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# HIGHEST COEFFICIENTS OF WEIGHTED EHRHART QUASI-POLYNOMIALS FOR A RATIONAL POLYTOPE

VELLEDA BALDONI, NICOLE BERLINE, AND MICHÈLE VERGNE

ABSTRACT. We describe a method for computing the highest degree coefficients of a weighted Ehrhart quasi-polynomial for a rational simple polytope.

## 1. INTRODUCTION

Let  $\mathfrak{p}$  be a rational polytope in  $V = \mathbb{R}^d$  and  $h(x)$  a polynomial function on  $V$ . A classical problem is to compute the sum of values of  $h(x)$  over the set of integral points of  $\mathfrak{p}$ ,

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

The function  $h(x)$  is called the weight. When  $\mathfrak{p}$  is dilated by an integer  $n \in \mathbb{N}$ , we obtain a function of  $n$  which is quasi-polynomial, the so-called weighted Ehrhart quasi-polynomial of the pair  $(\mathfrak{p}, h)$

$$S(n\mathfrak{p}, h) = \sum_{m=0}^{d+M} E_m n^m.$$

It has degree  $d+M$ , where  $N = \deg h$ . The coefficients  $E_m$  are periodic functions of  $n \in \mathbb{N}$ , with period the smallest integer  $q$  such that  $q\mathfrak{p}$  is a lattice polytope.

In [4], Barvinok obtained a formula relating the  $k$ th coefficient of the (unweighted) Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to  $k$ -dimensional faces of the polytope. As a consequence, he proved that the  $k_0$  highest degree coefficients of the unweighted Ehrhart quasi-polynomial of a *rational simplex* can be computed by a polynomial algorithm, when the dimension  $d$  is part of the input, but  $k_0$  is fixed.

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The sum  $S(\mathbf{p}, h)$  has natural generalizations, the *intermediate* sums  $S^L(\mathbf{p}, h)$ , where  $L \subseteq V$  is a rational vector subspace. For a polytope  $\mathbf{p} \subset V$  and a polynomial  $h(x)$

$$S^L(\mathbf{p}, h) = \sum_x \int_{\mathbf{p} \cap (x+L)} h(y) dy,$$

where the summation index  $x$  runs over the projected lattice in  $V/L$ . In other words, the polytope  $\mathbf{p}$  is sliced along lattice affine subspaces parallel to  $L$  and the integrals of  $h$  over the slices are added up. For  $L = V$ , there is only one term and  $S^V(\mathbf{p}, h)$  is just the integral of  $h(x)$  over  $\mathbf{p}$ , while, for  $L = \{0\}$ , we recover  $S(\mathbf{p}, h)$ . Barvinok's method was to introduce particular linear combinations of the intermediate sums,

$$\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathbf{p}, h).$$

It is natural to replace the polynomial weight  $h(x)$  with an exponential function  $x \mapsto e^{\langle \xi, x \rangle}$ , and consider the corresponding holomorphic functions of  $\xi$  in the dual  $V^*$ . Moreover, one can allow  $\mathbf{p}$  to be unbounded, then the sums

$$S^L(\mathbf{p})(\xi) = \sum_x \int_{\mathbf{p} \cap (x+L)} e^{\langle \xi, y \rangle} dy$$

still make sense as meromorphic functions on  $V^*$ . The map  $\mathbf{p} \mapsto S^L(\mathbf{p})(\xi)$  is a valuation. In [2], we proved that Barvinok's valuation  $\sum_{L \in \mathcal{L}} \rho(L) S^L(\mathbf{p})(\xi)$  approximates  $S(\mathbf{p})(\xi)$  in a sense which is made precise below. As a consequence, we recovered Barvinok's main theorem of [4] and we sketched a method for computing the highest degree coefficients of the Ehrhart quasipolynomial of a rational simplex which is hopefully easier to implement. The proof in [2] relied on our Euler-Maclaurin expansion of these functions.

The main interest of the present article is to give a simpler formulation and an elementary proof of the approximation result of [2], in the case of a **simple polytope**.

In a forthcoming article, in collaboration with J. De Loera and M. Koeppel, we plan to apply the results of this article to derive a polynomial algorithm for computing the  $k_0$  highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex, when  $k_0$  and the degree of the weight  $h(x)$  are fixed, but the dimension of the simplex is part of the input. The article [1] dealt with the integral  $\int_{\mathbf{p}} h(x) dx$ , which is of course the highest coefficient, if  $h(x)$  is homogeneous.

To explain the main idea, let us assume in this introduction that the vertices  $s$  of  $\mathbf{p}$  are integral. Using a theorem of Brion, one writes

$S(\mathbf{p})(\xi)$  as a sum of the generating functions of the supporting cones at the vertices  $s$  of  $\mathbf{p}$ ,

$$S(\mathbf{p})(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} S(s + \mathbf{c}_s)(\xi).$$

Then the dilated polytope  $n\mathbf{p}$  has vertices  $ns$  with the same cone of feasible directions  $\mathbf{c}_s$ , thus

$$S(n\mathbf{p})(\xi) = \sum_{s \in \mathcal{V}(\mathbf{p})} e^{n\langle \xi, s \rangle} S(\mathbf{c}_s)(\xi).$$

The generating function of a cone  $\mathbf{c}$  is a meromorphic function of a particular type, namely, near  $\xi = 0$ , it is a quotient of a holomorphic function by a product of linear forms. Hence, it admits a decomposition into homogeneous components. One shows that the lowest degree is  $\geq -d$ .

Let us fix an integer  $k_0$ ,  $0 \leq k_0 \leq d$ . Our goal is to compute the  $k_0+1$  highest degree coefficients of the Ehrhart quasi-polynomial of  $\mathbf{p}$  for the weight  $h(x) = \frac{\langle \xi, x \rangle^M}{M!}$ . As we explain in Section 3, this computation amounts to computing the lowest homogeneous components

$$S(\mathbf{c}_s)_{[-d+k]}(\xi), \quad k = 0, \dots, k_0,$$

of the generating functions of the cones  $\mathbf{c}_s$ .

Our main result, Theorem 16, is an expression, depending on  $k_0$ , for the components  $S(s + \mathbf{c})_{[-d+k]}(\xi)$ ,  $k = 0, \dots, k_0$  in the particular case where the cone  $s + \mathbf{c}$  is simplicial. For a unimodular cone, the generating function is given by a "short" formula, thus its lowest degree components are readily computed. In general, let  $v_i \in \mathbb{Z}^d$ ,  $i = 1, \dots, d$ , be integral generators of the edges of  $\mathbf{c}$ . The finite sum

$$f(\xi) = \sum_{x \in (\sum_{i=1}^d [0,1[v_i) \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}$$

is an analytic function of  $\xi$ . (If  $\mathbf{c}$  is unimodular and the generators  $v_i$  are primitive, then  $f(\xi) = 1$ ). The term of degree  $k$  of  $f(\xi)$  is given by

$$\sum_{x \in (\sum_{i=1}^d [0,1[v_i) \cap \mathbb{Z}^d} \frac{\langle \xi, x \rangle^k}{k!}.$$

A monomial of degree  $k$  in  $(\xi_1, \dots, \xi_d)$  can involve at most  $k$  variables among the  $\xi_i$ . From this elementary remark, we deduce that the terms of degree  $\leq k_0$  of  $f(\xi)$  can be computed using a Moebius type combination of sums similar to  $f(\xi)$ , in dimension  $\leq k_0$ , and determinants. As a consequence, we obtain an expression for the terms

$S(s + \mathbf{c})_{[-d+k]}(\xi)$ ,  $k = 0, \dots, k_0$ , which involves the generating functions of cones in dimension  $\leq k_0$  only. This feature is useful because, when the dimension is fixed, there is an efficient algorithm for decomposing a simplicial cone into a signed combination of unimodular cones, due to Barvinok.

## 2. NOTATIONS AND BASIC FACTS

2.1. We consider a rational vector space  $V$  of dimension  $d$ , that is to say a finite dimensional real vector space with a lattice denoted by  $\Lambda$ . We will need to consider subspaces and quotient spaces of  $V$ , this is why we cannot just let  $V = \mathbb{R}^d$  and  $\Lambda = \mathbb{Z}^d$ . The set  $\Lambda \otimes \mathbb{Q}$  of rational points in  $V$  is denoted by  $V_{\mathbb{Q}}$ . A subspace  $L$  of  $V$  is called rational if  $L \cap \Lambda$  is a lattice in  $L$ . If  $L$  is a rational subspace, the image of  $\Lambda$  in  $V/L$  is a lattice in  $V/L$ , so that  $V/L$  is a rational vector space. The image of  $\Lambda$  in  $V/L$  is called the projected lattice.

A rational space  $V$ , with lattice  $\Lambda$ , has a canonical Lebesgue measure  $dx$ , for which  $V/\Lambda$  has measure 1.

A convex rational polyhedron  $\mathbf{p}$  in  $V$  (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that  $\mathbf{p}$  is solid (in  $V$ ) if the affine span of  $\mathbf{p}$  is  $V$ .

In this article, a cone is a polyhedral cone (with vertex 0) and an affine cone is a translated set  $s + \mathbf{c}$  of a cone  $\mathbf{c}$ .

A polytope  $\mathbf{p}$  is a compact polyhedron. The set of vertices of  $\mathbf{p}$  is denoted by  $\mathcal{V}(\mathbf{p})$ . For each vertex  $s$ , the cone of feasible directions at  $s$  is denoted by  $\mathbf{c}_s$ .

A cone  $\mathbf{c}$  is called simplicial if it is generated by independent elements of  $V$ . A simplicial cone  $\mathbf{c}$  is called unimodular if it is generated by independent integral vectors  $v_1, \dots, v_k$  such that  $\{v_1, \dots, v_k\}$  can be completed to an integral basis of  $V$ . An affine cone  $\mathbf{a}$  is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

### 2.2. Generating functions.

**Definition 1.** We denote by  $\mathcal{H}(V^*)$  the ring of holomorphic functions defined around  $0 \in V^*$ . We denote by  $\mathcal{M}(V^*)$  the ring of meromorphic functions defined around  $0 \in V^*$  and by  $\mathcal{M}_{\ell}(V^*) \subset \mathcal{M}(V^*)$  the subring consisting of those meromorphic functions  $\phi(\xi)$  such that there exists a product of linear forms  $D(\xi)$  such that  $D(\xi)\phi(\xi)$  is holomorphic.

A function  $\phi(\xi) \in \mathcal{M}_\ell(V^*)$  has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m \geq m_0} \phi_{[m]}(\xi).$$

If  $P$  is a homogeneous polynomial on  $V^*$  of degree  $p$ , and  $D$  a product of  $r$  linear forms, then  $\frac{P}{D}$  is an element in  $\mathcal{M}_\ell(V^*)$  homogeneous of degree  $m = p - r$ . For instance,  $\frac{\xi_1}{\xi_2}$  is homogeneous of degree 0. On this example, we observe that a function in  $\mathcal{M}_\ell(V^*)$  which has no negative degree terms need not be analytic.

Let us recall the definition of the functions  $I(\mathfrak{p})$  and  $S(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$  associated to a polyhedron  $\mathfrak{p}$ , (see for instance the survey [5]).

**Proposition 2.** *There exists a unique map  $I$  which to every polyhedron  $\mathfrak{p} \subset V$  associates a meromorphic function  $I(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$ , so that the following properties hold:*

- (a) *If  $\mathfrak{p}$  is not solid or if  $\mathfrak{p}$  contains a straight line, then  $I(\mathfrak{p})=0$ .*
- (b) *If  $\xi \in V^*$  is such that  $e^{\langle \xi, x \rangle}$  is integrable over  $\mathfrak{p}$ , then*

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x \rangle} dx.$$

- (c) *For every point  $s \in V_{\mathbb{Q}}$ , one has*

$$I(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} I(\mathfrak{p})(\xi).$$

**Proposition 3.** *There exists a unique map  $S$  which to every polyhedron  $\mathfrak{p} \subset V$  associates a meromorphic function  $S(\mathfrak{p}) \in \mathcal{M}_\ell(V^*)$ , so that the following properties hold:*

- (a) *If  $\mathfrak{p}$  contains a straight line, then  $S(\mathfrak{p})=0$ .*
- (b) *If  $\xi \in V^*$  is such that  $e^{\langle \xi, x \rangle}$  is summable over the set of lattice points of  $\mathfrak{p}$ , then*

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle}.$$

- (c) *For every point  $s \in \Lambda$ , one has*

$$S(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} S(\mathfrak{p})(\xi).$$

Moreover the maps  $\mathfrak{p} \mapsto I(\mathfrak{p})$  and  $\mathfrak{p} \mapsto S(\mathfrak{p})$  have additivity properties, with consequence the fundamental Brion's theorem.

**Theorem 4** (Brion, [7]). *Let  $\mathfrak{p}$  be a polyhedron with set of vertices  $\mathcal{V}(\mathfrak{p})$ . For each vertex  $s$ , let  $\mathfrak{c}_s$  be the cone of feasible directions at  $s$ . Then*

$$S(\mathfrak{p}) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s) \text{ and } I(\mathfrak{p}) = \sum_{s \in \mathcal{V}(\mathfrak{p})} I(s + \mathfrak{c}_s).$$

### 2.3. Notations and basic facts in the case of a simplicial cone.

Let  $v_i, i = 1, \dots, d$  be linearly independent integral vectors and let  $\mathbf{c} = \sum_{i=1}^d \mathbb{R}^+ v_i$  be the cone they span.

**Definition 5.** *The semi-closed unit cell  $\mathbf{B}$  of the cone (with respect to the generators  $v_i, i = 1, \dots, d$ ) is the set*

$$\mathbf{B} = \sum_{i=1}^d [0, 1[v_i.$$

We recall the following elementary but crucial lemma.

**Lemma 6.** (i) *The affine cone  $(s + \mathbf{c}) \cap \Lambda$  is the disjoint union of the translated cells  $s + \mathbf{B} + v$ , for  $v \in \sum_{j=1}^d \mathbb{N}v_j$ .*

(ii) *The set of lattice points in the affine cone  $s + \mathbf{c}$  is the disjoint union of the sets  $x + \sum_{i=1}^d \mathbb{N}v_i$  when  $x$  runs over the set  $(s + \mathbf{B}) \cap \Lambda$ .*

(iii) *The number of lattice points in the cell  $s + \mathbf{B}$  is equal to the volume of the cell with respect to the Lebesgue measure defined by the lattice, that is*

$$\text{Card}((s + \mathbf{B}) \cap \Lambda) = |\det(v_i)|.$$

Let  $s \in V_{\mathbb{Q}}$ . We have immediately

$$(1) \quad I(s + \mathbf{c})(\xi) = e^{\langle \xi, s \rangle} \frac{(-1)^d |\det(v_i)|}{\prod_{i=1}^d \langle \xi, v_i \rangle}.$$

The study of the function  $S(s + \mathbf{c})(\xi)$  will be the main point of this article. It reduces to the study of the holomorphic function  $S(s + \mathbf{B})(\xi)$  defined by the following finite sum, over the lattice points of the unit cell.

**Definition 7.**

$$S(s + \mathbf{B})(\xi) = \sum_{x \in (s + \mathbf{B}) \cap \Lambda} e^{\langle \xi, x \rangle}.$$

**Lemma 8.**

$$(2) \quad S(s + \mathbf{c})(\xi) = S(s + \mathbf{B})(\xi) \frac{1}{\prod_{j=1}^d (1 - e^{\langle \xi, v_j \rangle})}.$$

*In particular,  $S(s + \mathbf{c}) \in \mathcal{M}_{\ell}(V^*)$ , thus it admits a decomposition into homogeneous components,*

$$(3) \quad S(s + \mathbf{c})(\xi) = S_{[-d]}(s + \mathbf{c})(\xi) + S_{[-d+1]}(s + \mathbf{c})(\xi) + \dots,$$

*and the lowest degree term  $S_{[-d]}(s + \mathbf{c})(\xi)$  is equal to  $I(\mathbf{c})(\xi)$*

*Proof.* (2) follows from Lemma 6 (ii). Next, we write

$$(4) \quad \prod_{j=1}^d \frac{1}{1 - e^{\langle \xi, v_j \rangle}} = \prod_{j=1}^d \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}} \frac{1}{\prod_{j=1}^d \langle \xi, v_j \rangle}.$$

The function  $\frac{x}{1-e^x}$  is holomorphic with value  $-1$  for  $x = 0$ . Thus  $S(s + \mathbf{c}) \in \mathcal{M}_\ell(V^*)$ . The value at  $\xi = 0$  of the sum over the cell is the number of lattice points of the cell, that is the volume  $|\det(v_i)|$ . This proves the last assertion.  $\square$

### 3. WEIGHTED EHRHART QUASIPOLYNOMIALS

Let  $\mathbf{p} \subset V$  be a rational polytope and let  $h(x)$  be a polynomial function of degree  $M$  on  $V$ . We consider the following weighted sum over the set of lattice points of  $\mathbf{p}$ ,

$$\sum_{x \in \mathbf{p} \cap \Lambda} h(x).$$

When  $\mathbf{p}$  is dilated by a non negative integer  $n$ , we obtain the quasi-polynomial of the pair  $(\mathbf{p}, h)$ ,

$$\sum_{x \in n\mathbf{p} \cap \Lambda} h(x) = \sum_{m=0}^{d+M} E_m n^m.$$

The coefficients  $E_m$  actually depend on  $n$ , but they depend only on  $n \bmod q$ , where  $q$  is the smallest integer such that  $q\mathbf{p}$  is a lattice polytope. If  $h(x)$  is homogeneous of degree  $M$ , the highest degree coefficient  $E_{d+M}$  is equal to the integral  $\int_{\mathbf{p}} h(x) dx$ .

Let us fix a number  $k_0$ . Our goal is to compute the  $k_0 + 1$  highest degree coefficients  $E_m$ , for  $m = M + d, \dots, M + d - k_0$ .

We concentrate on the special case where the polynomial  $h(x)$  is a power of a linear form

$$h(x) = \frac{\langle \xi, x \rangle^M}{M!}.$$

Of course, any polynomial can be written as a linear combination of powers of linear forms.

We will explain our results with the simplifying assumption that the vertices of the polytope are lattice points.

**Definition 9.** We define the coefficients  $E_m(\mathbf{p}, \xi, M)$ ,  $m = 0, \dots, M + d$  by

$$\sum_{x \in n\mathbf{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathbf{p}, \xi, M) n^m.$$



**Proposition 10.** *Let  $\mathfrak{p}$  be a lattice polytope. Then, for  $k \geq 0$ , we have*

$$(5) \quad E_{M+d-k}(\mathfrak{p}, \xi, M) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} S_{[-d+k]}(\mathbf{c}_s)(\xi).$$

*The highest degree coefficient is just the integral*

$$E_{M+d}(\mathfrak{p}, \xi, M) = \int_{\mathfrak{p}} \frac{\langle \xi, x \rangle^M}{M!} dx.$$

**Remark 11.** *As functions of  $\xi$ , the coefficients  $E_m(\mathfrak{p}, \xi, M)$  are polynomial, homogeneous of degree  $M$ . However, in (5), they are expressed as linear combinations of rational functions of  $\xi$ , whose poles cancel out.*

*Proof.* The starting point is Brion's formula. As the vertices are lattice points, we have

$$(6) \quad \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + \mathbf{c}_s)(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{\langle \xi, s \rangle} S(\mathbf{c}_s)(\xi).$$

When  $\mathfrak{p}$  is replaced with  $n\mathfrak{p}$ , the vertex  $s$  is replaced with  $ns$  but the cone  $\mathbf{c}_s$  does not change. We obtain

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{n\langle \xi, s \rangle} S(\mathbf{c}_s)(\xi).$$

We replace  $\xi$  with  $t\xi$ ,

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{t\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{nt\langle \xi, s \rangle} S(\mathbf{c}_s)(t\xi).$$

The decomposition into homogeneous components gives

$$S(\mathbf{c}_s)(t\xi) = t^{-d} I(\mathbf{c}_s)(\xi) + t^{-d+1} S_{[-d+1]}(\mathbf{c}_s)(\xi) + \cdots + t^k S_{[k]}(\mathbf{c}_s)(\xi) + \cdots.$$

Hence, the  $t^M$ -term in the right-hand side is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k} t^{-d+k} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} S_{[-d+k]}(\mathbf{c}_s)(\xi).$$

Thus we have

$$(7) \quad \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{s \in \mathcal{V}(\mathfrak{p})} n^{M+d} \frac{\langle \xi, s \rangle^{M+d}}{(M+d)!} I(\mathbf{c}_s)(\xi) \\ + n^{M+d-1} \frac{\langle \xi, s \rangle^{M+d-1}}{(M+d-1)!} S_{[-d+1]}(\mathbf{c}_s)(\xi) + \cdots + S_{[M]}(\mathbf{c}_s)(\xi).$$

On this relation, we read immediately that  $\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!}$  is a polynomial function of  $n$  of degree  $M + d$ , and that the coefficient of  $n^{M+d-k}$  is given by (5). The highest degree coefficient is given by

$$E_{M+d}(\mathfrak{p}, \xi, M) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s \rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi).$$

Applying Brion's formula for the integral, this is equal to the term of  $\xi$ -degree  $M$  in  $I(\mathfrak{p})(\xi)$ , which is indeed the integral  $\int_{\mathfrak{p}} \frac{\langle \xi, x \rangle^M}{M!} dx$ .  $\square$

From Proposition 10, we draw an important consequence: in order to compute the  $k_0 + 1$  highest degree terms of the weighted Ehrhart polynomial for the weight  $h(x) = \frac{\langle \xi, x \rangle^M}{M!}$ , we **need only** the  $k_0 + 1$  lowest degree homogeneous terms of the meromorphic function  $S(\mathfrak{c}_s)(\xi)$ , for every vertex  $s$  of  $\mathfrak{p}$ . We compute such an approximation in the next section.

#### 4. APPROXIMATION OF THE GENERATING FUNCTION OF A SIMPLICIAL AFFINE CONE

Let  $\mathfrak{c} \subset V$  be a simplicial cone with integral generators  $(v_j, j = 1, \dots, d)$ , and let  $s \in V_{\mathbb{Q}}$ . Let  $k_0 \leq d$ . In this section we will obtain an expression for the  $k_0 + 1$  lowest degree homogeneous terms of the meromorphic function  $S(s + \mathfrak{c})(\xi)$ . Recall that if  $\mathfrak{c}$  is unimodular, the function  $S(s + \mathfrak{c})(\xi)$  has a "short" expression, given by (2),

$$S(s + \mathfrak{c})(\xi) = e^{\langle \xi, \bar{s} \rangle} \prod_{j=1}^d \frac{1}{1 - e^{\langle \xi, v_j \rangle}},$$

where  $(v_i, i = 1, \dots, d)$  are the primitive integral generators of the edges and  $\bar{s}$  is the unique lattice point in the corresponding cell  $s + \mathbf{B}$ . Thus in the unimodular case, computing the lowest degree components is immediate.

When  $\mathfrak{c}$  is not unimodular, it is not possible to compute efficiently the Taylor expansion of the function  $S(s + \mathbf{B})(\xi)$  at order  $k_0$ , if the order is part of the input as well as the dimension  $d$ . In contrast, **if the order  $k_0$  is fixed**, we are going to obtain an expression for the terms of degree  $\leq k_0$  which involves discrete summation over cones in dimension  $\leq k_0$  only, and determinants. For example, for  $k_0 = 0$ , the constant term  $S(s + \mathbf{B})(0)$  is the number of lattice points in the cell, which is equal to a determinant, by Lemma 6 (iii).

We need some notations.

For  $I \subseteq \{1, \dots, d\}$ , we denote by  $L_I$  the linear span of the vectors  $(v_i, i \in I)$ . We denote by  $\mathbf{B}_I = \sum_{i \in I} [0, 1]v_i$  the unit cell in  $L_I$ .

We denote by  $I^c$  the complement of  $I$  in  $\{1, \dots, d\}$ . We have  $V = L_I \oplus L_{I^c}$ . For  $x \in V$  we denote the components by

$$x = x_I + x_{I^c}.$$

Thus we identify the quotient  $V/L_{I^c}$  with  $L_I$  and we denote the projected lattice by  $\Lambda_I \subset L_I$ . Note that  $L_I \cap \Lambda \subseteq \Lambda_I$ , but the inclusion is strict in general.

**Example 12.**  $v_1 = (1, 0), v_2 = (1, 2)$ . Take  $I = \{1\}$ . Then  $\Lambda_I = \mathbb{Z}\frac{v_1}{2}$ .

We denote by  $\mathbf{c}_I$  the projection of the cone  $\mathbf{c}$  on the space  $L_I$ . Its edges are generated by  $v_j, j \in I$ , and the corresponding unit cell  $\mathbf{B}_I$  is the projection of  $\mathbf{B}$ . Remark that  $v_j$  may be non primitive for the projected lattice  $\Lambda_I$ , even if it is primitive for  $\Lambda$ , as we see in the previous example. This is the reason why in Lemma 6, we did not make the (unnecessary) assumption that the generators  $v_j$  are primitive.

For  $u = (u_1, \dots, u_d)$ , we denote  $u_I = \sum_{i \in I} u_i$ .

We denote the binomial coefficient  $\frac{p!}{k!(p-k)!}$  by  $\binom{p}{k}$ .

**Definition 13.** Given a function  $I \mapsto \lambda(I)$  on the set of subsets  $I \subseteq \{1, \dots, d\}$  with cardinal  $|I| \leq k_0$ , we denote

$$T(s, \mathbf{c}, k_0, \lambda)(\xi) = \sum_{|I| \leq k_0} \lambda(I) \text{vol}(\mathbf{B}_{I^c}) S(s_I + \mathbf{c}_I)(\xi) (-1)^{d-|I|} \prod_{j \in I^c} \frac{1}{\langle \xi, v_j \rangle}.$$

**Remark 14.** The function  $S(s_I + \mathbf{c}_I)(\xi)$  is a meromorphic function on the space  $L_I^*$ . We extend it to  $V^*$  by the decomposition  $V = L_I \oplus L_{I^c}$ .

It is easy to see that the function  $T(s, \mathbf{c}, k_0, \lambda)(\xi)$  lies in  $\mathcal{M}_\ell(V^*)$ , and its expansion into homogeneous components has lowest degree  $-d$ . Thus

$$T(s, \mathbf{c}, k_0, \lambda)(\xi) = T_{[-d]}(s, \mathbf{c}, k_0, \lambda)(\xi) + T_{[-d+1]}(s, \mathbf{c}, k_0, \lambda)(\xi) + \dots$$

We will use functions  $I \mapsto \lambda(I)$  which have the following property.

**Definition 15.** A  $(d, k_0)$ -patchfunction is a function  $I \mapsto \lambda(I)$  on the set of subsets  $I \subseteq \{1, \dots, d\}$  of cardinal  $|I| \leq k_0$  which satisfies the following condition.

$$(8) \quad e^{u_1 + \dots + u_d} \equiv \sum_{|I| \leq k_0} \lambda(I) e^{u_I} \text{ mod terms of } u\text{-degree} \geq k_0 + 1.$$

**Theorem 16.** *Let  $I \mapsto \lambda(I)$  be a  $k_0$ -patchfunction. Then we have*

$$(9) \quad S(s + \mathbf{B})(\xi) \equiv \sum_{|I| \leq k_0} \lambda(I) \operatorname{vol}(\mathbf{B}_{I^c}) S(s_I + \mathbf{B}_I)(\xi)$$

*mod terms of  $\xi$ -degree  $\geq k_0 + 1$ .*

(10)

$$S(s + \mathbf{c})(\xi) \equiv T(s, \mathbf{c}, k_0, \lambda)(\xi) \text{ mod terms of } \xi\text{-degree } \geq -d + k_0 + 1.$$

*Proof.* Using (2), we write

$$S(s + \mathbf{c})(\xi) = \left( S(s + \mathbf{B})(\xi) \prod_{j=1}^d \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}} \right) \frac{1}{\prod_{j=1}^d \langle \xi, v_j \rangle}.$$

Thus we need only the terms of  $\xi$ -degree at most  $k_0$  in the Taylor expansion of the holomorphic function  $S(s + \mathbf{B})(\xi) \prod_{j=1}^d \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}}$ , and finally we need only the terms of  $\xi$ -degree at most  $k_0$  in the Taylor expansion of  $S(s + \mathbf{B})(\xi)$ . Applying (8) to each term  $e^{\langle \xi, x \rangle} = e^{\xi_1 x_1 + \dots + \xi_d x_d}$  of the finite sum  $S(s + \mathbf{B})(\xi)$ , we have

$$S(s + \mathbf{B})(\xi) \equiv \sum_{|I| \leq k_0} \lambda(I) \sum_{x \in (s + \mathbf{B}) \cap \Lambda} e^{\langle \xi, x_I \rangle} \text{ mod terms of } \xi\text{-degree } \geq k_0 + 1.$$

For each  $I$ , the term  $\sum_{x \in (s + \mathbf{B}) \cap \Lambda} e^{\langle \xi, x_I \rangle}$  is the sum, over  $x \in (s + \mathbf{B}) \cap \Lambda$ , of a function of  $x$  which depends only on the projection  $x_I$  of  $x$  in the decomposition  $x = x_I + x_{I^c} \in L^I \oplus L^{I^c}$ . When  $x$  runs over  $(s + \mathbf{B}) \cap \Lambda$ , its projection  $x_I$  runs over  $(s_I + \mathbf{B}_I) \cap \Lambda_I$ . Let us show that the fibers have the same number of points, equal to  $\operatorname{vol}(\mathbf{B}_{I^c})$ . For a given  $x_I \in (s_I + \mathbf{B}_I) \cap \Lambda_I$ , let us compute the fiber

$$\{y \in (s + \mathbf{B}) \cap \Lambda; y_I = x_I\}.$$

Fix a point  $x_I + x_{I^c}$  in this fiber. Then  $y = x_I + y_{I^c}$  lies in the fiber if and only if  $y_{I^c} - x_{I^c} \in (s_{I^c} - x_{I^c} + \mathbf{B}_{I^c}) \cap \Lambda$ . By Lemma 6(ii), the cardinal of the fiber is equal to  $\operatorname{vol}(\mathbf{B}_{I^c})$ . Thus, we have obtained (9).

Next we write the sum  $S(s_I + \mathbf{c}_I)(\xi)$  over the projected cone  $s_I + \mathbf{c}_I$  in terms of the sum over the projected cell  $s_I + \mathbf{B}_I$ . We obtain

$$\begin{aligned} S(s + \mathbf{c})(\xi) &\equiv \sum_{|I| \leq k_0} \lambda(I) \operatorname{vol}(\mathbf{B}_{I^c}) S(s_I + \mathbf{c}_I)(\xi) \prod_{j \in I^c} \frac{1}{(1 - e^{\langle \xi, v_j \rangle})} \\ &\equiv \sum_{|I| \leq k_0} \lambda(I) \operatorname{vol}(\mathbf{B}_{I^c}) S(s_I + \mathbf{c}_I)(\xi) (-1)^{d-|I|} \prod_{j \in I^c} \frac{1}{\langle \xi, v_j \rangle} \end{aligned}$$

*mod terms of  $\xi$ -degree  $\geq -d + k_0 + 1$ .*

□

Next we compute an explicit  $(d, k_0)$ -patchfunction.

**Lemma 17.** *If  $I \subseteq \{1, \dots, d\}$  has cardinal  $|I| \leq k_0$ , let*

$$\lambda_{d, k_0}(I) = (-1)^{k_0 - |I|} \binom{d - |I| - 1}{d - k_0 - 1}.$$

*Then  $\lambda_{d, k_0}$  satisfies Condition (8).*

*Proof.* The trick is to write  $e^u = 1 + t(e^u - 1)|_{t=1}$ . Thus

$$e^{u_1 + \dots + u_d} = \prod_1^d e^{u_i} = \prod_1^d (1 + t(e^{u_i} - 1))|_{t=1}$$

Let us consider  $P(t) := \prod_1^d (1 + t(e^{u_i} - 1)) = \sum_{q=0}^d C_q(u) t^q$  as a polynomial in the indeterminate  $t$ . As  $e^{u_i} - 1$  is a sum of terms of  $u_i$ -degree  $> 0$ , we have

$$(11) \quad e^{u_1 + \dots + u_d} \equiv \sum_{q=0}^{k_0} C_q(u) \pmod{\text{terms of } u\text{-degree} \geq k_0 + 1}.$$

In order to compute the coefficient  $C_q(u)$ , we write

$$P(t) = \prod_1^d (1 + t(e^{u_i} - 1)) = \prod_1^d ((1 - t) + te^{u_i}).$$

By expanding the product, we obtain

$$C_q(u) = \sum_{|I| \leq q} (-1)^{q - |I|} \binom{d - |I|}{q - |I|} e^{u_I}.$$

Summing up these coefficients for  $0 \leq q \leq k_0$ , we obtain

$$\sum_{q=0}^{k_0} C_q(u) = \sum_{|I| \leq k_0} \left( \sum_{q=|I|}^{k_0} (-1)^{q - |I|} \binom{d - |I|}{q - |I|} \right) e^{u_I}.$$

For  $m_0 \leq d_0$ , let us denote

$$F(m_0, d_0) = \sum_{j=0}^{m_0} (-1)^j \binom{d_0}{j}.$$

Thus

$$\sum_{q=0}^{k_0} C_q(u) = \sum_{|I| \leq k_0} F(k_0 - |I|, d - |I|) e^{u_I}.$$

The sum  $F(m_0, d_0)$  is easy to compute by induction on  $m_0$ , using the recursion relation

$$\binom{d_0}{j} = \binom{d_0 - 1}{j} + \binom{d_0 - 1}{j - 1}.$$

We obtain

$$F(m_0, d_0) = (-1)^{m_0} \binom{d_0 - 1}{m_0}.$$

Hence,

$$F(k_0 - |I|, d - |I|) = (-1)^{k_0 - |I|} \binom{d - |I| - 1}{k_0 - |I|} = (-1)^{k_0 - |I|} \binom{d - |I| - 1}{d - k_0 - 1}.$$

The claim follows now from Equation (11).  $\square$

**Remark 18.** *As promised, the main feature of Formula (10) is that the right-hand-side  $T(s, \mathbf{c}, k_0, \lambda)$  involves discrete summations in dimension  $|I| \leq k_0$  only.*

Theorem 16 can be reformulated in terms of the intermediate valuations introduced by Barvinok in [4]. The reformulation relies on the next lemma, which shows that the  $(d, k_0)$ -patchfunction condition is equivalent to a Moebius-type condition for the function  $I \mapsto \lambda(I)$ .

**Lemma 19.** *Let  $0 \leq k_0 \leq d$  be two integers. Let  $\lambda$  be a function on the set of subsets  $I \subseteq \{1, \dots, d\}$  of cardinal  $|I| \leq k_0$ . The following conditions are equivalent.*

(i) *For every  $I_0$  of cardinal  $|I_0| \leq k_0$ ,*

$$\sum_{\{I; |I| \leq k_0, I_0 \subseteq I\}} \lambda(I) = 1.$$

(ii) *For every integer  $k$  such that  $0 \leq k \leq k_0$ , we have the equality of polynomials*

$$(12) \quad (u_1 + \dots + u_d)^k = \sum_{|I| \leq k_0} \lambda(I) u_I^k$$

(iii) *The function  $\lambda$  is a  $(d, k_0)$ -patchfunction.*

*Proof.* Conditions (ii) and (iii) are clearly equivalent. Let us show that (i) and (ii) are equivalent. We expand  $(u_1 + \dots + u_d)^k$  into a sum of monomials. A monomial of degree  $k$  can involve at most  $k$  variables  $u_i$ , with  $k \leq k_0$ . Therefore we obtain

$$(13) \quad \frac{1}{k!} (u_1 + \dots + u_d)^k = \sum_{|I| \leq k_0} \sum_{\substack{(k_i)_{i \in I} \\ \sum k_i = k}} \prod_{i \in I} \frac{u_i^{k_i}}{k_i!}.$$

We expand similarly each term in the right-hand side of (12). A given monomial  $\prod_{i \in I_0} \frac{u_i^{k_i}}{k_i!}$ , with  $k_i \neq 0$  for all  $i \in I_0$ , occurs in the right-hand side of (12) exactly for the subsets  $I$  such that  $I_0 \subseteq I$ . Thus (i) implies (ii). Conversely, Equation (12) for  $k = k_0$  implies (i).  $\square$

## 5. COMPUTATION OF EHRHART QUASI-POLYNOMIALS

We now apply the approximation of the generating functions of the cones at vertices to the computation of the highest coefficients for a weighted Ehrhart polynomial, when the weight is a power of a linear form, as we explained in section 3.

**Corollary 20.** *Let  $\mathfrak{p}$  be a simple lattice polytope. Fix  $\xi \in \mathbb{R}^d$  and  $M \in \mathbb{N}$ . Let  $E_m(\mathfrak{p}, \xi, M)$ ,  $m = 0, \dots, d + M$ , be the coefficients of the weighted Ehrhart polynomial*

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M) n^m.$$

Fix  $0 \leq k_0 \leq d$ . Let  $\lambda$  be a  $(d, k_0)$ -patchfunction. Then, for  $k = 0, \dots, k_0$ , the Ehrhart coefficient  $E_{M+d-k}(\mathfrak{p}, \xi, M)$  is given by the following formula.

$$(14) \quad E_{M+d-k}(\mathfrak{p}, \xi, M) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} T(0, \mathbf{c}_s, k_0, \lambda)_{[-d+k]}(\xi).$$

In the general case, when the vertices are not assumed to be lattice points, we state the result without going through the details of the computation.

**Theorem 21.** *Let  $\mathfrak{p}$  be a simple polytope. For each vertex  $s$  of  $\mathfrak{p}$ , let  $q_s \in \mathbb{N}$  be the smallest integer such that  $q_s s \in \Lambda$ . For  $n \in \mathbb{N}$ , let  $n_s$  be the residue of  $n$  mod  $q_s$ , so that  $0 \leq n_s \leq q_s - 1$ . Fix  $\xi \in V^*$  and  $M$  a nonnegative integer. Fix  $0 \leq k_0 \leq d$ . Let  $\lambda$  be a  $(d, k_0)$ -patchfunction.*

*Then the Ehrhart quasi-polynomial*

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!}$$

*coincides in degree  $\geq M + d - k_0$  with the following quasi-polynomial*

$$(15) \quad \sum_{k=0}^{k_0} \sum_{s \in \mathcal{V}(\mathfrak{p})} (n - n_s)^{M+d-k} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} T_{[-d+k]}(n_s s, \mathbf{c}_s, k_0, \lambda)(\xi).$$

Observe that (15) is clearly a quasi-polynomial in  $n$  with coefficients which depend only on the residues  $(n \bmod q_s) = n_s, s \in \mathcal{V}(\mathfrak{p})$ .

**Remark 22.** *In practice, we first reduce the vertices  $s \bmod \Lambda$  by using*

$$S(s + \mathbf{c}_s)(\xi) = e^{\langle \xi, v \rangle} S(s - v + \mathbf{c}_s)(\xi), \text{ for } v \in \Lambda.$$

*Then we write an approximation similar to (15).*

## 6. CONCLUSION

Let  $\mathfrak{p} \subset \mathbb{R}^d$  be a rational simplex. Let  $\langle \xi, x \rangle$  be a rational linear form on  $\mathbb{R}^d$ , and consider a power  $\langle \xi, x \rangle^M$ . Let  $E_m(\mathfrak{p}, \xi, M), m = 0, \dots, d + M$ , be the coefficients of the weighted Ehrhart quasi-polynomial

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M) n^m.$$

Fix an integer  $k_0, 0 \leq k_0 \leq d$ . The main consequence of this study is a method for efficiently computing the  $k_0 + 1$  highest degree coefficients  $E_m(\mathfrak{p}, \xi, M)$ , for  $m = M + d, \dots, M + d - k_0$ . The method relies on expanding (15) in Theorem 21 as a power series in  $\xi$ .

Furthermore, one can write any homogeneous polynomial weight  $h(x)$  as a linear combination of powers of linear forms,

$$h(x) = \sum_k c_k \langle \xi_k, x \rangle^M.$$

In a forthcoming article by the authors of [1], we will show how to derive

- first, an algorithm for computing  $E_m(\mathfrak{p}, \xi, M)$ , for  $m = M + d, \dots, M + d - k_0$ . Hopefully this algorithm is polynomial, when the input consists of the dimension  $d$  and the degree  $M$ , the rational simplex  $\mathfrak{p} \subset \mathbb{R}^d$ , the rational linear form  $\xi$  on  $\mathbb{R}^d$ , provided  $k_0$  is fixed;
- second, an algorithm for computing the  $k_0 + 1$  highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex. Hopefully this algorithm is polynomial when  $k_0$  and the degree of the weight  $h(x)$  are fixed, but the dimension of the simplex is part of the input.

[1] dealt with the case of the highest Ehrhart coefficient which is just the integral of the weight over the simplex.

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