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First passage time law for some Lévy processes with compound Poisson: Existence of a density

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Let $(X_t, t \ge 0)$ be a Lévy process with compound Poisson process and τ_x be the first passage time of a fixed level x > 0 by $(X_t, t \ge 0)$. We prove that the law of τ_x has a density (defective when $\mathbb{E}(X_1) < 0$) with respect to the Lebesgue measure.

Keywords: first passage time law; jump process; Lévy process

1. Introduction

The main purpose of this paper is to show that the first passage time distribution associated with a Lévy process with compound Poisson process has a density with respect to the Lebesgue measure.

Let X be a cadlag process started at 0 and τ_x the first passage time of level x > 0 by X.

Lévy, in [15], computed the law of τ_x when X is a Brownian motion with drift. This result is extended by Alili *et al.* [1] and Leblanc [12] to the case where X is an Ornstein–Uhlenbeck process. The case where X is a Bessel process was studied by Borodin and Salminen in [4].

For the situation where the process X has jumps, the first results were obtained by Zolotarev [22] and Borokov [5] for X a spectrally negative Lévy process. Moreover, if X_t has probability density p(t,x) with respect to the Lebesgue measure, then the law of τ_x has density f(t,x) with respect to the Lebesgue measure, where xf(t,x) = tp(t,x) and $X_{\tau_x} = x$ almost surely.

If X is a spectrally positive Lévy process, Doney [7] gives an explicit formula for the joint Laplace transform of τ_x and the overshoot $X_{\tau_x} - x$. When X is a stable Lévy process, Peskir [16] and Bernyk *et al.* [2] obtain an explicit formula for the passage time density.

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The case where X has signed jumps has been studied more recently. In [9], the authors give the law of τ_x when X is the sum of a decreasing Lévy process and an independent compound process with exponential jump sizes. This result is extended by Kou and Wang in [11] to the case of a diffusion process with jumps where the jump sizes follow a double exponential law. They compute the Laplace transform of τ_x and derive an expression for the density of τ_x . For a more general jump-diffusion process, Roynette et~al. [19] show that the Laplace transform of $(\tau_x, x - X_{\tau_x}, X_{\tau_x} - x)$ is the solution of some kind of random integral.

For a general Lévy processes, Doney and Kyprianou [8] give the quintuple law of $(\bar{G}_{\tau_x}, \tau_x - \bar{G}_{\tau_x}, X_{\tau_x} - x, x - X_{\tau_x}, x - \bar{X}_{\tau_x})$ where $\bar{X}_t = \sup_{s \le t} X_s$ and $\bar{G}_t = \sup\{s < t, \bar{X}_s = X_s\}$.

Results are also available for some Lévy processes without Gaussian component; see Lefèvre et al. [13, 14, 17, 18]. Blanchet [3] considers a process satisfying the stochastic equation $\mathrm{d} X_t = X_{t_-}(\mu\,\mathrm{d} t + \sigma \mathbf{1}_{\tilde{\phi}(t)=0}\,\mathrm{d} W_t + \phi \mathbf{1}_{\tilde{\phi}(t)=\phi}\,\mathrm{d} \tilde{N}_t), t \leq T$, where T is a finite horizon, $\mu \in \mathbb{R}$, $\sigma > 0$, $\tilde{\phi}(\cdot)$ is a function taking two values, 0 or ϕ , W is a Brownian motion, N is a Poisson process with intensity $\frac{1}{\phi^2}\mathbf{1}_{\tilde{\phi}(t)=\phi}$ and \tilde{N} is the compensated Poisson process.

The aim of this paper is to add to these results the law of a first passage time by a Lévy process with compound Poisson process.

The paper is organized as follows: Section 2 contains the main result (Theorem 2.1) which gives the first passage time law by a jump Lévy process. We compute the derivative of the distribution function of τ_x at t=0 in Section 2.1 and at t>0 in Section 2.2. Section 2.2 contains the proofs of some useful results.

2. First passage time law

Let $m \in \mathbb{R}$ $(W_t, t \geq 0)$ be a standard Brownian motion $(N_t, t \geq 0)$ be a Poisson process with constant positive intensity a and $(Y_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with distribution function F_Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that the σ -fields $\sigma(Y_i, i \in \mathbb{N}^*)$, $\sigma(N_t, t \geq 0)$ and $\sigma(W_t, t \geq 0)$ are independent. Let $(T_n, n \in \mathbb{N}^*)$ be the sequence of the jump times of the process N and $(S_i, i \in \mathbb{N}^*)$ be a sequence of independent identically distributed random variables with exponential law of parameter a such that $T_n = \sum_{i=1}^n S_i$, $n \in \mathbb{N}^*$.

Let \tilde{X} be the Brownian motion with drift $m \in \mathbb{R}$ and for z > 0, $\tilde{\tau}_z = \inf\{t \ge 0 : mt + W_t \ge z\}$. By [10], formula (5.12), page 197, $\tilde{\tau}_z$ has the following law on $\mathbb{R}_+ : \tilde{f}(u,z) du + \mathbb{P}(\tilde{\tau}_z = \infty) \delta_{\infty}(du)$, where

$$\tilde{f}(u,z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z-mu)^2}{2u}\right] \mathbf{1}_{]0,\infty[}(u), \qquad u \in \mathbb{R}, \quad \text{and}$$

$$\mathbb{P}(\tilde{\tau}_z = \infty) = 1 - e^{mz - |mz|}.$$
(1)

The function $\tilde{f}(\cdot,z)$ and all its derivatives admit 0 as right limit at 0 and are \mathcal{C}^{∞} on \mathbb{R} .

Let X be the process defined by $X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, t \ge 0$, and τ_x be the first passage time of level x > 0 by $X : \tau_x = \inf\{u > 0 : X_u \ge x\}$. The main result of this paper is the following theorem.

Theorem 2.1. The distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0,\infty[$. The derivative, denoted $f(\cdot,x)$, is equal to

$$f(0,x) = \frac{a}{2}(2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4}(F_Y(x) - F_Y(x_-))$$

and for every t > 0,

$$f(t,x) = a\mathbb{E}(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)) + \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}}\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})).$$

Furthermore, $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

The proof of Theorem 2.1 is given in Sections 2.1 and 2.2.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the completed natural filtration generated by the processes $(W_t, t\geq 0)$, $(N_t, t\geq 0)$ and the random variables $(Y_i, i\in \mathbb{N}^*): \mathcal{F}_t = \sigma(W_s, s\leq t) \vee \sigma(N_s, s\leq t, Y_1, \ldots, Y_{N_t}) \vee \mathcal{N}$. Here, \mathcal{N} is the set of negligible sets of $(\mathcal{F}, \mathbb{P})$.

Remark 2.2. This result is already known when X has no positive jumps (see [20], Theorem 46.4, page 348), when X is a stable Lévy process with no negative jumps (see [2]) and when X is a jump diffusion where the jump sizes follow a double exponential law (see [11]).

According to [14] and [21], for all x > 0, the passage time τ_x is finite almost surely if and only if $m + a\mathbb{E}(Y_1) \ge 0$.

2.1. Existence of the right derivative at t=0

In this section, we show that the distribution function of τ_x has a right derivative at 0 and we compute this derivative. For this purpose, we split the probability $\mathbb{P}(\tau_x \leq h)$ according to the values of $N_h : \mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2)$.

Note that $\mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq 1 - e^{-ah} - ahe^{-ah}$ and thus $\lim_{h\to 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0$. It suffices to prove the following two properties:

$$\frac{\mathbb{P}(\tau_x \le h, N_h = 0)}{h} \xrightarrow{h \to 0} 0; \tag{2}$$

$$\frac{\mathbb{P}(\tau_x \le h, N_h = 1)}{h} \xrightarrow{h \to 0} \frac{a}{2} (2 - F_Y(x) - F_Y(x_-)) + \frac{a}{4} (F_Y(x) - F_Y(x_-)). \tag{3}$$

On the set $\{\omega: N_h(\omega) = 0\}$, the processes $(X_t, 0 \le t \le h)$ and $(\tilde{X}_t, 0 \le t \le h)$ are equal and \mathbb{P} -a.s. $\tau_x \wedge h = \tilde{\tau}_x \wedge h$. Since $\tilde{\tau}_x$ is independent of N, we have $\mathbb{P}(\tau_x \le h, N_h = 0) = 0$

 $e^{-ah}\mathbb{P}(\tilde{\tau}_x \leq h)$. The law of $\tilde{\tau}_x$ has a C^{∞} density (possibly defective) with respect to the Lebesgue measure, null on $]-\infty,0]$, Thus, (2) holds.

To prove (3), we use the same type of arguments as in [19] (for the proof of Theorem 2.4). We split the probability $\mathbb{P}(\tau_x \leq h, N_h = 1)$ into three parts according to the relative positions of τ_x and T_1 , the first jump time of the Poisson process N:

$$\mathbb{P}(\tau_x \le h, N_h = 1) = \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \le h, N_h = 1)$$
$$= A_1(h) + A_2(h) + A_3(h).$$

Step 1: As for (2), we easily prove that $\xrightarrow{A_1(h)} \xrightarrow{h \to 0} 0$.

Step 2: We prove that $\xrightarrow{A_2(h)} \xrightarrow{h \to 0} \frac{a}{2} (2 - F_Y(x) - F_Y(x_-)).$

Note that $A_2(h) = \mathbb{P}(\tilde{\tau}_x > T_1, \tilde{X}_{T_1} + Y_1 \geq x, T_1 \leq h < T_2)$. Using the independence of $(S_i, i \ge 1)$ and $(Y_1, \tilde{X}, \tilde{\tau}_x)$, we get $\mathbb{P}(\tau_x = T_1, N_h = 1) = ae^{-ah} \int_0^h \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x > s\}} \mathbf{1}_{\{Y_1 > x - \tilde{X}_s\}}) ds$. Integrating with respect to Y_1 , we obtain

$$\frac{\mathbb{P}(\tau_x = T_1, N_h = 1)}{a e^{-ah}} = \int_0^h \mathbb{E}((1 - F_Y)((x - \tilde{X}_s)_-)) \, \mathrm{d}s - \int_0^h \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le s\}}(1 - F_Y)((x - \tilde{X}_s)_-)) \, \mathrm{d}s.$$

On the one hand, since F_Y is a cadlag bounded function and $\tilde{X}_s = ms + W_s$, where W is continuous and symmetric, we get $\lim_{s\to 0} \mathbb{E}(F_Y((x-\tilde{X}_s)_-)) = \frac{F_Y(x) + F_Y(x_-)}{2}$. On the other hand, $\lim_{s\to 0} \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x\leq s\}}(1-F_Y)((x-\tilde{X}_s)_-))=0.$ We deduce that $\lim_{h\to 0}\frac{A_2(h)}{h}=\frac{a}{2}(2-F_Y(x)-F_Y(x_-)).$ Step 3: We prove that $\frac{A_3(h)}{h}\stackrel{h\to 0}{\longrightarrow}\frac{a}{4}(F_Y(x)-F_Y(x_-)).$ Note that $\mathbb{P}(T_1<\tau_x\leq h,N_h=1)=\mathbb{P}(T_1<\tau_x\leq h,T_1\leq h< T_2)$ and $T_2=T_1+S_2\circ\theta_{T_1},$

where θ is the translation operator.

Moreover, on $\{T_1 < \tau_x \leq h < T_2\}$, $X_s = X_{T_1} + \tilde{X}_{s-T_1} \circ \theta_{T_1}$, where $T_1 < s \leq h$ and $\tau_x = T_1 + \tilde{\tau}_{x-X_{T_1}} \circ \theta_{T_1}$. The strong Markov property gives, with $\mathbb{E}^{T_1}(\cdot)$ standing for $\mathbb{E}(\cdot \mid \mathcal{F}_{T_1}),$

$$\begin{split} A_3(h) &= \mathbb{E}(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \ge T_1\}} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}} \mathbf{1}_{\{h - T_1 < S_2\}})) \\ &= \mathbb{E}(\mathbf{1}_{\{\tau_x > T_1\}} \mathbf{1}_{\{h \ge T_1\}} \mathrm{e}^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})) \\ &= -\mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le T_1 \le h\}} \mathbf{1}_{\{X_{T_1} < x\}} \mathrm{e}^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})) \\ &+ \mathbb{E}(\mathbf{1}_{\{h \ge T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} \mathrm{e}^{-a(h - T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x - X_{T_1}} \le h - T_1\}})). \end{split}$$

Since the distribution function of $\tilde{\tau}_x$ has a null derivative at 0, we have

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}(\mathbf{1}_{\{\tilde{\tau}_x \le T_1 \le h\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \le h-T_1\}})) = 0.$$

It remains to show that $\lim_{h\downarrow 0} \frac{G(h)}{h} = \frac{a}{4} [F(x) - F(x^-)]$, where

$$G(h) = \mathbb{E}(\mathbf{1}_{\{h \geq T_1\}} \mathbf{1}_{\{X_{T_1} < x\}} e^{-a(h-T_1)} \mathbb{E}^{T_1}(\mathbf{1}_{\{\tilde{\tau}_{x-X_{T_1}} \leq h-T_1\}})).$$

Integrating with respect to T_1 and then using the fact that $\tilde{f}(\cdot,z)$ is the derivative of the distribution function of $\tilde{\tau}_z$, we get $G(h) = a \mathrm{e}^{-ah} \int_0^h \int_0^{h-s} \mathbb{E}[\mathbf{1}_{\{\tilde{X}_s + Y_1 < x\}} \tilde{f}(u, x - \tilde{X}_s - Y_1)] \, \mathrm{d}u \, \mathrm{d}s$.

We may apply Lemma A.1 to p=1, $\mu=x-ms-Y_1$ and $\sigma=\sqrt{s}$. Then,

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G)\mathbf{1}_{\{\mu + \sigma G > 0\}}] = \frac{1}{\sqrt{2\pi}}\mathbb{E}\left[e^{-(\mu - mu)^2/(2(\sigma^2 + u))}\left(\frac{\mu + \sigma^2 m}{(\sigma^2 + u)^{3/2}} + \frac{\sigma G}{\sqrt{u}(\sigma^2 + u)}\right)^+\right]$$

with $x^+ = \max\{0, x\}$ and G is a Gaussian $\mathcal{N}(0, 1)$ variable and we have

$$G(h) = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^{h-s} \mathbb{E}\left[e^{-(x-m(u+s)-Y_1)^2/(2(u+s))} \left(\frac{x-Y_1}{(u+s)^{3/2}} + \frac{G\sqrt{s}}{\sqrt{u}(u+s)}\right)^+\right] du ds.$$

We make the changes of variables s = th, u = hv. Then,

$$\frac{G(h)}{h} = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} \mathbb{E}\left[e^{-(x-mh(v+t)-Y_1)^2/(2h(v+t))} \left(\frac{x-Y_1}{\sqrt{h}(v+t)^{3/2}} + \frac{G\sqrt{t}}{\sqrt{v}(v+t)}\right)^+\right] dt dv.$$

However,

$$\lim_{h \to 0^+} e^{-(x - mh(T = v) - Y_1)^2/(2h(t + v))} \left(\frac{x - Y_1}{\sqrt{h(t + v)^{3/2}}} + \frac{G\sqrt{t}}{\sqrt{v(t + v)}} \right)^+ = \frac{\sqrt{t}}{\sqrt{v(t + v)}} G^+ \mathbf{1}_{\{x = Y_1\}}$$

and

$$\sup_{0 \le h \le 1} e^{-(x - mh(t + v) - Y_1)^2 / (2h(t + v))} \left(\frac{x - Y_1}{\sqrt{h}(t + v)^{3/2}} + \frac{G\sqrt{v}}{\sqrt{1 - v}} \right)^+ \\ \le \frac{\sup_{z \ge 0} z e^{-z^2/2} + |m|}{\sqrt{t + v}} + \frac{\sqrt{t}}{\sqrt{v}(t + v)} |G|.$$

From Lebesgue's dominated convergence theorem, we then obtain

$$\lim_{h \to 0} \frac{G(h)}{h} = \Delta F_Y(x) \frac{\mathbb{E}(G_+)}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} \frac{\sqrt{t}}{\sqrt{v}(t+v)} \, \mathrm{d}v \, \mathrm{d}t = \frac{1}{4} \Delta F_Y(x),$$

where $\Delta F_Y(z) = F_Y(z) - F_Y(z_-)$. This identity achieves the proof of step 3.

2.2. Existence of the derivative at t > 0

Our task now is to show that the distribution function of τ_x is differentiable on \mathbb{R}_+^* and to compute its derivative. For this purpose we split the probability $\mathbb{P}(t < \tau_x \le t + h)$, according to the values of $N_{t+h} - N_t$, into three parts:

$$\mathbb{P}(t < \tau_r < t + h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_r < t + h, N_{t+h} - N_t = 1)$$

$$+ \mathbb{P}(t < \tau_x \le t + h, N_{t+h} - N_t \ge 2)$$

= $B_1(h) + B_2(h) + B_3(h)$.

Since $B_3(h) \leq \mathbb{P}(N_{t+h} - N_t \geq 2)$, we have $\lim_{h \to 0} \frac{B_3(h)}{h} = 0$. By the Markov property at t, $B_2(h) = \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} \mathbb{P}^t(\tau_{x-X_t} \leq h, N_h = 1))$, where $\mathbb{P}^t(\cdot) = \mathbf{1}$ $\mathbb{P}(\cdot|\mathcal{F}_t)$.

By (3), $\frac{B_2(h)}{h}$ converges to $\frac{a}{2}[2-F_Y(x-X_t)-F_Y((x-X_t)_-)]+\frac{a}{4}[F_Y(x-X_t)-F_Y((x-X_t)_-)]$ and is upper bounded by $\frac{\mathbb{P}(N_h=1)}{h}=a\mathrm{e}^{-ah}\leq a$. The dominated convergence theorem gives

$$\lim_{h \to 0} \frac{B_2(h)}{h} = a \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} (1 - F_Y)(x - X_t)) + \frac{3a}{4} \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}} \Delta F_Y(x - X_t)).$$

However, the jumps set of F_Y is countable and X has a density (see [6], Proposition 3.12, page 90). Thus, $\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}\Delta F_Y(x-X_t))=0$ and $\lim_{h\to 0}\frac{B_2(h)}{h}=a\mathbb{E}(\mathbf{1}_{\{\tau_x>t\}}(1-t))$ F_Y) $(x-X_t)$).

It thus remain to prove that

$$\frac{B_1(h)}{h} \xrightarrow{h \to 0} \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})). \tag{4}$$

Since T_{N_t} is not a stopping time, we cannot apply the strong Markov property. We

$$B_1(h) = \mathbb{P}(t < \tilde{\tau}_x \le t + h < T_1) + \sum_{k=1}^{\infty} \mathbb{P}(t < \tau_x \le t + h, T_k < t < t + h < T_{k+1}).$$

On the set $\{T_k < t\}$, we have $X_t = X_{T_k} + X_{t-T_k} \circ \theta_{T_k}$, hence on the set $\{\tau_x > T_k\}$, we have $\tau_x = T_k + \tau_{x-X_{T_k}} \circ \theta_{T_k}$. Moreover, on the set $\{T_k < \min(t, \tau_x)\}$,

$$\mathbf{1}_{\{t < \tau_x \leq t+h, T_k < t < t+h < T_{k+1}\}} = \mathbf{1}_{\{T_k < t\}} \mathbf{1}_{\{t-T_k < \tilde{\tau}_x \leq t+h-T_k < S_{k+1}\}} \circ \theta_{T_k}$$

and the strong Markov property at T_k gives

$$B_{1}(h) = e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_{x} \le t+h)$$

$$+ \sum_{k=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{T_{k} < t\}} \mathbf{1}_{\{\tau_{x} > T_{k}\}} e^{-a(t+h-T_{k})} \mathbb{E}^{T_{k}}(\mathbf{1}_{\{t-T_{k} < \tilde{\tau}_{x-X_{T_{k}}} \le t+h-T_{k}\}})).$$

The \mathcal{F}_{T_k} -conditional law of $\tilde{\tau}_{x-X_{T_k}}$ has the density (possibly defective) $\tilde{f}(\cdot, x-X_{T_k})$, thus since $e^{-a(t-T_k)} = \mathbb{E}^{T_k}(\mathbf{1}_{\{T_{k+1}>t\}})$, we have

$$B_1(h) = e^{-ah} \int_t^{t+h} \mathbb{E}(\mathbf{1}_{\{0 \le t < T_1\}}) \tilde{f}(u, x) du$$

$$+ e^{-ah} \sum_{k=1}^{\infty} \int_{t}^{t+h} \mathbb{E}(\mathbf{1}_{\{T_{k} \le t < T_{k+1}\}} \mathbf{1}_{\{\tau_{x} > T_{k}\}} \tilde{f}(u - T_{k}, x - X_{T_{k}})) du$$

$$= e^{-ah} \int_{t}^{t+h} \mathbb{E}(\mathbf{1}_{\{T_{N_{t}} < \tau_{x}\}} \tilde{f}(u - T_{N_{t}}, x - X_{T_{N_{t}}})) du.$$
(5)

Since \tilde{f} is continuous with respect to u, for all t > 0, almost surely,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{t}^{t+h} \mathbf{1}_{\{T_{N_{t}} < \tau_{x}\}} \tilde{f}(u - T_{N_{t}}, x - X_{T_{N_{t}}}) \, \mathrm{d}u = \mathbf{1}_{\{T_{N_{t}} < \tau_{x}\}} \tilde{f}(t - T_{N_{t}}, x - X_{T_{N_{t}}}).$$

According to Proposition A.2 in the Appendix, the family of random variables $(\frac{1}{h}\int_t^{t+h} \tilde{f}(u-T_{N-t},x-X_{T_{N_t}}) du)_{0 < h \leq 1}$ is uniformly integrable. We then obtain

$$\lim_{h \to 0} \frac{B_1(h)}{h} = \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})).$$

Using (4), we deduce that

$$\frac{\mathbb{P}(t < \tau_x \le t + h)}{h} \xrightarrow{h \to 0} a \mathbb{E}(\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - X_t)) + \mathbb{E}(\mathbf{1}_{\{\tau_x > T_{N_t}\}}\tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})).$$

The proof of Theorem 2.1 is thus complete.

Appendix

We prove the following on \tilde{f} given in (1).

Lemma A.1. Let G be a Gaussian random variable $\mathcal{N}(0,1)$ and let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, $p \ge 1$ and $x^+ = \max\{x, 0\}$. Then, for every $u \in \mathbb{R}$,

$$\begin{split} & \mathbb{E} \big[\tilde{f}(u, \mu + \sigma G)^p \mathbf{1}_{\{\mu + \sigma G > 0\}} \big] \\ &= \frac{1}{\sqrt{2^p \pi^p}} \frac{u^{(1 - 2p)/2} \mathrm{e}^{-p(\mu - mu)^2/(2(p\sigma^2 + u))}}{(p\sigma^2 + u)^{(p+1)/2}} \\ & \times \mathbb{E} \Bigg[\Bigg(\sigma G + \sqrt{\frac{u}{p\sigma^2 + u}} (\mu - mu) + m\sqrt{u(p\sigma^2 + u)} \Bigg)_+^p \Bigg]. \end{split}$$

Proposition A.2. For every t > 0 and $1 \le p < 3/2$,

$$\sup_{0< h \leq 1} \mathbb{E}\left[\left(\frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du\right)^p\right] < +\infty.$$

Proof. Let I(h) be

$$I(h) = \frac{1}{h} \int_{t}^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) \, \mathrm{d}u.$$

Using Jensen's inequality, the following estimate holds:

$$\mathbb{E}(I(h)^p) \le \frac{1}{h} \int_t^{t+h} \mathbb{E}(\mathbf{1}_{\{x - X_{T_{N_t}} > 0\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}})^p) \, \mathrm{d}u.$$

Conditioning by the filtration generated by N and Y_i , $i \in \mathbb{N}$, it becomes, where G is a standard Gaussian random variable independent of N and Y_i , $i \in \mathbb{N}$,

$$\mathbb{E}(I(h)^{p}) \leq \frac{1}{h} \int_{t}^{t+h} \mathbb{E}\left(\mathbf{1}_{\{x-mT_{N_{t}}-\sum_{i=1}^{N_{t}}Y_{i}-\sqrt{T_{N_{t}}}G>0\}} \times \tilde{f}\left(u-T_{N_{t}}, x-mT_{N_{t}}-\sum_{i=1}^{N_{t}}Y_{i}-\sqrt{T_{N_{t}}}G\right)^{p}\right) du.$$

Note that for $u \in [t, t+h]$, $t-T_{N_t} \le u-T_{N_t} \le 1+t-T_{N_t}$, $pT_{N_t}+t-T_{N_t} > t$ and if $C_p = \sup_{x \in \mathbf{R}^+} \sqrt{x^p} \mathrm{e}^{-px/2}$, then, from Lemma A.1,

$$\mathbb{E}(I(h)^p) \le \frac{3^{p-1}}{\sqrt{2^p \pi^p}} \mathbb{E}\left(\frac{T_{N_t}^{p/2}}{(t - T_{N_t})^{p-1/2} t^{(p+1)/2}} \mathbb{E}(|G|^p) + \frac{1}{(t - T_{N_t})^{(p-1)/2} t^{1/2 + p}} C_p + |m|^p \frac{1}{t^{1/2} (t - T_{N_t})^{(p-1)/2}}\right).$$

Observe that for every t > 0 and $(\alpha, \gamma) \in]-1, 0] \times [0, +\infty[$, the random variables $(t - T_{N_t})^{\alpha} T_{N_t}^{\gamma}$ are integrable (see the details below), which completes the proof of Proposition $\Delta 2$

Note that

$$\mathbb{E}((t - T_{N_t})^{\alpha} T_{N_t}^{\gamma}) \le t^{\alpha} + \sum_{i=1}^{\infty} \mathbb{E}(\mathbf{1}_{t > T_i} (t - T_i)^{\alpha} T_i^{\gamma}) < +\infty.$$
(A.6)

However, for $i \ge 1$, T_i admits as density the function $u \mapsto \frac{a^i}{(i-1)!}u^{i-1}e^{-au}$, thus

$$\mathbb{E}(\mathbf{1}_{\{t>T_i\}}(t-T_i)^{\alpha}T_i^{\gamma}) = \frac{a^i}{(i-1)!} \int_0^t e^{-au}(t-u)^{\alpha}u^{\gamma+i-1} du \le \frac{a^i}{(i-1)!} \int_0^t (t-u)^{\alpha}u^{\gamma+i-1} du = \frac{a^i}{(i-1)!} t^{\gamma+i+\alpha} \frac{\Gamma(\gamma+i)\Gamma(\alpha+1)}{\Gamma(\gamma+i+\alpha+1)}.$$

Consequently, the sum in the right-hand term of inequality (A.6) is finite and the random variable $(t - T_{N_t})^{\alpha} T_{N_t}^{\gamma}$ is integrable.

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