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First passage time law for some Lévy processes with compound Poisson: Existence of a density

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Let $(X_t, t \geq 0)$ be a Lévy process with compound Poisson process and τ_x be the first passage time of a fixed level $x > 0$ by $(X_t, t \geq 0)$. We prove that the law of τ_x has a density (defective when $\mathbb{E}(X_1) < 0$) with respect to the Lebesgue measure.

Keywords: first passage time law; jump process; Lévy process

1. Introduction

The main purpose of this paper is to show that the first passage time distribution associated with a Lévy process with compound Poisson process has a density with respect to the Lebesgue measure.

Let X be a cadlag process started at 0 and τ_x the first passage time of level $x > 0$ by X.

Lévy, in [15], computed the law of τ_x when X is a Brownian motion with drift. This result is extended by Alili *et al.* [1] and Leblanc [12] to the case where X is an Ornstein–Uhlenbeck process. The case where X is a Bessel process was studied by Borodin and Salminen in [4].

For the situation where the process X has jumps, the first results were obtained by Zolotarev [22] and Borokov [5] for X a spectrally negative Lévy process. Moreover, if X_t has probability density $p(t, x)$ with respect to the Lebesgue measure, then the law of τ_x has density $f(t, x)$ with respect to the Lebesgue measure, where $xf(t, x) = tp(t, x)$ and $X_{\tau_x} = x$ almost surely.

If X is a spectrally positive Lévy process, Doney [7] gives an explicit formula for the joint Laplace transform of τ_x and the overshoot $X_{\tau_x} - x$. When X is a stable Lévy process, Peskir [16] and Bernyk *et al.* [2] obtain an explicit formula for the passage time density.

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The case where X has signed jumps has been studied more recently. In [9], the authors give the law of τ_x when X is the sum of a decreasing Lévy process and an independent compound process with exponential jump sizes. This result is extended by Kou and Wang in [11] to the case of a diffusion process with jumps where the jump sizes follow a double exponential law. They compute the Laplace transform of τ_x and derive an expression for the density of τ_x. For a more general jump-diffusion process, Roynette *et al.* [19] show that the Laplace transform of $(\tau_x, x - X_{\tau_{x\_}}, X_{\tau_x} - x)$ is the solution of some kind of random integral.

For a general Lévy processes, Doney and Kyprianou [8] give the quintuple law of $(\bar{G}_{\tau_{x\_}}, \tau_x - \bar{G}_{\tau_x -}, X_{\tau_x} - x, x - X_{\tau_x -}, x - \bar{X}_{\tau_x -})$ where $\bar{X}_t = \sup_{s\leq t} X_s$ and $\bar{G}_t = \sup\{s < t, \bar{X}_s = X_s\}$.

Results are also available for some Lévy processes without Gaussian component; see Lefèvre *et al.* [13, 14, 17, 18]. Blanchet [3] considers a process satisfying the stochastic equation $\mathrm{d}X_t = X_{t\_}(\mu\,\mathrm{d}t + \sigma\mathbf{1}_{\tilde{\phi}(t)=0}\,\mathrm{d}W_t + \phi\mathbf{1}_{\tilde{\phi}(t)=\phi}\,\mathrm{d}\tilde{N}_t), t \leq T$, where T is a finite horizon, $\mu \in \mathbb{R}$, $\sigma > 0$, $\tilde{\phi}(\cdot)$ is a function taking two values, 0 or ϕ, W is a Brownian motion, N is a Poisson process with intensity $\frac{1}{\phi^2}\mathbf{1}_{\tilde{\phi}(t)=\phi}$ and $\tilde{N}$ is the compensated Poisson process.

The aim of this paper is to add to these results the law of a first passage time by a Lévy process with compound Poisson process.

The paper is organized as follows: Section 2 contains the main result (Theorem 2.1) which gives the first passage time law by a jump Lévy process. We compute the derivative of the distribution function of τ_x at $t = 0$ in Section 2.1 and at $t > 0$ in Section 2.2. Section 2.2 contains the proofs of some useful results.

2. First passage time law

Let $m \in \mathbb{R}$ $(W_t, t \geq 0)$ be a standard Brownian motion $(N_t, t \geq 0)$ be a Poisson process with constant positive intensity a and $(Y_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with distribution function F_Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that the σ-fields $\sigma(Y_i, i \in \mathbb{N}^*)$, $\sigma(N_t, t \geq 0)$ and $\sigma(W_t, t \geq 0)$ are independent. Let $(T_n, n \in \mathbb{N}^*)$ be the sequence of the jump times of the process N and $(S_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with exponential law of parameter a such that $T_n = \sum_{i=1}^n S_i$, $n \in \mathbb{N}^*$.

Let $\tilde{X}$ be the Brownian motion with drift $m \in \mathbb{R}$ and for $z > 0$, $\tilde{\tau}_z = \inf\{t \geq 0 : mt + W_t \geq z\}$. By [10], formula (5.12), page 197, $\tilde{\tau}_z$ has the following law on $\overline{\mathbb{R}}_+ : \tilde{f}(u, z)\,\mathrm{d}u + \mathbb{P}(\tilde{\tau}_z = \infty)\delta_\infty(\mathrm{d}u)$, where

$$\begin{aligned} \tilde{f}(u, z) &= \frac{|\, z \,|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z - mu)^2}{2u}\right]\mathbf{1}_{]0,\infty[}(u), \qquad u \in \mathbb{R}, \quad \text{and} \\ \mathbb{P}(\tilde{\tau}_z = \infty) &= 1 - \mathrm{e}^{mz - |mz|}. \end{aligned} \tag{1}$$

The function $\tilde{f}(\cdot, z)$ and all its derivatives admit 0 as right limit at 0 and are $\mathcal{C}^\infty$ on $\mathbb{R}$.

Let X be the process defined by $X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, t \geq 0$, and τ_x be the first passage time of level $x > 0$ by $X: \tau_x = \inf\{u > 0 : X_u \geq x\}$. The main result of this paper is the following theorem.

Theorem 2.1. *The distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(\cdot, x)$, is equal to*

$$f(0,x) = \frac{a}{2}(2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4}(F_Y(x) - F_Y(x_-))$$

and for every $t > 0$,

$$f(t,x) = a\mathbb{E}(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)) + \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}}\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})).$$

Furthermore, $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

The proof of Theorem 2.1 is given in Sections 2.1 and 2.2.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the completed natural filtration generated by the processes $(W_t, t \geq 0)$, $(N_t, t \geq 0)$ and the random variables $(Y_i, i \in \mathbb{N}^*): \mathcal{F}_t = \sigma(W_s, s \leq t) \vee \sigma(N_s, s \leq t, Y_1, \ldots, Y_{N_t}) \vee \mathcal{N}$. Here, $\mathcal{N}$ is the set of negligible sets of $(\mathcal{F}, \mathbb{P})$.

Remark 2.2. This result is already known when X has no positive jumps (see [20], Theorem 46.4, page 348), when X is a stable Lévy process with no negative jumps (see [2]) and when X is a jump diffusion where the jump sizes follow a double exponential law (see [11]).

According to [14] and [21], for all $x > 0$, the passage time τ_x is finite almost surely if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

2.1. Existence of the right derivative at $t = 0$

In this section, we show that the distribution function of τ_x has a right derivative at 0 and we compute this derivative. For this purpose, we split the probability $\mathbb{P}(\tau_x \leq h)$ according to the values of $N_h: \mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2)$.

Note that $\mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq 1 - \mathrm{e}^{-ah} - ah\mathrm{e}^{-ah}$ and thus $\lim_{h\to 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0$.

It suffices to prove the following two properties:

$$\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} \overset{h\to 0}{\longrightarrow} 0; \tag{2}$$

$$\frac{\mathbb{P}(\tau_x \leq h, N_h = 1)}{h} \overset{h\to 0}{\longrightarrow} \frac{a}{2}(2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4}(F_Y(x) - F_Y(x_-)). \tag{3}$$

On the set $\{\omega : N_h(\omega) = 0\}$, the processes $(X_t, 0 \leq t \leq h)$ and $(\tilde{X}_t, 0 \leq t \leq h)$ are equal and $\mathbb{P}$-a.s. $\tau_x \wedge h = \tilde{\tau}_x \wedge h$. Since $\tilde{\tau}_x$ is independent of N, we have $\mathbb{P}(\tau_x \leq h, N_h = 0) =$

$e^{-ah}\mathbb{P}(\tilde{\tau}_x \le h)$. The law of $\tilde{\tau}_x$ has a C^∞ density (possibly defective) with respect to the Lebesgue measure, null on $]-\infty,0]$, Thus, (2) holds.

To prove (3), we use the same type of arguments as in [19] (for the proof of Theorem 2.4). We split the probability $\mathbb{P}(\tau_x \le h, N_h = 1)$ into three parts according to the relative positions of τ_x and T_1, the first jump time of the Poisson process N:

$$\begin{aligned}\mathbb{P}(\tau_x \le h, N_h = 1) &= \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \le h, N_h = 1)\\ &= A_1(h) + A_2(h) + A_3(h).\end{aligned}$$

Step 1: As for (2), we easily prove that $\frac{A_1(h)}{h} \overset{h\to 0}{\longrightarrow} 0$.

Step 2: We prove that $\frac{A_2(h)}{h} \overset{h\to 0}{\longrightarrow} \frac{a}{2}(2 - F_Y(x) - F_Y(x_-))$.

Note that $A_2(h) = \mathbb{P}(\tilde{\tau}_x > T_1, \tilde{X}_{T_1} + Y_1 \ge x, T_1 \le h < T_2)$. Using the independence of $(S_i, i \ge 1)$ and $(Y_1, \tilde{X}, \tilde{\tau}_x)$, we get $\mathbb{P}(\tau_x = T_1, N_h = 1) = a e^{-ah} \int_0^h \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x > s\}} \mathbf{1}_{\{Y_1 \ge x - \tilde{X}_s\}})\, ds$.

Integrating with respect to Y_1, we obtain

$$\frac{\mathbb{P}(\tau_x = T_1, N_h = 1)}{a e^{-ah}} = \int_0^h \mathbb{E}((1 - F_Y)((x - \tilde{X}_s)_-))\, ds - \int_0^h \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le s\}}(1 - F_Y)((x - \tilde{X}_s)_-))\, ds.$$

On the one hand, since F_Y is a cadlag bounded function and $\tilde{X}_s = ms + W_s$, where W is continuous and symmetric, we get $\lim_{s\to 0} \mathbb{E}(F_Y((x - \tilde{X}_s)_-)) = \frac{F_Y(x) + F_Y(x_-)}{2}$. On the other hand, $\lim_{s\to 0} \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le s\}}(1 - F_Y)((x - \tilde{X}_s)_-)) = 0$.

We deduce that $\lim_{h\to 0} \frac{A_2(h)}{h} = \frac{a}{2}(2 - F_Y(x) - F_Y(x_-))$.

Step 3: We prove that $\frac{A_3(h)}{h} \overset{h\to 0}{\longrightarrow} \frac{a}{4}(F_Y(x) - F_Y(x_-))$.

Note that $\mathbb{P}(T_1 < \tau_x \le h, N_h = 1) = \mathbb{P}(T_1 < \tau_x \le h, T_1 \le h < T_2)$ and $T_2 = T_1 + S_2 \circ \theta_{T_1}$, where θ is the translation operator.

Moreover, on $\{T_1 < \tau_x \le h < T_2\}$, $X_s = X_{T_1} + \tilde{X}_{s-T_1} \circ \theta_{T_1}$, where $T_1 < s \le h$ and $\tau_x = T_1 + \tilde{\tau}_{x - X_{T_1}} \circ \theta_{T_1}$. The strong Markov property gives, with $\mathbb{E}^{T_1}(\cdot)$ standing for $\mathbb{E}(\cdot \mid \mathcal{F}_{T_1})$,

$$\begin{aligned}A_3(h) &= \mathbb{E}(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \ge T_1\}} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}} \mathbf{1}_{\{h - T_1 < S_2\}}))\\ &= \mathbb{E}(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \ge T_1\}} e^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}}))\\ &= -\mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le T_1 \le h\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}}))\\ &\quad + \mathbb{E}(\mathbf{1}_{\{h \ge T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})).\end{aligned}$$

Since the distribution function of $\tilde{\tau}_x$ has a null derivative at 0, we have

$$\lim_{h\to 0} \frac{1}{h} \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le T_1 \le h\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})) = 0.$$

It remains to show that $\lim_{h\downarrow 0} \frac{G(h)}{h} = \frac{a}{4}[F(x) - F(x^-)]$, where

$$G(h) = \mathbb{E}(\mathbf{1}_{\{h \ge T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})).$$

Integrating with respect to T_1 and then using the fact that $\tilde{f}(\cdot,z)$ is the derivative of the distribution function of $\tilde{\tau}_z$, we get $G(h) = a\mathrm{e}^{-ah}\int_0^h\int_0^{h-s}\mathbb{E}[\mathbf{1}_{\{\tilde{X}_s+Y_1<x\}}\tilde{f}(u, x-\tilde{X}_s - Y_1)]\,\mathrm{d}u\,\mathrm{d}s$.

We may apply Lemma A.1 to $p=1$, $\mu = x - ms - Y_1$ and $\sigma = \sqrt{s}$. Then,

$$\mathbb{E}[\tilde{f}(u,\mu+\sigma G)\mathbf{1}_{\{\mu+\sigma G>0\}}] = \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[\mathrm{e}^{-(\mu-mu)^2/(2(\sigma^2+u))}\left(\frac{\mu+\sigma^2 m}{(\sigma^2+u)^{3/2}} + \frac{\sigma G}{\sqrt{u}(\sigma^2+u)}\right)^+\right]$$

with $x^+ = \max\{0,x\}$ and G is a Gaussian $\mathcal{N}(0,1)$ variable and we have

$$G(h) = \frac{a\mathrm{e}^{-ah}}{\sqrt{2\pi}}\int_0^h\int_0^{h-s}\mathbb{E}\left[\mathrm{e}^{-(x-m(u+s)-Y_1)^2/(2(u+s))}\left(\frac{x-Y_1}{(u+s)^{3/2}} + \frac{G\sqrt{s}}{\sqrt{u}(u+s)}\right)^+\right]\mathrm{d}u\,\mathrm{d}s.$$

We make the changes of variables $s = th$, $u = hv$. Then,

$$\frac{G(h)}{h} = \frac{a\mathrm{e}^{-ah}}{\sqrt{2\pi}}\int_0^1\int_0^{1-t}\mathbb{E}\left[\mathrm{e}^{-(x-mh(v+t)-Y_1)^2/(2h(v+t))}\left(\frac{x-Y_1}{\sqrt{h}(v+t)^{3/2}} + \frac{G\sqrt{t}}{\sqrt{v}(v+t)}\right)^+\right]\mathrm{d}t\,\mathrm{d}v.$$

However,

$$\lim_{h\to 0^+}\mathrm{e}^{-(x-mh(T=v)-Y_1)^2/(2h(t+v))}\left(\frac{x-Y_1}{\sqrt{h}(t+v)^{3/2}} + \frac{G\sqrt{t}}{\sqrt{v}(t+v)}\right)^+ = \frac{\sqrt{t}}{\sqrt{v}(t+v)}G^+\mathbf{1}_{\{x=Y_1\}}$$

and

$$\begin{aligned}&\sup_{0\le h\le 1}\mathrm{e}^{-(x-mh(t+v)-Y_1)^2/(2h(t+v))}\left(\frac{x-Y_1}{\sqrt{h}(t+v)^{3/2}} + \frac{G\sqrt{v}}{\sqrt{1-v}}\right)^+\\ &\quad\le \frac{\sup_{z\ge 0} z\mathrm{e}^{-z^2/2} + |m|}{\sqrt{t+v}} + \frac{\sqrt{t}}{\sqrt{v}(t+v)}|G|.\end{aligned}$$

From Lebesgue's dominated convergence theorem, we then obtain

$$\lim_{h\to 0}\frac{G(h)}{h} = \Delta F_Y(x)\frac{\mathbb{E}(G_+)}{\sqrt{2\pi}}\int_0^1\int_0^{1-t}\frac{\sqrt{t}}{\sqrt{v}(t+v)}\,\mathrm{d}v\,\mathrm{d}t = \frac{1}{4}\Delta F_Y(x),$$

where $\Delta F_Y(z) = F_Y(z) - F_Y(z_-)$. This identity achieves the proof of step 3.

2.2. Existence of the derivative at $t > 0$

Our task now is to show that the distribution function of τ_x is differentiable on $\mathbb{R}_+^*$ and to compute its derivative. For this purpose we split the probability $\mathbb{P}(t < \tau_x \le t+h)$, according to the values of $N_{t+h} - N_t$, into three parts:

$$\mathbb{P}(t < \tau_x \le t+h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_x \le t+h, N_{t+h} - N_t = 1)$$

$$+\mathbb{P}(t<\tau_x\le t+h, N_{t+h}-N_t\ge 2)$$
$$= B_1(h)+B_2(h)+B_3(h).$$

Since $B_3(h)\le \mathbb{P}(N_{t+h}-N_t\ge 2)$, we have $\lim_{h\to 0}\frac{B_3(h)}{h}=0$.

By the Markov property at t, $B_2(h)=\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}\mathbb{P}^t(\tau_{x-X_t}\le h, N_h=1))$, where $\mathbb{P}^t(\cdot)=\mathbb{P}(\cdot|\mathcal{F}_t)$.

By (3), $\frac{B_2(h)}{h}$ converges to $\frac{a}{2}[2-F_Y(x-X_t)-F_Y((x-X_t)_-)]+\frac{a}{4}[F_Y(x-X_t)-F_Y((x-X_t)_-)]$ and is upper bounded by $\frac{\mathbb{P}(N_h=1)}{h}=a\mathrm{e}^{-ah}\le a$. The dominated convergence theorem gives

$$\lim_{h\to 0}\frac{B_2(h)}{h}=a\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}(1-F_Y)(x-X_t))+\frac{3a}{4}\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}\Delta F_Y(x-X_t)).$$

However, the jumps set of F_Y is countable and X has a density (see [6], Proposition 3.12, page 90). Thus, $\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}\Delta F_Y(x-X_t))=0$ and $\lim_{h\to 0}\frac{B_2(h)}{h}=a\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}(1-F_Y)(x-X_t))$.

It thus remain to prove that

$$\frac{B_1(h)}{h}\xrightarrow{h\to 0}\mathbb{E}(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})). \tag{4}$$

Since T_{N_t} is not a stopping time, we cannot apply the strong Markov property. We split

$$B_1(h)=\mathbb{P}(t<\tilde{\tau}_x\le t+h<T_1)+\sum_{k=1}^{\infty}\mathbb{P}(t<\tau_x\le t+h, T_k<t<t+h<T_{k+1}).$$

On the set $\{T_k<t\}$, we have $X_t=X_{T_k}+X_{t-T_k}\circ\theta_{T_k}$, hence on the set $\{\tau_x>T_k\}$, we have $\tau_x=T_k+\tau_{x-X_{T_k}}\circ\theta_{T_k}$. Moreover, on the set $\{T_k<\min(t,\tau_x)\}$,

$$\mathbf{1}_{\{t<\tau_x\le t+h, T_k<t<t+h<T_{k+1}\}}=\mathbf{1}_{\{T_k<t\}}\mathbf{1}_{\{t-T_k<\tilde{\tau}_x\le t+h-T_k<S_{k+1}\}}\circ\theta_{T_k}$$

and the strong Markov property at T_k gives

$$B_1(h)=\mathrm{e}^{-a(t+h)}\mathbb{P}(t<\tilde{\tau}_x\le t+h)$$
$$+\sum_{k=1}^{\infty}\mathbb{E}(\mathbf{1}_{\{T_k<t\}}\mathbf{1}_{\{\tau_x>T_k\}}\mathrm{e}^{-a(t+h-T_k)}\mathbb{E}^{T_k}(\mathbf{1}_{\{t-T_k<\tilde{\tau}_{x-X_{T_k}}\le t+h-T_k\}})).$$

The $\mathcal{F}_{T_k}$-conditional law of $\tilde{\tau}_{x-X_{T_k}}$ has the density (possibly defective) $\tilde{f}(\cdot,x-X_{T_k})$, thus since $\mathrm{e}^{-a(t-T_k)}=\mathbb{E}^{T_k}(\mathbf{1}_{\{T_{k+1}>t\}})$, we have

$$B_1(h)=\mathrm{e}^{-ah}\int_t^{t+h}\mathbb{E}(\mathbf{1}_{\{0\le t<T_1\}})\tilde{f}(u,x)\,\mathrm{d}u$$

$$+\,\mathrm{e}^{-ah}\sum_{k=1}^{\infty}\int_t^{t+h}\mathbb{E}(\mathbf{1}_{\{T_k\le t<T_{k+1}\}}\mathbf{1}_{\{\tau_x>T_k\}}\tilde{f}(u-T_k,x-X_{T_k}))\,\mathrm{d}u \tag{5}$$

$$=\mathrm{e}^{-ah}\int_t^{t+h}\mathbb{E}(\mathbf{1}_{\{T_{N_t}<\tau_x\}}\tilde{f}(u-T_{N_t},x-X_{T_{N_t}}))\,\mathrm{d}u.$$

Since $\tilde{f}$ is continuous with respect to u, for all $t>0$, almost surely,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h}\mathbf{1}_{\{T_{N_t}<\tau_x\}}\tilde{f}(u-T_{N_t},x-X_{T_{N_t}})\,\mathrm{d}u=\mathbf{1}_{\{T_{N_t}<\tau_x\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}}).$$

According to Proposition A.2 in the Appendix, the family of random variables $(\frac{1}{h}\int_t^{t+h}\tilde{f}(u-T_{N-t},x-X_{T_{N_t}})\,\mathrm{d}u)_{0<h\le 1}$ is uniformly integrable. We then obtain

$$\lim_{h\to 0}\frac{B_1(h)}{h}=\mathbb{E}(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})).$$

Using (4), we deduce that

$$\frac{\mathbb{P}(t<\tau_x\le t+h)}{h}\overset{h\to 0}{\longrightarrow}a\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}(1-F_Y)(x-X_t))+\mathbb{E}(\mathbf{1}_{\{\tau_x>T_{N_t}\}}\tilde{f}(t-T_{N_t},x-X_{T_{N_t}})).$$

The proof of Theorem 2.1 is thus complete.

Appendix

We prove the following on $\tilde{f}$ given in (1).

Lemma A.1. *Let G be a Gaussian random variable $\mathcal{N}(0,1)$ and let $\mu\in\mathbb{R}$, $\sigma\in\mathbb{R}_+$, $p\ge 1$ and $x^+=\max\{x,0\}$. Then, for every $u\in\mathbb{R}$,*

$$\begin{aligned}&\mathbb{E}[\tilde{f}(u,\mu+\sigma G)^p\mathbf{1}_{\{\mu+\sigma G>0\}}]\\&\quad=\frac{1}{\sqrt{2^p\pi^p}}\frac{u^{(1-2p)/2}\mathrm{e}^{-p(\mu-mu)^2/(2(p\sigma^2+u))}}{(p\sigma^2+u)^{(p+1)/2}}\\&\qquad\times\mathbb{E}\left[\left(\sigma G+\sqrt{\frac{u}{p\sigma^2+u}}(\mu-mu)+m\sqrt{u(p\sigma^2+u)}\right)_+^p\right].\end{aligned}$$

Proposition A.2. *For every $t>0$ and $1\le p<3/2$,*

$$\sup_{0<h\le 1}\mathbb{E}\left[\left(\frac{1}{h}\int_t^{t+h}\mathbf{1}_{\{T_{N_t}<\tau_x\}}\tilde{f}(u-T_{N_t},x-X_{T_{N_t}})\,\mathrm{d}u\right)^p\right]<+\infty.$$

Proof. Let $I(h)$ be

$$I(h)=\frac{1}{h}\int_t^{t+h}\mathbf{1}_{\{T_{N_t}<\tau_x\}}\tilde f(u-T_{N_t},x-X_{T_{N_t}})\,\mathrm{d}u.$$

Using Jensen's inequality, the following estimate holds:

$$\mathbb{E}(I(h)^p)\le\frac{1}{h}\int_t^{t+h}\mathbb{E}(\mathbf{1}_{\{x-X_{T_{N_t}}>0\}}\tilde f(u-T_{N_t},x-X_{T_{N_t}})^p)\,\mathrm{d}u.$$

Conditioning by the filtration generated by N and Y_i, $i\in\mathbf{N}$, it becomes, where G is a standard Gaussian random variable independent of N and Y_i, $i\in\mathbf{N}$,

$$\mathbb{E}(I(h)^p)\le\frac{1}{h}\int_t^{t+h}\mathbb{E}\Bigg(\mathbf{1}_{\{x-mT_{N_t}-\sum_{i=1}^{N_t}Y_i-\sqrt{T_{N_t}}G>0\}}\times\tilde f\Bigg(u-T_{N_t},x-mT_{N_t}-\sum_{i=1}^{N_t}Y_i-\sqrt{T_{N_t}}G\Bigg)^p\Bigg)\,\mathrm{d}u.$$

Note that for $u\in[t,t+h]$, $t-T_{N_t}\le u-T_{N_t}\le 1+t-T_{N_t}$, $pT_{N_t}+t-T_{N_t}>t$ and if $C_p=\sup_{x\in\mathbf{R}^+}\sqrt{x^p}\mathrm{e}^{-px/2}$, then, from Lemma A.1,

$$\mathbb{E}(I(h)^p)\le\frac{3^{p-1}}{\sqrt{2^p\pi^p}}\mathbb{E}\Bigg(\frac{T_{N_t}^{p/2}}{(t-T_{N_t})^{p-1/2}t^{(p+1)/2}}\mathbb{E}(|G|^p)+\frac{1}{(t-T_{N_t})^{(p-1)/2}t^{1/2+p}}C_p+|m|^p\frac{1}{t^{1/2}(t-T_{N_t})^{(p-1)/2}}\Bigg).$$

Observe that for every $t>0$ and $(\alpha,\gamma)\in]-1,0]\times[0,+\infty[$, the random variables $(t-T_{N_t})^\alpha T_{N_t}^\gamma$ are integrable (see the details below), which completes the proof of Proposition A.2.

Note that

$$\mathbb{E}((t-T_{N_t})^\alpha T_{N_t}^\gamma)\le t^\alpha+\sum_{i=1}^{\infty}\mathbb{E}(\mathbf{1}_{t>T_i}(t-T_i)^\alpha T_i^\gamma)<+\infty. \tag{A.6}$$

However, for $i\ge1$, T_i admits as density the function $u\mapsto\frac{a^i}{(i-1)!}u^{i-1}\mathrm{e}^{-au}$, thus

$$\mathbb{E}(\mathbf{1}_{\{t>T_i\}}(t-T_i)^\alpha T_i^\gamma)=\frac{a^i}{(i-1)!}\int_0^t\mathrm{e}^{-au}(t-u)^\alpha u^{\gamma+i-1}\,\mathrm{d}u\le\frac{a^i}{(i-1)!}\int_0^t(t-u)^\alpha u^{\gamma+i-1}\,\mathrm{d}u=\frac{a^i}{(i-1)!}t^{\gamma+i+\alpha}\frac{\Gamma(\gamma+i)\Gamma(\alpha+1)}{\Gamma(\gamma+i+\alpha+1)}.$$

Consequently, the sum in the right-hand term of inequality (A.6) is finite and the random variable $(t-T_{N_t})^\alpha T_{N_t}^\gamma$ is integrable. □

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