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RELIABLE ROBUST PATH PLANNING

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This paper is devoted to path planning when the safety of the system considered has to be guaranteed in the presence of bounded uncertainty affecting its model. A new path planner addresses this problem by combining Rapidly-exploring Random Trees (RRT) and a set representation of uncertain states. An idealized algorithm is presented first, before a description of one of its possible implementations, where compact sets are wrapped into boxes. The resulting path planner is then used for nonholonomic path planning in robotics.

Keywords: Dynamical systems, Interval analysis, Path planning, Robust control.

1. Introduction

Consider a system described by a continuous-time state-space model. Designing some control input to drive this system from a possibly uncertain initial state to a desired final state is a well-known robust control problem (Ackerman, Barlett, Kaesbauer, Sienel and Steinhauser, 1993; Francis and Khargonekar, 1995). This problem is made more complicated when constraints on the control input and on the evolution of the state have also to be satisfied. To solve it, a model of the system is usually assumed to be available, where noise variables account for the fact that this model is only an approximation of reality. The control input has then to be chosen in such a way that the system reaches the desired final state, despite uncertainty on the initial state and the presence of noise, *i.e.*, the control input has to be *robust* to any type of uncertainty.

This paper focuses on applications in robotics, where the robust control problem becomes a reliable path-planning problem (Latombe, 1991). Consider, for example, a vehicle moving in a two-dimensionnal structured environment. This vehicle should be driven from an initial state or *configuration* (position and orientation of the vehicle with respect to a frame attached to the environment) to a final desired configuration, despite the presence of uncertainty related to the model of the vehicle, to imperfect embedded sensors, to approximately charted obstacles, *etc*. The control input and the corresponding paths (succession of states) achieving this goal without collision are said to be safe or *reliable*.

Path planners involving Rapidly-exploring Random Trees (RRT) (LaValle, 1998; LaValle and Kuffner, 2001b; LaValle and Kuffner, 2001a) represent the state-of-art in random search. They allow an efficient exploration of the configuration space but, to the best of our knowledge, do not provide any robustness to model uncertainty. When taken into account, configuration uncertainty is usually described probabilistically, *e.g.*, by a multivariate Gaussian probability density function (Lambert and Gruyer, 2003; Pepy and Lambert, 2006; Gonzalez and Stentz, 2005). The main drawback of path planners based on this description is that the reliability of the path obtained may be guaranteed at best up to a given confidence level.

To facilitate path planning in the presence of uncertainty, information allowing the vehicle to localize itself is sometimes assumed to be available. In (Lazanas and Latombe, 1995; Gonzalez and Stentz, 2004; Gonzalez and Stentz, 2007; Bouilly, Simeon and Alami, 1995; Fraichard and Mermond, 1998), for example, relocalization zones, in which the configurations become perfectly or at least much more accurately known are considered. This technique is rather efficient, but requires the preparation of these relocalization zones. In (Lambert and Gruyer, 2003; Pepy and Lambert, 2006), a complex model of exteroceptive sensors (sonars) and an extended Kalman filter are used. To provide distance measurements during path planning, sonars are simulated assuming that the vehicle is located at the mean of the multivariate Gaussian function that characterizes the location uncertainty. The resulting simulated measurements are then used to reduce uncer-

tainty. If this technique facilitates the obtention of a path, it of course does not allow any statement about the reliability of this path.

This paper presents a first conceptual reliable robust path planner, assuming that all uncertain quantities are *bounded* with *known bounds*. At each time instant, uncertain configurations are represented by possibly non-connected sets. The proposed path planner takes advantage of the ability of RRTs to explore the whole configuration space efficiently. Starting from some uncertain initial configuration (represented by a set), the planner aims at driving the vehicle to a final configuration *set* (it will not be possible to drive it accurately to a point final configuration). Provided that the assumptions on the error bounds are not violated, if a robust path is found using this new path planner, its reliability will be *guaranteed*.

This paper is organized as follows. In Section 2, the two types of robust path planning problems to be addressed are presented. The principle of path planners based on RRTs is described in Section 3. Section 4 provides a conceptual extension of these planners to sets, and Box-RRT, one of its implementable counterparts where these sets are represented by boxes (or interval vectors). Section 5 applies Box-RRT to path planning for non-holonomic vehicles. Examples of path planning tasks for a vehicle are given in Section 6, before drawing some conclusions.

2. Reliable robust path planning

Consider a system, the evolution of which is described by the continuous-time state equation

$$\frac{ds(t)}{dt} = f(s(t), u(t), w(t)), \tag{1}$$

where $s(t) \in \mathbb{S} \subset \mathbb{R}^n$ is the state of the system, u is some bounded input function with values in $[u]$ and w is some random bounded state perturbation function remaining in $[w]$. It is assumed that u belongs to $\mathcal{U}_{[u]}^{\Delta t}$, the set of piecewise-constant bounded functions over intervals of the form $[k\Delta t, (k+1)\Delta t[$, with $\Delta t > 0$ and $k \in \mathbb{N}$, and that w belongs to $\mathcal{W}_{[w]}$, the set of functions bounded in $[w]$. For all $t \in [k\Delta T, (k+1)\Delta T[$, $u \in \mathcal{U}_{[u]}^{\Delta t}$, and $w \in \mathcal{W}_{[w]}$, $g(s,t) = f(s, u(t), w(t))$ is assumed ℓ-Lipschitz over $\mathbb{S}$.

The state-space $\mathbb{S}$ is partitionned into $\mathbb{S}_{\text{free}}$, to which the state of the system is allowed to belong, and $\mathbb{S}_{\text{obs}} = \mathbb{S} \setminus \mathbb{S}_{\text{free}}$, to which it is not. $\mathbb{S}_{\text{obs}}$ results of constraints imposed on the system, *e.g.*, by its environment.

At time $t = 0$, $s(0)$ is assumed to belong to some known set $\mathbb{S}(0) = \mathbb{S}_{\text{init}} \subset \mathbb{S}_{\text{free}}$. The system has to be driven to a given set of goal states $\mathbb{S}_{\text{goal}} \subset \mathbb{S}_{\text{free}}$. The aim of *robust* path planning is then to design an input function $u \in \mathcal{U}_{[u]}^{\Delta t}$ such that the system reaches $\mathbb{S}_{\text{goal}}$, without entering $\mathbb{S}_{\text{obs}}$ at any time instant, whatever the initial state $s \in \mathbb{S}_{\text{init}}$ and the noise function $w \in \mathcal{W}_{[w]}$. A planned path is *reliable* when a given function $u \in \mathcal{U}_{[u]}^{\Delta t}$ can be *proved* robustly to drive the system from any $s \in \mathbb{S}_{\text{init}}$ to a final state in $\mathbb{S}_{\text{goal}}$.

As will be seen below, there may be several formulations of this robust path planning problem.

2.1. Problem 1: Path planning. A first formulation of the robust path planning problem amounts to determining whether

$$\begin{aligned} &\exists K > 0 \text{ and } \exists u \in \mathcal{U}_{[u]}^{\Delta t} \text{ such that} \\ &\forall s \in \mathbb{S}_{\text{init}} \text{ and } \forall w \in \mathcal{W}_{[w]},\ s(K\Delta t) \in \mathbb{S}_{\text{goal}} \text{ and} \\ &\forall t \in [0, K\Delta t],\ s(t) \in \mathbb{S}_{\text{free}}, \end{aligned} \tag{2}$$

where $s(t)$ is the solution of (1).

In (2), the *same* sequence of inputs has to drive the system robustly from its imprecisely known initial state to a final state belonging to $\mathbb{S}_{\text{goal}}$. When the initial uncertainty on the state, or the state perturbation is too large, or when $\mathbb{S}_{\text{free}}$ has a complex structure and the distance between $\mathbb{S}_{\text{init}}$ and $\mathbb{S}_{\text{goal}}$ is too long, it may become quite difficult to find such a sequence of inputs. It may then be convenient to relax Problem 1 into Problem 2, presented in the next section.

2.2. Problem 2: Reachability analysis. Even if a solution to (2) exists, actual control inputs are usually not applied in open loop. Instead, an observer is used to estimate the state evolution using measurements provided by sensors, see, *e.g.*, (Luenberger, 1966). With this improved knowledge, it may be very useful to update path planning from time to time. In such a context, determining whether there exists a unique sequence of inputs that drives the system to $\mathbb{S}_{\text{goal}}$ whatever the initial state in $\mathbb{S}_{\text{init}}$ is too stringent. It suffices to know whether for any initial state $s \in \mathbb{S}_{\text{init}}$, there exists a sequence of inputs that drives the system from s to $\mathbb{S}_{\text{goal}}$. This is typically a *reachability* problem: one has to determine whether $\mathbb{S}_{\text{goal}}$ is reachable from any state in $\mathbb{S}_{\text{init}}$ and for any $w \in \mathcal{W}_{[w]}$.

Formally, one has to determine whether

$$\begin{aligned} &\forall s \in \mathbb{S}_{\text{init}}, \exists K > 0 \text{ and } \exists u \in \mathcal{U}_{[u]}^{\Delta t} \text{ such that} \\ &\forall w \in \mathcal{W}_{[w]},\ s(K\Delta t) \in \mathbb{S}_{\text{goal}} \text{ and} \\ &\forall t \in [0, K\Delta t],\ s(t) \in \mathbb{S}_{\text{free}}, \end{aligned} \tag{3}$$

where $s(t)$ is again the solution of (1).

3. Rapidly-exploring Random Trees (RRT)

As for several non-reliable path planning algorithms, the RRT algorithm will be the corner-stone of the proposed reliable and robust path planner. The structure and properties of the RRT algorithm are thus now briefly recalled. In

Algorithm 1 RRT($s_{\text{init}} \in \mathbb{S}_{\text{free}}$, $\mathbb{S}_{\text{goal}} \subset \mathbb{S}_{\text{free}}$, $\Delta t \in \mathbb{R}^+$, $K \in \mathbb{N}$)

1: G.init(s_{init})
2: $i = 0$
3: **repeat**
4: $\quad s_{\text{rand}} \leftarrow$ random_vector($\mathbb{S}_{\text{free}}$)
5: $\quad s_{\text{new}} \leftarrow$ RRT_extend(G, s_{rand}, Δt)
6: **until** i++$> K$ **or** ($s_{\text{new}} \neq$**null and** $s_{\text{new}} \in \mathbb{S}_{\text{goal}}$)
7: **return** G

Algorithm 2 RRT_extend(G, s_{rand}, Δt)

1: $s_{\text{near}} \leftarrow$ nearest_neighbor(G, s_{rand})
2: $u \leftarrow$ select_input(s_{rand}, s_{near})
3: $s_{\text{new}} \leftarrow$ new_state(s_{near}, u, Δt)
4: **if** collision_free_path(s_{near},s_{new},u,Δt) **then**
5: $\quad G$.add_node(s_{new})
6: $\quad G$.add_edge(s_{near}, s_{new}, u)
7: $\quad$ **return** s_{new}
8: **end if**
9: **return null**

Algorithm 3 Set-RRT($\mathbb{S}_{\text{init}} \subseteq \mathbb{S}_{\text{free}}$, $\mathbb{S}_{\text{goal}} \subseteq \mathbb{S}_{\text{free}}$, $\Delta t \in \mathbb{R}^+$, $K \in \mathbb{N}$)

1: G.init($\mathbb{X}_{\text{init}}$)
2: $i \leftarrow 0$
3: **repeat**
4: $\quad \mathbb{S}_{\text{rand}} \leftarrow$ random_set($\mathbb{S}_{\text{free}}$)
5: $\quad \mathbb{S}_{\text{new}} \leftarrow$ Set-RRT_extend(G, $\mathbb{S}_{\text{rand}}$, Δt)
6: **until** i++$> K$ **or** ($\mathbb{S}_{\text{new}} \neq$**null and** $\mathbb{S}_{\text{new}} \subset \mathbb{S}_{\text{goal}}$)
7: **return** G

Algorithm 4 Set-RRT_extend(G, $\mathbb{S}_{\text{rand}}$, Δt)

1: $\mathbb{S}_{\text{near}} \leftarrow$ nearest_neighbor(G, $\mathbb{S}_{\text{rand}}$)
2: $u \leftarrow$ select_input($\mathbb{S}_{\text{rand}}$, $\mathbb{S}_{\text{near}}$)
3: $\mathbb{S}_{\text{new}} \leftarrow$ prediction($\mathbb{S}_{\text{near}}$, u, Δt)
4: **if** collision_free_path($\mathbb{S}_{\text{near}}$,$\mathbb{S}_{\text{new}}$,u,Δt) **then**
5: $\quad G$.add_guaranteed_node($\mathbb{S}_{\text{new}}$)
6: $\quad G$.add_guaranteed_edge($\mathbb{S}_{\text{near}}$, $\mathbb{S}_{\text{new}}$, u)
7: $\quad$ **return** $\mathbb{S}_{\text{new}}$
8: **end if**
9: **return null**

the remainder of this section, it is assumed that the initial state $s\,(0) = s_{\text{init}}$ is perfectly known, and that no perturbation affects the state equation (1).

3.1. Description. The RRT algorithm (Kuffner and LaValle, 2000; LaValle and Kuffner, 2001b) is an incremental method aimed at quickly exploring a given configuration space from a given starting configuration. Its is described in Algorithm 1. First, the tree G is initialised with a single node corresponding to s_{init}. Then, a state $s_{\text{rand}} \in \mathbb{S}_{\text{free}}$ is chosen at random. The `nearest_neighbor` function searches in the tree G for the node s_{near} that is the closest to s_{rand} according to some metric d. A control input $u \in [u]$ is then chosen (for instance at random). Integrating (1) over a time interval Δt with initial condition s_{near} and constant control input u results in a new state s_{new}. If it can be *proved* that all state values along the trajectory between s_{near} and s_{new} lie in $\mathbb{S}_{\text{free}}$, then the trajectory between s_{near} and s_{new} is *reliable* and s_{new} is added to G and connected to s_{near}. Otherwise, s_{new} is not added to G. A new random state is chosen to start the next iteration of the algorithm. A path is found when $s_{\text{new}} = s_{\text{goal}}$, or (more realistically) when $s_{\text{new}} \in \mathbb{S}_{\text{goal}}$.

Figure 1 illustrates the growth of the tree G with the number of iterations of the RRT algorithm when $\mathbb{S} = [0, 100]^2$, $\dot{s} = u$, with $s \in \mathbb{S} \subset \mathbb{R}^2$, $u \in [0, 1]^2$ and $\Delta t = 100$ ms.

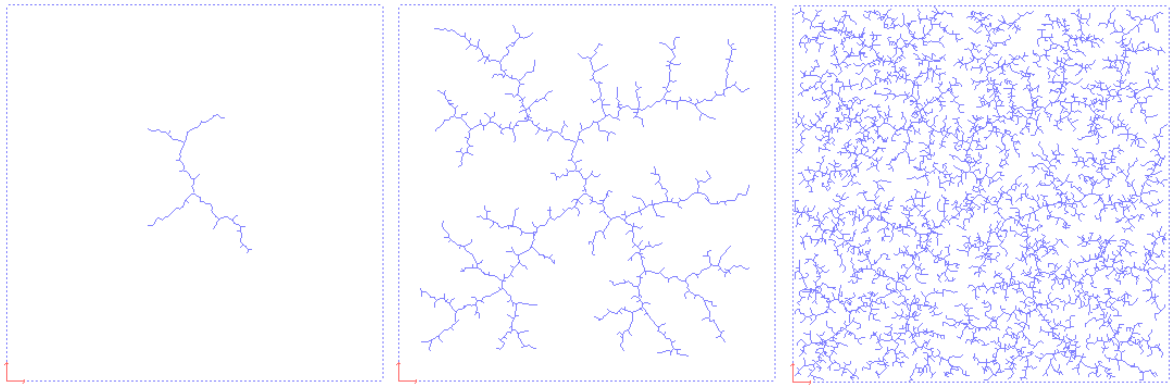
3.2. Improvements. Much attention has been dedicated to improving RRT. In (LaValle and Kuffner, 2001b), the generation of s_{rand} is modified by biasing the tree toward s_{goal}, which increases the planning speed for some specific $\mathbb{S}_{\text{free}}$. Instead of choosing s_{rand} in the whole $\mathbb{S}_{\text{free}}$, another option is to choose it with a probability $p > 0$ in a given subset $\mathbb{S}_{\text{rand}}$ of $\mathbb{S}_{\text{free}}$. When $\mathbb{S}_{\text{rand}} = \{s_{\text{goal}}\}$, one obtains the RRT-*Goalbias* algorithm and when $\mathbb{S}_{\text{rand}}$ is the circle centered on s_{goal} with a radius $\min_{s \in G} d(s, s_{\text{goal}})$, one gets the RRT-*GoalZoom* algorithm.

4. Set-RRT and Box-RRT

In order to cope with an uncertain initial configuration and bounded state pertubations, the classical RRT path planner has to be adapted to deal with sets. The first part of this section is devoted to the presentation of a new conceptual algorithm, before presenting one of its implementable counterparts.

4.1. Set-RRT. Set-RRT aims at generating a graph G consisting of nodes associated with *sets* in state space. The structure of Set-RRT is very close to that of the classical RRT algorithm, where nodes were associated with vectors. The main changes concern the metric required to evaluate distances between sets, the prediction function, which has to determine the evolution of uncertain states according to (1), and the collision test to determine whether all possible trajectories between two consecutive sets are reliable. The principle of Set-RRT is given in Algorithm 3.

At step 4, $\mathbb{S}_{\text{rand}}$ is most often chosen as a point vector, but making it a set allows replacement of $\mathbb{S}_{\text{rand}}$ by $\mathbb{S}_{\text{goal}}$ for the implementation of set variants of *Goalbias* and *GoalZoom*. Set-RRT stops when either the number of nodes



(a) 100 nodes (b) 600 nodes (c) 6000 nodes

Fig. 1. Growth of the tree built by the RRT algorithm

generated reaches its limit $\overline{K}$, or when the goal area is reached, *i.e.*, the tree includes a node associated with a set $\mathbb{S}_k$ such that $\mathbb{S}_k \subset \mathbb{S}_{\text{goal}}$.

4.2. Box-RRT. Dealing with general sets of $\mathbb{R}^n$ is very difficult, even for the simplest uncertain state equations. Wrappers (Jaulin, Kieffer, Didrit and Walter, 2001) guaranteed to contain the sets $\mathbb{S}_k$ have to be used to get an implementable counterpart to Set-RRT. Candidate wrappers are for example ellipsoids (Schweppe, 1973), zonotopes (Alamo, Bravo, Camacho and de Sevilla, 2003), interval vectors (Moore, 1979) or union of interval vectors (Kieffer, Jaulin and Walter, 2002; Kieffer, Jaulin, Braems and Walter, 2001). In what follows, interval vectors, or *boxes*, are used to represent uncertain states. These are quite simple sets, which may provide a very coarse description of complex-shaped sets. Using more accurate wrappers may increase the number of problems to which solutions may be found.

In what follows, a specialization of Algorithm 3 to boxes is called Box-RRT. In Box-RRT, the Hausdorff distance (Berger, 1987) between boxes may be used by the `nearest_neighbor` function. The box $[s_{\text{new}}] = [s_{k+1}]$ containing all possible state values at time $(k+1)\,\Delta t$ considering that the state is in $[s_{\text{near}}]$ at time $k\Delta t$ and that the input $u_k \in [u]$ is constant over $[k\Delta t, (k+1)\,\Delta t]$ must be computed while taking into account the bounded state perturbation. This may be performed by a set `prediction` function involving guaranteed numerical integration, as proposed, *e.g.*, in (Jaulin, 2002; Kieffer and Walter, 2003; Raissi, Ramdani and Candau, 2004; Kieffer and Walter, 2006). Finally, the set collision test that guarantees the reliability of every path between $[s_{\text{near}}]$ and $[s_{\text{new}}]$ implemented in `collision_free_path` requires to wrap all possible state trajectories between $[s_{\text{near}}]$ and $[s_{\text{new}}]$. This is again performed using guaranteed numerical integration. Note that wrapping may be so coarse that a path may not be deemed robustly reliable even if it actually is, see Figure 2. On the contrary, in situations such as that of Figure 3, the set of paths between $[s_{\text{near}}]$ and $[s_{\text{new}}]$ can be easily proved to be robustly reliable.

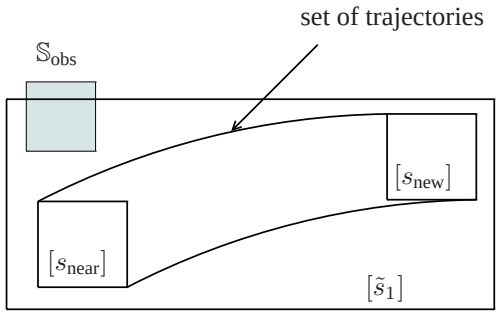


Fig. 2. The set of trajectories between $[s_{\text{near}}]$ and $[s_{\text{new}}]$, wrapped in $[\tilde{s}_1]$, is reliable, but this cannot be proved, since $[\tilde{s}_1]$ has a non-empty intersection with $\mathbb{S}_{\text{obs}}$.

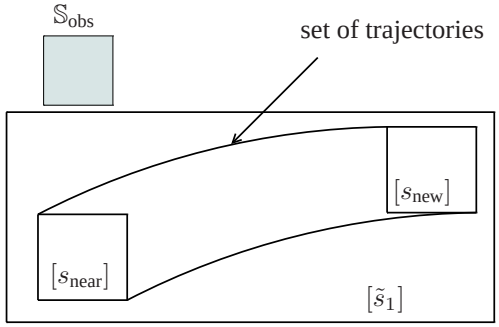


Fig. 3. The set of trajectories between $[s_{\text{near}}]$ and $[s_{\text{new}}]$, wrapped in $[\tilde{s}_1]$, is proved to be reliable, since $[\tilde{s}_1]$ has an empty intersection with $\mathbb{S}_{\text{obs}}$.

Figures 4(a) and 4(b) shows two paths planned for a system described by the two-dimensionnal uncertain state equation

$$\dot{s} = \frac{1}{1-w}u \tag{4}$$

where $s \in \mathbb{R}^2$, $w \in [-0.02, 0.02]$, $s_{\text{init}} \in [90, 90.1]^2$, $s_{\text{goal}} = [10, 20]^2$, $u \in [0, 1]^2$ and $\Delta t = 100$ ms. Figure 4(b) illustrates the performance of the *Goalbias* variant of the Box-RRT algorithm with $p = 0.1$.

These two first examples show the ability of Box-RRT to find a reliable path in a simple environment, accounting for uncertainty in the model of the system. Nevertheless, uncertainty is growing along the path, since no measurement is used to reduce it. The next section is devoted to a solution of Problem 2 described in Section 2.

4.3. Reach-RRT: Box-RRT and reachability analysis. In Box-RRT a unique series of constant control inputs over time intervals of width Δt is used to compute $[s_{\text{new}}]$ from $[s_{\text{near}}]$ satisfying (1). They are the same for all $s \in [s_{\text{near}}]$, which is natural for path planning, since the succession of values taken by the control input is important to actually drive the system from $[s_{\text{init}}]$ to $[s_{\text{goal}}]$.

If Box-RRT does not manage to find a unique input function $u \in \mathcal{U}_{[u]}^{\Delta t}$ to solve Problem 1, one may first try to split $[s_{\text{init}}]$ into subboxes and to apply Box-RRT on each of them. A solution to Problem 2 is then obtained, as the control input sequence is usually no longer the same for all $s \in [s_{\text{init}}]$. The main difficulty with this technique is that the number of boxes in which $[s_{\text{init}}]$ has to be split so that Box-RRT provides a solution for each of them may be difficult to determine *a priori*. Moreover, instead of getting a single tree, one obtains as many trees as subboxes in $[s_{\text{init}}]$.

We propose instead to generate a single tree, leading to a set of trajectories without branching leading from $[s_{\text{init}}]$ to $[s_{\text{goal}}]$. Between two consecutive boxes $[s_k]$ and $[s_{k+1}]$ of this set of trajectories, the control input may be adapted to each $s \in [s_k]$ to ensure that the system actually reaches $[s_{k+1}]$. This allows the size of $[s_{\text{new}}]$ to be reduced at each iteration with a simple modification of Box-RRT, entitled BoxReduction, executed just after Step 4 of the `extend` function of the Box-RRT algorithm. Algorithm 5 describes the proposed Reach-RRT algorithm.

Algorithm 5 Reach-RRT($[s_{\text{init}}] \subseteq \mathbb{S}_{\text{free}}$, $[s_{\text{goal}}] \subseteq \mathbb{S}_{\text{free}}$, $\Delta t \in \mathbb{R}^+$, $K \in \mathbb{N}$, $J \in \mathbb{N}$)

```
1: G.init([s_init])
2: i ← 0
3: repeat
4:    [s_rand] ← random_box(S_free)
5:    [s_new] ← Reach-RRT_extend(G, [s_rand], Δt, J)
6: until i++> K or ([s_new] ≠ ∅ and [s_new] ∈ [s_goal])
7: return G
```

In the `extend` function of Algorithm 4, assume that $[s_{\text{near}}]$ corresponds to time $k\Delta t$. After Step 4 of this function, one gets $[s_{\text{new}}]$ at $(k+1)\,\Delta t$ corresponding to the set of states consistent with $[s_{\text{near}}]$, the chosen control input u, and the noise $w \in \mathcal{W}_{[w]}$. The aim is to find some $[s'_{\text{new}}] \subset [s_{\text{new}}]$ with minimum width, such that $\forall s \in [s_{\text{near}}]$, $\exists u_k \in [u]$ satisfying

$$\forall w \in \mathcal{W}_{[w]}, s\left((k+1)\,\Delta t\right) \in [s'_{\text{new}}] \text{ and}$$
$$\forall t \in [k\Delta t, (k+1)\,\Delta t], s(t) \in \mathbb{S}_{\text{free}}. \quad (5)$$

This problem may be quite difficult to solve. The following sub-optimal algorithm aims only at finding a box $[s_{\text{red}}]$ that is smaller than $[s_{\text{new}}]$ and satisfies (5). It is inspired from (Jaulin and Walter, 1996).

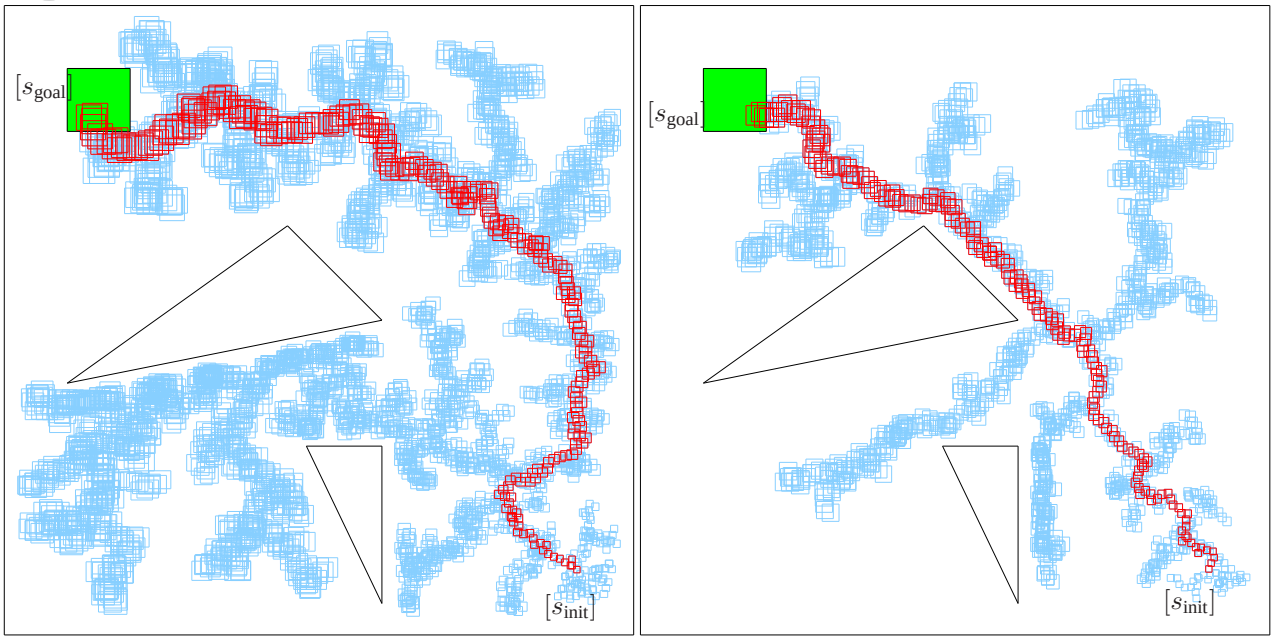
First, a box $[s_{\text{red}}] \subset [s_{\text{new}}]$ is chosen such that mid$\{[s_{\text{red}}]\}$ =mid$\{[s_{\text{new}}]\}$ and rad$\{[s_{\text{red}}]\}$ = $(1-\varepsilon)\cdot$rad$\{[s_{\text{new}}]\}$ with $\varepsilon \in]0, 1[$, see Figure 5. In this figure, $\phi([s], u_k, k\Delta t)$ represents a box containing the set of all solutions of (1) evaluated at time $(k+1)\,\Delta t$, obtained for an initial state $s \in [s]$ at $k\Delta t$, with a constant control input u_k. Then $[s_{\text{near}}]$ is split into J subboxes $[s_{\text{near}}]_j$, $j = 1 \ldots J$. For each $[s_{\text{near}}]_j$, one tries to find a constant input $u^j \in [u]$ that robustly drives all states from $[s_{\text{near}}]_j$ to $[s_{\text{red}}]$ (see Figure 5). For that purpose, one starts from $[u]$. If mid$\{[u]\}$ robustly drives $[s_{\text{near}}]_j$ to $[s_{\text{red}}]$, u^j =mid$\{[u]\}$. Else, $[u]$ is bisected and the midpoints of the two resulting boxes are tested again. The bisection procedure is repeated until a control input is found or until the resulting subboxes are too small to be further

Algorithm 6 Reach-RRT_extend(G, $[s_{\text{rand}}]$, Δt, J)

```
1: [s_near] ← nearest_neighbor(G, [s_rand])
2: u ← select_input([s_rand], [s_near])
3: [s_new] ← prediction([s_near], u, Δt)
4: if collision_free_path([s_near],[s_new],u,Δt) then
5:    [s_new] ← box_reduction([s_near], [s_new], J)
6:    G.add_guaranteed_node([s_new])
7:    G.add_guaranteed_edge([s_near], [s_new], u)
8:    return [s_new]
9: end if
10: return ∅
```

Algorithm 7 box_reduction($[s_{\text{near}}]$, $[s_{\text{new}}]$, J)

```
1: [s_toreturn] ← [s_new]
2: S ← cut([s_near], J)   {S = {[s_near]_1, [s_near]_2, ..., [s_near]_J}}
3: repeat
4:    [s_red] ← reduce([s_toreturn])
5:    for all [s_near]_j ∈ S do
6:       isReduced ← find_input([s_near]_j, [s_red])
7:       if (isReduced == FAILURE) then
8:          return [s_toreturn]
9:       end if
10:   end for
11:   [s_toreturn] ← [s_red]
12: until TRUE
```



(a) Path planned using the Box-RRT algorithm (b) Path planned using the *Goalbias* variant of Box-RRT

Fig. 4. Paths planned using the Box-RRT algorithm

Algorithm 8 find_input(IN: $[s]$, $[s_{\text{red}}]$, $[u]$, ε)

```
1:  𝔸 ← [u]
2:  while 𝔸 ≠ ∅ do
3:      [c] ← Pop(𝔸)
4:      [s_new] ← φ([s], mid([c]), k · Δt)
5:      if [s_new] ⊆ [s_reduced] and
        collision_free_path([s_near], [s_new], mid([c]),Δt)
        then
6:          return SUCCES
7:      else
8:          if w([c]_k)< ε then
9:              return FAILURE
10:         else
11:             {[c_left] , [c_right]} ← bissection([c])
12:             𝔸+ = [c_left]
13:             𝔸+ = [c_right]
14:         end if
15:     end if
16: end while
```

bisected. Algorithm 8 summarizes these operations. It has to be called for each subbox $[s_{\text{near}}]_j$ of $[s_{\text{near}}]$.

When control inputs satisfying (5) have been found for each $[s_{\text{near}}]_j$, one may try to reduce $[s_{\text{red}}]$ further.

5. Application in robotic

The proposed Box-RRT algorithm is now applied to path planning for nonholonomic vehicles in a structured 2D environment, where obstacles are described by polygons. One of the difficulties of path planning in this context is the characterization of $\mathbb{S}_{\text{free}}$, which may be quite complex. In (Jaulin, 2001), $\mathbb{S}_{\text{free}}$ is characterized first or constructed iteratively. Here, $\mathbb{S}_{\text{free}}$ is not explicitely determined: only the constraints of the environment are used to determine whether a set of paths is reliable. Apart from the model of the vehicle considered here, this section provides a description of a collision test to determine whether a set of paths between two consecutive sets of states is reliable.

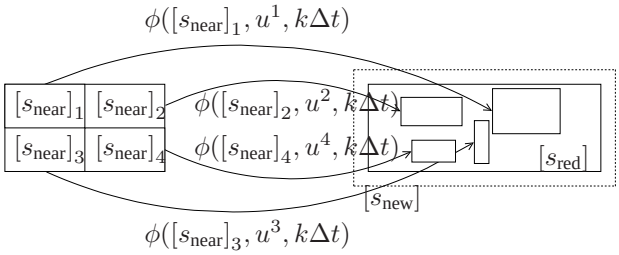


Fig. 5. $[s_{\text{near}}]$ is split into subboxes and for each subbox $[s_{\text{near}}]_j$, an input u^j is computed such that $\phi([s_{\text{near}}]_j, u^j, k\Delta t) \subseteq [s_{\text{red}}]$

5.1. Model of the vehicle. Various kinematic or dynamic models of vehicles (Pepy, Lambert and Mounier, 2006) could be used to test the Set-RRT path planner. Here, a model based on the classical *simple car* model (LaValle, 2006) evolving in a 2D environment is considered, see Figure 6. This model incorporates nonholo-

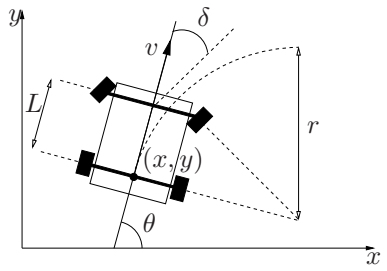


Fig. 6. The *simple car* model

nomic constraints and is given by

$$\begin{cases} \dot{x} = v(1+w_v)\cos\theta \\ \dot{y} = v(1+w_v)\sin\theta \\ \dot{\theta} = \frac{v(1+w_v)}{L}\tan\left(\delta(1+w_\delta)\right) \end{cases}, \tag{6}$$

where the state vector $s = (x, y, \theta)^{\mathrm{T}}$ specifies the position (x, y) and orientation θ of a frame $\mathcal{V}$ attached to the vehicle with respect to a world frame $\mathcal{W}$ attached to the environment. The control input vector is $u = (v, \delta)^{\mathrm{T}}$, with v the longitudinal speed and $\delta \in [-\delta_{\max}, \delta_{\max}]$ the steering angle. Here, u is assumed to belong to a set $\mathbb{U}$ with finite cardinality. L is the distance between the front and rear wheels. The noise components $w_v \in [-v_{\text{err}}, v_{\text{err}}]$ and $w_\delta \in [-\delta_{\text{err}}, \delta_{\text{err}}]$ account for the slipping of the vehicle and for the steering unprecision.

On the figures to follow, walls and obstacles to be avoided are represented by polygons.

5.2. Collision test. If $[s_{\text{init}}]$ and $[s_{\text{goal}}]$ are respectively the set of initial and final states, one has to show, before starting the path planner, that both sets of states belong to $\mathbb{S}_{\text{free}}$. In what follows, the collision tests for a box in the configuration space and for a set of paths between configuration boxes is described. These tests form the core of the `collision_free_path()` function of Algorithm 3.

5.2.1. Collision-free configuration. The projection of the shape of the vehicle onto the (x, y)-plane in $\mathcal{V}$ is wrapped in a convex polytope $\mathcal{C}$. Each vertex v_i, $i = 1 \ldots n_{\text{v}}$ of $\mathcal{C}$ is identified by its coordinates $(x_i^{\mathcal{V}}, y_i^{\mathcal{V}})$ in $\mathcal{V}'$, the projection onto the (x, y)-plane of $\mathcal{V}$. Assume that the state of the vehicle is $s = (x, y, \theta)^{\mathrm{T}}$ in $\mathcal{W}$. The boxes $\left([x_i^{\mathcal{W}}], [y_i^{\mathcal{W}}]\right)$ containing the set of coordinates of the n_{v} vertices of the polytope in $\mathcal{W}'$, the projection onto the (x, y)-plane of $\mathcal{W}$, are then

$$\begin{pmatrix} [x_i^{\mathcal{W}}] \\ [y_i^{\mathcal{W}}] \end{pmatrix} = \begin{pmatrix} [x] \\ [y] \end{pmatrix} + \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \begin{pmatrix} x_i^{\mathcal{V}} \\ y_i^{\mathcal{V}} \end{pmatrix}. \tag{7}$$

To determine the set containing all possible $\mathcal{C}$ in $\mathcal{W}$, one may build the convex envelope of $\left([x_i^{\mathcal{W}}], [y_i^{\mathcal{W}}]\right)$, $i = 1 \ldots n_{\text{v}}$. A polytope containing these n_{v} boxes is easily obtained by the Graham scan method (Graham, 1972) with time complexity $\mathrm{O}(n \log n)$.

Since this convex hull is an outer approximation of the union of all the possible locations of parts of the vehicle that are associated with a given configuration box, one may now test whether the vehicle is safely located. A collision may occur only if there exists a segment of the polygon that intersects a segment of the environment or when a segment of the environment is entirely included in the polygon.

Algorithm 9 collision_free_path($[s_{\text{near}}]$, $[s_{\text{new}}]$, u, Δt, Environment)

```
[s̃1] = [s_near] ⊔ [s_new]
while [s_near] + [0, Δt] f([s̃1], u) ⊄ [s̃1] do
    [s̃1] ← [s̃1] + ε [−1, 1]^{×3}
end while
if CollisionFreeConfiguration([s̃1], u, Δt, Environment) then
    return true;
else
    return false
end if
```

5.2.2. Collision-free path. The previous test is useful to determine whether $[s_{\text{init}}]$ and $[s_{\text{goal}}]$ are reliable. Now, one has to extend it to determine whether a collision may occur when the vehicle moves from $[s_{\text{near}}]$ to $[s_{\text{new}}]$. This is the aim of the `collision_free_path` function.

Guaranteed numerical integration (Moore, 1966; Lohner, 1987) has been used to obtain $[s_{\text{new}}]$ from $[s_{\text{near}}]$. To enclose the set of trajectories between $[s_{\text{new}}]$ and $[s_{\text{near}}]$, it suffices to find $[\tilde{s}_1]$ satisfying

$$[s_{\text{near}}] + [0, \Delta t] f([\tilde{s}_1], u) \subset [\tilde{s}_1]. \tag{8}$$

Then, the following holds true (Moore, 1966)

$$\forall s \in [s_0] \ \forall t \in [k\Delta t, (k+1)\Delta t], \ s(t) \in [\tilde{s}_1]. \tag{9}$$

The box $[\tilde{s}_1]$ is evaluated in the first step of guaranteed numerical integrators (Picard-Lindelöf iteration to prove the existence and uniqueness of solutions to ODEs). It is thus obtained as a byproduct of these integrators. Once $[\tilde{s}_1]$ is computed, it has to be tested for reliability with the same algorithm as for $[s_{\text{init}}]$ and $[s_{\text{goal}}]$. The collision test used with the Box-RRT algorithm is summed up in Algorithm 9.

When it is proved that no collision occurs between any two consecutive nodes of the tree, by induction one proves that the path between $[s_{\text{init}}]$ and $[s_{\text{goal}}]$ (if it exists in the tree) is robustly reliable.

6. Results

This section provides some results obtained with the Box-RRT algorithm considering the *simple car* model of Section 5.1. In all examples, $\Delta t = 100$ ms. Only projections of boxes onto the (x, y)-plane are represented to increase readability.

6.1. Successes. First, results obtained at low speed (lower than 1 m·s^{-1}) are presented; slipping is then negligible ($v_{\text{err}} = 0$), and it is also assumed that $\delta_{\text{err}} = 0$.

Figure 7(a) represents the solution of a simple path-planning problem using the Box-RRT algorithm. The width of each component of $[s_{\text{init}}]$ is 20 cm for the x and y components and 0.1 rad for θ. The box $[s_{\text{goal}}]$, with size 10 m$\times$10 m$\times 2\pi$ rad, has to be reached. The distance between the projections onto the (x, y)-plane of mid$\{[s_{\text{init}}]\}$ and mid$\{[s_{\text{goal}}]\}$ is around 100 m. About 30 000 nodes are built by Box-RRT to reach $[s_{\text{goal}}]$. In Figure 7(b) about 100 000 nodes are required to reach a smaller goal, with size 5 m$\times 5$ m$\times 2\pi$.

Other types of models could readily be used. For example, Figure 7(c) shows a path planned for a model only able to turn on the right. Dynamic vehicle models (Pepy *et al.*, 2006) or kinematic chains (Yakey, LaValle and Kavraki, 2001) could also be considered.

Harder problems may be solved, such as path planning in an environment with more obstacles, as in Figure 7(d). In this example, the size of $[s_{\text{goal}}]$ is 15m$\times$15m$\times 2\pi$ rad. Again, a guaranteed path between the beginning and the end of the labyrinth is found.

6.2. Challenges. In the previous section, between Figures 7(a) and 7(b), the size of $[s_{\text{goal}}]$ has been reduced, which made the problem harder to solve. If the size of $[s_{\text{goal}}]$ is reduced further, a path may no longer be found (see Figure 8(a)) even if it may still exist. Since only prediction is used, and considering the form of the dynamical equation describing the motion of simple car, the size of the box describing the uncertain state always grows up along the path. Thus, as soon as the size of $[s]$ at the end of a path becomes bigger than that of $[s_{\text{goal}}]$, there is no chance to reach $[s_{\text{goal}}]$ from this box.

The same problem appears when the skidding error is too large. This problem is illustrated in Figure 8(b), where the size of $[s_{\text{init}}]$ is 10 cm$\times 10$ cm$\times[1, 1.05]$ rad, the size of $[s_{\text{goal}}]$ is 10 m$\times$10 m$\times 2\pi$ rad, $v_{\text{err}} = 10^{-2}$ and $\delta_{\text{err}} = 10^{-3}$. Uncertainty then becomes exceedingly large and the vehicle no longer passes through the corridor. Thus, this problem cannot be solved using the present version of Box-RRT presented in Section 4.2, unless some exteroceptive measurements are used at some points along the path to reduce uncertainty.

6.3. Application of Reach-RRT. The same simulated conditions are considered as in Figure 8(b) of Section 6.2.

Results illustrated on Figures 9(a) and 9(b) show that the use of differentiated inputs allows the boxes to be reduced and guaranteed reachability to be proven.

In this example, adapting the input allows the size of the box $[s_{\text{new}}]$ at each iteration to be reduced by 17% in average. This rate is obtained at the price of splitting each $[s_{\text{near}}]$ in at least 64 subboxes, which increases significantly the computationnal load. Thus, the reduction step may be used with a period larger than Δt.

In Figure 9(a), box reduction is performed every second. The path planner find a path by generating about 10 000 nodes. Similarly, a path is found for the problem illustrated in Figure 9(b) with box reduction performed every two seconds on each path.

As mentionned earlier, with Reach-RRT, one proves that for each initial state, there exists a control input able to drive the system robustly to the goal area. It is then worth trying to divide the global path planning task into several local (short-term) planning tasks along the path obtained by Reach-RRT. This allows information provided by sensors to be taken into account, facilitating the task of Box-RRT.

7. Conclusions and perspectives

This paper has presented algorithms based on Rapidly-exploring Random Trees able to perform path planning tasks for models of systems including uncertainties. Uncertain quantities are assumed to belong to sets. A first conceptual Set-RRT path planner dealing with general sets has been presented, followed by an implementable Box-RRT dealing with boxes. The Box-RRT has also been adapted to perform reachability analysis.

Some algorithms presented in this paper are rather preliminary, but show the potential of the approach. For example, the choice of the control input in Box-RRT or Reach-RRT is not optimised yet. Better local (short-term) reachability analysis techniques could be used, see, *e.g.*, (Collins and Goldsztejn, 2008; Ramdani, Meslem and Candau, 2008).

In the present version of Set-RRT and Reach-RRT, $\mathbb{S}_{\text{free}}$ is assumed to be constant with time. One could easily adapt the proposed algorithms to $\mathbb{S}_{\text{free}}$ varying with time, to describe moving obstacles, to take into account the limited energy available to the system, *etc.*

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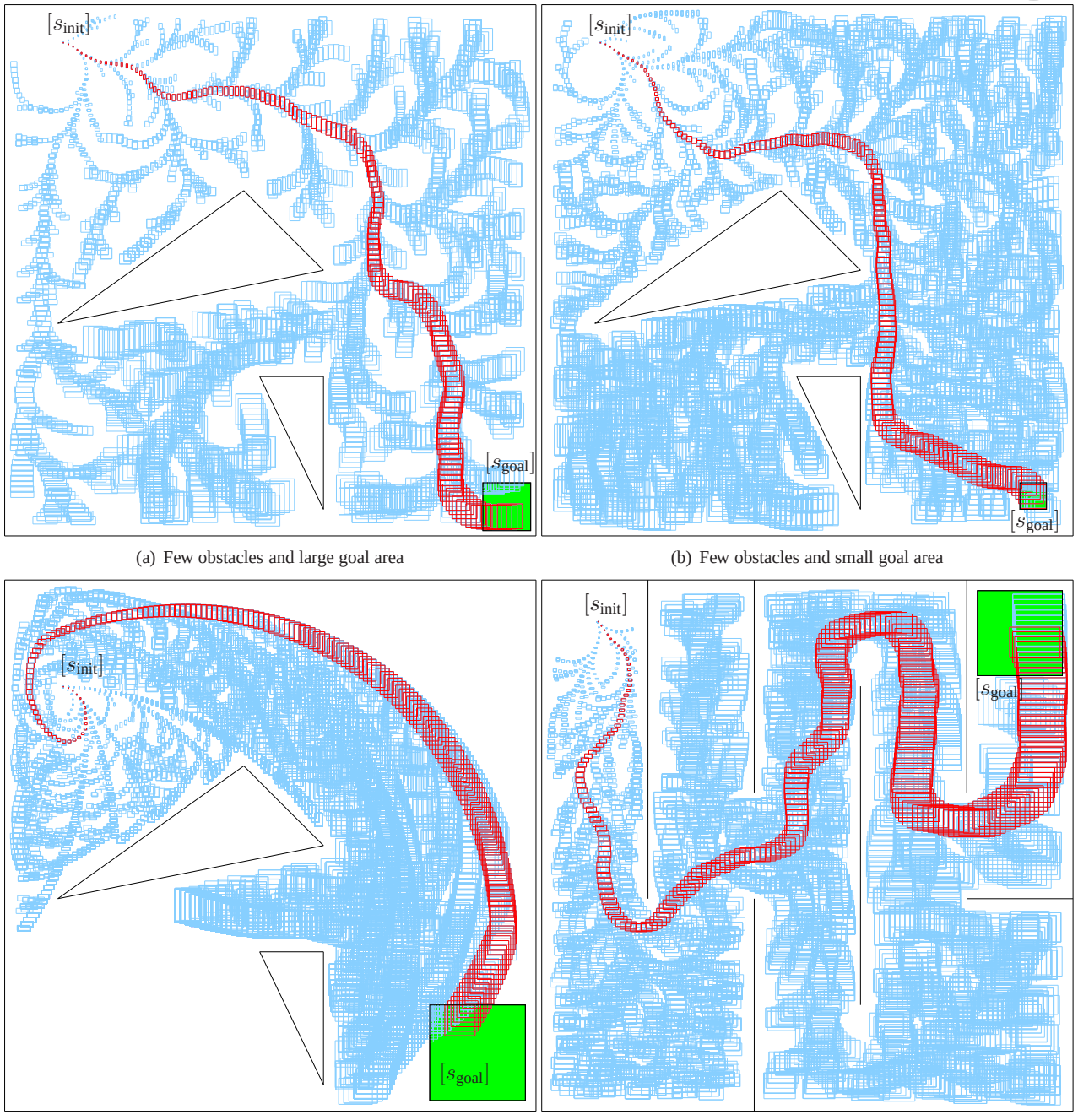

(a) Few obstacles and large goal area

(b) Few obstacles and small goal area

(c) A car which can only turn on right

(d) Labyrinthic environment

Fig. 7. Some path-planning problems solved by Box-RRT.

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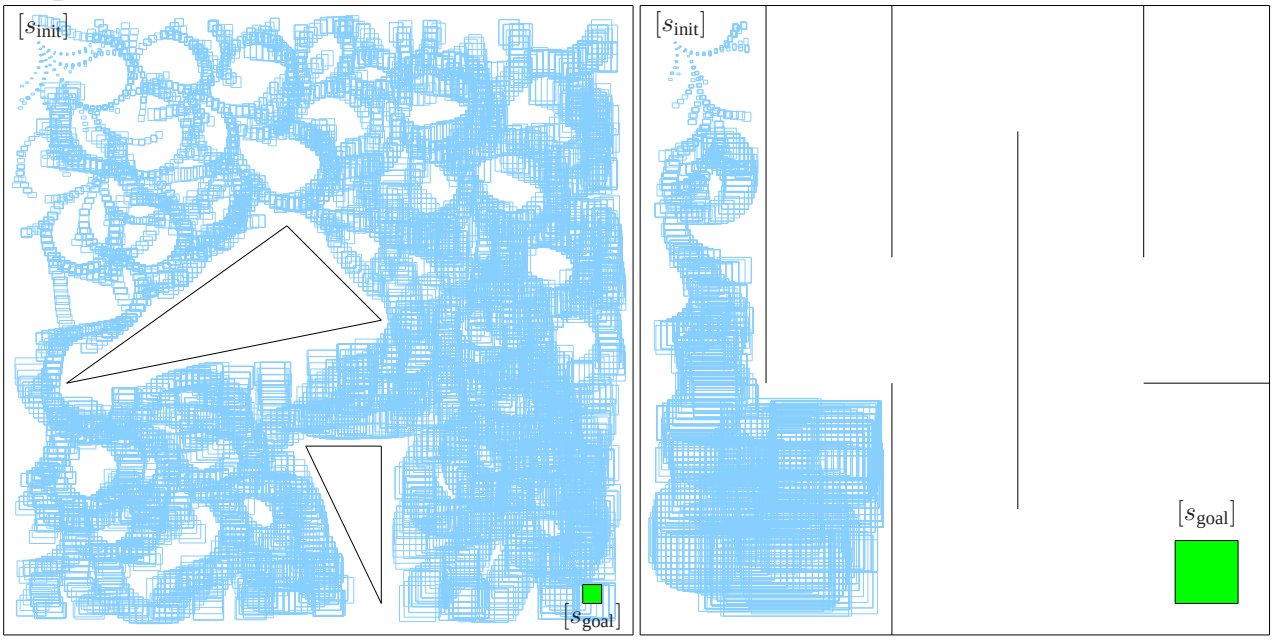
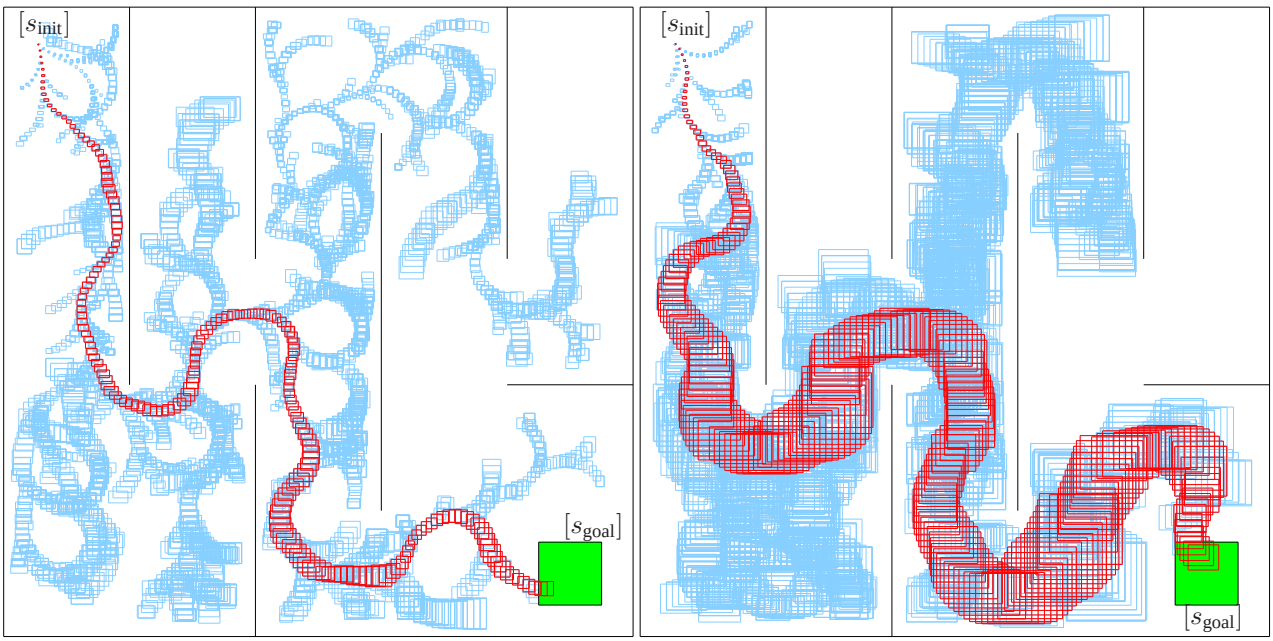


(a) Box-RRT fails to find a path (the goal area is too small)

(b) Box-RRT fails to find a path (skidding errors are too large)

Fig. 8. Some more difficult problems where Box-RRT fails to find a solution.



(a) Path found using box reduction every second

(b) Path found using box reduction every two seconds

Fig. 9. Results obtained using the FindInput algorithm

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