->Bt1[2]*Bc[P(k)[i]]-p*Bc[P(k)[i]],0*x[1]);;
    fi;; end;; return List([1..4],i->P(k)[n(i,k,p)]);; end;;
R := function(i,j) if i=1 and j=1 then return [13,16,19,28];;
    elif i=1 and j=-1 then return [17,20,23,26];;
    elif i=-1 and j=-1 then return [12,15,18,22];;
    elif i=-1 and j=1 then return [10,11,14,25];; fi;; end;;
List([1,-1],i->List([1,-1],j->R(i,j)=N(i,j)));
> [ [ true, true ], [ true, true ] ]
r := function(t) if t=1 then return 1;
    elif t=-1 then return 1;;
    elif t=0 then return 2;; fi;; end;;
Q := function(k,s,t) local q;;

```

```

q := function(j,k,s,t)
  if j=1 then return Position(List([1..4],
    i->Bt1[3]*Bc[R(k,s)[i]]-t*Bc[R(k,s)[i]]),0*x[1]);;
  else return q(j-1,k,s,t)+Position(List([q(j-1,k,s,t)+1..4],
    i->Bt1[3]*Bc[R(k,s)[i]]-t*Bc[R(k,s)[i]]),0*x[1]);;
  fi;; end;; return List([1..r(t)],i->R(k,s)[q(i,k,s,t)]);; end;;
W := [[1,1,1],[1,-1,1],[1,1,-1],[1,-1,-1],[1,1,0],[1,-1,0]];
S := 2*(1+Sum(List([1..Length(W)],i->Length(Q(W[i][1],W[i][2],W[i][3]))))
  +Dimension(C)-Dimension(c));
> 0
A := function(i) return List([1..r(W[i][3])],t->List([1..r(W[i][3])],
  s->Bc[Q(W[i][1],W[i][2],W[i][3])[s]]*Bc[Q(-W[i][1],-W[i][2],-W[i][3])[t]]));;
  end;;
A(1,1,1);A(1,-1,1);A(1,1,-1);A(1,-1,-1);A(1,1,0);A(1,-1,0);
> [ [ (-1)*v.63 ] ]
> [ [ v.63 ] ]
> [ [ v.63 ] ]
> [ [ (-1)*v.63 ] ]
> [ [ (-1)*v.57+(-1)*v.60, (-1)*v.63 ], [ (-1)*v.63, 0*v.1 ] ]
> [ [ (-1)*v.57+(-1)*v.60, (-1)*v.63 ], [ (-1)*v.63, 0*v.1 ] ]
a := Bc[M(2)[1]]*Bc[M(-2)[1]];
> v.133

```

In conclusion, Condition (1) of Proposition 6.8 is satisfied for $t := \mathbb{k}t$ and $t_1 := \text{span}(Bt_1)$.

(2) E_8 , $\dim \mathfrak{g}^e = 84$: In this case, $\dim \mathfrak{l}^e = 48$ and $\dim t_1 = 3$. The matrix $A(7)$ has order 5 and it is singular of rank 4. The order of the other matrices is at most 2.

```

L := SimpleLieAlgebra("E",8,Rationals);; R := RootSystem(L);;
x := PositiveRootVectors(R);; y := NegativeRootVectors(R);;
e := x[54]+x[61]+x[77]+x[97];;
c := LieCentralizer(L,Subspace(L,[e]));; Bc := BasisVectors(Basis(c));;
z := LieCentre(c);; Bz := BasisVectors(Basis(z));;
t := Bc[Dimension(c)];;
le := LieCentralizer(L,Subspace(L,[t],e));
> <Lie algebra of dimension 48 over Rationals>
n := function(k) if k=2 then return 1;; elif k=-2 then return 1;;
  elif k=1 then return 17;; elif k=-1 then return 17;; fi;; end;;
M := function(k) local m;; m := function(j,k)
  if j=1 then return Position(List([1..Dimension(c)],
    i->Bc[84]*Bc[i]-k*Bc[i]),0*x[1]);;
  else return m(j-1,k)+Position(List([m(j-1,k)+1..Dimension(c)],

```

```

        i->Bc[84]*Bc[i]-k*Bc[i]), 0*x[1]);;
    fi;; end;; return List([1..n(k)],i->m(i,k));; end;;
P := function(k) if k=1 then return
    [10,14,16,17,19,22,25,29,33,37,40,41,44,48,50,52,67];;
    elif k=-1 then return
    [8,11,12,13,15,18,21,24,28,32,35,36,39,43,47,49,66];; fi;; end;;
List([1,-1],i->M(i)=P(i));;
r := function(k,t) if k=1 and t=1 then return 4;;
    elif k=-1 and t=-1 then return 4;;
    elif k=1 and t=-1 then return 4;;
    elif k=-1 and t=1 then return 4;;
    elif k=1 and t=0 then return 9;;
    elif k=-1 and t=0 then return 9;;fi;; end;;
Bt1 := [Bc[84],Bc[83],Bt1[3]];;
N := function(k,t) local p;;
    p := function(j,k,t)
        if j=1 then return Position(List([1..n(k)],
            i->Bt1[2]*Bc[P(k)[i]]-t*Bc[P(k)[i]]),0*x[1]);;
        else return p(j-1,k,t)+Position(List([p(j-1,k,t)+1..n(k)],
            i->Bt1[2]*Bc[P(k)[i]]-t*Bc[P(k)[i]]),0*x[1]);;
        fi;; end;; return List([1..r(k,t)],i->P(k)[p(i,k,t)]);; end;;
R := function(k,t) if k=1 and t=1 then return [17,19,41,44];;
    elif k=-1 and t=-1 then return [11,12,32,35];;
    elif k=1 and t=-1 then return [14,16,37,40];;
    elif k=-1 and t=1 then return [13,15,36,39];;
    elif k=1 and t=0 then return [10,22,25,29,33,48,50,52,67];;
    elif k=-1 and t=0 then return [8,18,21,24,28,43,47,49,66];;
    fi;; end;;
v := [[1,1],[-1,-1],[1,-1],[-1,1],[1,0],[-1,0]];;
List([1..Length(v)],i->N(v[i][1],v[i][2])=R(v[i][1],v[i][2]));;
m := function(k,s,t) if k=1 and s=1 and t=-1 then return 2;;
    elif k=-1 and s=-1 and t=1 then return 2;;
    elif k=1 and s=1 and t=0 then return 2;;
    elif k=-1 and s=-1 and t=0 then return 2;;
    elif k=1 and s=-1 and t=1 then return 2;;
    elif k=-1 and s=1 and t=-1 then return 2;;
    elif k=1 and s=-1 and t=0 then return 2;;
    elif k=-1 and s=1 and t=0 then return 2;;
    elif k=1 and s=0 and t=1 then return 2;;
    elif k=-1 and s=0 and t=-1 then return 2;;
    elif k=1 and s=0 and t=-1 then return 2;;

```

```

        elif k=-1 and s=0 and t=1 then return 2;;
        elif k=1 and s=0 and t=0 then return 5;;
        elif k=-1 and s=0 and t=0 then return 5;;
        fi;; end;;
Q := function(k,s,t) local q;;
  q := function(j,k,s,t)
    if j=1 then return Position(List([1..r(k,s)],
      i->Bt1[3]*Bc[R(k,s)[i]]-t*Bc[R(k,s)[i]]),0*x[1]);;
    else return q(j-1,k,s,t)+Position(List([q(j-1,k,s,t)+1..r(k,s)],
      i->Bt1[3]*Bc[R(k,s)[i]]-t*Bc[R(k,s)[i]]),0*x[1]);;
    fi;; end;; return List([1..m(k,s,t)],i->R(k,s)[q(i,k,s,t)]);; end;;
W := [[1,1,-1],[1,1,0],[1,-1,1],[1,-1,0],[1,0,1],[1,0,-1],[1,0,0]];
S := 2 + 2*Sum(List([1..Length(W)],i->Length(Q(W[i][1],W[i][2],W[i][3]))))
  + Dimension(C)-Dimension(c);;
A := function(i) return List([1..m(W[i][1],W[i][2],W[i][3])],
  t->List([1..m(W[i][1],W[i][2],W[i][3])],
  s->Bc[Q(W[i][1],W[i][2],W[i][3])[s]]*Bc[Q(-W[i][1],-W[i][2],-W[i][3])[t]]));;
end;;
# A(7) is singular of order 5 of rank 4; its minor
List([1..4],s->List([1..4],t->
Bc[Q(W[7][1],W[7][2],W[7][3])[s]]*Bc[Q(-W[7][1],-W[7][2],-W[7][3])[t]]));;
# is not 0.
a := Bc[M(2)[1]]*Bc[M(-2)[1]];

```

In conclusion, Condition (2) of Proposition 6.8 is satisfied for $t := \mathbb{k}t$ and $t_1 := \text{span}(\text{Bt1})$.

REFERENCES

- [Be72] P. Bernat et al., *Représentations des groupes de Lie résolubles*, Monographies de la Société Mathématique de France **no 4** (1972), Dunod, Paris.
- [Bol91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicae, vol. **24** (1991), **n°1**, 253–274.
- [BoK79] W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reductiver Gruppen*, Commentarii Mathematici Helvetici, vol. **54** (1979), 61–104.
- [Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley*, Springer-Verlag, Berlin (2002).
- [BrK94] R.K. Brylinski and B. Kostant, *Nilpotent orbits, normality, and Hamiltonian group actions*, J. Amer. Math. Soc. **7** (1994), 269–298.
- [Ca85] R.W. Carter, *Finite groups of Lie Type. Conjugacy classes and complex characters*, Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs and Tracts. (1985), Chichester, New-York, Brisbane, Toronto, Singapore.
- [Ch04] J.-Y. Charbonnel, *Propriétés (Q) et (C). Variété commutante*, Bulletin de la Société Mathématique de France, **132** (2004), **no 4**, 477–508.
- [CMo08] J.-Y. Charbonnel and A. Moreau, *Nilpotent bicone and characteristic submodule of a reductive Lie algebra*, Transformation Groups, **14**, (2008).
- [CMa93] D. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Co. New York **65** (1993).

- [De08] W.A. DeGraaf, *Computing with nilpotent orbits in simple Lie algebras of exceptional type*, London Math. Soc. (2008), 1461–1570.
- [Di74] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars (1974).
- [Di75] J. Dixmier, *Polarisations dans les Algèbres de Lie II*, Bulletin de la Société Mathématique de France, **99** (1975), no **4**, 45–63.
- [Di79] J. Dixmier, *Champs de vecteurs adjoints sur les groupes et algèbres de Lie semi-simples*, Journal für die reine und angewandte Mathematik, Band. **309** (1979), 183–190.
- [Dv03] A. Dvorsky, *Index of parabolic and seaweed subalgebras of \mathfrak{so}_n* , Lin. Alg. Appl, **374** (2003), 127–142.
- [E85a] A.G. Elashvili, *On the index of orispherical sublagebras of semisimple Lie algebras*, Proc. Razmadze Math. Institute, Tiflis, **77** (1985), 116–126.
- [E85b] A.G. Elashvili, *Sheets of the exceptional Lie algebras*, in *Issledovaniya po algebre*, Tbilisi (1985), 171–194 (Russian).
- [H77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics n°**52** (1977), Springer-Verlag, Berlin Heidelberg New York.
- [J07] A. Joseph, *On semi-invariants and index for biparabolic (seaweed) algebras. II*, J. Algebra, **312** (2007), no. 1, 158–193.
- [Ko63] B. Kostant, *Lie group representations on polynomial rings*, American Journal of Mathematics n°**85** (1963), 327–404.
- [LT08] R. Lawther and D.M. Testerman, *Centres of centralizers of unipotent elements in simple algebraic groups*, Preprint.
- [LS79] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. **19** (1979), 41–52.
- [MF78] A.S. Mishchenko and A.T. Fomenko, *Euler equations on Lie groups*, Math. USSR-Izv. **12** (1978), 371–389.
- [Mo06a] A. Moreau, *Indice du normalisateur du centralisateur d'un élément nilpotent dans une algèbre de Lie semi-simple*, Bulletin de la Société Mathématique de France, vol. **134** (2006), 83–117.
- [Mo06b] A. Moreau, *Indice et décomposition de Cartan d'une algèbre de Lie semi-simple réelle*, J. Algebra, **303** (2006), no. 1, 382–406.
- [Mo06c] A. Moreau, *Quelques propriétés de l'indice dans une algèbre de Lie semi-simple*, PhD Thesis (2006), available on <http://www.institut.math.jussieu.fr/theses/2006/moreau/>.
- [Pa03a] D.I. Panyushev, *The index of a Lie algebra, the centraliser of a nilpotent element, and the normaliser of the centraliser*, Math. Proc. Camb. Phil. Soc. **134** (2003), 41–59.
- [Pa03b] D.I. Panyushev, *Some amazing properties of spherical nilpotent orbits*, Math. Z., (2003) 245, no.3, 557–580.
- [PPY07] D.I. Panyushev, A. Premet and O. Yakimova, *On symmetric invariants of centralizers in reductive Lie algebras*, Journal of Algebra (2007), n°**1**, 343–391.
- [PY08] D.I. Panyushev and O. Yakimova, *The argument shift method and maximal commutative subalgebras of Poisson algebras*, Mathematical Research Letters, vol. **15**, n°**2** (2008), 239–249.
- [Pe90] A.M. Perelomov, *Integrable systems of classical mechanics and Lie algebras*, Birkhuser Verlag (1990).
- [Ri87] R.W. Richardson, *Derivatives of invariant polynomials on a semisimple Lie algebra in "Microconference on harmonic analysis and operator algebras"*, Proceedings of the Centre of Mathematical Analysis, Australian National University 15, Australian National University, Camberra (1987), 228–241.
- [Ro63] M. Rosenlicht, *A remark on quotient spaces*, An. Acad. Brasil. Cienc. **35** (1963), 487–489.
- [Sh94] I.R. Shafarevich, *Basic algebraic geometry 2*, Springer-Verlag (1994), Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong-Kong, Barcelona, Budapest.
- [Sl80] P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, n°**815** (1980), Springer-Verlag, Berlin, Heidelberg, New York.
- [Sp82] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Springer-Verlag, Berlin (1982).
- [TY04] P. Tauvel and R.W.T. Yu, *Sur l'indice de certaines algèbres de Lie*, Annales de l'Institut Fourier, Grenoble, **54** (2004), n°**6**, 1793–1810.
- [TY05] P. Tauvel and R.W.T. Yu, *Lie algebras and algebraic groups*, Monographs in Mathematics (2005), Springer, Berlin Heidelberg New York.
- [V72] F.D. Veldkamp, *The center of the universal enveloping algebra of a Lie algebra in characteristic p*, Annales Scientifiques de L'École Normale Supérieure n°**5** (1972) , 217–240.
- [Y06a] O. Yakimova, *The index of centralisers of elements in classical Lie algebras*, Functional Analysis and its Applications **40** (2006), 42–51.
- [Y06b] O. Yakimova, *Centres of centralisers un the classical Lie algebras*, preprint available on <http://www.mccme.ru/yakimova/centre/centre.pdf> (2006).
- [Y07] O. Yakimova, *A counterexample to Premet's and Joseph's conjecture*, Bull. London Math. Soc. **39** (2007), 749–754.

UNIVERSITÉ PARIS 7 - CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU, THÉORIE DES GROUPES, CASE 7012, BÂTIMENT
CHEVALERET, 75205 PARIS CEDEX 13, FRANCE

E-mail address: `jyc@math.jussieu.fr`

ANNE MOREAU, ETH ZÜRICH, DEPARTEMENT MATHEMATIK, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND

E-mail address: `anne.moreau@math.ethz.ch`