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► **To cite this version:**

Matthieu Durnerin, Nadine Martin. Minimum Variance Filters and Mixed Spectrum Estimation. Signal Processing, 2000, 80 (12), pp.2597-2608. 10.1016/S0165-1684(00)00137-7 . hal-00374817v1

HAL Id: hal-00374817

<https://hal.science/hal-00374817v1>

Submitted on 10 Apr 2009 (v1), last revised 6 Dec 2010 (v2)

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Minimum Variance Filters and Mixed Spectrum Estimation

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Abstract

This paper presents a spectral density estimator based on a Normalized Minimum Variance (MV) estimator as the one proposed by Lagunas. With an equivalent frequency resolution, this new estimator preserves the amplitude estimation lost in Lagunas one. This proposition comes from a theoretical study of MV filters that highlights this amplitude lost. Two signal types are taken into account: periodic deterministic signals (narrow band spectral structures) and stationary random signals (broad band spectral structures). Without selecting a smoothing window, the proposed estimator is an alternative to Fourier based estimator and, without modeling the signal, is a concurrent to high resolution estimators.

Résumé

Cet article présente un estimateur de densité spectrale défini à partir d'un estimateur du Minimum de Variance (MV) Normalisé tel que celui proposé par Lagunas. Avec une résolution fréquentielle équivalente, l'objectif de ce nouvel estimateur est de préserver l'estimation de l'amplitude contrairement à l'estimateur de Lagunas. Cette proposition s'appuie sur l'étude de la fonction de transfert du filtre MV. Deux types de signaux sont considérés: des signaux déterministes périodiques (dont la structure spectrale est à bande étroite) et des signaux aléatoires stationnaires (dont la structure spectrale est à large bande). Sans avoir à choisir une fenêtre d'apodisation, l'estimateur proposé est une alternative aux estimateurs de Fourier, et, sans appliquer de modèle au signal, est un concurrent des estimateurs paramétriques

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| Number of pages | 14 |
| Number of tables | 0 |
| Number of figures | 5 |

Key words: spectral analysis, data filtering, minimum variance method.

Introduction

In spectral analysis, the Minimum Variance (MV) method holds an important place with regard to classic Fourier based methods. The method was first developed by J. Capon [2] for frequency wavenumber analysis in seismic array processing, then by R. T. Lacoss [7]. Capon's method estimates the signal power from a filter bank calculated from the signal itself. The filters are constrained in order to reject the signal power at all frequencies but the desired one. M.A. Lagunas [8] has proposed an estimation of the spectral density from Capon's power estimator by introducing the filter bandwidth.

The wide application of these methods comes from the fact that no model is applied to the signal. The filter signal dependency implies optimal properties for its design which is not the case for Fourier analysis [7]. The use of the MV estimator has been conclusive in many situations : in acoustics to characterize hydrodynamic fluids [12], in room acoustics to analyze the impulse response of concert halls [11] and in geophysics [10]. All these applications concern non-stationary signals. In this case, MV methods can be extended to non-stationary signals by mean of a gliding time window under the assumption of local stationarity [1]. Thus, the properties of these estimators are all deduced from those of the stationary case. This justifies the interest of the here presented study for finite duration signals.

Our paper copes with the analysis of filters induced by MV constraints. In particular, the designed filter nature is of interest to study the normalization proposed by Lagunas. Many statistical studies had been carried on the MV estimator [3], [13], [15], [5]. Little of them have considered it as a filter adapted to the data set [6]. Due to the signal dependency, this study relies on analytical developments and also on simulated filters from finite duration signals. From these results, we explain the behavior of these estimators, mainly for the amplitude estimation, and we deduce a new MV estimator. Its principle is close to Lagunas one but a different scaling significantly improves the performance. A statistical study illustrates the preservation of the amplitude value and the increase of the frequency resolution.

Section 1 briefly recalls the MV filters design and the derivation of the normalized estimator. In section 2, the frequency response of the MV filter is studied for a complex exponential signal

embedded in an additive white noise. This filter design is analyzed according to the signal to noise ratio and the exponential frequency value. These filters are empirically generalized for mixed spectrum. From these results, the properties of the normalized estimator are deduced in the third section. In a last section, we propose a new estimator which properties are illustrated on simulations from a mixed spectrum.

1- Minimum Variance (MV) and Normalized Minimum Variance (NMV) methods

1.1- Minimum Variance method (Capon's method)

The MV estimator here presented references to the data filtering concept [2] [5] [4] [7]. Let $\{x(n)\}$ be a wide-sense stationary random process sampled at t_e and $S_x(\nu)$ its power spectral density function (PSD) at the frequency ν . Let us apply $x(n)$ as a filter signal input, this filter is designed at a frequency ν_F in order to estimate the signal power at this frequency. Two constraints are assigned to the filter :

$$1 - \text{The frequency response } A_{\nu_F}(\nu) \text{ must be equal to 1 at frequency } \nu_F : \quad A_{\nu_F}(\nu_F) = 1 \quad (1)$$

2 - The power out of the filter due to other frequencies than ν_F must be minimized. This is equivalent

$$\text{to minimizing the global output power } P_{MV}(\nu_F) [6] : \quad P_{MV}(\nu_F) = \mathbf{A}_{\nu_F}^H \mathbf{R}_x \mathbf{A}_{\nu_F} \quad (2)$$

with $\mathbf{A}_{\nu_F}^T = (a(0), a(1), \dots, a(M-1))$ as the impulse response coefficients vector at ν_F ,

and \mathbf{R}_x as the $M \times M$ autocorrelation matrix of the input signal x .

$$\text{Notice that } A_{\nu_F}(\nu) = a(0) + a(1) e^{-2\pi j \nu t_e} + \dots + a(M-1) e^{-2\pi j (M-1) \nu t_e} = \mathbf{E}_{\nu_F}^H \mathbf{A}_{\nu_F} \quad (3)$$

$$\text{where } \mathbf{E}_{\nu_F}^T = (1, e^{2\pi j \nu_F t_e}, \dots, e^{2\pi j (M-1) \nu_F t_e})$$

The superscripts $()^T$ and $()^H$ denote transpose and Hermitian transpose.

The minimization of (2) under the constraint (1) by Lagrange multiplier technique [3] yields the impulse response and the output power. This defines the MV filter:

$$\mathbf{A}_{\nu_F} = \frac{\mathbf{R}_x^{-1} \mathbf{E}_{\nu_F}}{\mathbf{E}_{\nu_F}^H \mathbf{R}_x^{-1} \mathbf{E}_{\nu_F}} \quad (4) \quad \text{and} \quad P_{MV}(\nu_F) = \frac{1}{\mathbf{E}_{\nu_F}^H \mathbf{R}_x^{-1} \mathbf{E}_{\nu_F}} \quad (5)$$

This filter is evaluated at all frequencies ν_F lying between 0 and Shannon frequency $\nu_e = 1/t_e$. The autocorrelation matrix \mathbf{R}_x must be estimated but the choice of such an estimator is not the purpose of that paper.

This filter design assumes no hypothesis on the signal itself. No model and no a priori information are needed. In Fourier analysis, the impulse response at each frequency is equal to an exponential at this frequency. The exponential is limited in time by a window equal to the width of the analyzed signal. Capon's filter behaves as a window calculated from an estimate of the covariance matrix of the signal. This dependency upon the signal ensures better properties compared with Fourier analysis. A comparative study is discussed in [7] and [11].

1.2- Links with a Linear Minimum Variance Unbiased Estimator

For the particular case of a complex exponential embedded in a complex gaussian noise, a Linear Minimum Variance Unbiased (LMVU) estimator can be defined [6] and is closely connected to Capon's estimator. The exponential amplitude is estimated by the output of a filter which impulse response is also defined by equation (4) but by setting the noise instead of the signal correlation matrix. The variance of this estimator is given by equation (5) by setting the noise correlation matrix too. When the signal is composed by more than one exponential, the matrix in equations (4) and (5) for the LMVU estimator is the correlation matrix of the noise plus all the exponentials which are not at the frequency of the filter. To do that, the exponential frequency and the noise correlation matrix need to be known. The LMVU estimator leads to a Maximum Likelihood estimation of the amplitude which is not the case of Capon's one. Being defined for a stationary random process and evaluated at all frequencies, Capon is more general than LMVU. The filter output power gives an estimate of the input signal power at the filter frequency and whatever the signal is. As one might expect, this estimate at a defined frequency strongly depends on the signal content at the other frequencies, therefore on the noise, and on the number of filter coefficients.

1.3- Normalized Minimum Variance method (Lagunas method)

$P_{MV}(\nu_F)$ is homogeneous to power but not to a spectral density function since the area under the estimated function does not represent the total power of the analyzed signal. M.A. Lagunas [8]

proposed a method to derive the spectral density from the MV power. Assuming that the true power density $S_x(\nu)$ is flat around ν_F and is roughly equal to $S_x(\nu_F)$, equation (2) can be written:

$$P_{MV}(\nu_F) = \int_{-\nu_e/2}^{+\nu_e/2} |A_{\nu_F}(\nu)|^2 S_X(\nu) d\nu \approx S_X(\nu_F) \int_{-\nu_e/2}^{+\nu_e/2} |A_{\nu_F}(\nu)|^2 d\nu \quad (6)$$

Note that this formula assumes that the filter is a narrow bandpass one. So the MV filter output power is linked to the spectral density of the input signal by a factor which is the noise equivalent bandwidth B_e of the filter owing to the MV constraint equation (1). Using Parseval relation, this factor is written as:

$$\int_{-\nu_e/2}^{+\nu_e/2} |A_{\nu_F}(\nu)|^2 d\nu = \frac{1}{t_e} \mathbf{A}_{\nu_F}^H \mathbf{A}_{\nu_F} = B_e \quad (7)$$

Combining (6), (7), (4) and (5), the NMV PSD estimator $S_{NMV}(\nu_F)$ is finally written as:

$$S_{NMV}(\nu_F) = t_e \frac{\mathbf{E}_{\nu_F}^H \mathbf{R}_x^{-1} \mathbf{E}_{\nu_F}}{\mathbf{E}_{\nu_F}^H \mathbf{R}_x^{-2} \mathbf{E}_{\nu_F}} \quad (8)$$

2 - Analysis of the MV filter

A MV estimator study was already performed in [14]. It shows how the estimation converges to the true spectrum with regard to the filter order M and the noise level. We aim here to point out specific properties of the MV filter for explaining the NMN estimator behavior.

The MV filter depends on the signal so that the frequency response has a different shape and then different properties according to signal characteristics. An analytical derivation of the filter should have been considered for each type of signal. We derived it only for a complex exponential embedded in white noise. Finally, we discuss the case of a simulated mixed spectrum.

2.1 - Analytical study of the MV filter

The MV filter is examined for a signal $x(n)$ which is a complex exponential of frequency ν_{exp} in an additive complex white noise $w(n)$ of power σ^2 . Let the complex amplitude of the exponential be $Ce^{j\Phi}$ where Φ is a uniformly distributed random phase:

$$\mathbf{X} = C e^{j\Phi} \mathbf{E}_{\nu_{\text{exp}}} + \mathbf{W} \quad \text{where } \mathbf{X}^T = (x(0), x(1), \dots, x(M-1)) \text{ and } \mathbf{W}^T = (w(0), w(1), \dots, w(M-1))$$

The autocorrelation matrix of the signal X writes as :

$$\mathbf{R}_x = C^2 \mathbf{E}_{\nu_{\text{exp}}} \mathbf{E}_{\nu_{\text{exp}}}^H + \sigma^2 \mathbf{I}$$

where \mathbf{I} is the $M \times M$ identity matrix.

Using Sherman-Morrison formula [4], the inverse of the autocorrelation matrix turns out to be

$$\mathbf{R}_x^{-1} = \frac{1}{\sigma^2} \left(\mathbf{I} - \frac{C^2 / \sigma^2}{1 + M C^2 / \sigma^2} \mathbf{E}_{\nu_{\text{exp}}} \mathbf{E}_{\nu_{\text{exp}}}^H \right) \quad (10)$$

Let Q equal $\frac{C^2 / \sigma^2}{1 + M C^2 / \sigma^2}$ and note that C^2 / σ^2 represents the signal to noise ratio. Substituting

equation (10) into equation (4) yields the impulse response at frequency ν_F :

$$\mathbf{A}_{\nu_F} = \frac{\mathbf{E}_{\nu_F} - Q \mathbf{E}_{\nu_{\text{exp}}} \mathbf{E}_{\nu_{\text{exp}}}^H \mathbf{E}_{\nu_F}}{M - Q \left| \mathbf{E}_{\nu_F}^H \mathbf{E}_{\nu_{\text{exp}}} \right|^2}$$

The filter frequency response adapted to the signal $x(n)$ at frequency ν_F is then deduced:

$$A_{\nu_F}(\nu) = \mathbf{E}_{\nu}^H \mathbf{A}_{\nu_F} = \frac{\mathbf{E}_{\nu}^H \mathbf{E}_{\nu_F} - Q \mathbf{E}_{\nu}^H \mathbf{E}_{\nu_{\text{exp}}} \mathbf{E}_{\nu_{\text{exp}}}^H \mathbf{E}_{\nu_F}}{M - Q \left| \mathbf{E}_{\nu_F}^H \mathbf{E}_{\nu_{\text{exp}}} \right|^2} = \frac{D(\nu_F - \nu) - Q M D(\nu_{\text{exp}} - \nu) D(\nu_F - \nu_{\text{exp}})}{1 - Q M \left| D(\nu_F - \nu_{\text{exp}}) \right|^2} \quad (11)$$

where Dirichlet Kernel $D(\nu_k - \nu_i)$ is equal to:

$$D(\nu_k - \nu_i) = \frac{1}{M} \mathbf{E}_{\nu_i}^H \mathbf{E}_{\nu_k} = \frac{1}{M} \sum_{l=0}^{M-1} e^{2\pi j(\nu_k - \nu_i) l t_e} = e^{j(\nu_k - \nu_i)(M-1)t_e} \frac{\sin(\pi(\nu_k - \nu_i) M t_e)}{M \sin(\pi(\nu_k - \nu_i) t_e)}$$

If the filter frequency ν_F is equal to the exponential frequency ν_{exp} , equation (11) becomes:

$$\left| A_{\nu_{\text{exp}}}(\nu) \right|^2 = D(\nu_{\text{exp}} - \nu)^2 \quad (12)$$

The exponential is filtered by a narrow band filter centered at the exponential frequency with its maximum equal to 1 (Figure 1-b). In this particular case of one exponential in noise, MV method at the exponential frequency is reduced to Fourier estimator with a filter length fixed by the order M and a -3 dB bandwidth equal to $1/(M t_e)$. We can notice that this quantity is frequently used for scaling Capon power at all frequencies even though the signal is more complex than one exponential [5], [6].

If the signal to noise ratio is small ($C^2 / \sigma^2 \ll 1$), Q tends to 0 and the squared modulus of the

$$\text{frequency response (11) is: } \left| A_{\nu_F}(\nu) \right|^2 \approx \left| D(\nu_F - \nu) \right|^2 \quad (13)$$

In this latter case, the filter frequency response modulus is also a Dirichlet's kernel centered at the filter frequency. This case is similar to Fourier analysis. The MV filter rejection is the same at all frequencies since the signal power is high at all frequencies.

If the signal to noise ratio is greater than 1 ($C^2 / \sigma^2 \gg 1$), QM tends to 1 and equation (11) shows that the filter is no longer a Dirichlet's kernel as in Fourier analysis. In this simple case of a single exponential frequency, the structure of the frequency response clearly explains the MV principle. The denominator in (11) is only a constant whereas the numerator is the difference between two Dirichlet's kernels. The first is centered at the filter frequency ν_F and the second at the exponential frequency ν_{exp} . The second kernel is smoothed by a factor which depends on $(\nu_F - \nu_{exp})$. We developed the MV filter shape in two cases to highlight its behavior.

Figure 1

2.2 - Case 1 : The filter frequency ν_F is far from the exponential frequency ν_{exp}

The smoothing factor $D(\nu_F - \nu_{exp})$ tends to zero. Then, equation (11) is reduced to equation (13) when ν is different from ν_{exp} and vanishes when ν tends to ν_{exp} .

In this case, the response is a Dirichlet's kernel except at the exponential frequency where the response is minimized. This latter point makes the difference with Fourier estimator according to the MV constraint. This case is illustrated by figure 1-c which compares the theoretical curve to a simulated one.

2.3 - Case 2 : the filter frequency ν_F is close to the exponential frequency ν_{exp}

Given that $(\nu_F - \nu_{exp})$ is low, the smoothing factor $D(\nu_F - \nu_{exp})$ in (11) plays a prominent part and the frequency response can have two main lobes as shown in figure 1-d. Furthermore, as soon as $\nu < \min(\nu_F, \nu_{exp})$ and $\nu > \max(\nu_F, \nu_{exp})$, $|A_{\nu_F}(\nu)|^2$ is greater than 1 (cf. algebra details in appendix). The frequency response is not maximum at ν_F but at a frequency which lays outside the frequency range ν_{exp} to ν_F . This property is shown in figure 1-d. In this case, we would like to emphasize that the filter is no more a narrow band pass filter centered at ν_F and that the maximum of the frequency response can be more than unity.

2.4- Synthesis of the MV filter's analysis

The filter's behavior is the same whether its frequency belongs to a band where the signal power level is maximum or minimum (case 1). In the neighborhood of this filter frequency, the frequency response is that of a narrow band filter centered at this frequency and with its maximum equal to 1. Outside this neighborhood, the response is minimized at each frequency where the signal power level is high. This result was known, but, moreover, when the signal power is small, the frequency response can have secondary lobes with maxima much greater than one. The areas of these lobes could even be greater than the area of the lobe containing the filter frequency. Note that this situation could not appear in the analytical example, but such a behavior is encountered when the response is constraint at more than one frequency.

For illustrating this point of view, we present the analysis of a more complex signal composed of a sum of a noised sine wave and a broad band signal simulated by a white noise filtered by a low-pass filter. The MV power estimation of this simulated signal is plotted in figure 2-a. Figure 2-d shows the response in a frequency range where the signal power is small, figure 2-b that for the broad band signal and figure 2-f the response at the exponential frequency.

When the filter frequency is close to a maximum of the signal power, the filter is no longer narrow band. The constraint of unity at the filter frequency is maintained but there is no relation for constraining the filter to be maximum at this frequency. On the contrary, such a response is constrained to be minimum at a frequency close to this filter frequency. So a greater maximum should be at a frequency on the other side of that frequency. This behavior detailed in case 2 is due to the filter structure be the filter frequency close to an exponential frequency (figure 2-e) or close to a broad band (figure 2-c). The other parts of the frequency response are similar to case 1.

Figure 2

In conclusion, Capon's filters are not always narrow band filters centered at the filter frequency ν_F . In fact, the design constraints, the unit gain at ν_F and the rejection of the signal power at the other frequencies, are not the only relevant constraints to designing a narrow band filter. At frequencies where signal power is small, the filter is free and the frequency response can be very high. But, the output power is always representative of the signal power due to the fact that the product $|A(\nu)|^2 S_x(\nu)$ is almost equal to zero whatever the value of the frequency response may be.

This particular behavior of Capon's filter was not previously pointed out. More often, the filter was explicitly or implicitly considered as a narrow band filter [9 p.351], [6 p.370], [8]. Therefore, the normalization of the MV method by the bandwidth as defined in (7) is only actual in the case of a narrow band filter. In the other cases, the measurement of the bandwidth includes the area of huge lobes which are not significant. This conclusion explains why the noise equivalent bandwidth cannot be considered as a measure of the frequency resolution as in Fourier analysis.

3-Properties of NMV method

The main property of the NMV method is a refining of power peaks. Each power variation of the MV estimator is enhanced by unrealistic transformations. This estimation has an interest as a frequency estimator but not as an amplitude one. This spectral refinement property can be explained from the previous study by evaluating the amplitude ratio $\frac{P_{MV}(\nu)}{S_{NMV}(\nu)}$ at the exponential frequency ν_{exp} and at a close frequency $(\nu_{\text{exp}} + \varepsilon)$ when ε tends to 0.

According to Lagunas assumption (section 1.3), the power $P_{MV}(\nu)$ defined in (5) and the PSD $S_{NMV}(\nu)$ defined in (8) are linked by the noise equivalent bandwidth B_e defined in equation (7). When the filter frequency ν_F equals the exponential frequency ν_{exp} , the frequency response is given by equation (A2). So the bandwidth noticed $B_{e \text{ exp}}$ in this case is:

$$B_{e \text{ exp}} \approx \int_{-v_e/2}^{+v_e/2} |D(\nu_{\text{exp}} - \nu)|^2 d\nu$$

When the filter frequency ν_F equals $\nu_{\text{exp}} + \varepsilon$, the frequency response is given by equation (A2). The

bandwidth noticed $B_{e \varepsilon}$ becomes:
$$B_{e \varepsilon} = \int_{-v_e/2}^{+v_e/2} \left| \frac{D(\nu_{\text{exp}} + \varepsilon - \nu) - D(\nu_{\text{exp}} - \nu) \left(1 - (\pi M t_e)^2 \varepsilon^2 / 6\right)}{(\pi M t_e)^2 \varepsilon^2 / 3} \right|^2 d\nu$$

When ε tends to 0, we can show that:
$$B_{e \text{ exp}} \ll B_{e \varepsilon}$$

This fundamental relation means that the bandwidth greatly increases close to the maximum of a peak. So, the amplitude ratio of the two estimators can be deduced:

$$\frac{S_{NMV}(\nu + \varepsilon)}{P_{MV}(\nu + \varepsilon)} \ll \frac{S_{NMV}(\nu)}{P_{MV}(\nu)}$$

Close to each maximum signal peak, the NMV method refines the spectral peak without keeping up the amplitudes. These different points are illustrated in figure 3-a which shows the NMV estimate of the same simulated signal as in figure 2. We clearly see the refining of the sinusoid peak compared with the MV estimation in figure 2-a, but, in the broad band, some differences with the true PSD too. Figure 3-b shows the variations of the bandwidth versus frequency. We can observe an increase just around the maximum of the sinusoid frequency. These figures show the way the frequency resolution is improved.

Figure 3

4- A new Estimator

If we maintain the objective of defining a PSD estimator from a MV one, previous results must be considered when equation (6) is applied. Due to the basic relation between a PSD and a power, we can't pass round the notion of integral and of its integration support. But, due to the extreme particularity of the filter shape, we may not consider a classical bandwidth definition. The integration support must be adapted to the designed filter. In this section, we propose a definition of this support which leads us to define a new estimator. The performance of this estimator is then evaluated by a statistical study in order to compare it with MV and NMV ones.

4.1- Definition

The meaningful part of the frequency response at the filter frequency ν_F lies in a band where the first constraint of the MV filter is actual, so we propose to only consider the lobe containing ν_F . The other lobes are meaningless, since they are in frequency bands where there is no high power. Let ΔB be the bandwidth of the lobe including ν_F . If ν is outside ΔB , the affected part of the integral defined in (2) is equal to 0, because either $S_x(\nu) \approx 0$ or $|A_{\nu_F}(\nu)|^2 \approx 0$. We thus can write:

$$P_{MV}(\nu_F) \approx \int_{\Delta B} |A_{\nu_F}(\nu)|^2 S_x(\nu) d\nu \quad (14)$$

Let us assume $S_x(\nu)$ is flat over ΔB range *i.e.* $S_x(\nu) \approx S_x(\nu_F)$. Therefore, equation (14) writes as:

$$P_{MV}(\nu_F) \approx S_x(\nu_F) \int_{\Delta B} |A_{\nu_F}(\nu)|^2 d\nu \quad (15)$$

Substituting equation (5) into (15), we define a new estimator $S_{mod}(v_F)$ as:

$$S_{mod}(v_F) = \frac{1}{E_{v_F}^H R_X^{-1} E_{v_F} \cdot B_{mod}} \quad \text{where} \quad B_{mod} \approx \int_{\Delta B} |A_{v_F}(v)|^2 dv \quad (16)$$

The last integral must be evaluated. To be adapted to all signals, we propose a numerical definition:

$$B_{mod} = \Delta B \cdot |A_{v_F}(v)|_{\max}^2 \quad \text{where} \quad |A_{v_F}(v)|_{\max}^2 \text{ is the maximum of the lobe containing } v_F. \quad (17)$$

The band $\Delta B = [v_-, v_+]$ is precisely defined by $|A_{v_F}(v_-)|^2 = |A_{v_F}(v_+)|^2 = |A_{v_F}(v)|_{\max}^2 / 2$

The chosen solution is the most simple to carry out. It should be possible to use the noise equivalent bandwidth definition too, but only on the lobe including the filter frequency and without using Parseval's equality. In any case, a numerical solution cannot be avoided. The quantity B_{mod} must be considered as a support of integration more than an estimation of a bandwidth.

This definition is illustrated in figure 4 in two cases. When the filter frequency is equal to the exponential frequency, that definition turns out to be exactly to the definition of the -3 dB bandwidth because v_F is at the maximum of the focused lobe as shown in figure 4-a. When the filter frequency is closer to a signal power peak the lobe is no longer centered at v_F and $|A_{v_F}(v)|_{\max}^2$ is greater than 1 as shown in figure 4-b. So, the support B_{mod} adapts to this configuration.

Figure 4

We test this estimator with the same signal as the one simulated in section 2.4. Figure 3-c shows the PSD obtained with the new estimator. We observe that the sinusoid peak is always refined but, in the broad band, the estimation is closed to the true PSD all along the band. Figure 3-d shows the support B_{mod} versus frequency as computed from (17). We observe a value increase when the filter frequency lies beside the sinusoid frequency or close to the broad band. Compared with the NMV estimator (figure 3-a and 3-b), fluctuations in the broad band are suppressed. At a sinusoid frequency, there is always a local minimum which implies a peak refining.

Furthermore, at frequencies where the signal power is high, the support B_{mod} lies in the correct order of magnitude as we might expect. For instance:

- at the sinusoid frequency $B_{mod} = 0.127$ Hz,

- at $\nu_F = 0.111$ Hz within the broad band

$$B_{\text{mod}} = 0.068 \text{ Hz.}$$

Whereas with the NMV estimator, in the last case, B_e equals to 1.626 Hz, which is greater than 0.5, the full frequency range of the signal ! The filter shape and, more particularly, the value of the maximum which is no more equal to 1, induces that unrealistic number which has really no meaning.

It is important to notice that the support B_{mod} , such as we defined it, cannot be considered as an estimation of the frequency resolution. It also has no relation with the sampling frequency or the signal frequency range. If B_{mod} is large, it does not mean that the frequency resolution is small, but more often, that around this frequency the signal has no energy. This explains why B_{mod} at the sinusoid frequency is relatively large, whereas if we measure the -3 dB bandwidth of Capon's peak (figure 2-a), we get 0.008 Hz.

4.2- Statistical Study

In order to compare the different MV estimators, hundred realizations of always the same signal as in section 2.4 were simulated. Figure 5 -a, b and c shows the overlaid estimations of the MV power, the NMV DSP and the new NMV DSP. The obtained biases are identical to those presented in figure 2-a for MV, 3-a for NMV and 3-c for the new NMV. The normalized variances of these estimates are displayed in figures 5 -d, e and f. These simulations corroborate the previously described behavior. Compared with the MV estimator, the peak refining of the NMV estimator is also preserved in the new one. With regard to the estimate level in the broad band, the MV estimator has a bias of -10 dB with respect to the true level. The NMV estimate is even lower whereas the new NMV one tracks correctly the true amplitude. The variances of the MV and new NMV estimators are nearly the same and lower than the NMV one except at the exponential frequency and close to the broad band.

Figure 5

Conclusion

This paper showed the importance of the filtering interpretation of MV method. We saw how the constraints influence the filter design. Consequently, we deduced that MV filters are not always narrow band filters. This was not previously noticed.

Our study explained how the normalization introduced by Lagunas can produce interesting results but some artifacts too. The amplitude variations are unrealistically enhanced, leading to a strongly biased

PSD estimation. We can state that the NMV method is more adapted to narrow spectral band signals and that the frequency resolution is improved compared with the MV estimator. But the NMV estimator is not adapted to track the amplitude and to ensure a good PSD estimation. In contrast, the new estimator we propose in this paper increases the frequency resolution compared with the MV method, sometimes less efficiently than the NMV method, but always preserves the PSD with lower bias and variance for any kind of signals even for mixed spectrum.

It relies on a scaling more adapted to Capon's filter behavior. Note that computation is a little harder and more time-consuming than Lagunas' method given that Parseval's equality could no longer be used. It is not possible to deduce the frequency resolution from this scaling factor and, at the moment, the only way to get a value of the frequency resolution is to measure it directly from the estimated power, as R.T. Lacoss did with a noised single sine wave [7].

Appendix

In order to understand the effect of the smoothing factor in equation (11), this term is approximated at

$$\text{order 2: } D(\nu_F - \nu_{\text{exp}}) \approx 1 - \frac{(\pi M t_e)^2 (\nu_F - \nu_{\text{exp}})^2}{6} \quad \text{and} \quad D(\nu_F - \nu_{\text{exp}})^2 \approx 1 - \frac{(\pi M t_e)^2 (\nu_F - \nu_{\text{exp}})^2}{3} \quad (\text{A1})$$

Substituting (A1) into (11) gives the squared frequency response in case 2:

$$\left\{ \begin{array}{l} |A_{\nu_F}(\nu)|^2 \approx \left| \frac{D(\nu_F - \nu) - D(\nu_{\text{exp}} - \nu) \left(1 - (\pi M t_e)^2 (\nu_F - \nu_{\text{exp}})^2 / 6\right)}{(\pi M t_e)^2 (\nu_F - \nu_{\text{exp}})^2 / 3} \right|^2 \\ C^2 / \sigma^2 \gg 1, \quad Q \approx 1/M, \quad \nu_{\text{exp}} - \nu_F \rightarrow 0 \end{array} \right. \quad (\text{A2})$$

A stronger approximation can point out the property of this filter. In the frequency range close to ν_F and ν_{exp} , we can limit all Dirichlet's functions to second order given that $(\nu_{\text{exp}} - \nu_F) \rightarrow 0$,

$(\nu_{\text{exp}} - \nu) \rightarrow 0$ and $(\nu_F - \nu) \rightarrow 0$. So, equation (11) becomes:

$$\left\{ \begin{array}{l} |A_{\nu_F}(\nu)|^2 \approx \frac{1}{4} \left(1 + \left(\frac{\nu_{\text{exp}} + \nu_F - 2\nu}{\nu_{\text{exp}} - \nu_F} \right) \right)^2 \\ C^2 / \sigma^2 \gg 1, Q \approx 1/M, \nu_{\text{exp}} - \nu_F \rightarrow 0, \nu_{\text{exp}} - \nu \rightarrow 0, \nu_F - \nu \rightarrow 0 \end{array} \right.$$

This strong approximation is correct because the two MV constraints (1) and (2) are always respected

References

- [1] M. Basseville, P. Flandrin, N. Martin Eds, Signaux non stationnaires. Analyse temps-fréquence et segmentation, Traitement du Signal, Supplement Vol. 9, No. 1, 1992.
- [2] J. Capon, High resolution frequency wavenumber spectrum analysis, Proc. IEEE, Vol. 57, Aug. 1969, pp. 1408-1418.
- [3] J. Capon, N.R. Goodman, Probability distributions for estimates of the frequency-wavenumber spectrum, Proc. IEEE, Vol. 58, 1970, pp. 1785-1786.
- [4] G.H. Golub, C.F. Van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, Maryland, 1985.
- [5] E. E. Ioannidis, On the behavior of a Capon-type spectral density estimator, The annals of Statistics, Vol. 22, No. 4, 1994, pp. 2089-2114.
- [6] S.M. Kay, Modern Spectral Estimation, Prentice-Hall Theory and Applications, Englewood Cliffs, New Jersey, 1988.
- [7] R.T. Lacoss, Data adaptive spectral analysis methods, Geophys., Vol. 36, No. 4, Aug. 1971, pp. 661-675.
- [8] M.A. Lagunas, M.E. Santamaria, A. Gasull and A. Moreno, Maximum likelihood filters in spectral estimation problems, Signal Processing, Vol. 10, No. 1, Jan. 1986, pp. 19-34.
- [9] S.L. Jr. Marple, Digital Spectral Analysis With Applications, Prentice-Hall Signal Processing Series, A.V. Oppenheim Editor, 1987.
- [10] J. Mars, N. Martin, J.L. Lacoume, M. Dubesset, Analysis of signal over short time windows, Signal Processing, Vol. 26, Feb.1992, pp. 147-159.
- [11] N. Martin, J. Mars, J. Martin, C. Chorier, A Capon's Time-Octave Representation. Application in Room Acoustics, IEEE Trans. in Signal Processing, Vol. 43, No. 8, Aug. 1995, pp. 1842-1854.
- [12] V. Pierson, N. Martin, Comparison of shape descriptors for parameters extraction of a time-frequency image, Seventh SP Workshop on Statistical Signal & Array Processing, Québec, Canada, June 26-29 1994, pp 259-262.
- [13] V.F. Pisarenko, On the estimation of spectra by means of non-linear functions of the covariance matrix, Geophysical Journal of the Royal Astronomical Society, Vol. 28, 1972, pp. 511-531.
- [14] P.J. Sherman, On the family of ML spectral estimates for mixed spectrum identification, IEEE Transactions on Signal Processing, Vol. 39, No. 3, March 1991.
- [15] T. Subba Rao, M.M. Gabr, The estimation of the spectrum, inverse spectrum and inverse autocovariances of a stationary time series, Journal Time S. Anal., Vol. 10, pp. 183-202, 1989.