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First hitting time law for some jump-diffusion processes : existence of a density*

Laure Coutin[†] and Diana Dorobantu[‡]

Abstract. Let $(X_t, t \geq 0)$ be a diffusion process with jumps, sum of a Brownian motion with drift and a compound Poisson process. We consider τ_x the first hitting time of a fixed level $x > 0$ by $(X_t, t \geq 0)$. We prove that the law of τ_x has a density (defective when $\mathbb{E}(X_1) < 0$) with respect to the Lebesgue measure.

Keywords : Lévy process, jump-diffusion process, hitting time law.

1 Introduction

The main purpose of this paper is to show that the first hitting time distribution associated with a jump-diffusion process (sum of a Brownian motion with drift and an independent compound Poisson process) has a density with respect to the Lebesgue measure.

Let $(X_t, t \geq 0)$ be a càd-làg process started at 0 and τ_x the first hitting time of level $x > 0$ by X .

Lévy, in [16], computes the law of τ_x when X is a Brownian motion with drift. This result is extended by Alili, Patie and Petersen [1] or Leblanc [13] to the case where X is an Ornstein-Uhlenbeck process. The case where X is a Bessel process was studied by Borodin and Salminen in [4].

Some results are also available when the process X has jumps. The first results are obtained by Zolotarev [23] and Borokov [5] when X is a spectrally negative Lévy process. They give the law of τ_x . Moreover, if X_t has the probability density with respect to the Lebesgue measure $p(t, x)$, then the law of τ_x has the density with respect to the Lebesgue measure $f(t, x)$, where $xf(t, x) = tp(t, x)$. In this case where X has only negative jumps, $X_{\tau_x} = x$ almost surely.

If X is a spectrally positive Lévy process, Doney [7] gives an explicit formula for the joint Laplace transform of τ_x and the overshoot $X_{\tau_x} - x$. When X is a stable Lévy process, Peskir [17], and Bernyk, Dalang and Peskir [2] obtain an explicit formula for the hitting time density.

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[†]IMT, University of Toulouse, France, laure.coutin@math.univ-toulouse.fr

[‡]ISFA, University of Lyon, France, diana.dorobantu@adm.univ-lyon1.fr

The case where X has signed jumps is more recently studied. In [9], the authors give the law of τ_x when X is the sum of a decreasing Lévy process and an independent compound process with exponential jump sizes. This result is extended by Kou et Wang in [11] to the case of a diffusion process with jumps where the jump sizes follow a double exponential law. They compute the Laplace transform of τ_x and derive an expression for the density of τ_x . For a more general jump-diffusion process, Roynette, Vallois and Volpi [20] show that the Laplace transform of $(\tau_x, x - X_{\tau_x-}, X_{\tau_x} - x)$ is solution of some kind of random integral.

Doney and Kyprianou [8] studied the problem for general Lévy processes. They give the quintuple law of $(\bar{G}_{\tau_x-}, \tau_x - \bar{G}_{\tau_x-}, X_{\tau_x} - x, x - X_{\tau_x-}, x - \bar{X}_{\tau_x-})$ where $\bar{X}_t = \sup_{s \leq t} X_s$ and $\bar{G}_t = \sup\{s < t, \bar{X}_s = X_s\}$.

Results are also available for some Lévy processes without Gaussian component, see Lefèvre, Loisel and Picard [14, 15, 18, 19]. Blanchet [3] considers a process satisfying the following stochastic equation : $dS_t = S_{t-}(\mu dt + \sigma \mathbf{1}_{\tilde{\phi}(t)=0} dW_t + \phi \mathbf{1}_{\tilde{\phi}(t)=\phi} d\tilde{N}_t)$, $t \leq T$ where T is a finite horizon, $\mu \in \mathbb{R}$, $\sigma > 0$, $\tilde{\phi}(\cdot)$ is a function taking two values 0 or ϕ , W is a Brownian motion, N is a Poisson process with intensity $\frac{1}{\phi^2} \mathbf{1}_{\tilde{\phi}(t)=\phi}$ and \tilde{N} is the compensated Poisson process.

The aim of our paper is to add to these studies the law of a first hitting time by a Lévy process which is the sum of a Brownian motion with drift and a compound Poisson process. We do not limit our study to a particular distribution of the jumps sizes.

This paper is organized as follows : Section 2 contains the main result (Theorem 2.1) which gives the first hitting time law by a jump Lévy process. The following two sections (Section 2.1 and Section 2.2) are dedicated to the proof of Theorem 2.1. In these sections we compute the derivative at $t = 0$ (Section 2.1) and at $t > 0$ (Section 2.2) of the hitting time distribution function. Section 3 contains the proofs of some useful results.

2 Hitting time law

Let $m \in \mathbb{R}$, $(W_t, t \geq 0)$ be a standard Brownian motion, $(N_t, t \geq 0)$ be a Poisson process with constant positive intensity a and $(Y_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with distribution function F_Y . We suppose that the following σ -fields $\sigma(Y_i, i \in \mathbb{N}^*)$, $\sigma(N_t, t \geq 0)$ and $\sigma(W_t, t \geq 0)$ are independent. Let $(T_n, n \in \mathbb{N}^*)$ be the sequence of the jump times of the process N .

Let X be the process defined by

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \quad (1)$$

Let τ_x be the first hitting time of level $x > 0$ by the stochastic process $(X_t, t \geq 0)$: $\tau_x = \inf\{u > 0 : X_u \geq x\}$.

The main result of this paper is the following theorem. It gives the law of τ_x .

Theorem 2.1 *If there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta|Y_1|}) < \infty$, then the distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(\cdot, x)$, is equal to*

$$f(0, x) = \frac{a}{2} (2 - F_Y(x) - F_Y(x^-)) + \frac{a}{4} (F_Y(x) - F_Y(x^-))$$

and for every $t > 0$ $f(t, x) = a\mathbb{E}(\mathbf{1}_{\tau_x > t}(1 - F_Y)(x - X_t)) + \mathbb{E}(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}))$

where

$$\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z - mu)^2}{2u}\right] \mathbf{1}_{]0, \infty[}(u), \quad u \in \mathbb{R}, \quad z \in \mathbb{R}.$$

Furthermore, $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

The proof of Theorem 2.1 is given in Sections 2.1 and 2.2.

Let \tilde{X} be the m drifted Brownian motion defined by

$$\tilde{X}_t = mt + W_t, \quad \text{for all } t \geq 0,$$

and $\tilde{\tau}_z$ be the first hitting time of $z > 0$ by \tilde{X} . Then, according to [10], $\tilde{f}(\cdot, z)$ is the derivative of the distribution function of $\tilde{\tau}_z$ (see Section 3 for more details).

Let $(\mathcal{F}_t)_{t \geq 0}$ be the completed natural filtration generated by the processes $(W_t, t \geq 0)$, $(N_t, t \geq 0)$ and the random variables $(Y_i, i \in \mathbb{N}^*)$.

Remark 2.2 *This result is already known in the following cases :*

When X has no positive jumps, the law of τ_x is given in Theorem 46.4 page 348 of [21].

When X is a stable Lévy process, with no negative jumps, the law of τ_x is given in [2].

When X is a jump-diffusion where the jump sizes follow a double exponential law, the result is given in [11].

2.1 Existence of the right derivative at $t = 0$

In this section we prove the first part of Theorem 2.1. Our goal is to show that the distribution function of τ_x (i.e. $t \mapsto \mathbb{P}(\tau_x \leq t)$) has a right derivative at 0 and we compute this derivative.

For this purpose we split the probability $\mathbb{P}(\tau_x \leq h)$ according to the values of N_h :

$$\mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2).$$

Note that $\mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq \mathbb{P}(N_h \geq 2) = 1 - e^{-ah} - ahe^{-ah}$, thus

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0.$$

The proof of the first part of Theorem 2.1 will be complete when the following lemma and proposition are shown :

Lemma 2.3 *The term $\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h}$ converges to 0 when h goes to 0.*

Proposition 2.4 *If there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta|Y_1|}) < \infty$, then for every $x > 0$ the term $\frac{\mathbb{P}(\tau_x \leq h, N_h=1)}{h}$ converges to $\frac{a}{2}(2 - F_Y(x) - F_Y(x^-)) + \frac{a}{4}(F_Y(x) - F_Y(x^-))$ when h goes to 0.*

Proof of Lemma 2.3.

The essential observation is that on the set $\{\omega : N_h(\omega) = 0\}$, the processes $(X_t, 0 \leq t \leq h)$ and $(\tilde{X}_t, 0 \leq t \leq h)$ are equal, and \mathbb{P} -almost surely $\tau_x \wedge h = \tilde{\tau}_x \wedge h$. Since $\tilde{\tau}_x$ is independent of N , then

$$\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} = \frac{e^{-ah}\mathbb{P}(\tilde{\tau}_x \leq h)}{h}.$$

The law of $\tilde{\tau}_x$ has a C^∞ density (possibly defective) with respect to the Lebesgue measure, null on $] - \infty, 0]$. Thus the limit of $\frac{\mathbb{P}(\tau_x \leq h, N_h=0)}{h}$ exists and is equal to 0 when h goes to 0. \square

To prove Proposition 2.4, we use the same type of arguments as in [20] (for the proof of Theorem 2.4). In [20], the authors compute the joint Laplace transform of (a passage time of a Lévy process, overshoot) as solution of an integral equation.

Proof of Proposition 2.4.

We split the probability $\mathbb{P}(\tau_x \leq h, N_h = 1)$ into three parts according to the relative position of τ_x and T_1 , the first jump time of the Poisson process N :

$$\mathbb{P}(\tau_x \leq h, N_h = 1) = \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \leq h, N_h = 1). \quad (2)$$

Step 1 : We prove that the contribution to the limit $\lim_{h \rightarrow 0^+} h^{-1}\mathbb{P}(\tau_x \leq h, N_h = 1)$ of the first term on the right hand side of (2) is null.

Since the processes $(X_t, 0 \leq t < T_1)$ and $(\tilde{X}_t, 0 \leq t < T_1)$ are equal, then on the set $\{\omega, \tau_x(\omega) < T_1(\omega)\}$, the stopping time τ_x and $\tilde{\tau}_x$ are equal. Thus

$$\mathbb{P}(\tau_x < T_1, N_h = 1) = \mathbb{P}(\tilde{\tau}_x < T_1 \leq h, N_h = 1) \leq \mathbb{P}(\tilde{\tau}_x \leq h).$$

The law of $\tilde{\tau}_x$ has a C^∞ density (possibly defective) with respect to the Lebesgue measure, null on $] - \infty, 0]$. Thus $\frac{\mathbb{P}(\tau_x < T_1, N_h=1)}{h}$ converges to 0 when h goes to 0.

Step 2 : We prove that the contribution to the limit $\lim_{h \rightarrow 0^+} h^{-1}\mathbb{P}(\tau_x \leq h, N_h = 1)$ of the second term on the right hand side of (2) is $\frac{a}{2}(2 - F_Y(x) - F_Y(x^-))$.

Note that

$$\mathbb{P}(\tau_x = T_1, N_h = 1) = \mathbb{P}(\tilde{\tau}_x > T_1, \tilde{X}_{T_1} + Y_1 \geq x, T_1 \leq h < T_2).$$

Here, for every $n \in \mathbb{N}^*$, $T_n = S_1 + \dots + S_n$ where $(S_i, i \geq 1)$ is a sequence of independent identically distributed random variables with exponential distribution with parameter a . Using the independence between $(S_i, i \geq 1)$ and $(Y_1, \tilde{X}, \tilde{\tau}_x)$ we get :

$$\begin{aligned} \mathbb{P}(\tau_x = T_1, N_h = 1) &= \int_0^h a e^{-as_1} \int_{h-s_1}^\infty a e^{-as_2} \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x > s_1} \mathbf{1}_{Y_1 \geq x - \tilde{X}_{s_1}} \right) ds_2 ds_1 \\ &= a e^{-ah} \int_0^h \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x > s} \mathbf{1}_{Y_1 \geq x - \tilde{X}_s} \right) ds. \end{aligned}$$

Integrating with respect to Y_1 , we obtain :

$$\mathbb{P}(\tau_x = T_1, N_h = 1) = ae^{-ah} \int_0^h \mathbb{E} \left((1 - F_Y)((x - \tilde{X}_s)^-) \right) ds - ae^{-ah} \int_0^h \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x \leq s} (1 - F_Y)((x - \tilde{X}_s)^-) \right) ds.$$

On the one hand, since F_Y is a càdlàg bounded function and \tilde{X} is a Brownian motion with drift, we get

$$\lim_{s \rightarrow 0} \mathbb{E} \left(F_Y((x - \tilde{X}_s)^-) \right) = \frac{F_Y(x) + F_Y(x^-)}{2}.$$

On the other hand, since the distribution function of $\tilde{\tau}_x$ is differentiable, then

$$\lim_{s \rightarrow 0} \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x \leq s} (1 - F_Y)((x - \tilde{X}_s)^-) \right) = 0.$$

We deduce that

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(\tau_x = T_1, N_h = 1)}{h} = \frac{a}{2} (2 - F_Y(x) - F_Y(x^-)).$$

Step 3 : We prove that the contribution to the limit $\lim_{h \rightarrow 0^+} h^{-1} \mathbb{P}(\tau_x \leq h, N_h = 1)$ of the third term on the right hand side of (2) is $\frac{a}{4}(F_Y(x) - F_Y(x^-))$.

To this end, we state the following lemma :

Lemma 2.5 *The term $\frac{\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1)}{h}$ converges to $\frac{a}{4}(F_Y(x) - F_Y(x^-))$ when h goes to 0.*

Proof Note that

$$\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) = \mathbb{P}(T_1 < \tau_x \leq h, T_1 \leq h < T_2)$$

and $T_2 = T_1 + S_2 \circ \theta_{T_1}$ where θ is the translation operator. Here S_2 is a random variable with exponential distribution with parameter a , independent of $(T_1, (W_t, t \geq 0), (Y_i, i \in \mathbb{N}^*))$.

Moreover, on $\{\omega : T_1(\omega) < \tau_x(\omega) \leq h < T_2(\omega)\}$, $X_s = X_{T_1} + \tilde{X}_{s-T_1} \circ \theta_{T_1}$ when $T_1 < s \leq h$ and $\tau_x = T_1 + \tilde{\tau}_{x-X_{T_1}} \circ \theta_{T_1}$.

Strong Markov Property at the $(\mathcal{F}_t, t \geq 0)$ -stopping time T_1 gives :

$$\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) = \mathbb{E} \left(\mathbf{1}_{\tau_x > T_1} \mathbf{1}_{h \geq T_1} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \mathbf{1}_{h-T_1 < S_2} \right) \right)$$

where $\mathbb{E}^{T_1}(\cdot)$ is $\mathbb{E}(\cdot | \mathcal{F}_{T_1})$.

Integrating with respect to S_2 , we obtain :

$$\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) = \mathbb{E} \left(\mathbf{1}_{\tau_x > T_1} \mathbf{1}_{h \geq T_1} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \right) \right).$$

Remark that $\{\omega : \tau_x(\omega) > T_1(\omega)\} = \{\omega : \tilde{\tau}_x(\omega) > T_1(\omega)\} \cap \{\omega : X_{T_1}(\omega) < x\}$. Consequently

$$\begin{aligned} \mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) &= - \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x \leq T_1 \leq h} \mathbf{1}_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \right) \right) \\ &\quad + \mathbb{E} \left(\mathbf{1}_{h \geq T_1} \mathbf{1}_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \right) \right). \end{aligned}$$

Since the distribution function of $\tilde{\tau}_x$ has a null derivative at 0, then

$$\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x \leq T_1 \leq h} \mathbf{1}_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \right) \right) = 0.$$

It remains to show that

$$\lim_{h \rightarrow 0^+} \frac{G(h)}{h} = \frac{a}{4} [F(x) - F(x^-)] \quad (3)$$

where

$$G(h) = \mathbb{E} \left(\mathbf{1}_{h \geq T_1} \mathbf{1}_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1} \right) \right).$$

Integrating with respect to T_1 and then using the fact that $\tilde{f}(\cdot, z)$ is the derivative of the distribution function of $\tilde{\tau}_z$, we get :

$$\begin{aligned} G(h) &= \int_0^h a e^{-as} \mathbb{E} \left[\mathbf{1}_{\tilde{X}_s + Y_1 < x} e^{-a(h-s)} \mathbb{E}^s (\mathbf{1}_{\tilde{\tau}_{x-\tilde{X}_s-Y_1} \leq h-s}) \right] ds \\ &= a e^{-ah} \int_0^h \int_0^{h-s} \mathbb{E} \left[\mathbf{1}_{\tilde{X}_s + Y_1 < x} \tilde{f}(u, x - \tilde{X}_s - Y_1) \right] du ds. \end{aligned}$$

Since $\tilde{X}_s = ms + W_s$, we may apply Lemma 3.1 to $\mu = x - ms - Y_1$ and $\sigma = \sqrt{s}$, then

$$G(h) = \frac{a e^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^{h-s} \mathbb{E} \left[e^{-\frac{(x-m(u+s)-Y_1)^2}{2(u+s)}} \left(\frac{x-Y_1}{(u+s)^{3/2}} + \frac{G\sqrt{s}}{\sqrt{u}(u+s)} \right)_+ \right] du ds.$$

We make the following change of variable $r = u + s$.

$$G(h) = \frac{a e^{-ah}}{\sqrt{2\pi}} \int_0^h \int_s^h \mathbb{E} \left[e^{-\frac{(x-mr-Y_1)^2}{2r}} \left(\frac{x-Y_1}{r^{3/2}} + \frac{G\sqrt{s}}{r\sqrt{r-s}} \right)_+ \right] dr ds.$$

Firstly, we apply Fubini's Theorem and secondly, we make the following change of variable $v = \frac{s}{r}$.

$$\begin{aligned} G(h) &= \frac{a e^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^r \mathbb{E} \left[e^{-\frac{(x-mr-Y_1)^2}{2r}} \left(\frac{x-Y_1}{r^{3/2}} + \frac{G\sqrt{s}}{r\sqrt{r-s}} \right)_+ \right] ds dr \\ &= \frac{a e^{-ah}}{\sqrt{2\pi}} \int_0^h g(r) dr, \end{aligned}$$

where

$$g(r) = \int_0^1 \mathbb{E} \left[e^{-\frac{(x-mr-Y_1)^2}{2r}} \left(\frac{x-Y_1}{\sqrt{r}} + \frac{G\sqrt{v}}{\sqrt{1-v}} \right)_+ \right] dv.$$

But,

$$\lim_{r \rightarrow 0^+} e^{-\frac{(x-mr-Y_1)^2}{2r}} \left(\frac{x-Y_1}{\sqrt{r}} + \frac{G\sqrt{v}}{\sqrt{1-v}} \right)_+ = \frac{\sqrt{v}}{\sqrt{1-v}} G_+ \mathbf{1}_{x=Y_1},$$

and

$$\sup_{0 \leq r \leq 1} e^{-\frac{(x-mr-Y_1)^2}{2r}} \left(\frac{x-Y_1}{\sqrt{r}} + \frac{G\sqrt{v}}{\sqrt{1-v}} \right)_+ \leq \sup_{z \geq 0} z e^{-\frac{z^2}{2}} + |m| + \frac{\sqrt{v}}{\sqrt{1-v}} |G|.$$

Then from Lebesgue's Dominated Convergence Theorem we obtain :

$$\lim_{r \rightarrow 0} g(r) = \mathbb{P}(Y_1 = x) \mathbb{E}(G_+) \int_0^1 \frac{\sqrt{v}}{\sqrt{1-v}} dv = \frac{\sqrt{2\pi}}{4} \mathbb{P}(Y_1 = x).$$

We deduce the identity (3), i.e.

$$\lim_{h \rightarrow 0} \frac{G(h)}{h} = \frac{a}{4} \mathbb{P}(Y_1 = x),$$

which achieves the proof of Lemma 2.5 □

Proposition 2.4 is a consequence of the Steps 1 to 3. □

2.2 Existence of the derivative at $t > 0$

Our task is now to show that the distribution function of τ_x is differentiable on \mathbb{R}_+^* and to compute its derivative. For this purpose we split the probability $\mathbb{P}(t < \tau_x \leq t+h)$ according to the values of $N_{t+h} - N_t$:

$$\begin{aligned} \mathbb{P}(t < \tau_x \leq t+h) &= \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1) \\ &\quad + \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t \geq 2). \end{aligned} \quad (4)$$

The third term on the right hand side of (4) is upper bounded by

$$\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t \geq 2) \leq \mathbb{P}(N_{t+h} - N_t \geq 2) = 1 - e^{-ah} - ahe^{-ah}.$$

Therefore $\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t \geq 2)}{h} = 0$.

Let us study the second term on the right hand side of (4). Markov Property at t gives :

$$\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1) = \mathbb{E}(\mathbf{1}_{\tau_x > t} \mathbb{P}^t(\tau_{x-X_t} \leq h, N_h = 1)),$$

where $\mathbb{P}^t(\cdot) = \mathbb{P}(\cdot | \mathcal{F}_t)$.

In virtue of Lemma 2.4, $\frac{\mathbb{P}^t(\tau_{x-X_t} \leq h, N_h = 1)}{h}$ converges to

$$\frac{a}{2} [2 - F_Y(x - X_t) - F_Y((x - X_t)^-)] + \frac{a}{4} [F_Y(x - X_t) - F_Y((x - X_t)^-)]$$

and is upper bounded by $\frac{\mathbb{P}(N_h=1)}{h} = ae^{-ah} \leq a$. Dominated Convergence Theorem gives :

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1)}{h} = a \mathbb{E}(\mathbf{1}_{\tau_x > t} (1 - F_Y)(x - X_t)) + \frac{3a}{4} \mathbb{E}(\mathbf{1}_{\tau_x > t} \Delta F_Y(x - X_t))$$

where $\Delta F_Y(z) = F_Y(z) - F_Y(z^-)$.

However the jumps set of F_Y (the distribution function of Y) is countable and X has a density (cf. Proposition 3.12 page 90 of [6]). Thus $\mathbb{E}(\mathbf{1}_{\tau_x > t} \Delta F_Y(x - X_t)) = 0$: indeed $0 \leq \mathbb{E}(\mathbf{1}_{\tau_x > t} \Delta F_Y(x - X_t)) \leq \mathbb{E}(\mathbf{1}_{Y_1 = x - X_t}) = \mathbb{E}(\mathbf{1}_{X_t = x - Y_1}) = 0$. Therefore

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1)}{h} = a \mathbb{E}(\mathbf{1}_{\tau_x > t} (1 - F_Y)(x - X_t)).$$

The proof of the second part of Theorem 2.1 will be complete when the following proposition is shown :

Proposition 2.6 *If there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta|Y_1|}) < \infty$, then*

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0)}{h} = \mathbb{E} \left(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f} \left(t - T_{N_t}, x - X_{T_{N_t}} \right) \right),$$

where \tilde{f} is the function introduced in Theorem 2.1.

Proof Since T_{N_t} is not a stopping time, we can not apply Strong Markov Property. We split

$$\begin{aligned} \mathbb{P}(t < \tau \leq t+h, N_{t+h} - N_t = 0) &= \sum_{k=0}^{\infty} \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} = N_t = k) \\ &= \mathbb{P}(t < \tilde{\tau}_x \leq t+h < T_1) + \sum_{k=1}^{\infty} \mathbb{P}(t < \tau_x \leq t+h, T_k < t < t+h < T_{k+1}). \end{aligned}$$

On the set $\{(\omega, t) : T_k(\omega) < t\}$, we have $X_t(\omega) = X_{T_k}(\omega) + X_{t-T_k} \circ \theta_{T_k}(\omega)$, hence on the set $\{\omega, \tau_x(\omega) > T_k(\omega)\}$, $\tau_x = T_k + \tau_{x-X_{T_k}} \circ \theta_{T_k}$. Strong Markov Property at the stopping time T_k gives

$$\begin{aligned} \mathbb{P}(t < \tau \leq t+h, N_{t+h} - N_t = 0) &= e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t+h) \\ &\quad + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{T_k < t} \mathbf{1}_{\tau_x > T_k} \mathbb{E}^{T_k} \left(\mathbf{1}_{t-T_k < \tilde{\tau}_{x-X_{T_k}} \leq t+h-T_k} \mathbf{1}_{t+h-T_k < S_{k+1}} \right) \right). \end{aligned}$$

On the set $\{(\omega, t) : \tau_z(\omega) \leq t < S_{k+1}(\omega)\}$, we have $\tau_z = \tilde{\tau}_z$ for every $z < 0$. Therefore

$$\begin{aligned} \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) &= e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t+h) \\ &\quad + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{T_k < t} \mathbf{1}_{\tau_x > T_k} e^{-a(t+h-T_k)} \mathbb{E}^{T_k} \left(\mathbf{1}_{t-T_k < \tilde{\tau}_{x-X_{T_k}} \leq t+h-T_k} \right) \right). \end{aligned}$$

The \mathcal{F}_{T_k} -conditional law of $\tilde{\tau}_{x-X_{T_k}}$ has the density (possibly defective) $\tilde{f}(\cdot, x - X_{T_k})$, thus

$$\begin{aligned} \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) &= e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t+h) \\ &\quad + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{T_k < t} \mathbf{1}_{\tau_x > T_k} e^{-a(t+h-T_k)} \int_{t-T_k}^{t+h-T_k} \tilde{f}(u, x - X_{T_k}) du \right). \end{aligned}$$

Let us point out that $e^{-a(t-T_k)} = \mathbb{E}^{T_k}(\mathbf{1}_{T_{k+1} > t})$ then

$$\begin{aligned} \mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) &= e^{-ah} \int_t^{t+h} \mathbb{E}(\mathbf{1}_{0 \leq t < T_1}) \tilde{f}(u, x) du \\ &\quad + e^{-ah} \sum_{k=1}^{\infty} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{T_k \leq t < T_{k+1}} \mathbf{1}_{\tau_x > T_k} \tilde{f}(u - T_k, x - X_{T_k}) \right) du, \end{aligned}$$

or shortly

$$\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0) = e^{-ah} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{T_{N_t} < \tau_x} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) \right) du. \quad (5)$$

Since \tilde{f} is continuous with respect to u , then

$$\lim_{u \rightarrow t^+} \mathbf{1}_{T_{N_t} < \tau_x} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) = \mathbf{1}_{T_{N_t} < \tau_x} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}),$$

From Lemma 3.2, Propositions 3.5 and 3.6, $\tilde{f}(u - T_{N_t}, x - X_{T_{N_t}})$ is dominated uniformly in u by a integrable random variable

$$\tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) \leq c_{\varepsilon, M} (t - T_{N_t})^{1-\varepsilon} \left[\frac{1}{(x - X_{T_{N_t}})^{4\varepsilon}} + \exp \left(\frac{2m}{M} (x - X_{T_{N_t}}) \right) \right].$$

Here $0 < \varepsilon < 1/4$ and $M > \max(1, 2|m|\beta^{-1})$, (where $\mathbb{E}(e^{\beta|Y|}) < +\infty$), and $c_{\varepsilon, M}$, is a constant defined in Lemma 3.2 depending only on ε and M . Then, using Lebesgue's Dominated Convergence Theorem in equation (5) we obtain

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1)}{h} = \mathbb{E} \left(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) \right).$$

□

Using Proposition 2.6, the limit $\lim_{h \rightarrow 0} \frac{\mathbb{P}(t < \tau_x \leq t+h)}{h}$ exists and is equal to

$$a\mathbb{E}(\mathbf{1}_{\tau_x > t}(1 - F_Y)(x - X_t)) + \mathbb{E} \left(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}}) \right).$$

The following lemma allows to conclude the proof of Theorem 2.1. The results are known, for example in [15] and [22] but for sake of completeness we give a proof.

Lemma 2.7 *For all $x > 0$, the stopping time τ_x is finite almost surely if and only if $m + a\mathbb{E}(Y_1) \geq 0$.*

Proof Remark that

$$\mathbb{P}(\tau_x = \infty) = \mathbb{P}(\sup_{t \geq 0} X_t < x).$$

Thanks to Theorem 7.2 page 183 of [12] which is a consequence of Strong Law of Large Numbers,

$$\begin{aligned} &\text{if } m + a\mathbb{E}(Y_1) > 0, \text{ then } \lim_{t \rightarrow \infty} X_t = +\infty \text{ and} \\ &\text{if } m + a\mathbb{E}(Y_1) = 0, \text{ then } \limsup_{t \rightarrow \infty} X_t = -\liminf_{t \rightarrow \infty} X_t = \infty. \end{aligned}$$

Therefore (see Exercise 39.11 page 271 of [21]), if $m + a\mathbb{E}(Y_1) \geq 0$, then $\sup_{t \geq 0} X_t = +\infty$. This proves the first part of the lemma.

Conversely, let us suppose that $m + a\mathbb{E}(Y_1) < 0$. Then $\lim_{t \rightarrow \infty} X_t = -\infty$, and according to Theorem 48.1 page 363 of [21], $\sup_{t \geq 0} X_t < \infty$. Assume that there exists $x_0 > 0$, such that $\mathbb{P}(\tau_{x_0} < +\infty) = 1$. Then from all x , such that $x \leq x_0$ we have $\mathbb{P}(\tau_x < +\infty) = 1$.

Now we use recurrence reasoning. Assume that for $n \geq 1$ and for all $x \leq nx_0$, we have $\mathbb{P}(\tau_x < +\infty) = 1$. Let x such that $nx_0 < x \leq (n+1)x_0$, then using Strong Markov Property,

$$\mathbb{P}(\tau_x < +\infty) = \mathbb{E}(\mathbf{1}_{\tau_{nx_0} < +\infty} \mathbf{1}_{X_{\tau_{nx_0}} \geq x}) + \mathbb{E}(\mathbf{1}_{\tau_{nx_0} < +\infty} \mathbf{1}_{X_{\tau_{nx_0}} < x} \mathbb{P}^{\tau_{nx_0}}(\tau_{x-X_{\tau_{nx_0}}} < \infty)).$$

Using the recurrence hypothesis, since $x - X_{\tau_{nx_0}} < x_0$, almost surely $\mathbb{P}^{\tau_{nx_0}}(\tau_{x-X_{\tau_{nx_0}}} < \infty) = 1$ and then

$$\mathbb{P}(\tau_x < +\infty) = 1.$$

We have proved that if there exists $x_0 > 0$ such that $\mathbb{P}(\tau_{x_0} < \infty) = 1$, then for all $x > 0$, $\mathbb{P}(\tau_x < \infty) = \mathbb{P}(\sup_{t \geq 0} X_t \geq x) = 1$. This contradicts the fact that $\sup_{t \geq 0} X_t < \infty$. Then, if $m + a\mathbb{E}(Y_1) < 0$, for all $x > 0$, $\mathbb{P}(\tau_x < \infty) < 1$. \square

3 Appendix

A Brownian motion with drift is a process

$$\tilde{X}_t = mt + W_t, \quad t \geq 0$$

with $m \in \mathbb{R}$.

Let $z > 0$ and $\tilde{\tau}_z$ be the passage time defined by $\tilde{\tau}_z = \inf\{t \geq 0 : \tilde{X}_t \geq z\}$. By (5.12) page 197 of [10], $\tilde{\tau}_z$ has the following law on $\overline{\mathbb{R}}_+$:

$$\tilde{f}(u, z)du + \mathbb{P}(\tilde{\tau}_z = \infty)\delta_\infty(du)$$

where

$$\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi}u^3} \exp\left[-\frac{(z - mu)^2}{2u}\right] \mathbf{1}_{]0, \infty[}(u), \quad u \in \mathbb{R}, \quad \text{and } \mathbb{P}(\tilde{\tau}_z = \infty) = 1 - e^{mz - |mz|}. \quad (6)$$

For a fixed z , the function $\tilde{f}(\cdot, z)$ and all its derivatives admit 0 as limit at 0^+ . The function \tilde{f} admits an extension (denoted \tilde{f}) of class \mathcal{C}^∞ on \mathbb{R} , defined by $\tilde{f}(u, z) = 0$ for $u \leq 0$. Moreover it checks the two following lemmas :

Lemma 3.1 *Let G be a Gaussian random variable $\mathcal{N}(0, 1)$ and let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$. Then for every $u \in \mathbb{R}$*

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\mu + \sigma G > 0}] = \frac{1}{\sqrt{2\pi}} \mathbb{E} \left[e^{-\frac{(\mu - mu)^2}{2(\sigma^2 + u)}} \left(\frac{\mu + \sigma^2 m}{(\sigma^2 + u)^{3/2}} + \frac{\sigma G}{\sqrt{u}(\sigma^2 + u)} \right)_+ \right]$$

where $(x)_+ = \max(x, 0)$.

Proof Using the probability density function of G and the definition of \tilde{f} (see (6)), we get :

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\mu + \sigma G > 0}] = \frac{1}{\sqrt{2\pi}u^3} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} |\mu + \sigma g| e^{-\frac{(\mu + \sigma g - mu)^2}{2u}} e^{-\frac{g^2}{2}} \mathbf{1}_{\mu + \sigma g > 0} dg.$$

Since $|\mu + \sigma g| \mathbf{1}_{\mu + \sigma g > 0} = (\mu + \sigma g) \mathbf{1}_{\mu + \sigma g > 0} = (\mu + \sigma g)_+$, then

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\mu + \sigma G > 0}] = \frac{1}{\sqrt{2\pi}u^3} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} (\mu + \sigma g)_+ e^{-\frac{(\mu + \sigma g - mu)^2}{2u}} e^{-\frac{g^2}{2}} dg.$$

A simple computation shows that $\frac{(\mu + \sigma g - mu)^2}{u} + g^2 = \frac{\sigma^2 + u}{u} \left(g + \frac{\sigma(\mu - mu)}{\sigma^2 + u} \right)^2 + \frac{(\mu - mu)^2}{\sigma^2 + u}$. Therefore

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\mu + \sigma G > 0}] = \frac{e^{-\frac{(\mu - mu)^2}{2(\sigma^2 + u)}}}{\sqrt{2\pi}u^3} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} (\mu + \sigma g)_+ e^{-\frac{\sigma^2 + u}{2u} \left(g + \frac{\sigma(\mu - mu)}{\sigma^2 + u} \right)^2} dg.$$

Making the following change of variable $x = \sqrt{\frac{\sigma^2 + u}{u}} \left(g + \frac{\sigma(\mu - mu)}{\sigma^2 + u} \right)$, we conclude the proof

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G) \mathbf{1}_{\mu + \sigma G > 0}] = \frac{e^{-\frac{(\mu - mu)^2}{2(\sigma^2 + u)}}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left(\frac{\mu^2 + \sigma^2 m}{(\sigma^2 + u)^{3/2}} + \frac{\sigma x}{\sqrt{u}(\sigma^2 + u)} \right)_+ e^{-\frac{x^2}{2}} dx.$$

□

Lemma 3.2 Let $\tilde{f} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ be the function defined by

$$\tilde{f}(u, z) = \begin{cases} \frac{|z|}{\sqrt{2\pi}u^3} \exp\left[-\frac{(z - mu)^2}{2u}\right] & \text{si } u > 0 \\ 0 & \text{si } u \leq 0, \end{cases}$$

where $m \in \mathbb{R}$. Then for every $\varepsilon > 0$ and $M \geq 1$, there exists a constant $c_{\varepsilon, M} > 0$ such that

$$\tilde{f}(u, z) \leq c_{\varepsilon, M} u^{-1+\varepsilon} \left[\frac{1}{|z|^{4\varepsilon}} + \exp\left(\frac{2mz}{M}\right) \right].$$

Proof Let $\varepsilon > 0$ and $M \geq 1$ be fixed. Remark that it is enough to prove that there exists a constant $\tilde{c}_{\varepsilon, M} > 0$ such that

$$\tilde{f}(u, z) \leq \tilde{c}_{\varepsilon, M} u^{-1+\varepsilon} \frac{1}{|z|^{2\varepsilon}} \exp\left(\frac{mz}{M}\right).$$

Then we conclude the proof using the inequality $x_1 x_2 \leq \frac{x_1^2 + x_2^2}{2}$ for $x_1 = \frac{1}{|z|^{2\varepsilon}}$ and $x_2 = \exp\left(\frac{mz}{M}\right)$.

We now seek to upper bound the quotient $\frac{\tilde{f}(u, z)}{u^{-1+\varepsilon} \frac{1}{|z|^{2\varepsilon}} \exp\left[\frac{mz}{M}\right]}$, $z < 0$, $u \in \mathbb{R}_+$.

Since $\exp\left[-\frac{(z - mu)^2}{2u}\right] \leq \exp\left[-\frac{(z - mu)^2}{2uM}\right]$, then

$$\begin{aligned} \frac{\tilde{f}(u, z)}{u^{-1+\varepsilon} \frac{1}{|z|^{2\varepsilon}} \exp\left[\frac{mz}{M}\right]} &\leq \frac{\frac{|z|}{\sqrt{2\pi}u^3} \exp\left[-\frac{(z - mu)^2}{2uM}\right]}{u^{-1+\varepsilon} \frac{1}{|z|^{2\varepsilon}} \exp\left[\frac{mz}{M}\right]} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{z^2}{u} \right)^{\frac{1}{2}+\varepsilon} \exp\left[-\frac{z^2}{2uM}\right] \exp\left[-\frac{m^2 u}{2M}\right] \\ &\leq \frac{M^{\frac{1}{2}+\varepsilon} 2^\varepsilon}{\sqrt{\pi}} \left(\frac{z^2}{2uM} \right)^{\frac{1}{2}+\varepsilon} \exp\left[-\frac{z^2}{2uM}\right]. \end{aligned}$$

Since the function $x \mapsto x^{\frac{1}{2}+\varepsilon}e^{-x}$ is continuous, null at 0 and at $+\infty$, then it is bounded on \mathbb{R}_+ . Hence there exists $c_\varepsilon > 0$ such that $x^{\frac{1}{2}+\varepsilon}e^{-x} \leq c_\varepsilon$ for any $x \in \mathbb{R}_+$. We apply this inequality to $x = \frac{z^2}{2uM}$, so $(\frac{z^2}{2uM})^{\frac{1}{2}+\varepsilon} \exp[-\frac{z^2}{2uM}] \leq c_\varepsilon$ and the proof is complete. \square

Lemma 3.3 *Let G be a Gaussian random variable $\mathcal{N}(0, 1)$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $0 < \alpha < 1$. Then there exists two constants $k_{1,\alpha} > 0$, $k_{2,\alpha} > 0$, depending only on α , such that*

$$\mathbb{E}(|\mu + \sigma G|^{-\alpha}) \leq \frac{k_{1,\alpha}}{\sigma} |\mu|^{1-\alpha} + k_{2,\alpha} \sigma^{-\alpha}.$$

Proof Using the probability density function of G , we get :

$$\mathbb{E}(|\mu + \sigma G|^{-\alpha}) = \int_{-\frac{\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} (\mu + \sigma x)^{-\alpha} e^{-\frac{x^2}{2}} \mu dx + \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} (-\mu - \sigma x)^{-\alpha} e^{-\frac{x^2}{2}} dx.$$

Integration by parts gives :

$$\begin{aligned} \mathbb{E}(|\mu + \sigma G|^{-\alpha}) &= \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{(\mu + \sigma x)^{1-\alpha}}{\sigma(1-\alpha)} \right]_{-\frac{\mu}{\sigma}}^{\infty} + \frac{1}{\sigma(1-\alpha)} \int_{-\frac{\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} (\mu + \sigma x)^{1-\alpha} x e^{-\frac{x^2}{2}} dx \\ &\quad + \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{(-\mu - \sigma x)^{1-\alpha}}{-\sigma(1-\alpha)} \right]_{-\infty}^{-\frac{\mu}{\sigma}} - \frac{1}{\sigma(1-\alpha)} \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} (-\mu - \sigma x)^{1-\alpha} x e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sigma(1-\alpha)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} |\mu + \sigma x|^{1-\alpha} e^{-\frac{x^2}{2}} \left(x \mathbf{1}_{x > -\frac{\mu}{\sigma}} - x \mathbf{1}_{x \leq -\frac{\mu}{\sigma}} \right) dx \\ &= \frac{1}{\sigma(1-\alpha)} \mathbb{E} \left[|\mu + \sigma G|^{1-\alpha} G (\mathbf{1}_{G > -\frac{\mu}{\sigma}} - \mathbf{1}_{G \leq -\frac{\mu}{\sigma}}) \right]. \end{aligned}$$

However for every $\mu \in \mathbb{R}$ and $\sigma > 0$, we have

$$|G(\mathbf{1}_{G > -\frac{\mu}{\sigma}} - \mathbf{1}_{G \leq -\frac{\mu}{\sigma}})| \leq G - 2G\mathbf{1}_{G \leq 0} = |G|.$$

Consequently $\mathbb{E}(|\mu + \sigma G|^{-\alpha}) \leq \frac{1}{\sigma(1-\alpha)} \mathbb{E}(|G| |\mu + \sigma G|^{1-\alpha})$.

The inequality $|\mu + \sigma G|^{1-\alpha} \leq (|\mu| + \sigma |G|)^{1-\alpha} \leq |\mu|^{1-\alpha} + \sigma^{1-\alpha} |G|^{1-\alpha}$ concludes the proof. \square

Lemma 3.4 *For every $t > 0$ and $(\alpha, \gamma) \in]-1, \infty[^2$, the following two series*

$$\begin{aligned} &\sum_{i \geq 1}^{\infty} \mathbb{E}(\mathbf{1}_{t > T_i} (t - T_i)^\alpha T_i^\gamma) z^i \\ &\sum_{i \geq 1}^{\infty} \mathbb{E}(\mathbf{1}_{t > T_i} (t - T_i)^\alpha T_i^\gamma) i z^i, \end{aligned}$$

have an infinite radius of convergence.

Proof It is enough to prove that the first series has an infinite radius of convergence.

Note that for $i \geq 1$, T_i admits as density the function $u \mapsto \frac{a^i}{(i-1)!} u^{i-1} e^{-au}$, thus

$$\begin{aligned} \mathbb{E}(\mathbf{1}_{t > T_i} (t - T_i)^\alpha T_i^\gamma) &= \frac{a^i}{(i-1)!} \int_0^t e^{-au} (t-u)^\alpha u^{\gamma+i-1} du \\ &\leq \frac{a^i}{(i-1)!} \int_0^t (t-u)^\alpha u^{\gamma+i-1} du \\ &= \frac{a^i}{(i-1)!} t^{\gamma+i+\alpha} \frac{\Gamma(\gamma+i)\Gamma(\alpha+1)}{\Gamma(\gamma+i+\alpha+1)} \end{aligned}$$

and the conclusion holds. \square

As a consequence of Lemma 3.4, we have the following two propositions which are used to prove Theorem 2.1.

Proposition 3.5 *Assume that there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta|Y_1|}) < \infty$. For every $t > 0$, $\varepsilon > 0$ and $M \geq 1$ such that $|\frac{2m}{M}| \leq \beta$, the random variable $(t - T_{N_t})^{-1+\varepsilon} \exp \frac{2m}{M}(x - X_{T_{N_t}})$ \mathbb{P} -integrable.*

Proof Note that

$$\begin{aligned} (t - T_{N_t})^{-1+\varepsilon} \exp \frac{2m}{M}(x - X_{T_{N_t}}) &= t^{-1+\varepsilon} \exp \frac{2mx}{M} \mathbf{1}_{t < T_1} + \sum_{i=1}^{\infty} \mathbf{1}_{T_i \leq t < T_{i+1}} (t - T_i)^{-1+\varepsilon} \exp \frac{2m}{M}(x - X_{T_i}), \\ &\leq t^{-1+\varepsilon} \exp \frac{2mx}{M} + \sum_{i=1}^{\infty} \mathbf{1}_{t > T_i} (t - T_i)^{-1+\varepsilon} \exp \left[\frac{2m}{M}(x - X_{T_i}) \right]. \end{aligned}$$

Let $\sigma(T_i)$ be the σ -field generated by T_i , $i > 0$. Conditioning by T_i , we obtain that

$$\mathbb{E} \left[\exp \left[\frac{2m}{M}(x - X_{T_i}) \right] \middle| \sigma(T_i) \right] = e^{\frac{2m}{M}x} e^{-\frac{2m^2}{M}T_i + \frac{2m^2}{M^2}T_i} \mathbb{E} \left(e^{-\frac{2m}{M}Y_1} \right)^i.$$

Since $M \geq 1$, then $e^{-\frac{2m^2}{M}T_i + \frac{2m^2}{M^2}T_i} \leq 1$, therefore

$$\mathbb{E} \left[\exp \left[\frac{2m}{M}(x - X_{T_i}) \right] \middle| \sigma(T_i) \right] \leq e^{\frac{2m}{M}x} \mathbb{E} \left(e^{-\frac{2m}{M}Y_1} \right)^i.$$

Consequently

$$\mathbb{E} \left[(t - T_{N_t})^{-1+\varepsilon} \exp \left[\frac{2m}{M}(x - X_{T_{N_t}}) \right] \right] \leq e^{\frac{2m}{M}x} \sum_{i=1}^{\infty} \mathbb{E} [\mathbf{1}_{t > T_i} (t - T_i)^{-1+\varepsilon}] \mathbb{E} \left(e^{-\frac{2m}{M}Y_1} \right)^i.$$

By the choice of M , $\mathbb{E}(e^{-\frac{2m}{M\sigma^2}Y_1}) < \mathbb{E}(e^{\beta|Y_1|}) < \infty$. We use Lemma 3.4, and the conclusion holds. \square

Another useful result for the proof of Theorem 2.1 is the following :

Proposition 3.6 Assume that there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta|Y_1|}) < \infty$. For every $t > 0$ and $0 < \varepsilon < \frac{1}{4}$ the random variable $(t - T_{N_t})^{-1+\varepsilon}|X_{T_{N_t}} - x|^{-4\varepsilon}$ is \mathbb{P} -integrable.

Proof Note that

$$\begin{aligned} (t - T_{N_t})^{-1+\varepsilon}|X_{T_{N_t}} - x|^{-4\varepsilon} &= t^{-1+\varepsilon}|x|^{-4\varepsilon}\mathbf{1}_{t < T_1} + \sum_{i=1}^{\infty} \mathbf{1}_{T_i \leq t < T_{i+1}} (t - T_i)^{-1+\varepsilon}|X_{T_i} - x|^{-4\varepsilon}, \\ &\leq |x|^{-4\varepsilon}t^{-1+\varepsilon} + \sum_{i=1}^{\infty} \mathbf{1}_{T_i \leq t} (t - T_i)^{-1+\varepsilon}|X_{T_i} - x|^{-4\varepsilon}. \end{aligned}$$

We apply Lemma 3.3 to $\alpha = 4\varepsilon$, $G = \frac{W_{T_i}}{\sqrt{T_i}}$, $\mu = mT_i + \sum_{j=1}^i Y_j - x$ and $\sigma = \sqrt{T_i}$. There exists $k_{1,\varepsilon} > 0$, $k_{2,\varepsilon} > 0$ such that

$$\mathbb{E} [|X_{T_i} - x|^{-4\varepsilon} |\sigma(T_i, Y_j, j \leq i)] \leq \frac{k_{1,\varepsilon}}{\sqrt{T_i}} |mT_i + \sum_{j=1}^i Y_j - x|^{1-4\varepsilon} + k_{2,\varepsilon} T_i^{-4\varepsilon}.$$

Let us use the inequality $x_1^{1-\alpha} \leq 1 + x_1$, $0 < \alpha < 1$, $x_1 \geq 0$ for $x_1 = |mT_i + \sum_{j=1}^i Y_j - x|$ and $\alpha = 4\varepsilon$:

$$\mathbb{E} [|X_{T_i} - x|^{-4\varepsilon} |\sigma(T_i, Y_j, j \leq i)] \leq k_{1,\varepsilon} T_i^{-\frac{1}{2}} + k_{1,\varepsilon} T_i^{-\frac{1}{2}} |mT_i + \sum_{j=1}^i Y_j - x| + k_{2,\varepsilon} T_i^{-4\varepsilon}.$$

Since $|mT_i + \sum_{j=1}^i Y_j - x| \leq |m|T_i + \sum_{j=1}^i |Y_j| + |x|$, then

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{T_i < t} \frac{(t - T_i)^{-1+\varepsilon}}{|X_{T_i} - x|^{-4\varepsilon}} \right] &\leq (k_{1,\varepsilon} + |x|) \mathbb{E} \left[\mathbf{1}_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-\frac{1}{2}} \right] + k_{1,\varepsilon} |m| \mathbb{E} \left[\mathbf{1}_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{\frac{1}{2}} \right] \\ &\quad + k_{1,\varepsilon} i \mathbb{E}(|Y_1|) \mathbb{E} \left[\mathbf{1}_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-\frac{1}{2}} \right] + k_{2,\varepsilon} \mathbb{E} [\mathbf{1}_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-4\varepsilon}]. \end{aligned}$$

We conclude the proof using Lemma 3.4. \square

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