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First passage time law for some Lévy processes with compound Poisson : existence of a density*

Laure Coutin [†] and Diana Dorobantu[‡]

Abstract. Let $(X_t, t \geq 0)$ be a Lévy process with compound Poisson process and τ_x be the first passage time of a fixed level $x > 0$ by $(X_t, t \geq 0)$. We prove that the law of τ_x has a density (defective when $\mathbb{E}(X_1) < 0$) with respect to the Lebesgue measure.

Keywords : Lévy process, jump process, first passage time law.

1 Introduction

The main purpose of this paper is to show that the first passage time distribution associated with a Lévy process with compound Poisson process has a density with respect to the Lebesgue measure.

Let X be a càd-làg process started at 0 and τ_x the first passage time of level $x > 0$ by X .

Lévy, in [16], computes the law of τ_x when X is a Brownian motion with drift. This result is extended by Alili and al. [1] or Leblanc [13] to the case where X is an Ornstein-Uhlenbeck process. The case where X is a Bessel process was studied by Borodin and Salminen in [4].

When the process X has jumps, the first results are obtained by Zolotarev [23] and Borokov [5] for X a spectrally negative Lévy process. Moreover, if X_t has the probability density with respect to the Lebesgue measure $p(t, x)$, then the law of τ_x has the density with respect to the Lebesgue measure $f(t, x)$, where $xf(t, x) = tp(t, x)$ and $X_{\tau_x} = x$ almost surely.

If X is a spectrally positive Lévy process, Doney [7] gives an explicit formula for the joint Laplace transform of τ_x and the overshoot $X_{\tau_x} - x$. When X is a stable Lévy process, Peskir [17], and Bernyk and al. [2] obtain an explicit formula for the passage time density.

The case where X has signed jumps is more recently studied. In [9], the authors give the law of τ_x when X is the sum of a decreasing Lévy process and an independent compound process with exponential jump sizes. This result is extended by Kou et Wang in [11] to the case of a diffusion process with jumps where the jump sizes follow a double exponential law. They compute the Laplace transform of τ_x and derive an expression for the density of τ_x . For a

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more general jump-diffusion process, Roynette and al. [20] show that the Laplace transform of $(\tau_x, x - X_{\tau_x-}, X_{\tau_x} - x)$ is solution of some kind of random integral.

For a general Lévy processes, Doney and Kyprianou, in [8], give the quintuple law of $(\bar{G}_{\tau_x-}, \tau_x - \bar{G}_{\tau_x-}, X_{\tau_x} - x, x - X_{\tau_x-}, x - \bar{X}_{\tau_x-})$ where $\bar{X}_t = \sup_{s \leq t} X_s$ and $\bar{G}_t = \sup\{s < t, \bar{X}_s = X_s\}$.

Results are also available for some Lévy processes without Gaussian component, see Lefèvre and al. [14, 15, 18, 19]. Blanchet [3] considers a process satisfying the following stochastic equation : $dX_t = X_{t-}(\mu dt + \sigma \mathbf{1}_{\tilde{\phi}(t)=0} dW_t + \phi \mathbf{1}_{\tilde{\phi}(t)=\phi} d\tilde{N}_t)$, $t \leq T$ where T is a finite horizon, $\mu \in \mathbb{R}$, $\sigma > 0$, $\tilde{\phi}(\cdot)$ is a function taking two values 0 or ϕ , W is a Brownian motion, N is a Poisson process with intensity $\frac{1}{\phi^2} \mathbf{1}_{\tilde{\phi}(t)=\phi}$ and \tilde{N} is the compensated Poisson process.

The aim of our paper is to add to these studies the law of a first passage time by a Lévy process with compound Poisson process.

This paper is organized as follows : Section 2 contains the main result (Theorem 2.1) which gives the first passage time law by a jump Lévy process. We compute the derivative of the distribution function of τ_x at $t = 0$ in Section 2.1, at $t > 0$ in Section 2.2. Section 3 contains the proofs of some useful results.

2 First passage time law

Let $m \in \mathbb{R}$, $(W_t, t \geq 0)$ be a standard Brownian motion, $(N_t, t \geq 0)$ be a Poisson process with constant positive intensity a and $(Y_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with distribution function F_Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that the following σ -fields $\sigma(Y_i, i \in \mathbb{N}^*)$, $\sigma(N_t, t \geq 0)$ and $\sigma(W_t, t \geq 0)$ are independent. Let $(T_n, n \in \mathbb{N}^*)$ be the sequence of the jump times of the process N and $(S_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with exponential law of parameter a such that $T_n = \sum_{i=1}^n S_i, n \in \mathbb{N}^*$.

Let \tilde{X} be the Brownian motion with drift $m \in \mathbb{R}$ and for $z > 0$, $\tilde{\tau}_z = \inf\{t \geq 0 : mt + W_t \geq z\}$. By (5.12) page 197 of [10], $\tilde{\tau}_z$ has the following law on $\bar{\mathbb{R}}_+$: $f(u, z)du + \mathbb{P}(\tilde{\tau}_z = \infty)\delta_\infty(du)$ where

$$\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z - mu)^2}{2u}\right] \mathbf{1}_{]0, \infty[}(u), \quad u \in \mathbb{R}, \quad \text{and } \mathbb{P}(\tilde{\tau}_z = \infty) = 1 - e^{mz - |mz|}. \quad (1)$$

The function $\tilde{f}(\cdot, z)$ and all its derivatives admit 0 as right limit at 0 and is \mathcal{C}^∞ on \mathbb{R} .

Let X be the process defined by $X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i$, $t \geq 0$, and τ_x be the first passage time of level $x > 0$ by X : $\tau_x = \inf\{u > 0 : X_u \geq x\}$. The main result of this paper is the following theorem.

Theorem 2.1 *The distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(\cdot, x)$, is equal to*

$$f(0, x) = \frac{a}{2} (2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4} (F_Y(x) - F_Y(x_-))$$

and for every $t > 0$ $f(t, x) = a\mathbb{E}(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)) + \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}))$.

Furthermore, $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

The proof of Theorem 2.1 is given in Sections 2.1 and 2.2.

Let $(\mathcal{F}_t)_{t \geq 0}$ be the completed natural filtration generated by the processes $(W_t, t \geq 0)$, $(N_t, t \geq 0)$ and the random variables $(Y_i, i \in \mathbb{N}^*) : \mathcal{F}_t = \sigma(W_s, s \leq t) \vee \sigma(N_s, s \leq t, Y_1, \dots, Y_{N_t}) \vee \mathcal{N}$. Here \mathcal{N} is the set of negligible sets of $(\mathcal{F}, \mathbb{P})$.

Remark 2.2 *This result is already known when X has no positive jumps (see Theorem 46.4 page 348 of [21]), when X is a stable Lévy process, with no negative jumps (see [2]) and when X is a jump-diffusion where the jump sizes follow a double exponential law (see [11]). According to [15] or [22], for all $x > 0$, the passage time τ_x is finite almost surely if and only if $m + a \mathbb{E}(Y_1) \geq 0$.*

2.1 Existence of the right derivative at $t = 0$

In this section, we show that the distribution function of τ_x has a right derivative at 0 and we compute this derivative. For this purpose, we split the probability $\mathbb{P}(\tau_x \leq h)$ according to the values of N_h : $\mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2)$.

Note that $\mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq 1 - e^{-ah} - ahe^{-ah}$, thus $\lim_{h \rightarrow 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0$.

It suffices to prove the following two properties :

$$\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} \xrightarrow{h \rightarrow 0} 0, \quad (2)$$

$$\frac{\mathbb{P}(\tau_x \leq h, N_h = 1)}{h} \xrightarrow{h \rightarrow 0} \frac{a}{2} (2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4} (F_Y(x) - F_Y(x_-)). \quad (3)$$

On the set $\{\omega : N_h(\omega) = 0\}$, the processes $(X_t, 0 \leq t \leq h)$ and $(\tilde{X}_t, 0 \leq t \leq h)$ are equal and \mathbb{P} -a.s. $\tau_x \wedge h = \tilde{\tau}_x \wedge h$. Since $\tilde{\tau}_x$ is independent of N , then $\mathbb{P}(\tau_x \leq h, N_h = 0) = e^{-ah} \mathbb{P}(\tilde{\tau}_x \leq h)$. The law of $\tilde{\tau}_x$ has a C^∞ density (possibly defective) with respect to the Lebesgue measure, null on $] -\infty, 0]$, Thus (2) holds.

To prove (3), we use the same type of arguments as in [20] (for the proof of Theorem 2.4). We split the probability $\mathbb{P}(\tau_x \leq h, N_h = 1)$ into three parts according to the relative position of τ_x and T_1 , the first jump time of the Poisson process N :

$$\begin{aligned} \mathbb{P}(\tau_x \leq h, N_h = 1) &= \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) \\ &= A_1(h) + A_2(h) + A_3(h). \end{aligned}$$

Step 1 : As for (2), we easily prove that $\frac{A_1(h)}{h} \xrightarrow{h \rightarrow 0} 0$.

Step 2 : We prove that $\frac{A_2(h)}{h} \xrightarrow{h \rightarrow 0} \frac{a}{2} (2 - F_Y(x) - F_Y(x_-))$.

Note that $A_2(h) = \mathbb{P}(\tilde{\tau}_x > T_1, \tilde{X}_{T_1} + Y_1 \geq x, T_1 \leq h < T_2)$. Using the independence between $(S_i, i \geq 1)$ and $(Y_1, \tilde{X}, \tilde{\tau}_x)$ we get : $\mathbb{P}(\tau_x = T_1, N_h = 1) = ae^{-ah} \int_0^h \mathbb{E} \left(\mathbf{1}_{\{\tilde{\tau}_x > s\}} \mathbf{1}_{\{Y_1 \geq x - \tilde{X}_s\}} \right) ds$.

Integrating with respect to Y_1 , we obtain :

$$\frac{\mathbb{P}(\tau_x = T_1, N_h = 1)}{ae^{-ah}} = \int_0^h \mathbb{E} \left((1 - F_Y)((x - \tilde{X}_s)_-) \right) ds - \int_0^h \mathbb{E} \left(\mathbf{1}_{\{\tilde{\tau}_x \leq s\}} (1 - F_Y)((x - \tilde{X}_s)_-) \right) ds.$$

On the one hand, since F_Y is a càdlàg bounded function and $\tilde{X}_s = ms + W_s$ where W is continuous and symmetric, we get $\lim_{s \rightarrow 0} \mathbb{E} \left(F_Y((x - \tilde{X}_s)_-) \right) = \frac{F_Y(x) + F_Y(x_-)}{2}$. On the other hand, $\lim_{s \rightarrow 0} \mathbb{E} \left(\mathbf{1}_{\{\tilde{\tau}_x \leq s\}} (1 - F_Y)((x - \tilde{X}_s)_-) \right) = 0$.

We deduce that $\lim_{h \rightarrow 0} \frac{A_2(h)}{h} = \frac{a}{2} (2 - F_Y(x) - F_Y(x_-))$.

Step 3 : We prove that $\frac{A_3(h)}{h} \xrightarrow{h \rightarrow 0} \frac{a}{4} (F_Y(x) - F_Y(x_-))$.

Note that $\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) = \mathbb{P}(T_1 < \tau_x \leq h, T_1 \leq h < T_2)$ and $T_2 = T_1 + S_2 \circ \theta_{T_1}$ where θ is the translation operator.

Moreover, on $\{T_1 < \tau_x \leq h < T_2\}$, $X_s = X_{T_1} + \tilde{X}_{s-T_1} \circ \theta_{T_1}$ when $T_1 < s \leq h$ and $\tau_x = T_1 + \tilde{\tau}_{x-X_{T_1}} \circ \theta_{T_1}$. The Strong Markov Property gives, with $\mathbb{E}^{T_1}(\cdot)$ standing for $\mathbb{E}(\cdot | \mathcal{F}_{T_1})$:

$$\begin{aligned} A_3(h) &= \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \geq T_1\}} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \mathbf{1}_{\{h-T_1 < S_2\}} \right) \right) \\ &= \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \geq T_1\}} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \right) \right) \\ &= -\mathbb{E} \left(\mathbf{1}_{\{\tilde{\tau}_x \leq T_1 \leq h\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \right) \right) \\ &\quad + \mathbb{E} \left(\mathbf{1}_{\{h \geq T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \right) \right). \end{aligned}$$

Since the distribution function of $\tilde{\tau}_x$ has a null derivative at 0, then

$$\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left(\mathbf{1}_{\{\tilde{\tau}_x \leq T_1 \leq h\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \right) \right) = 0.$$

It remains to show that $\lim_{h \downarrow 0} \frac{G(h)}{h} = \frac{a}{4} [F(x) - F(x^-)]$, where

$$G(h) = \mathbb{E} \left(\mathbf{1}_{\{h \geq T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}} \right) \right).$$

Integrating with respect to T_1 and then using the fact that $\tilde{f}(\cdot, z)$ is the derivative of the distribution function of $\tilde{\tau}_z$, we get : $G(h) = ae^{-ah} \int_0^h \int_0^{h-s} \mathbb{E} \left[\mathbf{1}_{\{\tilde{X}_s + Y_1 < x\}} \tilde{f}(u, x - \tilde{X}_s - Y_1) \right] dud s$.

We may apply Lemma 3.1 to $p = 1$, $\mu = x - ms - Y_1$ and $\sigma = \sqrt{s}$. Then

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\{\mu + \sigma G > 0\}}] = \frac{1}{\sqrt{2\pi}} \mathbb{E} \left[e^{-\frac{(\mu - mu)^2}{2(\sigma^2 + u)}} \left(\frac{\mu + \sigma^2 m}{(\sigma^2 + u)^{3/2}} + \frac{\sigma G}{\sqrt{u}(\sigma^2 + u)} \right)^+ \right]$$

with $x^+ = \max\{0, x\}$ and G is a Gaussian $\mathcal{N}(0, 1)$ variable and

$$G(h) = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^{h-s} \mathbb{E} \left[e^{-\frac{(x-m(u+s)-Y_1)^2}{2(u+s)}} \left(\frac{x - Y_1}{(u+s)^{3/2}} + \frac{G\sqrt{s}}{\sqrt{u}(u+s)} \right)^+ \right] dud s.$$

We make the following change of variable $s = th$, $u = hv$. Then

$$\frac{G(h)}{h} = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} \mathbb{E} \left[e^{-\frac{(x-mh(v+t)-Y_1)^2}{2h(v+t)}} \left(\frac{x-Y_1}{\sqrt{h(v+t)^{3/2}} + \frac{G\sqrt{t}}{\sqrt{v(v+t)}} \right)^+ \right] dt dv.$$

But $\lim_{h \rightarrow 0^+} e^{-\frac{(x-mh(T=v)-Y_1)^2}{2h(t+v)}} \left(\frac{x-Y_1}{\sqrt{h(t+v)^{3/2}} + \frac{G\sqrt{t}}{\sqrt{v(t+v)}} \right)^+ = \frac{\sqrt{t}}{\sqrt{v(t+v)}} G^+ \mathbf{1}_{\{x=Y_1\}}$, and

$$\sup_{0 \leq h \leq 1} e^{-\frac{(x-mh(t+v)-Y_1)^2}{2h(t+v)}} \left(\frac{x-Y_1}{\sqrt{h(t+v)^{3/2}} + \frac{G\sqrt{v}}{\sqrt{1-v}}} \right)^+ \leq \frac{\sup_{z \geq 0} z e^{-\frac{z^2}{2}} + |m|}{\sqrt{t+v}} + \frac{\sqrt{t}}{\sqrt{v(t+v)}} |G|.$$

Then from Lebesgue's Dominated Convergence Theorem we obtain :

$$\lim_{h \rightarrow 0} \frac{G(h)}{h} = \Delta F_Y(x) \frac{\mathbb{E}(G_+)}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} \frac{\sqrt{t}}{\sqrt{v(t+v)}} dv dt = \frac{1}{4} \Delta F_Y(x),$$

where $\Delta F_Y(z) = F_Y(z) - F_Y(z_-)$. This identity achieves the proof of Step 3.

2.2 Existence of the derivative at $t > 0$

Our task is now to show that the distribution function of τ_x is differentiable on \mathbb{R}_+^* and to compute its derivative. For this purpose we split the probability $\mathbb{P}(t < \tau_x \leq t+h)$ according to the values of $N_{t+h} - N_t$ into three parts

$$\begin{aligned} & \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1) + \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t \geq 2) \\ & = B_1(h) + B_2(h) + B_3(h). \end{aligned}$$

Since $B_3(h) \leq \mathbb{P}(N_{t+h} - N_t \geq 2)$, we have $\lim_{h \rightarrow 0} \frac{B_3(h)}{h} = 0$.

By the Markov Property at t , $B_2(h) = \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} \mathbb{P}^t(\tau_x - X_t \leq h, N_h = 1))$, where $\mathbb{P}^t(\cdot) = \mathbb{P}(\cdot | \mathcal{F}_t)$. By (3), $\frac{B_2(h)}{h}$ converges to $\frac{a}{2}[2 - F_Y(x - X_t) - F_Y((x - X_t)_-)] + \frac{a}{4}[F_Y(x - X_t) - F_Y((x - X_t)_-)]$ and is upper bounded by $\frac{\mathbb{P}(N_h=1)}{h} = ae^{-ah} \leq a$. The Dominated Convergence Theorem gives :

$$\lim_{h \rightarrow 0} \frac{B_2(h)}{h} = a \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} (1 - F_Y)(x - X_t)) + \frac{3a}{4} \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} \Delta F_Y(x - X_t)).$$

However the jumps set of F_Y is countable and X has a density (cf. Proposition 3.12 page 90 of [6]). Thus $\mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} \Delta F_Y(x - X_t)) = 0$ and $\lim_{h \rightarrow 0} \frac{B_2(h)}{h} = a \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} (1 - F_Y)(x - X_t))$.

It thus remain to prove that

$$\frac{B_1(h)}{h} \xrightarrow{h \rightarrow 0} \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) \right). \quad (4)$$

Since T_{N_t} is not a stopping time, we can not apply the Strong Markov Property. We split

$$B_1(h) = \mathbb{P}(t < \tilde{\tau}_x \leq t+h < T_1) + \sum_{k=1}^{\infty} \mathbb{P}(t < \tau_x \leq t+h, T_k < t < t+h < T_{k+1}).$$

On the set $\{T_k < t\}$, we have $X_t = X_{T_k} + X_{t-T_k} \circ \theta_{T_k}$, hence on the set $\{\tau_x > T_k\}$, $\tau_x = T_k + \tau_{x-X_{T_k}} \circ \theta_{T_k}$. Moreover, on the set $\{T_k < \min(t, \tau_x)\}$,

$$\mathbf{1}_{\{t < \tau_x \leq t+h, T_k < t < t+h < T_{k+1}\}} = \mathbf{1}_{\{T_k < t\}} \mathbf{1}_{\{t-T_k < \tilde{\tau}_x \leq t+h-T_k < S_{k+1}\}} \circ \theta_{T_k}$$

and the Strong Markov Property at T_k gives

$$B_1(h) = e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t+h) + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{\{T_k < t\}} \mathbf{1}_{\{\tau_x > T_k\}} e^{-a(t+h-T_k)} \mathbb{E}^{T_k} \left(\mathbf{1}_{\{t-T_k < \tilde{\tau}_{x-X_{T_k}} \leq t+h-T_k\}} \right) \right).$$

The \mathcal{F}_{T_k} -conditional law of $\tilde{\tau}_{x-X_{T_k}}$ has the density (possibly defective) $\tilde{f}(\cdot, x - X_{T_k})$, thus since $e^{-a(t-T_k)} = \mathbb{E}^{T_k}(\mathbf{1}_{\{T_{k+1} > t\}})$, we have

$$\begin{aligned} B_1(h) &= e^{-ah} \int_t^{t+h} \mathbb{E}(\mathbf{1}_{\{0 \leq t < T_1\}}) \tilde{f}(u, x) du + e^{-ah} \sum_{k=1}^{\infty} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{\{T_k \leq t < T_{k+1}\}} \mathbf{1}_{\{\tau_x > T_k\}} \tilde{f}(u - T_k, x - X_{T_k}) \right) du \\ &= e^{-ah} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) \right) du. \end{aligned} \quad (5)$$

Since \tilde{f} is continuous with respect to u , for all $t > 0$, almost surely,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du = \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}).$$

According to Proposition 3.2 in the appendix, the family of r.v. $(\frac{1}{h} \int_t^{t+h} \tilde{f}(u - T_{N-t}, x - X_{T_{N_t}}) du)_{0 < h \leq 1}$ is uniformly integrable. Then, we obtain

$$\lim_{h \rightarrow 0} \frac{B_1(h)}{h} = \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) \right).$$

Using (4), we deduce

$$\frac{\mathbb{P}(t < \tau_x \leq t+h)}{h} \xrightarrow{h \rightarrow 0} a \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)) + \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) \right).$$

The proof of Theorem 2.1 is complete.

3 Appendix

We prove the following on \tilde{f} given in (1).

Lemma 3.1 *Let G be a Gaussian random variable $\mathcal{N}(0, 1)$ and let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, $p \geq 1$ and $x^+ = \max\{x, 0\}$. Then for every $u \in \mathbb{R}$*

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G)^p \mathbf{1}_{\{\mu + \sigma G > 0\}}] = \frac{1}{\sqrt{2^p \pi^p}} \frac{u^{\frac{1-2p}{2}} e^{-\frac{p(\mu-mu)^2}{2(p\sigma^2+u)}}}{(p\sigma^2+u)^{\frac{p+1}{2}}} \mathbb{E} \left[\left(\sigma G + \sqrt{\frac{u}{p\sigma^2+u}}(\mu - mu) + m\sqrt{u(p\sigma^2+u)} \right)_+^p \right].$$

Proposition 3.2 For every $t > 0$ and $1 \leq p < 3/2$

$$\sup_{0 < h \leq 1} \mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du \right)^p \right] < +\infty.$$

Proof

Let $I(h)$ be

$$I(h) = \frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du.$$

Using Jensen inequality, the following estimate holds

$$\mathbb{E}(I(h)^p) \leq \frac{1}{h} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{\{x - X_{T_{N_t}} > 0\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}})^p \right) du.$$

Conditionning by the filtration generated by N and $Y_i, i \in \mathbf{N}$, it becomes where G is a standard Gaussian r.v. independent of N and $Y_i, i \in \mathbf{N}$,

$$\mathbb{E}(I(h)^p) \leq \frac{1}{h} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{\{x - mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} G > 0\}} \tilde{f}(u - T_{N_t}, x - mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} G)^p \right) du.$$

Note that for $u \in [t, t+h]$, $t - T_{N_t} \leq u - T_{N_t} \leq 1 + t - T_{N_t}$, $pT_{N_t} + t - T_{N_t} > t$, and if $C_p = \sup_{x \in \mathbf{R}^+} \sqrt{x^p} e^{-\frac{px}{2}}$, then from Lemma 3.1

$$\mathbb{E}(I(h)^p) \leq \frac{3^{p-1}}{\sqrt{2^p \pi^p}} \mathbb{E} \left(\frac{T_{N_t}^{\frac{p}{2}}}{(t - T_{N_t})^{p-\frac{1}{2}} t^{\frac{p+1}{2}}} \mathbb{E}(|G|^p) + \frac{1}{(t - T_{N_t})^{\frac{p-1}{2}} t^{\frac{1}{2}+p}} C_p + |m|^p \frac{1}{t^{\frac{1}{2}}(t - T_{N_t})^{\frac{p-1}{2}}} \right).$$

Observe that for every $t > 0$ and $(\alpha, \gamma) \in]-1, 0] \times [0, +\infty[$, the r.v. $(t - T_{N_t})^\alpha T_{N_t}^\gamma$ are integrable (see details below), which achieve the proof of Proposition 3.2.

Note that

$$\mathbb{E} \left((t - T_{N_t})^\alpha T_{N_t}^\gamma \right) \leq t^\alpha + \sum_{i=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{\{t > T_i\}} (t - T_i)^\alpha T_i^\gamma \right) < +\infty. \quad (6)$$

But for $i \geq 1$, T_i admits as density the function $u \mapsto \frac{a^i}{(i-1)!} u^{i-1} e^{-au}$, thus

$$\begin{aligned} \mathbb{E} \left(\mathbf{1}_{\{t > T_i\}} (t - T_i)^\alpha T_i^\gamma \right) &= \frac{a^i}{(i-1)!} \int_0^t e^{-au} (t-u)^\alpha u^{\gamma+i-1} du \leq \frac{a^i}{(i-1)!} \int_0^t (t-u)^\alpha u^{\gamma+i-1} du \\ &= \frac{a^i}{(i-1)!} t^{\gamma+i+\alpha} \frac{\Gamma(\gamma+i)\Gamma(\alpha+1)}{\Gamma(\gamma+i+\alpha+1)}. \end{aligned}$$

Consequently, the sum in the right term of inequality (6) is finite and the r.v. $(t - T_{N_t})^\alpha T_{N_t}^\gamma$ is integrable. \square

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