Central Limit Theorems for the Brownian motion on large unitary groups

Florent Benaych-Georges

CENTRAL LIMIT THEOREMS FOR THE BROWNIAN MOTION ON LARGE UNITARY GROUPS

FLORENT BENAYCH-GEORGES

Abstract. In this paper, we are concerned with the large N limit of linear combinations of entries of Brownian motions on the group of $N \times N$ unitary matrices. We prove that the process of such a linear combination converges to a Gaussian one. Various scales of time are concerned, giving rise to various limit processes, in relation to the geometric construction of the unitary Brownian motion. As an application, we recover certain results about linear combinations of the entries of Haar distributed random unitary matrices.

Introduction

There is a natural definition of Brownian motion on any compact Lie group, which distribution is sometimes called the heat kernel measure. Mainly due to its relations with the object from free probability theory called the free unitary Brownian motion and with the two-dimentional Yang-Mills theory, the Brownian motion on large unitary groups has appeared in several papers during the last decade. Rains, in [R97], Xu, in [X97], Biane, in [B97a, B97b] and Lévy, in [L08], are all concerned with the asymptotics of the spectral distribution of large random matrices distributed according to the heat kernel measure. Also, in [D08], Demni makes use of the unitary Brownian motion in the study of Jacobi processes, and, in [BL08], Lévy and the author of the present paper construct a continuum of convolutions between the classical and free ones based on the conjugation of random matrices with a unitary Brownian motion. In this paper, we are concerned with the asymptotic distributions of linear combinations of the entries of an $N \times N$ unitary Brownian motion as N tends to infinity.

We first give the joint limit distribution, as N tends to infinity, of (possibly rescaled) random processes of the type $(\mathrm{Tr}[A(V_t - I)])_{t \geq 0}$ for $(e^{-t/2} V_t)_{t \geq 0}$ a Brownian motion staring at I on the group of unitary $N \times N$ matrices and A a random $N \times N$ matrix, independent of $(V_t)_{t \geq 0}$ (Theorem 1.2). This theorem is the key-result of the paper, since specifying the choice of the matrices A will then allow us to prove all other results. As a first example, it allows us to find out, for any sequence $(\lambda_N)_N$ of positive integers with a limit $\lambda \in [0, +\infty]$, the limit distribution of any upper-left corner of $\sqrt{N/\lambda_N}(V_{\log(\lambda_N t+1)} - I)_{t \geq 0}$ (Corollary

Date: April 10, 2009.

2000 *Mathematics Subject Classification.* 15A52, 60B15, 60F05, 46L54.

Key words and phrases. Unitary Brownian Motion, Heat Kernel, Random Matrices, Central Limit Theorem, Haar measure.

This work was partially supported by the *Agence Nationale de la Recherche* grant ANR-08-BLAN-0311-03.

1.3): it appears that for small scales of time (i.e. when $\lambda = 0$), the limit process is purely skew-Hermitian, whereas for large scales of time ($\lambda = +\infty$), the limit process is a standard complex matricial Brownian motion. For intermediate scales of time ($0 < \lambda < +\infty$), the limit process is an interpolation between these extreme cases. The existence of these three asymptotic regimes can be explained by the fact that the unitary Brownian motion is the "wrapping", on the unitary group, of a Brownian motion on the tangent space at I (which is the space of skew-Hermitian matrices), and that as the time goes to infinity, its distribution tends to the Haar measure (for which, as stated by Corollary 1.11, the upper-left corners are asymptotically distributed as standard complex Gaussian random matrices).

Secondly, we consider a unitary Brownian motion $(e^{-t/2}V_t)_{t\geq 0}$ which is uniformly distributed on the group of permutation matrices at time zero: its entries are exchangeable at any value of t. In this case, for any positive sequence (λ_N) and any positive integer p, the $p \times p$ upper left corner of $(\sqrt{N/\lambda_N}V_{\log(\lambda_N t+1)})_{t\geq 0}$ converges to a standard complex matricial Brownian motion (Corollary 1.8).

As an application, since the unitary Brownian motion distributed according to the Haar measure at time zero has a stationary distribution, our results allow us to recover certain results of Diaconis *et al.* about the asymptotic normality of linear combinations of the entries of uniform random unitary matrices (Theorem 1.10 and Corollary 1.11).

Let us now present briefly what problems underlie the asymptotics of linear combinations of the entries of a unitary Brownian motion.

Asymptotic normality of random unitary vectors and unitary matrices: The historical first result in this direction is due to Émile Borel, who proved a century ago, in [B1906], that, for a uniformly distributed point $(X_1, \ldots, X_N)$ on the unit euclidian sphere $\mathbb{S}^{N-1}$, the scaled first coordinate $\sqrt{N}X_1$ converges weakly to the standard Gaussian distribution as the dimension N tends to infinity. As explained in the introduction of the paper of Diaconis *et al.*, [ADN03], this says that the features of the "microcanonical" ensemble in a certain model for statistical mecanics (uniform measure on the sphere) are captured by the "canonical" ensemble (Gaussian measure). Since then, a long list of further-reaching results about the entries of uniformly distributed random orthogonal or unitary matrices have been obtained. The most recent ones are the previously cited paper of Diaconis *et al.*, the papers of Meckes and Chatterjee [M08, CM08], the paper of Collins and Stolz [CS08] and the paper of Jiang [J06], where the point of view is slightly different. In the present paper, we give a new, quite short, proof of the asymptotic normality of the linear combinations of the entries of uniformly distributed random unitary matrices, but we also extend these investigations to the case where the distribution of the matrices is not the Haar measure but the heat kernel measure, with any initial distribution and any rescaling of the time.

Second order freeness: A theory has been developed these last five years about Gaussian fluctuations (called *second order limits*) of traces of large random matrices around their limits, the most emblematic articles in this theory being [MN04, MS06, MSS07, CMSS07].

The results of this paper can be related to this theory, even though, technically speaking, we do not consider the powers of the matrices here[1].

Brownian motion on the Lie algebra and Itô map: The unitary Brownian motion is a continuous random process taking values on the unitary group, which has independent and stationary multiplicative increments. The most constructive way to define it is to consider a standard Brownian motion $(B_t)_{t\geq 0}$ on the tangent space of the unitary group at the identity matrix and to consider its image by the *Itô map* (which inverse is sometimes called the *Cartan application*), i.e. to wrap[2] it around the unitary group: the process $(U_t)_{t\geq 0}$ obtained is a unitary Brownian motion starting at I. Our results give us an idea of the way the Itô map adulterates the Brownian feature of the entries of $(B_t)_{t\geq 0}$ at different scales of time. Moreover, the question of the choice of a rescaling of the time (depending on the dimension) raises interesting questions: as exposed above, according to whether we choose as small, a large, or an intermediate rescaling of the time, $(e^{t/2}U_t - I)_{t\geq 0}$ is asymptotically distributed as a skew-Hermitian Brownian motion, as a standard matricial Brownian motion, or as an interpolation between these extreme cases.

Notation. For each $N \geq 1$, $\mathbb{U}_N$ shall denote the group of $N \times N$ unitary matrices. The identity matrix will always be denoted by I. For each complex matrix M, M^* will denote the adjoint of M. Tr will denote the trace and tr will denote the normalized trace: if M has size $N \times N$, $\mathrm{tr}(M) = \frac{1}{N}\mathrm{Tr}(M)$. We shall call a standard complex Brownian motion a complex-valued process which real and imaginary parts are independent standard real Brownian motions divided by $\sqrt{2}$. For all $k \geq 1$, the space of continuous functions from $[0, +\infty)$ to $\mathbb{C}^k$ will be denoted by $\mathcal{C}([0, +\infty), \mathbb{C}^k)$ and will be endowed with the topology of the uniform convergence on every compact interval.

1. Statement of the results

1.1. Brief presentation of the Brownian motion on the unitary group.

There are several ways to construct the Brownian motion on the unitary group[3]. For the one we choose here, all facts can easily be recovered by the use of the matricial Itô calculus, as exposed in Section 2.1.

Let N be a positive integer and ν_0 a probability measure on the group of unitary $N \times N$ matrices. We shall call a *unitary Brownian motion with initial law* ν_0 any random process $(U_t)_{t\geq 0}$ with values on the space of $N \times N$ complex matrices such that U_0 is ν_0-distributed

[1]The reason is that the constant matrices we consider here, like $\sqrt{N}\times$*elementary* $N \times N$ *matrices*, have no bounded moments of order higher than two: our results are the best ones that one could obtain with such matrices.

[2]For G a matricial Lie group with tangent space $\mathfrak{g}$ at I, the "wrapping" w_γ, on G, of a continuous, piecewise smooth, path $\gamma : [0, +\infty) \to \mathfrak{g}$ such that $\gamma(0) = 0$ is defined by $w_\gamma(0) = I$ and $w'_\gamma(t) = w_\gamma(t) \times \gamma'(t)$. It B is a Brownian motion on $\mathfrak{g}$ and $(B_n)_{n\geq 1}$ is a sequence of continuous, piecewise affine interpolations of B with a step tending to zero as n tends to infinity, then the sequence w_{B_n} converges in probability to a process which doesn't depend on the choice of the interpolations and which is a Brownian motion on G [IW81, VI.7], [FO09].

[3]See [H56, SV73, IW81, RW87]. A very concise and elementary definition is also given in [R97].

and $(U_t)_{t\geq 0}$ is a strong solution of the stochastic differential equation

$$\mathrm{d}U_t = i\mathrm{d}H_t U_t - \frac{1}{2}U_t \mathrm{d}t, \tag{1}$$

where $(H_t)_{t\geq 0}$ is a Brownian motion[4] on the space of $N \times N$ Hermitian matrices endowed with the scalar product $\langle A, B\rangle = N \operatorname{Tr}(AB)$.

It can be proved that for such a process $(U_t)_{t\geq 0}$, for any $t_0 \geq 0$, U_{t_0} is almost surely unitary and both processes $(U_{t_0+t}U_{t_0}^*)_{t\geq 0}$ and $(U_{t_0}^* U_{t_0+t})_{t\geq 0}$ are unitary Brownian motions starting at I_N and independent of the σ-algebra generated by $(U_s)_{0\leq s\leq t_0}$.

Remark 1.1 (Communicated by Thierry Lévy)**.** There are other ways to scale the time for the Brownian motion on the unitary group. Each of them amounts to a rescaling of the scalar product on the space of Hermitian matrices. For example, for the scalar product $\langle A, B\rangle = \operatorname{Tr}(AB)$, the Brownian motion obtained is $(U_{Nt})_{t\geq 0}$. Our scaling of the time is the one for which the three limit regimes correspond respectively to small values of t, finite values of t and large values of t and for which the limit non commutative distribution of $(U_t)_{t\geq 0}$ is the one of a free unitary Brownian motion. It also has a heuristic geometrical meaning: with this scaling, the diameter[5] of the group has the same order as the distance[6] between U_0 and U_t. It means that for all values of N, for all $t > 0$, U_t is probably no longer too close to its departure point, while it also probably hasn't "orbited" the unitary group too many times.

1.2. The three asymptotic regimes for the unitary Brownian motion starting at I_N.

The following theorem is the key-result of this article.

Let $(N_n)_{n\geq 1}$ be an increasing sequence of positive integers and let $(\lambda_n)_{n\geq 1}$ be a sequence of positive numbers with a limit $\lambda \in [0, +\infty]$. Let us fix a positive integer k and let, for each $n \geq 1$, $A_{n,1}, \ldots, A_{n,k}$ be a family of random $N_n \times N_n$ matrices and $(e^{-t/2}V_{n,t})_{t\geq 0}$ be a Brownian motion on $\mathbb{U}_{N_n}$ starting at I, independent of the $A_{n,l}$'s. Suppose that there exists nonrandom complex matrices $[a_l]_{l=1}^k$, $[p_{l,l'}]_{l,l'=1}^k$ and $[q_{l,l'}]_{l,l'=1}^k$ such that we have

$$\forall\, l, \qquad \operatorname{tr}(A_{n,l}) \underset{n\to\infty}{\longrightarrow} a_l \qquad \text{(convergence in probability)}, \tag{2}$$

$$\forall\, l, l', \qquad \operatorname{tr}(A_{n,l}A_{n,l'}) \underset{n\to\infty}{\longrightarrow} p_{l,l'} \qquad \text{(convergence in probability)}, \tag{3}$$

$$\forall\, l, l', \qquad \operatorname{tr}(A_{n,l}A_{n,l'}^*) \underset{n\to\infty}{\longrightarrow} q_{l,l'} \qquad \text{(convergence in probability)}. \tag{4}$$

For each n, let us define, for $t \geq 0$, $V_{n,t} = e^{t/2}U_{n,t}$ and

$$X_{n,t} = \lambda_n^{-1/2}(\operatorname{Tr}[A_{n,1}(V_{n,\log(\lambda_n t+1)} - I)], \ldots, \operatorname{Tr}[A_{n,k}(V_{n,\log(\lambda_n t+1)} - I)]).$$

[4] $(iH_t)_{t\geq 0}$ is in fact the skew-Hermitian Brownian motion that the process $(U_t U_0^*)_{t\geq 0}$ wraps around the unitary group, as explained in Footnote 2.

[5] By definition, the diameter of the group is the supremum, over pairs U, V of unitary matrices, of the length of the shortest geodesic between U and V. Here, it is equal the length of the geodesic $t \in [0, N\pi] \mapsto \exp(itI_N/N)$ between I_N and $-I_N$, i.e. to $N\pi$.

[6] As explained in Footnotes 2 and 4, $(U_t)_{t\geq 0}$ is the wrapping of $(iH_t)_{t\geq 0}$ on the unitary group, hence the distance between U_0 and U_t has the same order as the one between 0 and iH_t, which, by the Law of Large Numbers, has order $\sqrt{t}N$.

Let μ be the probability measure on $\mathcal{C}([0,+\infty),\mathbb{C}^k)$ defined by the fact that any process $X_t = (X_{t,1},\dots,X_{t,k})$ distributed according to μ is a Gaussian centered process with independent increments such that for all $t>0$, for all $l,l'=1,\dots,k$,

$$(5)\quad \mathbb{E}(X_{t,l}\overline{X_{t,l'}}) = q_{l,l'}t\,,\ \mathbb{E}(X_{t,l}X_{t,l'}) = \begin{cases} -p_{l,l'}t & \text{if } \lambda=0,\\ -p_{l,l'}\frac{\log(\lambda t+1)}{\lambda} + a_l a_{l'}\frac{\log^2(\lambda t+1)}{2\lambda} & \text{if } 0<\lambda<+\infty,\\ 0 & \text{if } \lambda=+\infty. \end{cases}$$

Theorem 1.2. *As n tends to infinity, the distribution of the process $(X_{n,t})_{t\geq 0}$ converges weakly to μ.*

Corollary 1.3. *Let us fix $p\geq 1$ and let (H_t), (S_t) be two independent standard Brownian motions on the euclidian spaces of $p\times p$ respectively Hermitian and skew-Hermitian matrices for the respective scalar products $\langle X,Y\rangle = \mathrm{Tr}(XY)/2$, $\langle X,Y\rangle = -\mathrm{Tr}(XY)/2$. Then, as n tends to infinity, the distribution of the $\mathbb{C}^{p\times p}$-valued process of the entries of the $p\times p$ upper left corner of $\sqrt{N_n/\lambda_n}(V_{n,\log(\lambda_n t+1)}-I)$ converges to the one of the random process $(H_{t-f_\lambda(t)}+S_{t+f_\lambda(t)})_{t\geq 0}$, where*

$$f_\lambda(t) = \begin{cases} t & \text{if } \lambda=0,\\ \frac{\log(\lambda t+1)}{\lambda} & \text{if } 0<\lambda<+\infty,\\ 0 & \text{if } \lambda=+\infty. \end{cases}$$

Remark 1.4. Note that when $\lambda=0$, the limit process is simply a standard Brownian motion on the space of $p\times p$ skew-Hermitian matrices, whereas, as λ grows from zero to $+\infty$, the Hermitian part of the limit process keeps growing, and at last, when $\lambda=+\infty$, the limit process is a standard Brownian motion on the space of $p\times p$ complex matrices. As said in the introduction, the existence of these three asymptotic regimes can be explained by the fact that the unitary Brownian motion is the "wrapping", on the unitary group, of a Brownian motion on the tangent space at I (which is the space of skew-Hermitian matrices), and that as the time goes to infinity, its distribution tends to the Haar measure (for which, as stated by Corollary 1.11, the upper-left corners are asymptotically distributed as standard complex Gaussian random matrices).

The following corollary is an immediate consequence of Theorem 1.2. Again, as λ passes from 0 to $+\infty$, the limit distribution passes from a Brownian motion on the imaginary line to a standard Brownian motion on the complex plane.

Corollary 1.5. *As n tends to infinity, the distribution of the process $(\lambda_n^{-1/2}\,\mathrm{Tr}[V_{n,t}-I])_{t\geq 0}$ converges weakly to the one of the random process $(B_{t/2-u_\lambda(t)}+iB'_{t/2+u_\lambda(t)})_{t\geq 0}$ for B,B' independent standard real Brownian motions and*

$$u_\lambda(t) = \begin{cases} t/2 & \text{if } \lambda=0,\\ \frac{\log(\lambda t+1)}{2\lambda} - \frac{\log^2(\lambda t+1)}{4\lambda} & \text{if } 0<\lambda<+\infty,\\ 0 & \text{if } \lambda=+\infty. \end{cases}$$

1.3. **The particular case of unitary Brownian motions with exchangeable entries.** Let $(N_n)_{n\geq 1}$ be an increasing sequence of positive integers and let $(\lambda_n)_{n\geq 1}$ be a

sequence of positive numbers (no hypothesis is made on its convergence). Let us fix a positive integer k and let, for each $n \geq 1$, $A_{n,1}, \ldots, A_{n,k}$ be a family of random $N_n \times N_n$ matrices and $(e^{-t/2}V_{n,t})_{t\geq 0}$ be a Brownian motion on $\mathbb{U}_{N_n}$, independent of the $A_{n,l}$'s, such that $V_{n,0}$ is uniformly distributed on the group of matrices of permutations of $\{1, \ldots, N_n\}$. Suppose that there exists a nonrandom complex matrix $[q_{l,l'}]_{l,l'=1}^k$ and a probability measure μ_0 on $\mathbb{C}^k$ such that

$$\forall\, l, l', \qquad \mathrm{tr}[A_{n,l}A_{n,l'}^*] \underset{n\to\infty}{\longrightarrow} q_{l,l'} \qquad \text{(conv. in probability)}, \tag{6}$$

$$\forall\, l, \qquad \frac{1}{N_n^2}\sharp\{(i,j)\,;\,((i,j)\text{-th entry of } A_{n,l}) \neq 0\} \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{(conv. in probability)}, \tag{7}$$

$$\lambda_n^{-1/2}(\mathrm{Tr}[A_{n,1}U_{n,0}], \ldots, \mathrm{Tr}[A_{n,k}U_{n,0}]) \underset{n\to\infty}{\longrightarrow} \mu_0 \qquad \text{(conv. in distribution)}. \tag{8}$$

For each n, let us define, for $t \geq 0$,

$$X_{n,t} = \lambda_n^{-1/2}(\mathrm{Tr}(A_{n,1}V_{n,\log(\lambda_n t+1)}), \ldots, \mathrm{Tr}(A_{n,k}V_{n,\log(\lambda_n t+1)})).$$

Let P be an Hermitian matrix such that $P^2 = [q_{l,l'}]_{l,l'=1}^k$, $(Z_1, \ldots, Z_k)$ be an independent family of standard complex Brownian motions and C be a μ_0-distributed random variable, independent of the Z_l's. Let us define μ to be the distribution, on $\mathcal{C}([0,+\infty), \mathbb{C}^k)$, of the process $(C + (Z_{1,t}, \ldots, Z_{k,t})P)_{t\geq 0}$.

Theorem 1.6. *As n tends to infinity, the distribution of X_n converges weakly to μ.*

Remark 1.7. a) Several papers, like [H51, C78, S88], give simple hypotheses on the $A_{n,l}$'s to have (8) for a certain probability measure μ_0.
b) Suppose that $(\lambda_n)_n$ is bounded from below by a positive constant, that (6) holds and that for all l, $N_n^{-1}\sharp\{(i,j)\,;\,((i,j)\text{-th entry of } A_{n,l}) \neq 0\}$ converges in probability to zero. Then, as an application of Lemma 2.7 bellow, (8) holds for μ_0 the Dirac mass at zero.

In the two following corollaries, for each N, $(e^{-t/2}V_{N,t})_{t\geq 0}$ is a Brownian motion on $\mathbb{U}_N$ such that $U_{N,0}$ is uniformly distributed on the group of matrices of permutations of $\{1, \ldots, N\}$. Both are direct applications of the previous theorem, the second one using implicitly the fact that the distribution of the number of fixed points of a uniform random permutation of $\{1, \ldots, N\}$ converges weakly, as N tends to infinity, to the Poisson distribution with mean one [DS94].

Corollary 1.8. *Let (λ_N) be a sequence of positive integers. For any $p \geq 1$, as N tends to infinity, the distribution of the $p \times p$ upper left corner of*

$$\left((N/\lambda_N)^{1/2}\, V_{N,\log(\lambda_N t+1)}\right)_{t\geq 0}$$

converges weakly to the one of an independent family of standard Brownian motions on the complex plane.

Corollary 1.9. *As N tends to infinity, the distribution of $(\mathrm{Tr}(V_{N,\log(t+1)}))_{t\geq 0}$ converges weakly to the one of $(C + Z_t)_{t\geq 0}$, where C is a Poisson random variable with mean one and $(Z_t)_{t\geq 0}$ is a standard complex Brownian motion, independent of C.*

1.4. **Application to the asymptotics of the uniform measure on the unitary group.** Since the Brownian motion on the unitary group distributed according to the Haar measure at time zero has a stationary distribution, our results allow us to recover certain results of asymptotic normality of linear combinations of the entries of uniform random unitary matrices.

The following theorem is not really new [CS08, CM08], except for the minor improvement that we allow to "randomize" the matrices $A_{n,l}$'s. However, our method allows to give a very direct proof, even under these very general hypotheses.

Theorem 1.10. *Let (N_n) be an increasing sequence of positive integers. Let us fix $k \geq 1$ and let, for each $n \geq 1$, $A_{n,1}, \ldots, A_{n,k}$ be $N_n \times N_n$ random matrices and U_n be a random matrix with uniform distribution on the group of $N_n \times N_n$ unitary matrices, independent of the $A_{n,l}$'s. Suppose that for all l, l', $\mathrm{tr}(A_{n,l}A_{n,l'}^*)$ converges in probability to a nonrandom complex number $q_{l,l'}$ as n tends to infinity. Then as n tends to infinity, the distribution of the random vector $(\mathrm{Tr}[A_{n,1}U_n], \ldots, \mathrm{Tr}[A_{n,k}U_n])$ converges weakly to the one of a Gaussian centered family $(Z_1, \ldots, Z_k)$ of complex random variables for all $l, l' = 1, \ldots, k$, $\mathbb{E}(Z_l Z_{l'}) = 0$ and $\mathbb{E}(Z_l \overline{Z_{l'}}) = q_{l,l'}$.*

The following corollary is immediate.

Corollary 1.11. *Let us fix $k \geq 1$ and let, for each N, $Z_{N,1}, \ldots, Z_{N,k}$ be k different entries of a random $N \times N$ matrix with uniform distribution on the unitary group. Then the joint distribution of $\sqrt{N}(Z_{N,1}, \ldots, Z_{N,k})$ converges weakly, as N tends to infinity, to the one of a family of independent standard complex Gaussian random variables.*

2. Proofs

2.1. **Preliminaries on matricial Itô calculus.** Let N be a positive integer. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(H_{N,t})_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-standard Brownian motion on the space of $N \times N$ Hermitian matrices endowed with the scalar product $\langle A, B\rangle = N\,\mathrm{Tr}(AB)$, i.e. a process with values in the space of $N \times N$ Hermitian matrices such that the diagonal and upper diagonal entries of $(\sqrt{N}H_{N,t})_{t\geq 0}$ are independent random processes, the ones on the diagonal being standard real Brownian motions and the ones above the diagonal being standard complex Brownian motions. If one considers two matrix-valued semimartingales X, Y such that

$$\mathrm{d}X_t = A_t(\mathrm{d}H_{N,t})B_t + C_t\mathrm{d}t, \qquad \mathrm{d}Y_t = D_t(\mathrm{d}H_{N,t})E_t + F_t\mathrm{d}t,$$

for some $(\mathcal{F}_t)_{t\geq 0}$-adapted matrix-valued processes A, B, C, D, E, F, then, by Itô's formula,

$$\begin{aligned} \mathrm{d}(XY)_t &= (\mathrm{d}X_t)Y_t + X_t\mathrm{d}Y_t + \mathrm{tr}(B_tD_t)A_tE_t\mathrm{d}t, \\ \mathrm{d}\langle \mathrm{Tr}(X), \mathrm{Tr}(Y)\rangle_t &= \mathrm{tr}(B_tA_tE_tD_t)\mathrm{d}t. \end{aligned}$$

We shall use these formulas many times in the paper, without citing them every time.

2.2. **A preliminary lemma.** Because it concerns the "convergence" of sequences of random variables which do not all take values on the same space (like random matrices), the following lemma has a quite intricate formulation. However, it shall sharply shorten

the proofs of most theorems stated above. Roughly, it says that if, for all N, A_N, U_N are independent $N \times N$ random matrices such that a certain functional $\varphi_N(A_N)$ converges in probability to a constant vector $v_\infty \in \mathbb{C}^d$, then in order to prove the convergence of the conditional distribution, with respect to the σ-algebra generated by A_N, of a certain functional $g_N(A_N, U_N)$ toward a law μ, it suffices to prove it along any subsequence, under the additional hypothesis that the A_N's are nonrandom.

Lemma 2.1. *Let $\mathcal{X}$ be a polish space, μ be a probability measure on $\mathcal{X}$, d be a positive integer and v_∞ be a fixed vector in $\mathbb{C}^d$. Let us consider, for each N, two measurable spaces $\mathcal{M}_N$, $\mathcal{U}_N$, an $\mathcal{U}_N$-valued random variable U_N and two measurable functions*

$$g_N : \mathcal{M}_N \times \mathcal{U}_N \to \mathcal{X}, \quad \varphi_N : \mathcal{M}_N \to \mathbb{C}^d.$$

Suppose that for any increasing sequence $(K_n)_{n\geq 0}$ of positive integers, for any sequence (M_n) of nonrandom elements of the $\mathcal{M}_{K_n}$'s such that $\varphi_{K_n}(M_n)$ tends to v_∞ as n goes to infinity, $g_{K_n}(M_n, U_{K_n})$ converges in distribution to μ as n goes to infinity.

Consider an increasing sequence (N_n) of positive integers, a sequence (A_n) of random elements of the $\mathcal{M}_{N_n}$'s such that for all n, A_n and U_{N_n} are defined on the same probability space and are independent, and a sequence (Σ_n) of σ-algebras such that for all n, A_n is Σ_n-measurable, whereas U_{N_n} is independent of Σ_n. Suppose that $\varphi_{N_n}(A_n)$ converges in probability to v_∞ as n goes to infinity.

Then for all continuous bounded functions f on $\mathcal{X}$, $\mathbb{E}[f \circ g_{N_n}(A_n, U_{N_n}) \,|\, \Sigma_n]$ converges in probability, as n tends to infinity, to $\int f(x)\mathrm{d}\mu(x)$. Moreover, the law of $g_{N_n}(A_n, U_{N_n})$ converges weakly to μ.

Proof. We shall only prove the first statement (the second one follows directly). Fix a continuous bounded function f on $\mathcal{X}$. Note firstly that for all n,

$$\mathbb{E}[f \circ g_{N_n}(A_n, U_{N_n}) \,|\, \Sigma_n] = \Lambda_{N_n}(A_n),$$

where for all N, Λ_N is the function defined on $\mathcal{M}_N$ by $\Lambda_N(M) = \mathbb{E}[f \circ g_N(M, U_N)]$ for all $M \in \mathcal{M}_N$.

Secondly, recall that a sequence (X_n) of real random variables defined on the same probability space converges in probability to a constant x if and only if any subsequence $(X_{n_p})_{p\geq 1}$ admits a subsequence $(X_{n_{p_k}})_{k\geq 1}$ converging almost surely to x (it is a consequence of Borel-Cantelli's Lemma and of Dominated Convergence Theorem).

Now, note that we can suppose that the A_n's are all defined on the same probability space (indeed, if it is not the case, one can consider the tensor product of their probability spaces). Thus, it suffices to prove that for all increasing sequences $(n_p)_{p\geq 1}$ of positive integers, $(\Lambda_{N_{n_p}}(A_{n_p}))_{p\geq 1}$ admits a subsequence converging almost surely to $\int f(x)\mathrm{d}\mu(x)$. So let us consider such a sequence $(n_p)_{p\geq 1}$. Since $\varphi_{N_n}(A_n)$ converges in probability to v_∞ as n goes to infinity, there is an increasing sequence $(p_k)_{k\geq 1}$ of positive integers such that $\varphi_{N_n}(A_n)$ converges to v_∞ almost surely along the subsequence $(n_{p_k})_{k\geq 1}$. Then by hypothesis, $(\Lambda_{N_n}(A_n))$ converges almost surely to $\int f(x)\mathrm{d}\mu(x)$ along the subsequence $(n_{p_k})_{k\geq 1}$. □

2.3. **Proof of Theorem 1.2.** Let us first state some matricial inequalities we shall often refer to in the following. Let X, Y be two complex matrices and G, H be two Hermitian nonnegative matrices. Then we have

$$|\operatorname{Tr}(XY)| \leq \sqrt{\operatorname{Tr}(XX^*)}\sqrt{\operatorname{Tr}(YY^*)}, \tag{9}$$

$$\operatorname{Tr}(G^2) \leq (\operatorname{Tr} G)^2, \tag{10}$$

$$|\operatorname{Tr}(GH)| \leq \operatorname{Tr}(G)\operatorname{Tr}(H). \tag{11}$$

Inequality (11) follows from (9) and (10), which are obvious.

Let us also recall an elementary formula: for any complex centered random variable Z,

$$\mathbb{E}\{[\Re(e^{i\theta}Z)]^2\} = \frac{1}{2}\{\mathbb{E}(|Z|^2) + \Re[e^{2i\theta}\mathbb{E}(Z^2)]\} \qquad (\theta \in \mathbb{R}). \tag{12}$$

Lemma 2.2. *Let us fix $N \geq 3$, an $N \times N$ matrix A and a Brownian motion $(e^{-t/2}V_t)_{t\geq 0}$ on $\mathbb{U}_N$ starting at I. Then there exists some real constants C_1, C_2, C_3, C_4 which are bounded from above by $100(\operatorname{tr} AA^*)^2$ and such that for all $t \geq 0$,*

$$\begin{aligned}
\mathbb{E}[\operatorname{tr}(AV_tAV_t)] &= \operatorname{tr}(A^2)\cos(t/N) - N(\operatorname{tr} A)^2\sin(t/N),\\
\mathbb{E}[\operatorname{tr}(AV_t)\operatorname{tr}(A^*V_t^*)] &= \frac{1}{N^2}(e^t-1)\operatorname{tr}(AA^*) + |\operatorname{tr} A|^2,\\
\mathbb{E}[|\operatorname{tr}(AV_tAV_t)|^2] &= |\operatorname{tr}(AA)|^2 + N^2\left\{\frac{|\operatorname{tr} A|^4}{2} + \frac{C_1}{N^{3/2}}\right\}\left\{\cosh\left(\frac{2t}{N}\right) - 1\right\}\\
&\quad -N\left\{\Re[(\operatorname{tr} A)^2\overline{\operatorname{tr}(AA)}] + \frac{C_2}{N^{3/2}}\right\}\sinh\left(\frac{2t}{N}\right)\\
&\quad +\frac{C_3}{N}(e^t-1) + \frac{C_4}{N^2}(e^{2t}-1).
\end{aligned}$$

Proof. Since the formulas we have to state are invariant under multiplication of A by a scalar, we can suppose that $\operatorname{tr}(AA^*) = 1$.

Note that for $H_{N,t}$ as in section 2.1, (V_t) is a strong solution of $\mathrm{d}V_t = iH_{N,t}\mathrm{d}V_t$. Hence by the matricial Itô calculus,

$$\begin{aligned}
\mathrm{d}\operatorname{tr}(AV_tAV_t) &= \mathrm{d}(\text{martingale term}) - \operatorname{tr}(AV_t)\operatorname{tr}(AV_t)\mathrm{d}t,\\
\mathrm{d}\operatorname{tr}(AV_t)\operatorname{tr}(AV_t) &= \mathrm{d}(\text{martingale term}) - N^{-2}\operatorname{tr}(AV_tAV_t)\mathrm{d}t.
\end{aligned}$$

It follows that for $x(t) = \mathbb{E}[\operatorname{tr}(AV_tAV_t)]$ and $y(t) = \mathbb{E}[\operatorname{tr}(AV_t)\operatorname{tr}(AV_t)]$, we have $x' = -y$ and $y' = -N^{-2}x$. The first formula follows.

Now, let us define, for C, D some $N \times N$ matrices, $u_{C,D}(t) = \mathbb{E}[\operatorname{tr}(V_tCV_t^*D)]$ and $v_{C,D}(t) = \mathbb{E}[\operatorname{tr}(V_tC)\operatorname{tr}(V_t^*D)]$. By the matricial Itô calculus again, one has

$$u_{C,D}(t) = (e^t-1)\operatorname{tr} C\operatorname{tr} D + \operatorname{tr}(CD),\ v_{C,D}(t) = \frac{1}{N^2}(e^t-1)\operatorname{tr}(CD) + \operatorname{tr}(C)\operatorname{tr}(D). \tag{13}$$

The second formula is proved. Let us now prove the third one. We introduce the functions

$$\begin{aligned}
f(t) &= \mathbb{E}(|\operatorname{tr}(AV_tAV_t)|^2),\\
g(t) &= \Re\{\mathbb{E}[\operatorname{tr}(AV_t)\operatorname{tr}(AV_t)\operatorname{tr}(A^*V_t^*A^*V_t^*)]\},\\
h(t) &= \mathbb{E}[|\operatorname{tr}(AV_t)|^4].
\end{aligned}$$

By the matricial Itô calculus again (using the hypothesis $\mathrm{tr}(AA^*)=1$),

$$\begin{aligned} f'(t) &= -2g(t)+\frac{4e^t}{N^2}u_{A^*A,AA^*}(t), \\ g'(t) &= -\frac{1}{N^2}f(t)-h(t)+4\frac{e^t}{N^2}\Re[v_{A,A^*AA^*}(t)], \\ h'(t) &= -\frac{2}{N^2}g(t)+\frac{4e^t}{N^2}v_{A,A^*}(t). \end{aligned} \tag{14}$$

It follows, by (13), that $g''(t)-\frac{4}{N^2}g(t)=\frac{8e^{2t}}{N^4}\kappa+\frac{4e^t}{N^2}\theta$, for

$$\begin{aligned} \kappa &= \mathrm{tr}(AA^*AA^*)-1, \\ \theta &= N^{-2}[2-\mathrm{tr}(AA^*AA^*)-\mathrm{tr}(AAA^*A^*)]-|\,\mathrm{tr}\,A|^2+\Re\{\mathrm{tr}\,A\,\mathrm{tr}(A^*AA^*)\}, \end{aligned}$$

hence $g(t)=\mu\cosh\left(\frac{2t}{N}\right)+\nu\sinh\left(\frac{2t}{N}\right)+\frac{2\kappa}{N^2(N^2-1)}e^{2t}+\frac{4\theta}{N^2-4}e^t$, with

$$\begin{aligned} \mu &= \Re[\mathrm{tr}\,A\,\mathrm{tr}\,A\,\mathrm{tr}(A^*A^*)]-\frac{2\kappa}{N^2(N^2-1)}-\frac{4\theta}{N^2-4}, \\ \nu &= -\frac{1}{2N}|\,\mathrm{tr}\,A^2|^2-\frac{N}{2}|\,\mathrm{tr}\,A|^4+\frac{2}{N}\Re[\mathrm{tr}\,A\,\mathrm{tr}(A^*AA^*)]-\frac{2\kappa}{N(N^2-1)}-\frac{2N\theta}{N^2-4}. \end{aligned}$$

From (14), it follows that

$$f(t)=-|\,\mathrm{tr}\,A^2|^2-N\mu\sinh\left(\frac{2t}{N}\right)-N\nu\left(\cosh\left(\frac{2t}{N}\right)-1\right)+w(t),$$

where $w(t)=-\frac{2\kappa}{N^2(N^2-1)}(e^{2t}-1)-\frac{8\theta}{N^2-4}(e^t-1)+\frac{2}{N^2}(e^{2t}-1)+\frac{4(\mathrm{tr}(AA^*AA^*)-1)}{N^2}(e^t-1)$.

Now, the conclusion follows from the fact that since $\mathrm{tr}(AA^*)=1$, the inequalities (9), (10) and (11) allow to prove that $|\,\mathrm{tr}\,A|,|\,\mathrm{tr}\,A^2|\leq 1$, $\mathrm{tr}(AA^*AA^*),|\,\mathrm{tr}(AAA^*A^*)|\leq N$ and $|\,\mathrm{tr}(A^*AA^*)|\leq\sqrt{N}$. □

Lemma 2.3. *Let $(N_n),(a_n),(b_n),(c_n)$ be sequences of real numbers such that (N_n) and (a_n) both tend to $+\infty$ and (b_n) and (c_n) are both bounded. Then we have*

$$u_n:=\frac{N_n^2b_n}{a_n}\left(\cosh\left(\frac{\log a_n}{N_n}\right)-1\right)+\frac{N_nc_n}{a_n}\sinh\left(\frac{\log a_n}{N_n}\right)\underset{n\to\infty}{\longrightarrow}0. \tag{15}$$

Proof. Let us define

$$K=\max\left\{\sup_{n\geq1}|a_n|\,,\ \sup_{n\geq1}|b_n|\,,\ \sup_{0<x\leq1}\frac{\cosh(x)-1}{x^2}\,,\ \sup_{0<x\leq1}\frac{\sinh(x)}{x}\right\}.$$

Then (15) follows from the fact that, since u_n can also be written

$$u_n=\frac{N_n^2b_n}{2a_n^{1-\frac{1}{N_n}}}+\frac{N_n^2b_n}{2a_n^{1+\frac{1}{N_n}}}-\frac{N_n^2b_n}{a_n}+\frac{N_nc_n}{2a_n^{1-\frac{1}{N_n}}}-\frac{N_nc_n}{2a_n^{1+\frac{1}{N_n}}},$$

we have

$$|u_n|\leq 5K\frac{N_n^2}{e^{N_n-1}}1_{a_n>e^{N_n}}+2K^2\frac{\log^2a_n+\log a_n}{a_n}1_{a_n\leq e^{N_n}}.$$

□

For the two following lemmas, we introduce an increasing sequence $(N_n)_{n\geq 1}$ of positive integers and a sequence $(\tau_n)_{n\geq 1}$ of positive numbers which admits a limit $\tau \in [0, +\infty]$. For each n, we consider an $N_n \times N_n$ matrix A_n and a unitary Brownian motion $(e^{-t/2}V_{n,t})_{t\geq 0}$ on $\mathbb{U}_{N_n}$ starting at I. Suppose also that there exists $a, p, q \in \mathbb{C}$ such that we have:

$$\operatorname{tr} A_n \underset{n\to\infty}{\longrightarrow} a, \qquad \operatorname{tr}(A_n^2) \underset{n\to\infty}{\longrightarrow} p, \qquad \operatorname{tr}(A_n A_n^*) \underset{n\to\infty}{\longrightarrow} q. \tag{16}$$

As a consequence of Lemma 2.2 (and of Lemma 2.3 if $\tau = +\infty$), one can easily verify the following lemma.

Lemma 2.4. *We have*

$$e^{-\tau_n/2} \operatorname{tr}(A_n V_{n,\tau_n}) \xrightarrow[n\to\infty]{L^2} a e^{-\tau/2}, \quad e^{-\tau_n} \operatorname{tr}((A_n V_{n,\tau_n})^2) \xrightarrow[n\to\infty]{L^2} p e^{-\tau} - a^2 \tau e^{-\tau},$$

where if $\tau = +\infty$, each right hand side term has to be replaced by its limit at $+\infty$, i.e. by 0.

Lemma 2.5. *Suppose now, for each n, A_n to be taken at random independently of $(U_{n,t})$ and the convergences of (16) to hold for the convergence in probability. Consider a σ-algebra Σ_n such that A_n is Σ_n-measurable, whereas $(U_{n,t})_{t\geq 0}$ is independent of Σ_n. Then the conditional distribution of*

$$\frac{1}{(e^{\tau_n} - 1)^{1/2}} \operatorname{Tr}[A_n(V_{n,\tau_n} - I)]$$

with respect to Σ_n converges in probability to the one of a complex centered Gaussian random variable Z such that $\mathbb{E}(|Z|^2) = q$ and $\mathbb{E}(Z^2) = -p\frac{\tau}{e^\tau - 1} + a^2 \frac{\tau^2}{2(e^\tau - 1)}$, where if $\tau = 0$ or $+\infty$, each term has to be replaced by its limit at τ.

Proof. Let us define, for each n, $\lambda_n = e^{\tau_n} - 1$. Firstly, following Lemma 2.1, one can suppose that for each n, A_n is nonrandom. Still following this lemma, we have to prove that for $t = 1$, the distribution of $Z_{n,t} := \lambda_n^{-1/2} \operatorname{Tr}[A_n(V_{n,\log(\lambda_n t+1)} - I)]$ converges weakly to the one of Z. We also introduce for $\xi, \theta \in \mathbb{R}$,

$$f_n(t, \xi, \theta) = \mathbb{E}\{\exp[2i\xi \Re(e^{i\theta} Z_{n,t})]\}.$$

Let us fix $\xi, \theta \in \mathbb{R}$ and prove that $f_n(1, \xi, \theta)$ converges to $\mathbb{E}\{\exp[2i\xi\Re(e^{i\theta}Z)]\}$. Set

$$\beta(t) = \begin{cases} -pt & \text{if } \tau = 0, \\ -p\frac{\log((e^\tau - 1)t+1)}{e^\tau - 1} + a^2 \frac{\log^2((e^\tau - 1)t+1)}{2(e^\tau - 1)} & \text{if } 0 < \tau < +\infty, \\ 0 & \text{if } \tau = +\infty, \end{cases}$$

and $\Psi(t) = -\xi^2\{qt + \Re[e^{2i\theta}\beta(t)]\}$. By (12), $\mathbb{E}\{\exp[2i\xi\Re(e^{i\theta}Z)]\} = \exp(\Psi(1))$. Thus we have to prove that $f_n(1, \xi, \theta)\exp\{-\Psi(1)\}$ tends to one as n tends to infinity. Note that $f_n(0, \xi, \theta)\exp\{-\Psi(0)\} = 1$ for all n. So it suffices to prove that $\partial_t[f_n(t, \xi, \theta)\exp\{-\Psi(t)\}]$ is uniformly bounded on every compact set and converges to zero almost everywhere.

Let us introduce $X_{n,t} = \operatorname{Tr}[A_n(V_{n,t} - I)]$ and, for $\xi, \theta \in \mathbb{R}$,

$$g_n(t, \xi, \theta) = \mathbb{E}\{\exp[2i\xi\Re(e^{i\theta}X_{n,t})]\}.$$

Since $\mathrm{d}V_{n,t} = i\mathrm{d}H_{N_n,t}V_{n,t}$, for $(H_{N_n,t})_{t\geq 0}$ as in Section 2.1, $\Re(e^{i\theta}X_{n,t})$ is a martingale and, by the matricial Itô calculus,

$$\mathrm{d}\langle \Re(e^{i\theta}X_{n,t}), \Re(e^{i\theta}X_{n,t})\rangle \;=\; \frac{1}{2}\{e^t \operatorname{tr}(A_nA_n^*) - \Re[e^{2i\theta}\operatorname{tr}(A_nV_{n,t}A_nV_{n,t})]\}\mathrm{d}t$$

Thus, by Itô's formula, we have

$$\partial_t g_n = -\xi^2 e^t \operatorname{tr}(A_nA_n^*)g_n + \xi^2\mathbb{E}\{\Re[e^{2i\theta}\operatorname{tr}(A_nV_{n,t}A_nV_{n,t})]e^{2i\xi\Re[e^{i\theta}X_{n,t}]}\},$$

and, since $f_n(t,\xi,\theta) = g_n(\log(\lambda_n t+1), \lambda_n^{-1/2}\xi, \theta)$, we get, for $U_{n,t} := e^{-t/2}V_{n,t}$,

$$\partial_t f_n = -\xi^2 \operatorname{tr}(A_nA_n^*)f_n + \xi^2\mathbb{E}\{\Re[e^{2i\theta}\operatorname{tr}((A_nU_{n,\log(\lambda_n t+1)})^2)]e^{2i\xi\Re[e^{i\theta}Z_{n,t}]}\}.$$

Hence we have

$$\partial_t[f_n(t,\xi,\theta)\exp\{-\Psi(t)\}] = e^{\Psi(t)}\partial_t f_n + \xi^2 e^{\Psi(t)}\{q + \Re[e^{2i\theta}\beta'(t)]\}f_n = \tag{17}$$
$$\xi^2 e^{\Psi(t)}[q - \operatorname{tr}(A_nA_n^*)]f_n + \xi^2 e^{\Psi(t)}\mathbb{E}\{\Re[e^{2i\theta}(\operatorname{tr}((A_nU_{n,\log(\lambda_n t+1)})^2) + \beta'(t))]e^{2i\xi\Re[e^{i\theta}Z_{n,t}]}\}$$

Since $|f_n| \leq 1$, by (16), $[q - \operatorname{tr}(A_nA_n^*)]f_n \underset{n\to\infty}{\longrightarrow} 0$. Moreover, by Lemma 2.4, for all $t > 0$,

$$\operatorname{tr}((A_nU_{n,\log(\lambda_n t+1)})^2) \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} -\beta'(t)$$

It follows that $\partial_t[f_n(t,\xi,\theta)\exp\{-\Psi(t)\}]$ converges to zero almost everywhere. Besides, since $\operatorname{tr}(A_nA_n^*)$ is bounded, it follows from (17) and (9) that $\partial_t[f_n(t,\xi,\theta)\exp\{-\Psi(t)\}]$ is uniformly bounded on every compact set, hence the lemma is proved. □

Lemma 2.6. *Let us fix $N \geq 1$, an $N \times N$ matrix A and a Brownian motion $(e^{-t/2}V_t)_{t\geq 0}$ on $\mathbb{U}_N$ starting at I. Then for all $t \geq 0$,*

$$\mathbb{E}\{|\operatorname{Tr}[A(V_t - I)]|^4\} \leq 3(e^t-1)^2(\operatorname{tr}(AA^*))^2. \tag{18}$$

Proof. We can suppose that $\operatorname{tr}(AA^*) = 1$. We have $\mathrm{d}V_t = i\mathrm{d}H_tV_t$, for (H_t) as the $(H_{N,t})$ of Section 2.1. Thus, the process $X_t := \operatorname{Tr}[A(V_t - I)]$ satisfies $\mathrm{d}X_t = i\operatorname{Tr}(\mathrm{d}H_tV_tA)$. Hence, by the matricial Itô calculus,

$$\mathrm{d}|X_t|^2 = e^t\mathrm{d}t - iX_t\operatorname{Tr}(\mathrm{d}H_tA_N^*V_t^*) + i\overline{X_t}\operatorname{Tr}(\mathrm{d}H_tV_tA_N). \tag{19}$$

It follows that $\partial_t\mathbb{E}(|X_t|^2) = e^t$, and that for all $t \geq 0$, $\mathbb{E}(|X_t|^2) = e^t - 1$. It also follows from (19) that

$$\begin{aligned}\mathrm{d}|X_t|^4 &= 2|X_t|^2\mathrm{d}|X_t|^2 + \mathrm{d}\langle |X_t|^2, |X_t|^2\rangle \\ &= \mathrm{d}(\text{martingale term}) + 4e^t|X_t|^2\mathrm{d}t - 2\Re[\overline{X_t}^2\operatorname{tr}(AV_tAV_t)]\mathrm{d}t.\end{aligned}$$

As a consequence,

$$\partial_t\mathbb{E}(|X_t|^4) = 4e^t\mathbb{E}(|X_t|^2) - 2\Re\{\mathbb{E}[\overline{X_t}^2\operatorname{tr}(AV_tAV_t)]\} = 4(e^{2t}-e^t) - 2\Re\{\mathbb{E}[\overline{X_t}^2\operatorname{tr}(AV_tAV_t)]\}.$$

Since $\operatorname{tr}(AA^*) = 1$, $|\operatorname{tr}(AV_tAV_t)| \leq e^t$, hence

$$|\Re\{\mathbb{E}[\overline{X_t}^2\operatorname{tr}(AV_tAV_t)]\}| \leq e^t\mathbb{E}(|X_t|^2) = (e^{2t}-e^t).$$

Thus

$$\partial_t\mathbb{E}(|X_t|^4) \leq 6(e^{2t}-e^t) = 3\partial_t[(e^t-1)^2].$$

Since $X_0 = 0$, (18) follows. □

Proof of Theorem 1.2. *Step I.* Firstly, following the second statement of Lemma 2.1, we suppose that the $A_{n,l}$'s are nonrandom matrices satisfying equations (2), (3) and (4).

Following [B68, Th. 8.1], we shall prove that the finite-dimentional marginals of the law of X_n converge to the ones of μ and that the sequence is tight.

Step II. Let us first prove the convergence of the finite-dimentional marginals. Let $X_t = (X_{t,1}, \ldots, X_{t,k})$ $(t \geq 0)$ be a random process distributed according to μ. For $l = 1, \ldots, k$, we denote by $X_{n,t,l} = \lambda_n^{-1/2} \operatorname{Tr}[A_{n,l}(V_{n,\log(\lambda_n t+1)} - I)]$ the l-th coordinate of $X_{n,t}$. Since X starts at zero and X_n does so for all n, we only have to prove that for all nonnegative integers r, for all $0 \leq t_0 < t_1 < \cdots < t_r$, for all $\lambda_1, \ldots, \lambda_r \in \mathbb{C}^k$,

$$\begin{aligned} &\mathbb{E}\left\{\prod_{j=1}^{r} \exp\left[i\Re\left(\sum_{l=1}^{k} \lambda_{j,l}(X_{n,t_j,l} - X_{n,t_{j-1},l})\right)\right]\right\} \\ &\underset{n\to\infty}{\longrightarrow} \prod_{j=1}^{r} \mathbb{E}\left\{\exp\left[i\Re\left(\sum_{l=1}^{k} \lambda_{j,l}(X_{t_j,l} - X_{t_{j-1},l})\right)\right]\right\}, \end{aligned} \tag{20}$$

Let us prove it by induction on r. If $r = 0$, there is nothing to prove. Now, let us fix $r \geq 1$ such that the result holds at the rank $r-1$. We fix $0 \leq t_0 < \cdots < t_r$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{C}^k$. Let us introduce, for each n,

$$\tau_n = \log\left(\tfrac{\lambda_n t_r + 1}{\lambda_n t_{r-1} + 1}\right),\ \tau_n' = \log(\lambda_n t_{r-1} + 1),\ C_n = (t_r - t_{r-1})^{1/2} \textstyle\sum_{l=1}^{k} \lambda_{r,l} A_{n,l} e^{-\tau_n'/2} V_{n,\tau_n'},$$
$$\tilde{V}_{n,t} = V_{n,\tau_n'}^{-1} V_{n,\tau_n'+t} \quad (t \geq 0).$$

Note that the left hand term of (20) can be written

$$\mathbb{E}\Bigg\{ \underbrace{\prod_{j=1}^{p-1} \exp\left[i\Re\Big(\sum_{l=1}^{k} \lambda_{j,l}(X_{n,t_j,l} - X_{n,t_{j-1},l})\Big)\right]}_{:=R_n} \underbrace{\exp\left[i\Re\left((e^{\tau_n} - 1)^{-1/2} \operatorname{Tr}[C_n(\tilde{V}_{n,\tau_n} - I)]\right)\right]}_{:=S_n} \Bigg\}.$$

If for all l, $\lambda_{r,l} = 0$, then the result follows from the induction hypothesis. In the other case, one can suppose that $\lambda_{r,1} \neq 0$ and, up to a replacement, for each n, of $A_{n,1}$ by $\sum_{l=1}^{k} \lambda_{r,l} A_{n,l}$, one can suppose that $\lambda_{r,1} = 1$ and $\lambda_{r,2} = \cdots = \lambda_{r,k} = 0$.

Now, let us denote by Σ_n the σ-algebra generated by $(V_{n,t})_{0 \leq t \leq \tau_n'}$. By the induction hypothesis, we have

$$\mathbb{E}(R_n) \underset{n\to\infty}{\longrightarrow} \prod_{j=1}^{r-1} \mathbb{E}\left\{\exp\left[i\Re\left(\sum_{l=1}^{k} \lambda_{j,l}(X_{t_j,l} - X_{t_{j-1},l})\right)\right]\right\}.$$

Thus, since R_n is Σ_n-measurable, $|R_n| \leq 1$ and $|S_n| \leq 1$, it suffices to prove that $\mathbb{E}[S_n \,|\, \Sigma_n]$ converges in probability to $\mathbb{E}\{\exp\left[i\Re(X_{t_r,1} - X_{t_{r-1},1})\right]\}$.

By (12) and (5), $\mathbb{E}\{\exp\left[i\Re(X_{t_r,1}-X_{t_{r-1},1})\right]\}=\exp(-\Gamma/4)$, where

$$\Gamma=\begin{cases}(q_{1,1}-\Re(p_{1,1}))(t_r-t_{r-1}) & \text{if } \lambda=0,\\ q_{1,1}(t_r-t_{r-1})-\frac{\Re(p_{1,1})}{\lambda}\log\left(\frac{\lambda t_r+1}{\lambda t_{r-1}+1}\right)+\frac{\Re(a_1^2)}{2\lambda}[\log^2(\lambda t_r+1)-\log^2(\lambda t_{r-1}+1)] & \text{if } 0<\lambda<\infty,\\ q_{1,1}(t_r-t_{r-1}) & \text{if } \lambda=+\infty.\end{cases}$$

Moreover, by (4), $\mathrm{tr}(C_nC_n^*)$ converges to $q:=(t_r-t_{r-1})q_{1,1}$ and by Lemma 2.4, the random variables $\mathrm{tr}(C_n)$ and $\mathrm{tr}(C_n^2)$ converge in the L^2 sense to respectively

$$a:=a_1\left(\frac{t_r-t_{r-1}}{\lambda t_{r-1}+1}\right)^{1/2},\quad p:=\frac{t_r-t_{r-1}}{\lambda t_{r-1}+1}p_{1,1}-\frac{t_r-t_{r-1}}{\lambda t_{r-1}+1}a_1^2\log(\lambda t_{r-1}+1),$$

where if $\lambda=+\infty$, each term has to be replaced by its limit as $\lambda\to+\infty$.

Besides,

$$\tau_n\underset{n\to\infty}{\longrightarrow}\tau:=\begin{cases}\log\left(\frac{\lambda t_r+1}{\lambda t_{r-1}+1}\right) & \text{if } \lambda<\infty,\\ \log(t_r/t_{r-1}) & \text{if } \lambda=+\infty,\ t_{r-1}>0,\\ +\infty & \text{if } \lambda=+\infty,\ t_{r-1}=0.\end{cases}$$

By definition of the unitary Brownian motion, $(e^{-t/2}\tilde{V}_{n,t})$ is a unitary Brownian motion starting at I and independent of Σ_n. Hence by Lemma 2.5, it follows that $\mathbb{E}[S_n\,|\,\Sigma_n]$ converges in probability to $\exp(-\Theta/4)$, where

$$\Theta=q-\Re(p)\frac{\tau}{e^\tau-1}+\Re(a^2)\frac{\tau^2}{2(e^\tau-1)}\quad\text{(computed with (12))}.$$

It can easily be verified that $\Gamma=\Theta$, thus (20) is proved. Note that another consequence of this part of the proof is the existence of a probability measure μ on $\mathcal{C}([0,+\infty),\mathbb{C}^k)$ such that any process distributed according to μ is a Gaussian centered process with independent increments satisfying (5). The uniqueness of such a distribution is due to the fact that the probability measures on $\mathcal{C}([0,\infty),\mathbb{C}^k)$ are characterized by their finite dimentional marginals.

Step III. Let us prove that the sequence $(Law(X_n))_{n\geq1}$ is tight in the space of probability measures on $\mathcal{C}([0,+\infty),\mathbb{C}^k)$ endowed with the weak topology. Following [B68, Th. 12.3] (adapted to $\mathbb{C}^k$-valued processes indexed by $[0,+\infty)$), it suffices to prove that for a certain norm $||\cdot||$ on $\mathbb{C}^k$, there is a constant C independent of n such that for all $t_0,t\geq0$,

$$\mathbb{E}\{||X_{n,t_0+t}-X_{n,t_0}||^4\}\leq Ct^2. \tag{21}$$

If we choose the norm $||\cdot||$ on $\mathbb{C}^k$ defined by $||x||=(\sum|x_i|^4)^{1/4}$, it amounts to prove it for $k=1$. So let us suppose that $k=1$ and fix $t_0\geq0$. We define, for all n,

$$X_n=(\lambda_nt_0+1)^{-1/2}V_{n,\log(\lambda_nt_0+1)},\quad \bar{V}_{n,t}=V_{n,\log(\lambda_nt_0+1)}^{-1}V_{n,\log(\lambda_nt_0+1)+t}\qquad(t\geq0).$$

Equation (21) is equivalent to the existence of a constant C such that for all t_0,t, for all $n\geq1$,

$$\left(\frac{\lambda_nt_0+1}{\lambda_n}\right)^2\mathbb{E}\{|\,\mathrm{Tr}[A_{n,1}X_n(\bar{V}_{n,\log\left(\frac{\lambda_n(t_0+t)+1}{\lambda_nt_0+1}\right)}-I)]|^4\}\leq Ct^2.$$

Since $X_n X_n^* = I$ and the process $(\bar{V}_{n,t})_{t\geq 0}$ has the same distribution as $(V_{n,t})_{t\geq 0}$ and is independent of (X_n), by Lemma 2.6, the constant $C = 3\sup_n\{\mathrm{tr}[A_{n,1}A_{n,1}^*]\}^2$ is convenient. It closes the proof of the theorem.

2.4. **Proof of corollary 1.3.** By Theorem 1.2 and formula (5), applied with matrices $A_{n,l}$ of the type $\sqrt{N_n}$ times an elementary matrix, the distribution of the $p\times p$ upper left corner of $\sqrt{N_n/\lambda_n}(V_{n,\log(\lambda_n t+1)} - I)_{t\geq 0}$ converges weakly to the one of the random process $(M_t)_{t\geq 0}$ with independent increments such that for all $p\times p$ complex matrices X, Y and all $t \geq 0$,

$$\mathbb{E}[\mathrm{Tr}(M_t X)\overline{\mathrm{Tr}(M_t Y)}] = t\,\mathrm{Tr}(XY^*), \quad \mathbb{E}[\mathrm{Tr}(M_t X)\mathrm{Tr}(M_t Y)] = -\frac{\log(\lambda t+1)}{\lambda}\,\mathrm{Tr}(XY),$$

where if $\lambda = 0$ or $+\infty$, the second right-hand term has to be replaced by respectively $-t\,\mathrm{Tr}(XY)$, 0. Since a standard Brownian (B_t) motion on an euclidian space $(E, \langle\cdot,\cdot\rangle)$ satisfies, for all $u, v \in E$, $\mathbb{E}(\langle B_t, u\rangle\langle B_t, v\rangle) = \langle u, v\rangle$, the result can easily be verified.

2.5. **Proof of Theorem 1.6.**

Lemma 2.7. *Let us fix $N \geq 2$, let S be the matrix of a uniform random permutation of $\{1,\dots,N\}$ and let A, B be $N\times N$ nonrandom matrices. Then we have*

$$\begin{aligned}\mathbb{E}\{|\,\mathrm{Tr}(AS)|\} &\leq \frac{1}{N}\sqrt{C_A}\sqrt{\mathrm{Tr}(AA^*)},\\ \mathbb{E}\{|\,\mathrm{Tr}(ASBS)|\} &\leq \frac{N-1+\sqrt{C_A C_B}}{N(N-1)}\sqrt{\mathrm{Tr}(AA^*)}\sqrt{\mathrm{Tr}(BB^*)},\end{aligned}$$

where for each matrix $X = [x_{i,j}]_{i,j=1}^N$, C_X denotes $\sharp\{(i,j)\,;\, x_{i,j}\neq 0\}$.

Proof. Let us denote by respectively $[a_{i,j}], [b_{i,j}], [s_{i,j}]$ the entries of A, B, S. By Hölder's inequality, we have

$$\mathbb{E}\{|\,\mathrm{Tr}(AS)|\} \leq \sum_{i,j}|a_{i,j}|\mathbb{E}(s_{i,j}) = \frac{1}{N}\sum_{i,j}|a_{i,j}| \leq \frac{1}{N}\sqrt{C_A}\sqrt{\mathrm{Tr}(AA^*)}.$$

Moreover, we have

$$\begin{aligned}\mathbb{E}\{|\,\mathrm{Tr}(ASBS)|\} &\leq \sum_{j,k}|a_{k,j}b_{k,j}|\mathbb{E}(s_{j,k}^2) + \sum_{\substack{i,j,k,l\\ j\neq l, k\neq i}}|a_{i,j}b_{k,l}|\mathbb{E}(s_{j,k}s_{l,i})\\ &\leq \frac{1}{N}\sum_{j,k}|a_{k,j}b_{k,j}| + \frac{1}{N(N-1)}\sum_{i,j,k,l}|a_{i,j}b_{k,l}|\end{aligned}$$

The conclusion follows from Hölder's inequality again. □

Proof of Theorem 1.6. *Step I.* Note that no hypothesis is made on the convergence of (λ_n). However, one can suppose that (λ_n) admits a limit $\lambda \in [0, +\infty]$. Indeed, by definition of the convergence of a sequence, we have to prove that any subsequence of (X_n) admits a subsequence converging in distribution to μ. Hence, since (N_n) is already an increasing sequence of positive integers and $[0, +\infty]$ is compact, one can suppose that (λ_n) admits a limit.

Step II. Let us prove the following convergences in probability:

$$\forall\, l, \qquad \mathrm{tr}(A_{n,l}U_{n,0}) \underset{n\to\infty}{\longrightarrow} 0, \tag{22}$$

$$\forall\, l, l' \qquad \mathrm{tr}(A_{n,l}U_{n,0}A_{n,l'}U_{n,0}) \underset{n\to\infty}{\longrightarrow} 0. \tag{23}$$

Let us fix $\varepsilon > 0$. For each $n \geq 1$, let us define the event

$$E_n := \big[\forall\, l = 1, \ldots, k, \mathrm{tr}(A_{n,l}A_{n,l}^*) \leq q_{l,l} + 1 \\ \text{and } \sharp\{(i,j)\,;\,((i,j)\text{-th entry of } A_{n,l}) \neq 0\} \leq \varepsilon N_n^2\big].$$

Using Lemma 2.7 and conditioning the expectation with respect to the σ-algebra generated by the $A_{n,l}$'s, one can easily prove that for all $l, l' = 1, \ldots, k$,

$$\begin{aligned} \mathbb{E}\{1_{E_n}|\,\mathrm{tr}[A_{n,l}U_{n,0}]|\} &\leq \sqrt{\varepsilon/N_n}\sqrt{q_{l,l}+1}, \\ \mathbb{E}\{1_{E_n}|\,\mathrm{tr}[A_{n,l}U_{n,0}A_{n,l'}U_{n,0}]|\} &\leq \frac{N_n - 1 + \varepsilon N_n^2}{N_n(N_n-1)}\sqrt{q_{l,l}+1}\sqrt{q_{l',l'}+1}. \end{aligned}$$

Since, by (6) and (7), the probability of the event E_n tends to one as n tends to infinity, the convergences in probability of (22) and (23) follow.

Step III. We consider $C, Z_1, \ldots, Z_k, P$ as introduced above the statement of the theorem. For each $t \geq 0$, let us define $K_t = (Z_{1,t}, \ldots, Z_{k,t})P$. Let us prove that, as n tends to infinity, the joint distribution of $(X_{n,0}, (X_{n,t} - X_{n,0})_{t\geq 0})$ converges weakly to the one of $(C, (K_t)_{t\geq 0})$, i.e. that for all continuous bounded functions Φ, Ψ respectively on the complex plane and on $\mathcal{C}([0,+\infty), \mathbb{C}^k)$,

$$\mathbb{E}\{\Phi(X_{n,0})\Psi[(X_{n,t} - X_{n,0})_{t\geq 0}]\} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\{\Phi(C)\}\mathbb{E}\{\Psi[(K_t)_{t\geq 0}]\}. \tag{24}$$

Note that for all n, $(e^{-t/2}V_{n,0}^*V_{n,t})_{t\geq 0}$ is a unitary Brownian motion starting at I. Hence by Theorem 1.2, if, for all n, $B_{n,1}, \ldots, B_{n,k}$ are nonrandom matrices such that

$$\forall\, l, \qquad \mathrm{tr}(B_{n,l}) \underset{n\to\infty}{\longrightarrow} 0, \tag{25}$$

$$\forall\, l, l', \qquad \mathrm{tr}(B_{n,l}B_{n,l'}) \underset{n\to\infty}{\longrightarrow} 0, \tag{26}$$

$$\forall\, l, l', \qquad \mathrm{tr}(B_{n,l}B_{n,l'}^*) \underset{n\to\infty}{\longrightarrow} q_{l,l'}, \tag{27}$$

then the law of $\mathbb{C}^k$-valued random process

$$\lambda_n^{-1/2}(\mathrm{Tr}[B_{n,1}(V_{n,0}^*V_{n,\log(\lambda_n t+1)} - I)], \ldots, \mathrm{Tr}[B_k(N)(V_{n,0}^*V_{n,\log(\lambda_n t+1)} - I)])_{t\geq 0}$$

converges weakly, as n tends to infinity, to the distribution which appears to be exactly the one of $(K_t)_{t\geq 0}$. Thus by Lemma 2.1 and Step II, if Σ_n denotes the σ-algebra generated by the random variables $A_{n,1}$, …, $A_{n,k}, U_{n,0}$, for all continuous bounded functions Ψ on $\mathcal{C}([0,\infty), \mathbb{C}^k)$, $\mathbb{E}\{\Psi[(X_{n,t} - X_{n,0})_{t\geq 0}]\,|\,\Sigma_n\}$ converges in probability to $\mathbb{E}\{\Psi((K_t)_{t\geq 0})\}$. By (8), (24) follows for all Φ, Ψ. It closes the proof of the theorem.

2.6. **Proof of Theorem 1.10.**

Lemma 2.8. *Let U be Haar-distributed on $\mathbb{U}_N$ and A be an $N \times N$ matrix. Then*

$$\mathbb{E}\{|\operatorname{Tr}(AU)|^2\} = \operatorname{tr}(AA^*), \tag{28}$$

$$\mathbb{E}\{|\operatorname{Tr}(AUAU)|^2\} \leq 4(\operatorname{tr}(AA^*))^2. \tag{29}$$

Proof. Set $U = [u_{i,j}]_{i,j=1}^N$. By [HJ91, Th. 3.1.1], one can write $A = VA'W$, with V, W unitary matrices and A' a diagonal matrix which diagonal entries $a_1, \ldots, a_N$ are nonnegative real numbers (which are the eigenvalues of $\sqrt{AA^*}$). Since the law of U is invariant under the left and right actions of the unitary group, one can suppose that $A = A'$. These invariances of the law of U also imply that for all i, j, $\mathbb{E}(u_{i,i}\overline{u_{j,j}}) = \delta_i^j/N$. Equation (28) follows. By the same principle, for all i, j, k, l, $\mathbb{E}(u_{i,j}u_{k,l}\overline{u_{m,n}u_{o,p}}) \neq 0$ implies $\{i, k\} = \{m, p\}$ and $\{j, l\} = \{n, p\}$. It follows that

$$\mathbb{E}\{|\operatorname{Tr}((AU)^2)|^2\} = \sum_i a_i^4\mathbb{E}\{|u_{i,j}|^4\} + 2\sum_{i\neq j} a_i^2a_j^2\mathbb{E}\{|u_{i,j}|^2|u_{j,i}|^2\} \leq 2\sum_{i,j} a_i^2a_j^2\mathbb{E}\{|u_{i,j}|^2|u_{j,i}|^2\}.$$

Inequality (29) follows, since, by [HP00, Prop. 4.2.3], for all i, j, k, l, $\mathbb{E}\{|u_{i,j}|^2|u_{k,l}|^2\} \leq 2/N^2$. □

Proof of Theorem 1.10. *Step I.* Firstly, following the second statement of Lemma 2.1, we suppose that the $A_{n,l}$'s are nonrandom. By Lemma 2.8, for all l, $\operatorname{tr}(A_{n,l}U_n)$ converges to zero as n tends to infinity in the L^2 sense, hence in probability. In the same way, for all l, l', $\operatorname{tr}(A_{n,l}U_nA_{n,l'}U_n)$ converge in probability to zero as N tends to infinity (for the case $l \neq l'$, use the equation $2\operatorname{tr}(XY) = \operatorname{tr}[(X+Y)^2] - \operatorname{tr}(X^2) - \operatorname{tr}(Y^2)$).

Step II. Following [DZ98, Th. D8], we fix a bounded real function f on $\mathbb{C}^k$ which is 1-Lipschitz for the canonical hermitian norm $||\cdot||$ on $\mathbb{C}^k$, and we shall prove that

$$\mathbb{E}\{f(\operatorname{Tr}[A_{n,1}U_n], \ldots, \operatorname{Tr}[A_{n,k}U_n])\} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\{f(Z_1, \ldots, Z_k)\}. \tag{30}$$

Let us fix $\varepsilon > 0$.

a) Let, for each $t \geq 0$, $(Z_{1,t}, \ldots, Z_{k,t})$ be a Gaussian family of centered complex random variables such that for all l, l', $\mathbb{E}[Z_{l,t}Z_{l',t}] = 0$ and $\mathbb{E}[Z_{l,t}\overline{Z_{l',t}}] = q_{l,l'}(1 - e^{-t})$ (such a family exists because the matrix $[q_{l,l'}]_{l,l'}$ is Hermitian and nonnegative). The distribution of $(Z_{1,t}, \ldots, Z_{k,t})$ tends to the one of $(Z_1, \ldots, Z_k)$ as t tends to infinity. Hence there is $t_0 > 0$ such that

$$|\mathbb{E}\{f(Z_1, \ldots, Z_k)\} - \mathbb{E}\{f(Z_{1,t_0}, \ldots, Z_{k,t_0})\}| \leq \varepsilon, \tag{31}$$

$$e^{-\frac{t_0}{2}} \sup_{N\geq 1}\{\operatorname{tr}[A_{n,1}A_{n,1}^* + \cdots + A_{n,k}A_{n,k}^*]\}^{1/2} \leq \varepsilon. \tag{32}$$

b) For each n, up to an extension of the probability space where U_n is defined, one can suppose that there exists a unitary Brownian motion $(U_{n,t})_{t\geq 0}$, starting at I, independent of U_n. Let us define, for each n,

$$\begin{aligned} X_n &= (\operatorname{Tr}[A_{n,1}U_nU_{n,t_0}], \ldots, \operatorname{Tr}[A_{n,k}U_nU_{n,t_0}]) \\ Y_n &= e^{-\frac{t_0}{2}}(\operatorname{Tr}[A_{n,1}U_n], \ldots, \operatorname{Tr}[A_{n,k}U_n]) \\ D_n &= X_n - Y_n = e^{-\frac{t_0}{2}}(\operatorname{Tr}[A_{n,1}U_n(e^{\frac{t_0}{2}}U_{n,t_0} - I)], \ldots, \operatorname{Tr}[A_{n,k}U_n(e^{\frac{t_0}{2}}U_{n,t_0} - I)]) \end{aligned}$$

By Step I and Theorem 1.2, as n tends to infinity, the distribution of D_n converges to the one of $(Z_{1,t_0}, \ldots, Z_{k,t_0})$. It follows that for n large enough,

$$|\mathbb{E}\{f(Z_{1,t_0}, \ldots, Z_{k,t_0})\} - \mathbb{E}\{f(D_n)\}| \leq \varepsilon. \tag{33}$$

c) At last, since f is 1-Lipschitz for $||\cdot||$, for all n, we have

$$\begin{aligned} |\mathbb{E}\{f(D_n)\} - \mathbb{E}\{f(X_n)\}| &\leq \mathbb{E}\{||D_n - X_n||\} \\ &\leq (e^{-t_0}\mathbb{E}\{|\operatorname{Tr}[A_{n,1}U_n]|^2 + \cdots + |\operatorname{Tr}[A_{n,k}U_n]|^2\})^{1/2} \\ &\leq e^{-t_0/2}(\operatorname{tr}[A_{n,1}A_{n,1}^*] + \cdots + \operatorname{tr}[A_{n,k}A_{n,k}^*])^{1/2}, \end{aligned}$$

the last inequality following from (28). By (32), it allows us to claim that for all n,

$$|\mathbb{E}\{f(D_n)\} - \mathbb{E}\{f(X_n)\}| \leq \varepsilon. \tag{34}$$

d) To conclude, note that by the right invariance of the law of U_n, X_n and the random vector $(\operatorname{Tr}[A_{n,1}U_n], \ldots, \operatorname{Tr}[A_{n,k}U_n])$ have the same distribution. Hence

$$\mathbb{E}[f(Z_1, \ldots, Z_k)] - \mathbb{E}[f(\operatorname{Tr}[A_{n,1}U_n], \ldots, \operatorname{Tr}[A_{n,k}U_n])] = \{\mathbb{E}[f(Z_1, \ldots, Z_k)]$$
$$-\mathbb{E}[f(Z_{1,t_0}, \ldots, Z_{k,t_0})]\} + \{\mathbb{E}[f(Z_{1,t_0}, \ldots, Z_{k,t_0})] - \mathbb{E}[f(D_n)]\} + \{\mathbb{E}[f(D_n] - \mathbb{E}[f(X_n)]\}$$

By equations (31), (33) and (34), it follows that for N large enough,

$$|\mathbb{E}\{f(Z_1, \ldots, Z_k)\} - \mathbb{E}\{f(\operatorname{Tr}[A_{n,1}U_n], \ldots, \operatorname{Tr}[A_{n,k}U_n])\}| \leq 3\varepsilon.$$

It closes the proof of the theorem.

References

[ADN03] A. D'Aristotile, P. Diaconis, C. Newman *Brownian motion and the classical groups*. With Probability, Statisitica and their applications: Papers in Honor of Rabii Bhattacharaya. Edited by K. Athreya, et al. 97-116. Beechwood, OH: Institute of Mathematical Statistics, 2003.

[BL08] F. Benaych-Georges, T. Lévy, *A continuous semigroup of notions of independence between the classical and the free one*. Submitted.

[B97a] P. Biane, *Brownian motion, free stochastic calculus and random matrices*. Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.

[B97b] P. Biane, *Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems*. J. Funct. Anal. 144 (1997), no. 1, 232–286.

[B68] P. Billingsley, *Convergence of probability measures* Wiley, 1968.

[B1906] É. Borel, *Sur les principes de la théorie cinétique des gaz*. Annales de l'École Normale Supérieure 23 (1906), 9–32.

[CM08] S. Chatterjee, E. Meckes, *Multivariate normal approximation using exchangeable pairs* ALEA 4 (2008).

[C78] L.H.Y. Chen, *Two central limit problems for dependent random variables*. Z. Wahrscheinlichkeits-theor. Verw. Geb. 43, 223–243 (1978).

[CMSS07] B. Collins, J.A. Mingo, P. Śniady, R. Speicher, *Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants*. Doc. Math. 12 (2007), 1–70.

[CS08] B. Collins, M. Stolz, *Borel theorems for random matrices from the classical compact symmetric spaces*. Annals of Probability Volume 36, Number 3 (2008), 876-895.

[DZ98] A. Dembo, O. Zeitouni, *Large deviations techniques and applications* Second edition, 1998.

[D08] N. Demni, *Free Jacobi process*. J. Theo. Prob., 21: 118-143 (2008).

[DS94] P. Diaconis, M. Shahshahani, *On the eigenvalues of random matrices*, Jour. Appl. Prob, Special 31A, 49–62. (1994).

[FO09] P.K. Friz, H. Oberhauser, *Rough path limits of Wong-Zakai type with modified drift term*, Journal of Functional Analysis 256 (2009), pp. 3236–3256.

[HP00] F. Hiai, D. Petz, *The semicircle law, free random variables, and entropy*, Amer. Math. Soc., Mathematical Surveys and Monographs Volume 77, 2000.

[H51] W. Hoeffding, *A combinatorial central limit theorem* Ann. Math. Statist. 22, 558–566 (1951).

[HJ91] R.A. Horn, C.R. Johnson, *Topics in matrix analysis*. Cambridge University Press, 1991.

[H56] G.A. Hunt, *Semigroups of measures on Lie groups*. Trans. Am. Math. Soc. 81, 264-293. 1956.

[IW81] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffussion processes*. North-Holland, 1981.

[J06] T. Jiang, *How many entries of a typical orthogonal matrix can be approximated by independent normals?* Ann. Probab. 34(4), 1497-1529. 2006.

[L08] T. Lévy, *Schur-Weyl duality and the heat kernel measure on the unitary group*. Adv. Math. 218 (2008), no. 2, 537–575.

[M08] E. Meckes, *Linear functions on the classical matrix groups*. Trans. Amer. Math. Soc. 360 (2008), no. 10, 5355–5366.

[MN04] J.A. Mingo, A. Nica, *Annular noncrossing permutations and partitions, and second-order asymptotics for random matrices*. Int. Math. Res. Not. 2004, no. 28, 1413–1460.

[MS06] J.A. Mingo, R. Speicher, *Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces*. J. Funct. Anal. 235 (2006), no. 1, 226–270.

[MSS07] J.A. Mingo, P. Śniady, R. Speicher, *Second order freeness and fluctuations of random matrices. II. Unitary random matrices*. Adv. Math. 209 (2007), no. 1, 212–240.

[R97] E.M. Rains, *Combinatorial properties of Brownian motion on compact classical groups*, J. Theo. Prob., Vol. 10, No. 3, 259-279. 1997.

[RW87] L.C.G. Rogers, D. Williams, *Diffusions, Markov processes and martingales: Volume 2, Ito calculus*, J. Wiley & sons, 1987.

[S88] W. Schneller, *A short proof of Motoo's combinatorial Central Limit Theorem using Stein's method.* Probab. Th. Rel. Fields 78, 249-252 (1988).

[SV73] D.W. Stroock, S.R.S. Varadhan, *Limit thoerems for random walks on Lie groups*, Sankhyā, Series A 35, 277-294. 1973.

[X97] F. Xu, *A random matrix model from two-dimentional Yang-Mills theory*. Comm. Math. Phys. 190 (2) 287-307. 1997.

Florent Benaych-Georges, LPMA, UPMC Univ Paris 6, Case courier 188, 4, Place Jussieu, 75252 Paris Cedex 05, France, and CMAP, École Polytechnique, route de Saclay, 91128 Palaiseau Cedex, France.

E-mail address: florent.benaych@upmc.fr