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# Ramified Random Apollonian Networks

Nicolas Bonnel and Pierre-Francois Marteau and Gildas Ménier

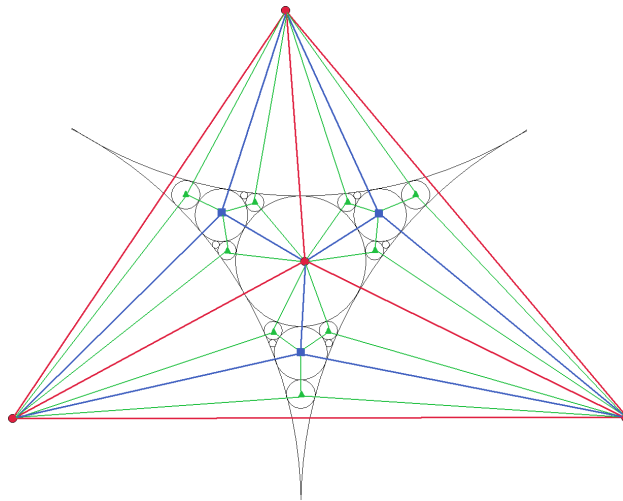
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**Abstract.** We present a simple algorithm that produces a so called Ramified Random Apollonian Networks (R-RAN) in any dimension  $d$ . Analytical derivations show that these networks still exhibit small-world and scale-free characteristics. To characterize further the structure of R-RAN, we introduce new parameters that we refer to as the parallel degree and the ramification coefficient, that determine locally and in average the number of vertices inside the  $(d+1)$ -cliques composing the network. We provide analytical derivations for the computation of the degree and parallel degree distributions, ramification and clustering coefficients and the average path lengths for R-RAN. We finally show that these derivations are in very good agreement with our simulations.

## 1. Introduction

During the last decade, the study of network topologies has become a useful way to tackle the understanding of information flow within complex natural systems. The applications range from sociology, logistics, epidemiology, immunology, neural networks characterization, granular packing analysis, P2P networking, etc. Among a multitude of proposed models, scale-free and small world networks have been widely addressed, essentially because many empirical or real life networks display such properties [5, 8]. This is the case for random graphs, social networks, the web and for gene networks for instance. Basically, scale free networks display a power-law degree distribution,  $p(k) \sim k^{-\gamma}$ , where  $k$  is connectivity (degree) and  $\gamma$  the degree exponent [2], while in small world networks, most vertices can be reached from any other by a small number of hops or steps. Small world networks are characterized by a high clustering coefficient, e.g. a high level of vertices interconnection, and small average path length, namely small minimum path length in average between any pairs of vertices in the network.



**Figure 1.** 2D Apollonian gasket and corresponding network

Among the topologies that display scale free and small world properties, Apollonian networks [1] have recently attracted much attention [7, 6]. Apollonian networks are constructed from a fractal generated from a set of hyper-spheres, where any hyper-sphere is tangent to the others. This fractal is also known as the Apollonian gasket, named after Greek mathematician Apollonius of Perga. The 2D Apollonian network, or Deterministic Apollonian Network (DAN) [1], is obtained by connecting the centers of touching spheres (intrepreted as vertices) in a three-dimensional Apollonian gasket by edges, as shown in Fig 1. The first generation for this fractal network is characterized by red disks vertices, the second generation is characterized by blue square vertices and the third generation is characterized by green triangle vertices. Extension to higher dimension have been provided in [9].

Random Apollonian Networks (RAN) [13], differ from the recursive construction of the DAN, as RAN starts from a  $(d+1)$ -clique (a triangle in dimension 2) containing  $d + 1$  vertices. Then, at each time step, a single  $(d+1)$ -clique is randomly selected from the set of  $(d+1)$ -clique in the network that do not already contain a vertex connected to all the vertices composing the  $(d+1)$ -clique. The selected  $(d+1)$ -clique is then used to insert a new node linking to all of the  $d + 1$  vertices of the selected  $(d+1)$ -clique. 2D Random Apollonian Networks (RAN) have been extensively studied in [13, 12], and extension to high dimension RAN (HDRAN) provided in [10].

In this paper, we extend the RAN or HDRAN topologies to what we call Ramified RAN (R-RAN). This new topology that differs slightly from RAN or HDRAN allows to insert several vertices inside a  $(d+1)$ -clique, each inserted vertices being fully connected to all the vertices composing the clique. This extension constructs networks of ramified random Apollonian structures that we study through out the paper.

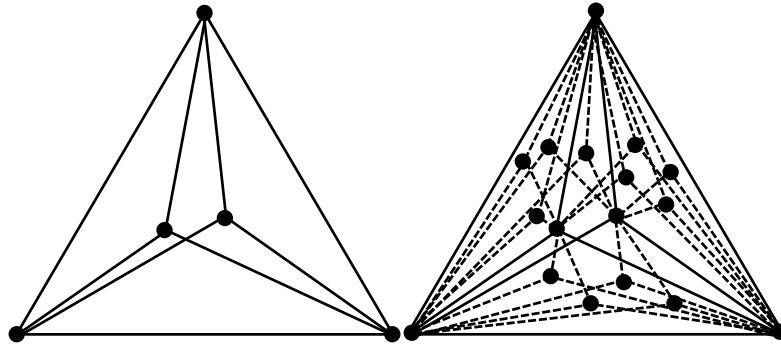
After a short presentation of Ramified Deterministic Apollonian Networks (R-DAN) and Ramified Random Apollonian Networks (R-RAN) in the first two sections, we introduce in the third section the parallel degree distribution and ramification coefficient for such networks and study their asymptotic statistical properties for any dimension. The fourth, fifth and sixth sections give the derivations respectively for the degree distribution and the degree exponent, the clustering coefficient and the average path length in R-RAN. Extensive simulation results are provided through out these sections to validate as far as possible the analytical derivations. A short conclusion ends the paper.

## 2. Ramified Deterministic Apollonian Networks

A ramified deterministic Apollonian network in dimension  $d$  is constructed recursively from an initial  $(d+1)$ -clique allowing to insert at step  $t$  more than one vertex into  $(d+1)$ -cliques composing the network at step  $t - 1$ . Various rules can be adopted for the construction of Ramified Apollonian networks. Some of them lead to Expanded Apollonian networks [11] or recursive clique tree [3] for which at each time step, a new vertex is inserted in every  $(d+1)$ -clique composing the network. In the following subsection, as an example, we propose other rules that lead to a different topology. To characterize the ramification nature of this kind of networks, we introduce what we call the parallel degree  $m \geq 0$  of a  $(d+1)$ -cliques that characterizes the number of vertices inside the clique and fully connected to the vertices composing the clique.

### 2.1. Constructing algorithm

Initially, a network containing  $d + 2$  vertices and  $d + 2$   $(d+1)$ -cliques is created. At each time step  $t$ ,  $m \geq 1$  vertices are added into all existing  $(d+1)$ -cliques created at time step  $t - 1$  in the current network and each new vertex is connected to each vertices of the embedding  $(d+1)$ -clique, creating  $m.(d + 1)$  new  $(d+1)$ -cliques.



**Figure 2.** 2-dimensionnal R-DAN at  $t=0$  (left) and  $t=1$  (right)

### 3. Ramified Random Apollonian Networks

We define Ramified Random Apollonian Networks as RAN in which a new vertex can be inserted at time step  $t$  in any  $(d+1)$ -clique composing the network, whatever its creation time step is. This means that a  $(d+1)$ -clique can contain in its inside more than one vertex fully connected to the vertices composing the clique. To our knowledge, no previous work have been reported specifically on R-RAN. Nevertheless, some similarity can be found for simple topologies described in one dimension in [4]. We study in the following sections R-RAN for any dimensions.

#### 3.1. Constructing algorithm

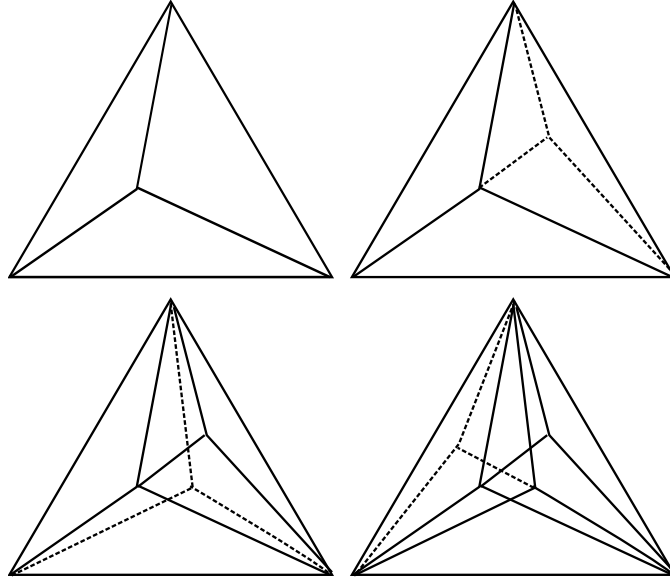
Initially, a network containing  $d + 2$  vertices and  $d + 2$   $(d+1)$ -cliques is created. At each time step, a vertex is added into a  $(d+1)$ -clique selected at random. The new vertex is connected to each node of the selected clique, creating  $d + 1$  new  $(d+1)$ -cliques. Thus, comparatively to RAN for which new vertices are inserted into  $(d+1)$ -cliques that contain no vertex inside, for R-RAN, any  $(d+1)$ -clique can be selected to insert a new node, what ever the number of inside vertices is.

### 4. Parallel degree distribution and ramification coefficient

The parallel degree is a characteristic that applies to  $(d+1)$ -cliques. We define the parallel degree of a  $(d+1)$ -clique as the number of vertices “inside” the  $(d+1)$ -clique, e.g. the number of vertices that are connected to every vertices of the  $(d+1)$ -clique but are not in the set of vertices that compose the  $(d+1)$ -clique.

#### 4.1. Estimating the parallel degree distribution

At time  $t = 0$ , the networks is composed with  $d + 2$  vertices forming  $d + 2$   $(d+1)$ -cliques. Each time a new vertex is inserted into the network, the number of  $(d+1)$ -cliques increases by  $d + 1$ . If  $N_{c_t}$  is the number of  $(d+1)$ -cliques at time  $t$ , we have  $N_{c_t} = d + 2 + t.(d + 1)$ .



**Figure 3.** 2-dimensionnal R-RAN. One vertex is added to a randomly chosen 3-clique at each time steps. Edges added at each time step are dashed

Furthermore, each time a  $(d+1)$ -clique  $c_j$  is selected for the insertion of a new vertex, its parallel degree  $m_j$  increases by 1. Thus, if  $N_{c_t}(m)$  is the number of  $(d+1)$ -cliques having a parallel degree equal to  $m$  at time  $t$  we get the following growth rate for  $N_{c_t}(m)$

$$N_{c_t}(m) = N_{c_{t-1}}(m) + \frac{N_{c_{t-1}}(m-1)}{d+2+(d+1)(t-1)} - \frac{N_{c_{t-1}}(m)}{d+2+(d+1)(t-1)} \quad (1)$$

Let  $P_{c_t}(m)$  be the probability to select a  $(d+1)$ -clique with parallel degree  $m$  at time  $t$ .  $P_{c_t}(m)$  can be approximated by the ratio  $\frac{N_{c_t}(m)}{d+2+t(d+1)}$ . Thus  $N_{c_t}(m) = N_{c_t} \cdot P_{c_t}(m) = (d+2+t(d+1)) \cdot P_{c_t}(m)$  and we get from Eq.1

$$(d+2+t(d+1)) \cdot P_{c_t}(m) = (d+2+(t-1)(d+1)) \cdot P_{c_{t-1}}(m) + P_{c_{t-1}}(m-1) - P_{c_{t-1}}(m) \quad (2)$$

Thus

$$P_{c_t}(m) = \frac{t(d+1)}{d+2+t(d+1)} \cdot P_{c_{t-1}}(m) + \frac{P_{c_{t-1}}(m-1)}{d+2+t(d+1)} \quad (3)$$

Eq.3 shows that  $\lim_{t \rightarrow +\infty} P_{c_t}(m) = P_c(m)$  exists and that for large  $t$ ,  $P_{c_t}(m) \sim P_{c_{t-1}}(m) \sim P_c(m)$ . Rewriting the previous equation for large  $t$  we get

$$P_c(m) \sim \frac{P_c(m-1)}{d+2} = \frac{P_c(0)}{(d+2)^m} \quad (4)$$

It is easy to show by induction on  $t$  that the probability to select at any time  $t$  a  $(d+1)$ -clique having a null parallel degree is  $P_c(0) = (d+1)/(d+2)$ . Thus for large  $t$

$$P_c(m) \sim \frac{d+1}{(d+2)^{(m+1)}} \quad (5)$$

This last result shows that the parallel degrees for R-RAN scales as a geometric series whose ratio is  $1/(d+2)$ .

#### 4.2. Average parallel degree and ramification coefficient

To measure the average degree of ramification of a R-RAN, we consider first the average parallel degree as follows

$$M = \sum_{m=1}^{\infty} m \cdot Pc(m) = \sum_{m=1}^{\infty} m \cdot \frac{d+1}{(d+2)^{(m+1)}} \quad (6)$$

$M$  measures the average number of nodes inside a  $(d+1)$ -clique.

We show easily that

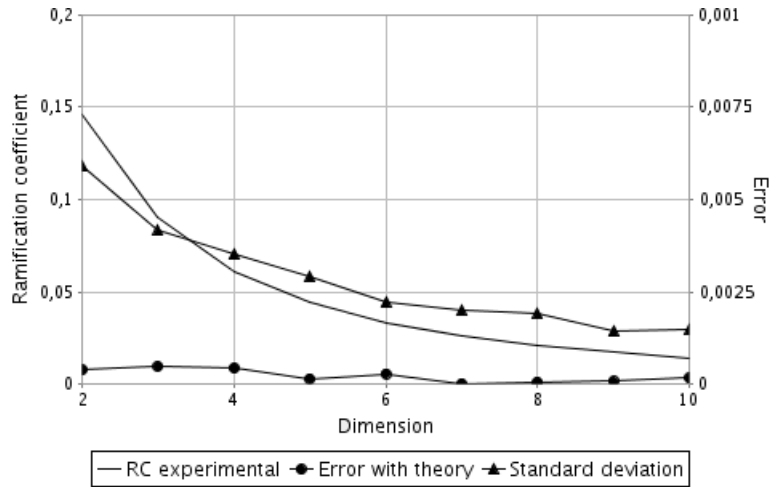
$$\begin{aligned} M &= \frac{1}{d+2} + \frac{1}{(d+2)^2} + \frac{1}{(d+2)^3} + \dots \\ &= \frac{1}{d+1} \end{aligned} \quad (7)$$

For  $d=2$ ,  $M=1/3$  for R-RAN which could be compared to the  $M=2/15$  for RAN.

Finally, we define the ramification coefficient  $\rho$  as the  $M - Pc(1)$ , i.e.

$$\rho = \sum_{m=2}^{\infty} m \cdot \frac{d+1}{(d+2)^{(m+1)}} = \frac{1}{d+1} - \frac{d+1}{(d+2)^2} = \frac{2d+3}{(d+1)(d+2)^2} \quad (8)$$

For  $d=2$ ,  $\rho=7/48$  for R-RAN while  $\rho=0$  for RAN.



**Figure 4.** Ramification coefficient of a R-RAN as a function of the dimension

## 5. Estimating the degree distribution

The degree of a vertex in a network is the number of connections it shares with other vertices and the degree distribution is the probability distribution of these degrees over the whole network.

### 5.1. Determining the degree distribution

We note that, after a vertex is added into the R-RAN network, the number of  $(d+1)$ -cliques available for the insertion of a new vertex is increased by  $d+1$ . After  $t$  iterations, the number of  $(d+1)$ -cliques available for the insertion of a new vertex is  $d+2+t(d+1)$ .

Thus, given a vertex  $v_i$ , when its degree increases by 1 the number of  $(d+1)$ -cliques that contain vertex  $v_i$  increases by  $d$ . So the number of  $(d+1)$ -cliques available for selection containing vertex  $v_i$  with degree  $k_i$  is  $(k_i - (d+1)).d + d + 1 = d.k_i - d^2 + 1$ , since at  $t = t_i$ , the creation time of vertex  $v_i$  there is  $d+1$   $(d+1)$ -cliques that contain vertex  $v_i$ .

Let  $N_t$  be the total number of vertices into the R-RAN at step  $t$  ( $N_t = d+2+t$ ) and let  $N_t(k)$  be the number of vertices having a degree  $k$  at time  $t$ . We can write the following difference equation

$$N_t(k) = N_{t-1}(k) + \frac{d.(k-1)-d^2+1}{d+2+(t-1).(d+1)} N_{t-1}(k-1) - \frac{d.k-d^2+1}{d+2+(t-1).(d+1)} N_{t-1}(k) \quad (9)$$

Let  $P_t(k)$  be the probability to select a vertex with degree  $k$  at time  $t$ .  $P_t(k)$  can be approximated by the ratio  $\frac{N_t(k)}{d+2+t}$ . Thus  $N_{C_t}(k) = (d+2+t).P_t(k)$  and we get from Eq.9

$$P_t(k).(d+2+t) = P_{t-1}(k).(d+2+(t-1)) + \frac{d(k-1)-d^2+1}{d+2+(t-1)(d+1)}.P_{t-1}(k-1).(d+2+(t-1)) - \frac{dk-d^2+1}{d+2+(t-1)(d+1)}.P_{t-1}(k).(d+2+(t-1)) \quad (10)$$

From Eq.10 we get that  $\lim_{t \rightarrow +\infty} P_t(k) = P(k)$  exists and that for large  $t$ ,  $P_t(k) \sim P_{t-1}(k) \sim P(k)$ . Rewriting the previous equation for large  $t$  we get

$$P(k). \left(1 + \frac{d.k-d^2+1}{d+1}\right) \sim \frac{d.k-d^2-d+1}{d+1}.P(k-1) \quad (11)$$

and finally

$$P(k) \sim \frac{d.k - d^2 - d + 1}{d.k - d^2 + d + 2} \cdot P(k-1) \quad (12)$$

This recursive equation is defined for  $k \geq d+1$ . We show in the following subsection that  $P(d+1) = 1/2$  for all dimensions.

### 5.2. Determining $P(d+1)$

- Let  $N_{d+1,t}$  be the expected number of vertices into the network having a degree equal to  $d+1$  at time  $t$ ,
- let  $n_t$  be the expected total number of  $(d+1)$ -cliques having a parallel degree equal to 0,



- let  $n'_t$  be the expected total number of  $(d+1)$ -cliques having a parallel degree equal to 0 for which all vertices have a degree  $k > (d + 1)$  at time  $t$ ,
- let  $n''_t$  be the expected total number of  $(d+1)$ -cliques having a parallel degree equal to 0 for which all vertices have a degree  $k > d + 1$  except one vertex that has a degree  $k = d + 1$  at time  $t$ .

For sufficiently large  $t$ , every vertex  $v_i$  in the network has a degree  $k_i \geq d + 1$ , and every  $(d+1)$ -clique  $v_j$  has either all its vertices with a degree  $k > d + 1$  or only one vertex with a degree  $k = 3$ . Thus, when we insert a new vertex in a  $(d+1)$ -clique  $c_j$ , only three cases arise for the  $(d+1)$ -clique selected for the insertion:

- If the clique has a parallel degree  $m > 0$  then  $N_{d+1}(t)$  is increase by one,  $n'_t$  is unchanged and  $n''_t$  is increased by  $d + 1$
- If the clique has a parallel degree  $m = 0$  and all its  $d + 1$  vertices has a degree  $k > d + 1$ , in this case the  $N_{d+1}(t)$  is increased by one,  $n'_t$  is decreased by one and  $n''_t$  is increased by  $d + 1$
- If the clique has a parallel degree  $m = 0$  and all its  $d + 1$  vertices has a degree  $k > d + 1$  except one with a degree equal to  $d + 1$ , in this case the  $N_{d+1}(t)$  is unchanged,  $n'_t$  is increased by  $d$  and  $n''_t$  is unchanged.

In Section 4 we have shown that the probability to select randomly a  $(d+1)$ -clique with a parallel degree  $m = 0$  is  $P(m = 0) = (d + 1)/(d + 2)$  and  $n_t \sim t \cdot \frac{(d+1)^2}{d+2}$ . Previous statements lead to the following equations

$$P(d + 1) \sim \frac{1}{d + 2} + \frac{d + 1}{d + 2} \cdot \frac{n'_t}{n_t} \quad (13)$$

$$\begin{aligned} n'_t &= n'_{t-1} + \left( d \cdot \frac{n''_{t-1}}{n_{t-1}} - \frac{n'_{t-1}}{n_{t-1}} \right) \cdot \frac{d+1}{d+2} \\ &= n'_{t-1} + \left( d \cdot \frac{n_{t-1} - n'_{t-1}}{n_{t-1}} - \frac{n'_{t-1}}{n_{t-1}} \right) \cdot \frac{d+1}{d+2} \\ &= n'_{t-1} \left( 1 - \frac{d+1}{n_{t-1}} \cdot \frac{d+1}{d+2} \right) + d \cdot \frac{d+1}{d+2} \end{aligned} \quad (14)$$

Assuming that  $\lim_{t \rightarrow +\infty} n'_t/n_t$  exists (this is obviously the case since  $P(k = d + 1)$  exists),  $n'_t \sim a \cdot t$  where  $a$  is a constant. Replacing  $n'_t$  in Eq.13 we get

$$a \cdot t = a \cdot (t - 1) \left( 1 - \frac{1}{t - 1} \right) + d \cdot \frac{d + 1}{d + 2} \quad (15)$$

leading to  $a = \frac{d}{2} \cdot \frac{(d+1)}{(d+2)}$ . Thus,

$$\frac{n'_t}{n_t} \sim \frac{a}{\frac{d+1}{d+2} \cdot (d + 1)} = \frac{d}{2 \cdot (d + 1)} \quad (16)$$

Finally,  $P(d + 1) = P(k = d + 1) \sim \frac{1}{d+2} + \frac{d+1}{d+2} \cdot \frac{n'_t}{n_t} = \frac{d+1}{d+2} + \frac{d}{2 \cdot (d+1)} = 1/2$ . Note that  $P(d + 1)$  is independent from the dimension  $d$ .

Thus, the recursive equation that gives the degree spectrum is

$$\begin{cases} P(k) \sim \frac{d \cdot k - d^2 - d + 1}{d \cdot k - d^2 + d + 2} \cdot P(k-1) & \text{for } k > d + 1 \\ P(k) \sim \frac{1}{2} & \text{for } k = d + 1 \end{cases} \quad (17)$$

To our knowledge, there is no simple analytical expression for  $P(k)$  in any dimension. Nevertheless, for  $d = 1$ , we get

$$P(k) \sim \frac{12}{(k+2) \cdot (k+1) \cdot k} \quad (18)$$

This result in dimension one has already been reported in [4].

### 5.3. Degree exponent

To show that the degree distribution follows a power law, we evaluate the asymptotic value of the following ratio

$$R(k) = \frac{\log(P(k)) - \log(P(k-1))}{\log(k) - \log(k-1)} = \frac{\log(P(k)/P(k-1))}{\log(k/(k-1))} \quad (19)$$

Thus

$$\begin{aligned} R(k) &= \frac{\log\left(\frac{d \cdot k - d^2 - d + 1}{d \cdot k - d^2 + d + 2}\right)}{\log(k/(k-1))} = \frac{\log\left(\frac{d \cdot k - d^2 - d + 1}{d \cdot k - d^2 + d + 2}\right)}{\log(k/(k-1))} \\ &= \frac{\log\left(\frac{1 + \frac{-d^2 - d + 1}{d \cdot k}}{1 + \frac{-d^2 + d + 2}{d \cdot k}}\right)}{-\log\left(1 - \frac{1}{k}\right)} \end{aligned} \quad (20)$$

and for large  $k$

$$\begin{aligned} R(k) &\sim k \cdot \left( \frac{-d^2 - d + 1}{d \cdot k} - \frac{-d^2 + d + 2}{d \cdot k} \right) \\ &\sim -\frac{2 \cdot d + 1}{d} \end{aligned} \quad (21)$$

This shows that for large  $k$   $P(k) \sim k^{-\gamma}$  with  $\gamma = \frac{2 \cdot d + 1}{d}$ . For  $d = 2$ , we get  $\gamma = 5/2$ .

## 6. Clustering coefficients

The clustering coefficient  $C_i$  for a vertex  $v_i$  is the proportion of links between the vertices within its neighborhood ( $v_i$  excluded) divided by the number of links that could possibly exist between them. For undirected graph, considering two vertices  $v_i$  and  $v_j$ , the edges  $v_i \rightarrow v_j$  and  $v_j \rightarrow v_i$  are considered identical. Therefore, if a vertex  $v_i$  has  $k_i$  neighbors,  $\frac{k_i(k_i-1)}{2}$  edges could exist among the vertices within its neighborhood. The clustering coefficient for the whole network is the average of the clustering coefficients  $C_i$  over the set of vertices composing the network.

When a vertex is inserted into the network, it is connected to all the vertices of a selected  $(d+1)$ -clique. It follows that every vertex having a degree  $k_i = d + 1$  has a clustering coefficient of one. Furthermore, when a vertex  $v_i$  having a degree  $k_i$  belongs

to a  $(d+1)$ -clique in which a new vertex is inserted, its degree increases by one and the new inserted neighbor connects to  $d$  among the  $k_i$  vertices of its previous neighborhood. This leads to the following clustering coefficient for a vertex having a degree  $k$

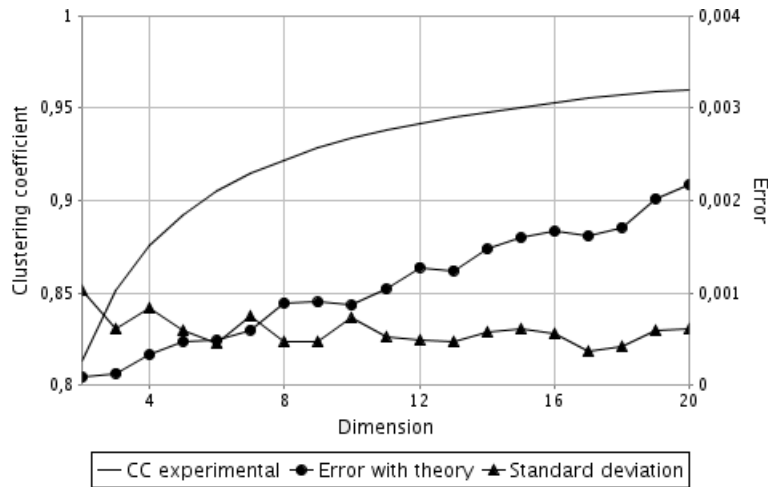
$$C(k) = \frac{\frac{d.(d+1)}{2} + d.(k-d-1)}{\frac{k.(k-1)}{2}} = \frac{d.(2k-d-1)}{k.(k-1)} \quad (22)$$

This local clustering coefficient is exactly the same as the one obtained for vertices in RAN []. Eq.22 shows that the local clustering coefficient scales as  $C(k) \sim k^{-1}$

We average these coefficients using the discrete degree distribution (Eq.17) as follows

$$C = \sum_{k_i=d+1}^{\infty} \left( \frac{d.(2k_i-d-1)}{k_i.(k_i-1)} . P(k_i) \right) \quad (23)$$

For  $d = 2$ , we get  $C = 0.813$ . Figure 6 shows that the clustering coefficient increases from 0.813 for  $d = 2$  to 1 as  $d$  tends towards infinity. Comparatively, HDRANs have a significantly lower clustering coefficient at low dimension, e.g. for  $d = 2$ , a RAN has a clustering coefficient  $C = .768$



**Figure 5.** Clustering coefficient of a R-RAN as a function of the dimension

## 7. Average path length

The average path length (APL) is a characteristic of the network topology that is defined as the average number of edges along the shortest paths for all possible pairs of network vertices. Following the very similar derivations already presented in [13, 12] for RAN, we address below the average path length for R-RAN.

First, we suppose that any vertex of the R-RAN network is ordered according to its insertion time stamp  $t$  that we consider discrete ( $t \in \mathbb{N}$ ). It is straightforward to establish that for R-RAN the following property holds (as well as for DAN or RAN)

For any two arbitrary vertices  $i$  and  $j$  all shortest paths from  $i$  to  $j$  does not pass through a vertex  $k$  if  $k > \max(i, j)$ .

Let  $d(i, j)$  denotes the distance between vertices  $i$  and  $j$ , namely the length of a shortest path between vertices  $i$  and  $j$ . Let  $\sigma(N)$  be the sum of all distances between all the pairs of vertices into the network with order  $N$ , e.g. containing  $N$  vertices.

$$\sigma(N) = \sum_{1 \leq i < j \leq N} d(i, j) \tag{24}$$

and let  $L(N)$  be the average path length of the R-RAN of order  $N$

$$L(N) = \frac{2\sigma(N)}{N \cdot (N - 1)} \tag{25}$$

Following exactly the approach given in [11] we get the following recursive inequality for  $\sigma(N)$

$$\sigma(N + 1) < \sigma(N) + N + \frac{2\sigma(N)}{N} \tag{26}$$

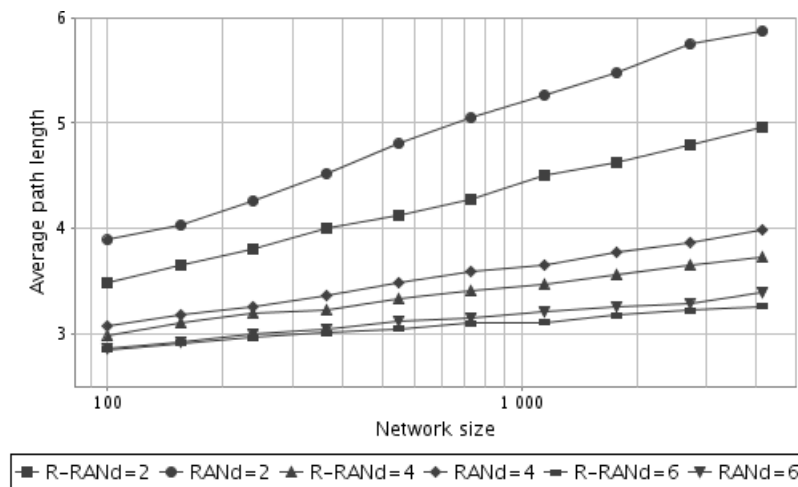
Considering the inequality Eq.26 as an equation we get the same upper bound for the variation of  $\sigma(N)$  than for RAN

$$\frac{d\sigma(N)}{dN} = N + \frac{2\sigma(N)}{N} \tag{27}$$

which leads to

$$\sigma(N) \leq N^2 \cdot \log(N) + S \tag{28}$$

where  $S$  is a constant. As  $\sigma(N)$  is asymptotically upper bonded by  $\sim N^2 \cdot \log(N)$ ,  $L(N)$  is asymptotically upper bounded by  $\log(N)$ , e.g.  $L(N)$  increases at most as  $\log(N)$  with  $N$ .



**Figure 6.** Average path length in RANs and R-RANs

Fig. 7 compares for dimensions 2, 4 and 6 average path lengths for HDRANs and R-RANs and shows that, for a given dimension, average path lengths are shorter for R-RANs than for HDRANs. Nevertheless, as the dimension increases, differences between path lengths vanish. This result was expected since R-RANs have a higher clustering coefficient than RANs.

## 8. Conclusion

From previous works on Apollonian Networks, mainly RAN and HDRAN networks, we have introduced what we call Ramified Deterministic or Random Apollonian Networks. These topologies, for which  $(d+1)$ -cliques may accept in their inside more than one vertex and fully connected to the vertices composing the clique, are still small world and scale free. This paper study statistical properties of R-RANs. For such networks, the degree exponent is in between 2 and 2.5. We have shown analytically that, comparatively to RAN or HDRAN, R-RAN networks are characterized with higher clustering coefficients and shorter average path lengths. R-RAN are also characterized by their parallel degree distribution and ramification coefficient that quantifies the number of vertices inside the  $(d+1)$ -cliques that compose R-RAN networks. We have provided extensive simulations that are in very good conformance with the analytical expectations.

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