Geometric Interpretation of m-th Elliptic Integrable System

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Introduction

In this paper, we give a geometric interpretation of all the m-th elliptic integrable systems associated to a k'-symmetric space $N = G/G_0$ (in the sense of C.L. Terng [32]).

This system can be written as a zero curvature equation

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{C}^*,$$

where $\alpha_\lambda = \sum_{j=0}^{m} \lambda^{-j} u_j + \lambda^j \bar{u}_j = \sum_{j=-m}^{m} \lambda^j \hat{\alpha}_j$ is a 1-form on a Riemann surface L taking values in the Lie algebra $\mathfrak{g}$. The "coefficient" u_j is a $(1,0)$-type 1-form on L with values in the eigenspace $\mathfrak{g}_{-j}$ of the automorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}$ of order k' (defining the (locally) k'-symmetric space $N = G/G_0$) with respect to the eigenvalue $\omega_{k'}^{-j}$. We denote by $\omega_{k'}$ a k'-th primitive root of unity. Moreover, we call the integer m the *order* of the system.
First, we remark that any solution of the system of order m is a solution of the system of order m', if $m \leq m'$ (and the automorphism τ is fixed). In other words, the system of order m is a reduction of the system of order m', if $m \leq m'$. Moreover, it turns out that we have to introduce the integer $m_{k'}$ defined by

$$m_{k'} = \left[\frac{k'+1}{2}\right] = \begin{cases} k \text{ if } k' = 2k \\ k+1 \text{ if } k' = 2k+1 \end{cases} \qquad \text{if } k' > 1, \text{ and } m_1 = 0.$$

Then the general problem splits into three cases : the primitive case ($m < m_{k'}$), the determined case ($m_{k'} \leq m \leq k'-1$) and the underdetermined case ($m \geq k'$).

The primitive systems have an interpretation in terms of F-holomorphic maps, with respect to an f-struture F ($F^3 + F = 0$). More precisely:

• *In the even case* ($k' = 2k$), we have a fibration $G/G_0 \to G/H$ over a k-symmetric space $M = G/H$ (defined by the square of the automorphism τ of order k' defining $N = G/G_0$). We also have a G-invariant splitting $TN = \mathcal{H} \oplus \mathcal{V}$ corresponding to this fibration (i.e. a connection $\mathcal{H}$ on this fibration), and then N is naturally endowed with an f-structure F which defines a complex structure on the horizontal subbundle $\mathcal{H}$ and vanishes on the vertical subbundle

$\mathcal{V}$. Moreover the eigenspace decomposition of the order k' automorphism τ gives us some G-invariant decomposition $\mathcal{H} = \oplus_{j=1}^{k-1}[\mathfrak{m}_j]$, where $\mathfrak{m}_j \subset \mathfrak{g}$ is defined by $\mathfrak{m}_j = \mathfrak{g}_{-j} \oplus \mathfrak{g}_j$, and $[\mathfrak{m}_j] \subset TN$ the corresponding G-invariant subbundle. This allows to define, by multiplying F on the left by the projections on the subbundles $\mathcal{H}^m = \oplus_{j=1}^{m}[\mathfrak{m}_j]$, a family of f-structures $F^{[m]}, 1 \leq m \leq k-1$. Then the primitive system of order m ($m < m_{k'} = k$) associated to G/G_0 is exactly the equation for $F^{[m]}$-holomorphic maps. In particular any solution of a primitive system is F-holomorphic.

• *In the odd case* ($k' = 2k+1$), $N = G/G_0$ is naturally endowed with an almost complex structure $\underline{J}$. Then the solutions of the primitive systems are exactly the $\underline{J}$-holomorphic curves. Moreover, in the same way as for the even case, the eigenspace decomposition of τ provides a G-invariant decomposition $TN = \oplus_{j=1}^{k}[\mathfrak{m}_j]$, which allows to define a family of f-structures $\underline{J}^{[m]}, 1 \leq m \leq k$, with $\underline{J}^{[k]} = \underline{J}$. Then the primitive system of order m ($m < m_{k'} = k+1$) associated to G/G_0 is exactly the equation for $\underline{J}^{[m]}$-holomorphic maps. In other words, the solutions of the primitive system of order m are exactly the integral holomorphic curves of the complex Pfaff system $\oplus_{j=1}^{m}[\mathfrak{m}_j] \subset TN$ in the almost complex manifold $(N, \underline{J})$.

The determined case We call "the minimal determined system" the determined system of minimal order $m_{k'}$, and "the maximal determined system" the determined system of maximal order $k'-1$.

• First, the determined system has two model cases:

In the even case, this model is the first elliptic integrable system associated to a symmetric space ($m = 1, k' = 2$) which is - as it is well known - exacly the equation of harmonic maps from the Riemann surface L into the symmetric space under consideration. This is the "smallest" determined system, i.e. with lowest order of symmetry in the target space $N = G/G_0$. In this case -N is symmetric- the determined case is reduced to one system, the one of order 1. All the further determined systems associated to target spaces N with higher order of symmetry will be modeled on this system i.e. on harmonic maps into a symmetric space.

In the odd case, this model is the second elliptic integrable system associated to a 3-symmetric space. This is "smallest" determined system in the odd case, i.e. with lowest odd order of symmetry in the Target space $N = G/G_0$. We prove that this system is exactly the equation for holomorphically harmonic maps into the almost complex manifold $(N, \underline{J})$ with respect to the anticanonical connection $\nabla^1 = \nabla^0 + [\ ,\ ]_{[\mathfrak{m}]}$, where ∇^0 is the canonical connection. Or equivalently this is the equation for holomorphically harmonic maps into the almost complex manifold $(N, -\underline{J})$ with respect to the canonical connection ∇^0. First given a general almost complex manifold (N, J) with a connection ∇, we

define holomorphically harmonic maps $f\colon L \to N$ as the solution of the equation

$$\left[\bar{\partial}^{\nabla}\partial f\right]^{1,0} = 0 \tag{1}$$

where $[\ ]^{1,0}$ denotes the $(1,0)$-component according to the splitting $TN^{\mathbb{C}} = T^{1,0}N \oplus T^{0,1}N$ defined by J. This equation is equivalent to

$$d^{\nabla} df + J d^{\nabla} * df = 0$$

or, equivalently, using any Hermitian metric g on L

$$T_g(f) + J\tau_g(f) = 0$$

where $T_g(f) = *f^*T = f^*T(e_1, e_2)$, with (e_1, e_2) an orthonormal basis of TL, and $\tau_g(f) = *d^{\nabla} * df = \mathrm{Tr}_g(\nabla df)$ is the tension field of f. Of course Tr_g denotes the trace with respect to g, and the expression ∇df denotes the covariant derivative of df with respect to the connection induced in $T^*L \otimes f^*TN$ by ∇ and the Levi-Civita connection in L.
In particular, we see that if ∇ is torsion free or more generally if f is torsion free, i.e. $f^*T = 0$, then holomorphic harmonicity is equivalent to harmonicity. Therefore, this new notion is interesting only in the case of a non torsion free connection ∇.
We prove that we can also interpret the holomorphic harmonicity in terms of *J-twisted harmonic maps*. Let us define this notion. Let (E, J) be a complex vector bundle over an almost complex manifold (M, j_M). Then let ∇ be a connection on E. Then we can decompose it in an unique way in the form

$$\nabla = \nabla^0 + A$$

where $\nabla^0 J = 0$ and $A \in \mathcal{C}(T^*M \otimes \mathrm{End}(E))$, $AJ = -JA$. Then we set

$$\nabla^J = \nabla^0 + (A \circ j_M)J.$$

We say that a map $f\colon (L, j_L) \to (N, J, \nabla)$ is J-twisted harmonic if and only if

$$\mathrm{Tr}_g((f^*\nabla)^J df) = 0.$$

Now, let us suppose that ∇ is *almost complex*, i.e. $\nabla J = 0$. Then, according to equation (1), we see that any holomorphic curve $f\colon (L, j_L) \to (N, J)$ is anti-holomorphically harmonic, i.e. holomorphically harmonic with respect to $-J$. In particular, this allows to recover that a 1-primitive solution (i.e. of order $m = 1$) of the elliptic system associated to a 3-symmetric space is also solution of the second elliptic system associated to this space.

Moreover, the holomorphically harmonic maps admit a formulation very analogous to that of harmonic maps in term of the vanishing of some $\bar{\partial}\partial$-derivative, which implies a well kown caracterisation in term of holomorphic 1-forms. Indeed we prove that $f\colon (L, j_L) \to (N, J, \nabla)$ is holomorphically harmonic if and only if

$$\overline{\hat{\partial}}^{\widehat{\nabla}}\,\hat{\partial} f = 0, \tag{2}$$

i.e. $\hat{\partial} f$ is a holomorphic section of $T^*_{1,0}L \otimes_{\mathbb{C}} f^*TN$. Here the hat " $\hat{}$ " means that we extend a 1-form on TL, like d or ∇, by $\mathbb{C}$-linearity as a linear map from $TL^{\mathbb{C}}$ into the complex bundle (TN, J). In other words instead of extending these 1-forms as $\mathbb{C}$-linear maps from $TL^{\mathbb{C}}$ into $TN^{\mathbb{C}}$ as it is usual, we use the already existing structure of complex vector bundle in (TN, J) and extend these very naturally as $\mathbb{C}$-linear map from $TL^{\mathbb{C}}$ into the complex bundle (TN, J). Therefore we can conclude that holomorphically harmonic maps have the same formulation as harmonic maps with the difference that instead of working in the complex vector bundle $TN^{\mathbb{C}}$, we stay in TN which is already a complex vector bundle in which we work.

Finally, let us suppose that N is endowed with a ∇-parallel Hermitian metric h. Therefore (N, J, h) is an almost Hermitian manifold with a Hermitian connection ∇. Suppose also that J anticommutes with the torsion T of ∇ i.e.

$$T(X, JY) = -JT(X, Y)$$

which is equivalent to

$$T = \frac{1}{4} N_J \quad \text{and} \quad T(JX, Y) = T(X, JY)$$

where N_J denotes the torsion of J i.e its Nijenhuis tensor.
Suppose also that the torsion of ∇ is totally skew-symmetric i.e.

$$T^*(X, Y, Z) = \langle T(X, Y), Z \rangle$$

is a 3-form. Lastly, we suppose that the torsion is closed, i.e. $\nabla T^* = 0$ which is equivalent to $\nabla T = 0$. Then we prove that this implies that the 3-form

$$H(X, Y, Z) = T^*(X, Y, JZ) = \langle T(X, Y), JZ \rangle$$

is closed $dH = 0$.[1]
Then *the equation for holomorphically harmonic maps $f \colon L \to N$ is the equation of motion (i.e. Euler-Lagrange equation) for the sigma model in N with the Wess-Zumino term defined by the closed 3-form H.* The action functional is given by

$$S(f) = E(f) + S^{WZ}(f) = \frac{1}{2} \int_L |df|^2 d\mathrm{vol}_g + \int_B H,$$

where B is 3-submanifold (or indeed a 3-chain) in N whose boundary is $f(L)$. Then since $dH = 0$, the variation of the Wess-Zumino term is a boundary term

$$\delta S^{WZ} = \int_B L_{\delta f} H = \int_B d\imath_{\delta f} H = \int_{f(L)} \imath_{\delta f} H,$$

[1]Let us point out that in general T^* is not closed even if it is ∇-parallel. For example, in a Riemannian naturally reductive homogeneous space G/H, endowed with its canonical connection ∇^0, we have $\nabla^0 T = 0$ but $dT^*(X, Y, Z, V) = -2\langle \mathrm{Jac}_{\mathfrak{m}}(X, Y, Z), V \rangle$ where $\mathrm{Jac}_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of the Jacobi identity (i.e. the sum of the circular permutations of $[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}}$). Of course $\mathfrak{m}$ is the $\mathrm{Ad}H$-invariant summand in the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

whence its contribution to the Euler-Lagrange equation involves only the original map $f\colon L \to N$.
In particular, applying this result to the case we are interseted in, i.e. N is 3-symmetric, we obtain:
The second elliptic system associated to 3-symmetric space $N = G/G_0$ is the equation of motion for the sigma model in N with the Wess-Zumino term defined by the closed 3-form H, corresponding to the canonical almost complex structure $\underline{J}$ and the canonical connection ∇^0.[2]

• The minimal determined system has an interpretation in terms of horizontally holomorphic vertically harmonic maps $f\colon L \to N = G/G_0$. It also has an equivalent interpretation in terms of vertically harmonic twistor lifts in some twistor space.

• To the best of our understanding, the maximal determined system does not have a satisfactory geometric interpretation in terms of the geometric map f into the homogeneous space N, but only in terms of twistor lifts J^f in some twistor spaces (which are subbundles of the bundle of endormorphisms in $M = G/H$ in the even case and in $N = G/G_0$ in the odd case; more precisely these are subbundles of finite order isometries). In fact, contrary to what happens in the minimal determined system - where the interpretations are similar in the homogeneous space N and in the twistor space: in terms of vertical harmonicity -, in the maximal determined system, the interpretation in the twistor space is in terms of $\underline{\tilde{J}}$-twisted harmonic twistor lift J^f but the equation of $\underline{J}$-twisted harmonicity of $f\colon L \to N$ does not give us the maximal determined elliptic system. So it seems that the general (satisfactory) interpertation of the general determined system is only in terms of twistor lifts.

• For the intermediate determined systems ($m_{k'} < m < k' - 1$), these are obtained from the maximal determined case by adding holomorphicity in the subbundle $\mathcal{H}^{2m_{k'}-1-m} \subset \mathcal{H}$ (defined above). It means that the m-th determined system has a geometric interpretation in terms of $\underline{\tilde{J}}$-twisted harmonic twistor lifts J^f which are $\widetilde{\mathcal{H}}^{2m_{k'}-1-m}$-holomorphic, i.e. $[\nabla J^f \circ j_L]^{\widetilde{\mathcal{H}}^{2m_{k'}-1-m}} = \underline{J}[\nabla J^f]^{\widetilde{\mathcal{H}}^{2m_{k'}-1-m}}$. Let us precise that $\widetilde{\mathcal{H}}^{2m_{k'}-1-m}$ is the lift of $\mathcal{H}^{2m_{k'}-1-m}$ as a subbundle of the horizontal subbundle of the twistor space. Moreover ∇ is a connection on the twistor space.

[2] In fact, we need a naturally reductive metric on N to ensure that T^*, and thus H, is closed. But if we allow Pseudo-Riemannian metric and if $\mathfrak{g}$ is semisimple then the metric defined by the Killing form is naturally reductive. In fact, the elliptic integrable system is an affine system, i.e. its natural geometric interpretation takes place in the context of affine geometry in terms of the affine connections $\nabla^t = \nabla^0 + t[\ ,\ ]_{[\mathfrak{m}]}$. If we want that this interpretation takes place in the context of Riemannian geometry we need, of course, to add some hypothesis of compactness, like the compactness of $\mathrm{Ad}_{\mathfrak{m}} G_0$ and the natural reductivity. But we do not need these hypothesis if we work in the Pseudo-Riemannian context.

The underdetermined case We prove that the m-th underdetermined system ($m > k' - 1$) is in fact equivalent to some m-th determined or primitive system associated to some new automorphism $\tilde{\tau}$ defined in a product $\mathfrak{g}^{q+1}$ of the initial lie algebra $\mathfrak{g}$. More precisely, we write

$$m = qk' + r, \quad 0 \leq r \leq k' - 1$$

the Euclidean division of m by k'. Then we consider the automorphism in $\mathfrak{g}^{q+1}$ defined by

$$\tilde{\tau}(a_1, \ldots, a_{q+1}) \in \mathfrak{g}^{q+1} \longmapsto (a_{q+1}, \tau(a_1), \ldots, a_q) \in \mathfrak{g}^{q+1}.$$

Then $\tilde{\tau}$ is of order $(q+1)k'$. We prove that the initial m-th system associated to $(\mathfrak{g}, \tau)$ is in fact equivalent to the m-th system associated to $(\mathfrak{g}^{q+1}, \tilde{\tau})$.

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Index of notations

0.1 Generalities

• Let $k \in \mathbb{N}$. Then we will often confuse - when it is convenient to do it- an element in $\mathbb{Z}_k$ with one of its representants. For example, let $(a_i)_{i \in \mathbb{Z}_k}$ be a family of elements in some vector space E, and $0 \leq m < k/2$ an integer. Then we will write

$$a_i = a_{-i} \quad 1 \leq i \leq m$$

to say that this equality holds for all $i \in \{1 + k\mathbb{Z}, \ldots, m + k\mathbb{Z}\} \subset \mathbb{Z}_k$.
• Let us suppose that a vector space E admits some decomposition $E = \oplus_{i \in I} E_i$. Then, for any vector $v \in E$ we denote by $[v]_{E_i}$ its component in E_i.

0.2 Almost complex geometry

Let E be a real vector space endowed with a complex structure: $J \in \mathrm{End}(E)$, $J^2 = -\mathrm{Id}$. Then we denote by $E^{1,0}$ and $E^{0,1}$ respectively the eigenspaces of J associated to the eigenvalues $\pm i$ respectively. Then we have the following eigenspace decomposition

$$E^{\mathbb{C}} = E^{1,0} \oplus E^{0,1} \tag{3}$$

and the following equalities

$$\begin{array}{l} E^{1,0} = \ker(J - i\mathrm{Id}) = (J + i\mathrm{Id})E^{\mathbb{C}} \\ E^{0,1} = \ker(J + i\mathrm{Id}) = (J - i\mathrm{Id})E^{\mathbb{C}} \end{array} \tag{4}$$

so that remarking that $(J \pm i\mathrm{Id})iE = (\mathrm{Id} \mp iJ)E = (\mathrm{Id} \mp iJ)JE = (J \pm i\mathrm{Id})E$, we can also write

$$\begin{array}{l} E^{1,0} = (J + i\mathrm{Id})E = (\mathrm{Id} - iJ)E = \{X - iJX, X \in E\} \\ E^{0,1} = (J - i\mathrm{Id})E = (\mathrm{Id} + iJ)E = \{X + iJX, X \in E\} \end{array} \tag{5}$$

In the same way we denote by

$$(E^*)^{\mathbb{C}} = E^*_{1,0} \oplus E^*_{0,1}$$

the decomposition induced on the dual E^* by the complex structure $J^*\colon \eta \in E^* \to \eta J \in E^*$. Besides, given a vector $Z \in E^{\mathbb{C}}$, we denote by

$$Z = [Z]^{1,0} + [Z]^{0,1}$$

its decomposition according to (3). Let us remark that

$$[Z]^{1,0} = (\mathrm{Id} - iJ)Z \quad \text{and} \quad [Z]^{0,1} = (\mathrm{Id} + iJ)Z.$$

Moreover, given η a n-form on E, we denote by $\eta^{(p,q)}$ its component in $\Lambda^{p,q}E^*$ according to the decomposition

$$\Lambda^n E^* = \oplus_{p+q=n}\Lambda^{p,q}E^*,$$

where $\Lambda^{p,q}E^* = \left(\Lambda^p E^*_{1,0}\right) \wedge \left(\Lambda^q E^*_{0,1}\right)$. However for 1-forms, we will often prefer the notation $\eta = \eta' + \eta''$, where η' and η'' denote respectively $\eta^{(1,0)}$ and $\eta^{(0,1)}$. More generally, all what precedes holds naturally when E is a real vector bundle over a manifold M, endowed with a complex structure J.
We will write

$$d = \partial + \bar{\partial}$$

the decompostion of the exterior derivative of differential forms on an almost complex manifold (M, J), according to the decomposition $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$.

In this paper, we will use the following definitions.

Definition 0.1 *Let E be a real vector bundle. A f-structure in E is an endomorphism $F \in \mathcal{C}(\mathrm{End}E)$ such that $F^3 + F = 0$. An f-structure on a manifold M is a f-structure in TM.*

An f-structure F in a vector bundle E is determined by its eigenspaces decomposition that we will denote by

$$E^{\mathbb{C}} = E^+ \oplus E^- \oplus E^0$$

where $E^{\pm} = \ker(F \mp i\mathrm{Id})$ and $E^0 = \ker F$. In particular if $E = TM$, then we will set $T^iM = (TM)^i$, $\forall i \in \{0, \pm 1\}$.

Definition 0.2 *Let (M, J^M) be an almost complex manifold and N a manifold with a splliting $TN = \mathcal{H} \oplus \mathcal{V}$. Let us suppose that the subbundle $\mathcal{H}$ is endowed with a complex structure $J^{\mathcal{H}}$. Then we will say that a map $f\colon (M, J^M) \to N$ is $\mathcal{H}$-holomorphic if it satisfies the equation*

$$\left[df \circ J^M\right]^{\mathcal{H}} = J^{\mathcal{H}}[df]^{\mathcal{H}}.$$

Moreover, if for some reason, $\mathcal{H}$ inherits the name of horizontal subbundle, then we will say that f is horizontally holomorphic.

This situation occurs for example if N is endowed with a f-structure F which leaves invariant $\mathcal{H}$ and $F_{|\mathcal{H}}$ is a complex structure (i.e. $T^0N \cap \mathcal{H} = \{0\}$).

1 Invariant connections on reductive homogeneous spaces

The references for this section where we recall some results that we will need in this paper, are [27], [29], [9], and to a lesser extent [21] and [1].

1.1 Linear isotropy representation

Let $M = G/H$ be a homogeneous space with G a Lie group and H a closed subgroup of G. G acts transitively on M in a natural manner which defines a natural representation: $\phi\colon g \in G \mapsto (\phi_g\colon p \in M \mapsto g.p) \in \mathrm{Diff}(M)$. Then $\ker\phi$ is the maximal normal subgroup of G included in H. Further, let us consider the linear isotropy representation:

$$\rho\colon h \in H \mapsto d\phi_h(p_0) \in GL(T_{p_0}M)$$

where $p_0 = 1.H$ is the reference point in M. Then we have $\ker\rho \supset \ker\phi$. Moreover the linear isotropy representation is faithful (i.e. ρ is injective) if and only if G acts freely on the bundle of linear frame $L(M)$.
We can always suppose without loss of generality that the action of G on M is effective (i.e. $\ker\phi = \{1\}$) but it does not imply in general that the linear isotropy representation is faithful. However if there exists on M a G-invariant affine connection, then the linear isotropy representation is faithful provided that G acts effectively on M. (Indeed, given a manifold M with an affine connection, and $p \in M$, an affine transformation of M is determined by $(f(p), df(p))$, i.e. f is the identity if and only if it leaves one linear frame fixed).

1.2 Reductive homogeneous space

Let us suppose now that G/H is reductive, i.e. there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is $\mathrm{Ad}H$-invariant: $\forall h \in H, \mathrm{Ad}h(\mathfrak{m}) = \mathfrak{m}$. Then the surjective map $\xi \in \mathfrak{g} \mapsto \xi.p_0 \in T_{p_0}M$ has $\mathfrak{h}$ as kernel and so its restriction to $\mathfrak{m}$ is an isomorphism $\mathfrak{m} \cong T_{p_0}M$. This provides an isomorphism of the associated bundle $G \times_H \mathfrak{m}$ with TM by:

$$[g, \xi] \mapsto g.(\xi.p_0) = \mathrm{Ad}g(\xi).p \tag{6}$$

where $p = \pi(g) = g.p_0$.
Moreover, we have a natural inclusion $G \times_H \mathfrak{m} \mapsto G \times_H \mathfrak{g}$ and the associated bundle $G \times_H \mathfrak{g}$ is canonically identified with the trivial bundle $M \times \mathfrak{g}$ via

$$[g, \xi] \mapsto (\pi(g), \mathrm{Ad}g(\xi)). \tag{7}$$

Thus we have an identification of TM with a subbundle $[\mathfrak{m}]$ of $M \times \mathfrak{g}$, which we may view as a $\mathfrak{g}$-valued 1-form β on M given by:

$$\beta_p(\xi.p) = \mathrm{Ad}g[\mathrm{Ad}g^{-1}(\xi)]_{\mathfrak{m}},$$

where $\pi(g) = p, \xi \in \mathfrak{g}$ and $[\ ]_{\mathfrak{m}}$ is the projection on $\mathfrak{m}$ along $\mathfrak{h}$. Equivalently, for all $X \in T_pM$, $\beta(X)$ is the unique element $\xi \in [\mathfrak{m}]_p$ $(= \mathrm{Ad}g(\mathfrak{m})$, with $\pi(g) = p)$ such that $X = \xi.p$, in other words $\beta(X)$ is caracterized by

$$\beta(X) \in [\mathfrak{m}]_p \subset \mathfrak{g} \quad \text{and} \quad X = \beta(X).p\ .$$

In fact, β is nothing but the projection on M of the H-equivariant 1-form on G, $\theta_{\mathfrak{m}}$ (i.e. $\theta_{\mathfrak{m}}$ is the H-equivariant lift of β), defined as the $\mathfrak{m}$-component of the left invariant Maurer-Cartan form of G, which can be written

$$(\pi^*\beta)_g = \mathrm{Ad}g(\theta_{\mathfrak{m}})\ \forall g \in G \tag{8}$$

with $\theta_g(\xi_g) = g^{-1}.\xi_g$ for all $g \in G$, $\xi_g \in T_gG$.

Notation For any $\mathrm{Ad}H$-invariant subspace $\mathfrak{l} \subset \mathfrak{m}$, we will denote by $[l]$ the subundle of $[\mathfrak{m}] \subset M \times \mathfrak{g}$ defined by $[l]_{g.p_0} = \mathrm{Ad}g(\mathfrak{l})$.

1.3 The (canonical) invariant connection

On a reductive homogeneous space $M = G/H$, the $\mathrm{Ad}(H)$-invariant summand $\mathfrak{m}$ provides by left translation in G, a G-invariant distribution $\mathcal{H}(\mathfrak{m})$, given by $\mathcal{H}(\mathfrak{m})_g = g.\mathfrak{m}$ which is horizontal for $\pi\colon G \to M$ and right H-invariant and thus defines a G-invariant connection in the principal bundle $\pi\colon G \to M$. In fact this procedure defines a bijective correspondance between reductive summands $\mathfrak{m}$ and G-invariant connections in $\pi\colon G \to M$ (see [27], chap. 2, Th 11.1). Then the corresponding $\mathfrak{h}$-valued connection 1-form ω on G (of this G-invariant connection) is the $\mathfrak{h}$-component of the left invariant Maurer-Cartan form of G:

$$\omega = \theta_{\mathfrak{h}}.$$

1.4 Associated covariant derivative

The connection ω induces a covariant derivative in the associated bundle $G \times_H \mathfrak{m} \cong TM$ and thus a G-invariant covariant derivative ∇^0 in the tangent bundle TM. In particular, we can conclude according to section 1.1 that if G/H is reductive then the linear isotropy representation is faithful (provided that G acts effectively) or equivalently that $\ker \mathrm{Ad}_{\mathfrak{m}} = \ker \rho = \ker \phi$. We will suppose in the following that, without explicit or implicit reference to the contrary, the action of G is effective and (thus) the linear isotropy representation is faithful.
One can compute explicitely ∇^0.

Lemma 1.1 [9]

$$\beta(\nabla^0_X Y) = X.\beta(Y) - [\beta(X), \beta(Y)], \quad X, Y \in \Gamma(TM).$$

Let us write (locally) $\beta(X) = \mathrm{Ad}U(X_\mathfrak{m})$, $\beta(Y) = \mathrm{Ad}U(Y_\mathfrak{m})$ where U is a (local) section of π and $X_\mathfrak{m}, Y_\mathfrak{m} \in C^\infty(M, \mathfrak{m})$ then we have (using the previous lemma)

$$\begin{aligned}\beta(\nabla^0_X Y) &= \mathrm{Ad}U\left(dY_\mathfrak{m}(X) + [\alpha(X), Y_\mathfrak{m}] - [X_\mathfrak{m}, Y_\mathfrak{m}]\right)\\ &= \mathrm{Ad}U\left(dY_\mathfrak{m}(X) + [\alpha_\mathfrak{h}(X), Y_\mathfrak{m}] + [\alpha_\mathfrak{m}(X) - X_\mathfrak{m}, Y_\mathfrak{m}]\right)\end{aligned}$$

where $\alpha = U^{-1}.dU$. Besides since U is a section of π ($\pi \circ U = \mathrm{Id}$), then pulling back (8) by U, we obtain $\beta = \mathrm{Ad}U(\alpha_\mathfrak{m})$ and then $\alpha_\mathfrak{m}(X) = X_\mathfrak{m}$, so that

$$\beta(\nabla^0_X Y) = \mathrm{Ad}U\left(dY_\mathfrak{m}(X) + [\alpha_\mathfrak{h}(X), Y_\mathfrak{m}]\right) \tag{9}$$

Remark 1.1 We could also say that $X_\mathfrak{m}, Y_\mathfrak{m}$ are respectively the pullback by U of the H-equivariant lifts $\tilde{X}, \tilde{Y}$ of X, Y (given by $\beta(X_{\pi(g)}) = \mathrm{Ad}g(\tilde{X}(g))$).
Then $\nabla^0_X Y$ lifts as the $\mathfrak{m}$-valued H-equivariant map on G:

$$\widetilde{\nabla^0_X Y} = d\tilde{Y}(\tilde{X}) + [\theta_\mathfrak{h}(\tilde{X}), \tilde{Y}]$$

and then taking the U-pullback we obtain the previous result (without using lemma 1.1).
Moreover, we can express ∇^0 in term of the flat differentiation in the trivial bundle $M \times \mathfrak{g}$ ($\supset [\mathfrak{m}]$). Let us differentiate the equation $Y = \mathrm{Ad}U(Y_\mathfrak{m})$ (we do the identification $TM = [\mathfrak{m}] \subset M \times \mathfrak{g}$)

$$dY = \mathrm{Ad}U\left(dY_\mathfrak{m} + [\alpha, Y_\mathfrak{m}]\right) = \mathrm{Ad}U\left(dY_\mathfrak{m} + [\alpha_\mathfrak{h}, Y_\mathfrak{m}]\right) + \mathrm{Ad}U\left([[\alpha_\mathfrak{m}, Y_\mathfrak{m}]]\right) = \nabla^0 Y + [\beta, Y].$$

Finally,

$$dY = \nabla^0 Y + [\beta, Y] \tag{10}$$

and we recover lemma 1.1.

1.5 G-invariant affine connections in terms of equivariant bilinear maps

Now let us recall the following results about invariant connections on reductive homogeneous spaces.

Theorem 1.1 [27] *Let $\pi_P \colon P \to M$ be a K-principal bundle over the reductive homogeneous space $M = G/H$ and suppose that G acts on P as a group of automorphisms and let $u_0 \in P$ be a fixed pont in the fibre of $p_0 \in M$ ($\pi_P(u_0) = p_0$). There is a bijective correspondance between the set of G-invariant connections ω in P and the set of linear maps $\Lambda_\mathfrak{m} \colon \mathfrak{m} \to \mathfrak{k}$ such that*

$$\Lambda_\mathfrak{m}(hXh^{-1}) = \lambda(h)\Lambda_\mathfrak{m}(X)\lambda(h)^{-1} \quad \textit{for } X \in \mathfrak{m} \textit{ and } h \in H \tag{11}$$

where $\lambda \colon H \to K$ is the morphism defined by $hu_0 = u_0\lambda(h)$ (H stabilizes the fibre $P_{p_0} = u_0.K$). The correspondance is given by

$$\Lambda(X) = \omega_{u_0}(\tilde{X}), \quad \forall X \in \mathfrak{g} \tag{12}$$

where $\tilde{X}$ is the vector field on P induced by X (i.e. $\forall u \in P$, $\phi_t^{\tilde{X}}(u) = \exp(tX).u$) and $\Lambda\colon \mathfrak{g} \to \mathfrak{k}$ is defined by $\Lambda_{|\mathfrak{m}} = \Lambda_{\mathfrak{m}}$ and $\Lambda_{|\mathfrak{h}} = \lambda$ (hence completely determined by $\Lambda_{\mathfrak{m}}$).

Corollary 1.1 *In the previous theorem, let us suppose that P is a K-structure on $M = G/H$, i.e. P is a subbundle of the bundle $L(M)$ of linear frame on M with structure group $K \subset GL(n,\mathbb{R}) = GL(\mathfrak{m})$ (we identify as usual $\mathfrak{m}$ to $T_{p_0}M$ by $\xi \mapsto \xi.p_0$, and $T_{p_0}M$ to $\mathbb{R}^n$ via the linear frame $u_0 \in P \subset L(M)$). Then in terms of the G-invariant covariant derivative ∇ corresponding to the G-invariant affine connection in P, ω, the previous bijective correspondance may be given by*

$$\Lambda(X)(Y) = \nabla_{\tilde{X}}\tilde{Y}$$

where $\tilde{X}, \tilde{Y}$ are any (local) left G-invariant vector field extending X, Y i.e. there exists a local section of $\pi\colon G \to M$, $g\colon U \subset M \to G$ such that $\tilde{X} = \mathrm{Ad}_{g(p)}(X).p$.

Remark 1.2 In theorem 1.1, the G-invariant connection in P defined by $\Lambda_{\mathfrak{m}} = 0$ is called the canonical connection (with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$). If we set $P(M,K) = G(G/H,H)$ with group of automorphisms G, the G-invariant connection defined by the horizontal distribution $\mathcal{H}(\mathfrak{m})$ is the canonical connection.
Now, let P be a G-invariant K-structure on $M = G/H$ as in corollary 1.1. Let P' be an G-invariant subbundle of P with structure group $K' \subset K$, then the canonical connection in P' defined by $\Lambda_{\mathfrak{m}} = 0$ is (the restriction of) the canonical connection in P which is itself the restriction to P of the canonical connection in $L(M)$. In particular, if we set $P' = G.u_0$, this is a subbundle of P with group H, which is isomorphic to the bundle $G(G/H,H)$. Then the canonical affine connection in P' corresponds to the invariant connection in $G(G/H,H)$ defined by the distribution $\mathcal{H}(\mathfrak{m})$.

Theorem 1.2 *Let $P \subset L(M)$ be a K-structure on $M = G/H$. Then the canonical affine connection ($\Lambda_{\mathfrak{m}} = 0$) in P defines the covariant derivative ∇^0 in TM (obtained from $\mathcal{H}(\mathfrak{m})$ in the associated bundle $G \times_H \mathfrak{m} \cong TM$). Moreover there is a bijective correpondance between the set of of G-invariant affine connections on M, ∇, determined by a connection in P, and the set of linear maps $\Lambda_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{k} \subset \mathfrak{gl}(\mathfrak{m})$ such that*

$$\Lambda_{\mathfrak{m}}(hXh^{-1}) = \mathrm{Ad}_{\mathfrak{m}}(h)\Lambda_{\mathfrak{m}}(X)\mathrm{Ad}_{\mathfrak{m}}(h)^{-1} \quad \forall X \in \mathfrak{m}, \forall h \in H, \tag{13}$$

given by

$$\nabla = \nabla^0 + \bar{\Lambda}_{\mathfrak{m}}$$

i.e. $\nabla_X Y = \nabla^0_X Y + \bar{\Lambda}_{\mathfrak{m}}(X)Y$ for any vector fields X, Y on M, where with the help of (13) we extended the $\mathrm{Ad}(H)$-equivariant map $\Lambda_{\mathfrak{m}}\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ to the bunlde $G \times_H \mathfrak{m} = TM$ to obtain a map $\bar{\Lambda}_{\mathfrak{m}}\colon TM \times TM \to TM$.

Example 1.1 *Let us suppose that M is Riemannian (i.e. $\mathrm{Ad}_{\mathfrak{m}}H$ is compact and $\mathfrak{m}$ is endowed with an $\mathrm{Ad}H$ invariant inner product which defines a G-invariant metric on M) and let us take $P = O(M)$ the bundle of orthonormal frames on M, the previous correspondance is between the set of G-invariant metric affine connection and the set of $\mathrm{Ad}(H)$-equivariant linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$.*
In particular the canonical connection ∇^0 is metric (for any G-invariant metric on M).

Theorem 1.3 • *G-invariant tensors on the reductive homogeneous space $M = G/H$ (or more generally G-invariant sections of associated bundles) are parallel with respect to the canonical connection.*

- *The canonical connection is complete (the geodesics are exactly the curves $x_t = \exp(tX).p_0$, for $X \in \mathfrak{m}$).*
- *Let P be a G-invariant K-structure on $M = G/H$, then the G-invariant connection defined by $\Lambda : \mathfrak{m} \to \mathfrak{k}$ has the same geodesics as the canonical connection if and only if*

$$\Lambda_{\mathfrak{m}}(X)X = 0, \quad \forall X \in \mathfrak{m}$$

Theorem 1.4 *The torsion tensor T and the curvature tensor R of the G-invariant connection corresponding to $\Lambda_{\mathfrak{m}}$ is given at the origin point p_0 as follows:*

1. $T(X,Y) = \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}$,
2. $R(X,Y) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}_{\mathfrak{m}}([X,Y]_{\mathfrak{h}})$,

for $X, Y \in \mathfrak{m}$.
In particular, for the canonical connection we have $T(X,Y) = -[X,Y]_{\mathfrak{m}}$ and $R(X,Y) = -\mathrm{ad}_{\mathfrak{m}}([X,Y]_{\mathfrak{h}})$, for $X, Y \in \mathfrak{m}$; moreover we have $\nabla T = 0$, $\nabla R = 0$.

1.6 A Family of connections on the reductive space M

We take in what precede (i.e. in section 1.5) $P = L(M)$. Then let us consider the one parameter family of connections ∇^t, $0 \leq t \leq 1$ defined by

$$\Lambda^t_{\mathfrak{m}}(X)Y = t[X,Y]_{\mathfrak{m}}, \quad 0 \leq t \leq 1.$$

For $t = 0$, we obtain the canonical connnection ∇^0. Since for any $t \in [0,1]$, $\Lambda^t_{\mathfrak{m}}(X)X = 0$, $\forall X \in \mathfrak{m}$, ∇^t has the same geodesics as ∇^0 and in particular is complete. The torsion tensor is given (at p_0) by

$$T^t(X,Y) = (2t-1)[X,Y]_{\mathfrak{m}}. \tag{14}$$

In particular $\nabla^{\frac{1}{2}}$ is the unique torsion free G-invariant affine connection having the same geodesics as the canonical connection (according to theorems 1.3 and

1.4).

If M is Riemannian, then let us take $P = O(M)$, then ∇^t is metric if and only if $\Lambda^t_{\mathfrak{m}}$ takes values in $\mathfrak{k} = \mathfrak{so}(\mathfrak{m})$ if and only if (for $t \neq 0$)M is naturally reductive (which means by definition that $\forall X \in \mathfrak{m}$, $[X, \cdot]_{\mathfrak{m}}$ is skew symmetric). Now (still in the Riemannian case) let us construct a family of affine connections, $\overset{\text{met}}{\nabla^t}$, $0 \leq t \leq 1$, which are always metric:

$$\overset{\text{met}}{\nabla^t} = \nabla^0 + t\left([\ ,\ ]_{[\mathfrak{m}]} + \mathcal{U}^M\right)$$

where $\mathcal{U}^M : TM \oplus TM \to TM$ is the naturally reductivity term which is the symmetric bilinear map defined by[3]

$$\langle \mathcal{U}^M(X,Y), Z\rangle = \langle [Z,X]_{[\mathfrak{m}]}, Y\rangle + \langle X, [Z,Y]_{[\mathfrak{m}]}\rangle \tag{15}$$

for all $X, Y, Z \in [\mathfrak{m}]$. Since $\mathcal{U}^M$ is symmetric, the torsion of $\overset{\text{met}}{\nabla^t}$ is once again given by

$$T^t(X,Y) = (2t-1)[X,Y]_{[\mathfrak{m}]}$$

and thus $\overset{\text{met}}{\nabla^{\frac{1}{2}}}$ is torsion free and metric and we recover that $\overset{\text{met}}{\nabla^{\frac{1}{2}}}$ is the Levi-Civita connection

$$\overset{\text{met}}{\nabla^{\frac{1}{2}}} = \nabla^{\text{L.C.}}.$$

Obviously if $M$is naturally reductive then $\overset{\text{met}}{\nabla^t} = \nabla^t$, $\forall t \in [0,1]$. Moreover if $M$is (locally) symmetric, i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, then all the connections coincide and are equal to the Levi-Civita connection: $\overset{\text{met}}{\nabla^t} = \nabla^t = \nabla^0 = \nabla^{\text{L.C.}}$.

Remark 1.3 ∇^1 is interesting since it is nothing but the flat differentiation in the trivial bundle $M \times \mathfrak{g}$ followed by the projection onto $[\mathfrak{m}]$ (along $[\mathfrak{h}]$) (see remark 1.1). So this connection is very natural and following [1], we will call it **the anticanonical connection**.

1.7 Differentiation in $\mathrm{End}(T(G/H))$

According to section 1.2, we have

$$\mathrm{End}(T(G/H)) = G \times_H \mathrm{End}(\mathfrak{m}) \subset (G/H) \times \mathrm{End}(\mathfrak{g}),$$

the previous inclusion being given by $[g, A] \mapsto (\pi(g), \mathrm{Ad}g \circ A \circ \mathrm{Ad}g^{-1})$ and we embedd $\mathrm{End}(\mathfrak{m})$ in $\mathfrak{g}$ by extending the an endomorphism in $\mathfrak{m}$ by 0 in $\mathfrak{h}$. In other words $\mathrm{End}(T(G/H))$ can be identified to the subbundle $[\mathrm{End}(\mathfrak{m})]$ of the trivial bundle $(G/H) \times \mathrm{End}(\mathfrak{g})$, with fibers $[\mathrm{End}(\mathfrak{m})]_{g.p_0} = \mathrm{End}(\mathrm{Ad}g(\mathfrak{m})) = \mathrm{Ad}g(\mathrm{End}(\mathfrak{m}))\mathrm{Ad}g^{-1} = \mathrm{Ad}g(\mathrm{End}(\mathfrak{m}) \oplus \{0\})\mathrm{Ad}g^{-1}$.

[3] $\mathcal{U}^M$ is the $\mathrm{Ad}H$-invariant extension of $\mathcal{U}^{\mathfrak{m}} : \mathfrak{m} \oplus \mathfrak{m} \to \mathfrak{m}$, its restriction to $\mathfrak{m} \oplus \mathfrak{m}$.

Now, let us compute in terms of the Lie algebra setting, the derivative of the inclusion map $\mathfrak{I}\colon \mathrm{End}(T(G/H)) \to M \times \mathrm{End}(\mathfrak{g})$ or more concretely the flat derivative in $M\times\mathrm{End}(\mathfrak{g})$ of any section of $\mathrm{End}(T(G/H))$; to do that, we compute the derivative of

$$\tilde{\mathfrak{J}}\colon (g, A_\mathfrak{m}) \in G \times \mathrm{End}(\mathfrak{m}) \longmapsto (g.p_0,\ \mathrm{Ad}g \circ A_\mathfrak{m} \circ \mathrm{Ad}g^{-1}) \in M \times \mathrm{End}(\mathfrak{g}),$$

we obtain

$$d\tilde{\mathfrak{J}}(g, A_\mathfrak{m}) = \left(\mathrm{Ad}g(\theta_\mathfrak{m})\,.\,\pi(g),\ \mathrm{Ad}g\,(dA_\mathfrak{m} + [\mathrm{ad}\theta, A_\mathfrak{m}])\,\mathrm{Ad}g^{-1}\right).$$

Then let us decompose the endomorphisms in $\mathfrak{g}$ by blocs (following $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$):

$$\mathrm{End}(\mathfrak{g}) = \begin{pmatrix} \mathrm{End}(\mathfrak{h}) & \mathrm{End}(\mathfrak{m}, \mathfrak{h}) \\ \mathrm{End}(\mathfrak{h}, \mathfrak{m}) & \mathrm{End}(\mathfrak{m}) \end{pmatrix}, \tag{16}$$

and by regrouping terms, we obtain the following splitting

$$\mathrm{End}(\mathfrak{g}) = \mathrm{End}(\mathfrak{m}) \oplus (\mathrm{End}(\mathfrak{m}, \mathfrak{h}) \oplus \mathrm{End}(\mathfrak{h}, \mathfrak{m}) \oplus \mathrm{End}(\mathfrak{h}))\,,$$

which applied to $d\tilde{\mathfrak{J}}(g, A_\mathfrak{m})$, gives us the decomposition

$$\begin{aligned} d\tilde{\mathfrak{J}}(g, A_\mathfrak{m}) \quad = \quad & \left(0,\ \mathrm{Ad}g\,(dA_\mathfrak{m} + [\mathrm{ad}_\mathfrak{m}\theta_\mathfrak{h}, A_\mathfrak{m}] + [[\mathrm{ad}_\mathfrak{m}\theta_\mathfrak{m}]_\mathfrak{m}, A_\mathfrak{m}])\,\mathrm{Ad}g^{-1}\right) \\ & + \left(\mathrm{Ad}g(\theta_\mathfrak{m})\,.\,\pi(g),\ \mathrm{Ad}g\,([\mathrm{ad}_\mathfrak{m}\theta_\mathfrak{m}]_\mathfrak{h} \circ A_\mathfrak{m} - A_\mathfrak{m} \circ \mathrm{ad}_\mathfrak{h}\theta_\mathfrak{m})\,\mathrm{Ad}g^{-1}\right). \end{aligned} \tag{17}$$

The first term is in the vertical space $\mathcal{V}_{\tilde{\mathfrak{J}}(g,A_\mathfrak{m})} = \mathrm{Ad}g(\mathrm{End}(\mathfrak{m}))\mathrm{Ad}g^{-1} = \mathrm{End}(T_{\pi(g)}M)$ and the previous decomposition (17) provides us with a splitting $T\mathrm{End}(M) = \mathcal{V} \oplus \mathcal{H} = \pi_M^*(\mathrm{End}(M)) \oplus \mathcal{H}$, i.e. a connection on $\mathrm{End}(M)$. Let us determine this connection: we see that the projection on the vertical space (along the horizontal space) corresponds to the projection on $[\mathrm{End}(\mathfrak{m})]$ following (16) so that according to remark 1.3, we can conclude that the horizontal distribution $\mathcal{H}$ defines the connection ∇^1 on $\mathrm{End}(TM) = TM^* \otimes TM$.

Remark 1.4 We can recover this fact directly from the first term of (17) and the definition of ∇^1. Indeed, first recall that given two affine connections ∇, ∇' on M, we can write $\nabla' = \nabla + F$, where F is a section of $TM^* \otimes \mathrm{End}(TM)$, and then for any section A in $\mathrm{End}(TM)$,

$$\nabla' A = \nabla A + [F, A].$$

Besides $\nabla^1 = \nabla^0 + [\ ,\ ]_{[\mathfrak{m}]}$, and moreover if we write (locally) $A = (\pi(U),\ \mathrm{Ad}U \circ A_\mathfrak{m} \circ \mathrm{Ad}U^{-1})$ where U is a local section of π and $A_\mathfrak{m} \in C^\infty(M, \mathrm{End}(\mathfrak{m}))$, then according to (9),

$$\nabla^0 A = \mathrm{Ad}U\,(dA_\mathfrak{m} + [\mathrm{ad}_\mathfrak{m}\alpha_\mathfrak{h}, A_\mathfrak{m}])\,, \tag{18}$$

so that we conclude that

$$\nabla^1 A = \mathrm{Ad}U\,(dA_\mathfrak{m} + [\mathrm{ad}_\mathfrak{m}\alpha_\mathfrak{h}, A_\mathfrak{m}] + [[\mathrm{ad}_\mathfrak{m}\alpha_\mathfrak{m}]_\mathfrak{m}, A_\mathfrak{m}])\,\mathrm{Ad}U^{-1}$$

which is the (pullback of) the first term of (17).

Furthermore if G/H is (locally) symmetric (i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$), then $\nabla^{L.C.} = \nabla^0 = \nabla^1$ and in particular

$$\nabla^{L.C.} A = \mathrm{Ad}U\,(dA_\mathfrak{m} + [\mathrm{ad}_\mathfrak{m}\alpha_\mathfrak{h}, A_\mathfrak{m}])\,. \tag{19}$$

2 m-th elliptic integrable system associated to a k'-symmetric space

2.0.1 Definition of G^τ (even when τ does not integrate in G)

Here, we will extend the notion of subgroup fixed by an automorphism of Lie group to the situation where only a Lie algebra automorphism is provided. Indeed, let $\tau\colon G \to G$ be a Lie group automorphism, then usually one can define $G^\tau = \{g \in G |\, \tau(g) = g\}$ the subgroup fixed by τ. Now, we want to extend this definition to the situation where we only have a Lie algebra automorphism, and so that the two definitions coincide when the Lie algebra automorphism integrates in G.

Let $\mathfrak{g}$ be a Lie algebra and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism. Then let us denote by

$$\mathfrak{g}_0 = \mathfrak{g}^\tau := \{\xi \in \mathfrak{g} |\, \tau(\xi) = \xi\} \tag{20}$$

the subalgebra of $\mathfrak{g}$ fixed by τ. Let us assume that τ defines in $\mathfrak{g}$ a τ-invariant reductive decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}, \quad [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{n}, \quad \tau(\mathfrak{n}) = \mathfrak{n}.$$

Without loss of generality, we assume that $\mathfrak{g}_0$ does not contain non-trivial ideal of $\mathfrak{g}$, i.e. that $\mathrm{ad}_\mathfrak{n}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{n})$ is injective (the kernel is a τ-invariant ideal of $\mathfrak{g}$ that we factor out). We then have

$$\mathfrak{g}_0 = \{\xi \in \mathfrak{g} |\, \tau_{|\mathfrak{n}} \circ \mathrm{ad}_\mathfrak{n}\xi \circ \tau_{|\mathfrak{n}}^{-1} = \mathrm{ad}_\mathfrak{n}\xi\} \tag{21}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then since $\mathfrak{g}_0$ satisfies: $\forall \xi \in \mathfrak{g}_0$, $\mathrm{ad}\xi(\mathfrak{n}) = \mathfrak{n}$ and $\tau_\mathfrak{n} \circ \mathrm{ad}_\mathfrak{n}\xi \circ \tau_\mathfrak{n}^{-1} = \mathrm{ad}_\mathfrak{n}\xi$, then the (connected) subgroup G_0^0 generated in G by $\mathfrak{g}_0$ satisfies: $\forall g \in G_0^0$, $\mathrm{Ad}g(\mathfrak{n}) = \mathfrak{n}$ and $\tau_\mathfrak{n} \circ \mathrm{Ad}_\mathfrak{n} g \circ \tau_\mathfrak{n}^{-1} = \mathrm{Ad}_\mathfrak{n} g$. Now, let us consider the subgroup

$$G_0 = \{g \in G |\, \mathrm{Ad}g(\mathfrak{n}) = \mathfrak{n} \text{ and } \tau_\mathfrak{n} \circ \mathrm{Ad}_\mathfrak{n} g \circ \tau_\mathfrak{n}^{-1} = \mathrm{Ad}_\mathfrak{n} g\}.$$

Then G_0 is a closed subgroup of G and $G_0 \supset G_0^0$, so that $\mathrm{Lie}\, G_0 \supset \mathfrak{g}_0$. Conversely, differentiating the second relation defining G_0, we obtain, according to (21), $\mathrm{Lie}\, G_0 \subset \mathfrak{g}_0$ and thus $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$, and $G_0^0 = (G_0)^0$.
Moreover, without loss of generality, we will suppose that G_0 does not contain non-trivial normal subgroup of G, i.e. that $\mathrm{Ad}_\mathfrak{n} G_0 \to GL(\mathfrak{n})$ is injective (see section 1). Now, we want to prove that if τ integrates in G, then we have $G_0 = G^\tau$, where G^τ is the subgroup fixed by $\tau\colon G \to G$. First, we have $\forall g \in G^\tau$, $\mathrm{Ad}g(\mathfrak{n}) = \mathfrak{n}$ and $\tau_\mathfrak{n} \circ \mathrm{Ad}_\mathfrak{n} g \circ \tau_\mathfrak{n}^{-1} = \mathrm{Ad}\tau(g)_{|\mathfrak{n}} = \mathrm{Ad}_\mathfrak{n} g$, thus $G^\tau \subset G_0$. Conversely, $\forall g \in G_0$, $\mathrm{Ad}_\mathfrak{n} g = \tau_\mathfrak{n} \circ \mathrm{Ad}_\mathfrak{n} g \circ \tau_\mathfrak{n}^{-1} = \mathrm{Ad}\tau(g)_{|\mathfrak{n}} = \mathrm{Ad}_\mathfrak{n}\tau(g)$ and thus $g = \tau(g)$ since $\mathrm{Ad}_\mathfrak{n}\colon G_0 \to GL(\mathfrak{n})$ is injective. We have proved $G^\tau = G_0$. This allows us to make the following:

Definition 2.1 *Let $\mathfrak{g}$ be a Lie algebra and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism, and G a Lie group with Lie algebra $\mathfrak{g}$. Let us assume that τ defines in $\mathfrak{g}$ a τ-invariant reductive decomposition: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}$. Then we will set*

$$G^\tau := \{g \in G |\, \mathrm{Ad}g(\mathfrak{n}) = \mathfrak{n} \text{ and } \tau_{\mathfrak{n}} \circ \mathrm{Ad}_{\mathfrak{n}} g \circ \tau_{\mathfrak{n}}^{-1} = \mathrm{Ad}_{\mathfrak{n}} g\}.$$

Let us conclude this subsection by some notations:

Notation and convention In all the paper, when a Lie algebra $\mathfrak{g}$ and an automorphism τ will be given, then $\mathfrak{g}_0$ will denote the Lie subalgebra defined by (20), G will denote a connected Lie group with Lie algebra $\mathfrak{g}$ and $G_0 \subset G$ a closed subgroup such that $(G^\tau)^0 \subset G_0 \subset G^\tau$ (which implies that its Lie algebra is $\mathfrak{g}_0$).
Moreover, without loss of generality, we will always suppose that $\mathfrak{g}_0$ does not contain non-trivial ideal of $\mathfrak{g}$ - we will then say that $(\mathfrak{g}, \mathfrak{g}_0)$ is effective - and also suppose that G^τ does not contain non-trivial normal subgroup of G (by factoring out, if needed, by some discrete subgroup of G). Consequently, when τ can be integrated in G, then G^τ will coincide with the subgroup of G fixed by $\tau\colon G \to G$.

2.1 Finite order Lie algebra automorphisms

Let $\mathfrak{g}$ be a real Lie algebra and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism of order k'. Then we have the following eigenspace decomposition:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}/k'\mathbb{Z}} \mathfrak{g}_j^{\mathbb{C}}, \qquad [\mathfrak{g}_j^{\mathbb{C}}, \mathfrak{g}_l^{\mathbb{C}}] \subset \mathfrak{g}_{j+l}^{\mathbb{C}}$$

where $\mathfrak{g}_j^{\mathbb{C}}$ is the $e^{2ij\pi/k'}$-eigenspace of τ.
We then have to distinguish two cases.

2.1.1 The even case: $k' = 2k$

Then we have $\mathfrak{g}_0^{\mathbb{C}} = (\mathfrak{g}_0)^{\mathbb{C}}$. Moreover let us remark that

$$\overline{\mathfrak{g}_j^{\mathbb{C}}} = \mathfrak{g}_{-j}^{\mathbb{C}}, \quad \forall j \in \mathbb{Z}/k'\mathbb{Z}. \tag{22}$$

Therefore $\overline{\mathfrak{g}_k^{\mathbb{C}}} = \mathfrak{g}_{-k}^{\mathbb{C}} = \mathfrak{g}_k^{\mathbb{C}}$ so that we can set $\mathfrak{g}_k^{\mathbb{C}} = (\mathfrak{g}_k)^{\mathbb{C}}$ with

$$\mathfrak{g}_k = \{\xi \in \mathfrak{g} |\, \tau(\xi) = -\xi\}.$$

Moreover, owing to (22), we can define $\mathfrak{m}_j$ as the unique real subspace in $\mathfrak{g}$ such that its complexified is given by

$$\mathfrak{m}_j^{\mathbb{C}} = \mathfrak{g}_j^{\mathbb{C}} \oplus \mathfrak{g}_{-j}^{\mathbb{C}} \text{ for } j \neq 0, k,$$

and $\mathfrak{n}$ as the unique real subspace such that

$$\mathfrak{n}^{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}'_k \setminus \{0\}} \mathfrak{g}_j^{\mathbb{C}},$$

that is $\mathfrak{n} = (\oplus_{j=1}^{k-1}\mathfrak{m}_j) \oplus \mathfrak{g}_k$. In particular τ defines a τ-invariant reductive decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}$.
Hence the eigenspace decomposition is written:

$$\mathfrak{g}^{\mathbb{C}} = \left(\mathfrak{g}^{\mathbb{C}}_{-(k-1)} \oplus \ldots \oplus \mathfrak{g}^{\mathbb{C}}_{-1}\right) \oplus \mathfrak{g}^{\mathbb{C}}_0 \oplus \left(\mathfrak{g}^{\mathbb{C}}_1 \oplus \ldots \oplus \mathfrak{g}^{\mathbb{C}}_{k-1}\right) \oplus \mathfrak{g}^{\mathbb{C}}_k$$

so that by grouping

$$\begin{aligned} \mathfrak{g}^{\mathbb{C}} &= \mathfrak{g}^{\mathbb{C}}_0 \oplus \mathfrak{g}^{\mathbb{C}}_k \oplus \left[\oplus_{j=1}^{k-1}\mathfrak{m}^{\mathbb{C}}_j\right] \\ &= \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} \end{aligned}$$

where $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_k$ and $\mathfrak{m} = \oplus_{j=1}^{k-1}\mathfrak{m}_j$. Considering the automorphism $\sigma = \tau^2$, we have $\mathfrak{h} = \mathfrak{g}^\sigma$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the reductive decomposition defined by the order k automorphism σ. Without loss of generality, and according to our convention applied to $\mathfrak{g}$ and σ, we will suppose in the following that $(\mathfrak{g}, \mathfrak{h})$ is effective i.e. $\mathfrak{h}$ does not contain non trivial ideal of $\mathfrak{g}$. This implies in particular that $(\mathfrak{g}, \mathfrak{g}_0)$ is also effective.

Now let us integrate our setting: let G be a Lie group with Lie algebra $\mathfrak{g}$ and we choose G_0 such that $(G^\tau)^0 \subset G_0 \subset G^\tau$. Then G/G_0 is a (locally) $2k$-symmetric space (it is globally $2k$-symmetric if τ integrates in G) and is in particular a reductive homogeneous space (reductive decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}$).
Moreover since $\sigma = \tau^2$ is an order k automorphism, then for any subgroup H, such that $(G^\sigma)^0 \subset H \subset G^\sigma$, G/H is a (locally) k-symmetric space. In all the following we will always do this choice for H and suppose that $H \supset G_0$ (it is already true up to covering since $\mathfrak{h} \supset \mathfrak{g}_0$) so that $N = G/G_0$ has a structure of associated bundle over $M = G/H$ with fibre H/G_0: $G/G_0 \cong G \times_H H/G_0$. We can add that on $\mathfrak{h}$, τ is an involution: $(\tau_{|\mathfrak{h}})^2 = \mathrm{Id}_{\mathfrak{h}}$, whose symmetric decomposition is $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_k$, and gives rise to the (locally) symmetric space H/G_0. The fibre H/G_0 is thus (locally) symmetric (and globally symmetric if the inner automorphism $\mathrm{Int}\tau_{|\mathfrak{m}}$ stabilizes $\mathrm{Ad}_{\mathfrak{m}}H$). Owing to the effectivity of $(\mathfrak{g}, \mathfrak{h})$, we have the following caracterisation:

$$\mathfrak{g}_0 = \{\xi \in \mathfrak{h} | [\mathrm{ad}_{\mathfrak{m}}\xi, \tau_{|\mathfrak{m}}] = 0\} \quad (23)$$

$$\mathfrak{g}_k = \{\xi \in \mathfrak{h} | \{\mathrm{ad}_{\mathfrak{m}}\xi, \tau_{|\mathfrak{m}}\} = 0\} \quad (24)$$

$\{\}$ being the anticommutator.

Besides $(\tau_{|\mathfrak{m}})^k$ is an involution hence there exists two invariant subspaces, sum of $\mathfrak{m}_j$'s, $\mathfrak{m}'$ and $\mathfrak{m}''$, such that

$$(\tau_{|\mathfrak{m}})^k = -\mathrm{Id}_{\mathfrak{m}'} \oplus \mathrm{Id}_{\mathfrak{m}''}$$

with $\mathfrak{m}' = \oplus_{j=0}^{[\frac{k-2}{2}]}\mathfrak{m}_{2j+1}$ and $\mathfrak{m}'' = \oplus_{j=1}^{[\frac{k-1}{2}]}\mathfrak{m}_{2j}$, or in other words

$$\mathfrak{m}'^{\mathbb{C}} = \bigoplus_{\substack{z^k=-1 \\ z \neq -1}} \ker(\tau - z\mathrm{Id}), \quad \mathfrak{m}''^{\mathbb{C}} = \bigoplus_{\substack{z^k=1 \\ z \neq \pm 1}} \ker(\tau - z\mathrm{Id}).$$

At this stage, there is two possibilities:

- if $\mathfrak{m}'' = 0$ then $(\tau_{|\mathfrak{m}})^k = -\mathrm{Id}_{\mathfrak{m}}$ and $\tau_{|\mathfrak{m}}$ admits eigenvalues only on the set $\{z^k = -1, z \neq -1\}$.
- if $\mathfrak{m}'' \neq 0$ then $(\tau_{|\mathfrak{m}})^k \neq -\mathrm{Id}_{\mathfrak{m}}$ and $\tau_{|\mathfrak{m}}$ admits eigenvalues in both the sets $\{z^k = 1, z \neq \pm 1\}$ and $\{z^k = -1, z \neq -1\}$.

These two cases give rise to types of $2k$-symmetric spaces (see section 3.5).

Now, let us suppose that $M = G/H$ is Riemannian (i.e. $\mathrm{Ad}_{\mathfrak{m}}H$ is compact) then we can choose an $\mathrm{Ad}H$-invariant inner product on $\mathfrak{m}$ for which $\tau_{|\mathfrak{m}}$ is an isometry (see the Appendice for the proof of the existence of such a inner product). We will always do this choice. Therefore, $\tau_{|\mathfrak{m}}$ is an order $2k$ isometry. We will study this kind of endomorphisms in section 3.
Moreover, let us remark that if G/H is Riemannian then so is G/G_0. Further, since the elliptic system we will study in this paper is given in the Lie algebra setting it is useful to know how the fact that G/H is Riemannian can be read in the Lie algebra setting. In fact, under our hypothesis of effectivity, G/H is Riemannian if and only if $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$.

2.1.2 the odd case: $k' = 2k + 1$

As in the even case we have $\mathfrak{g}_0^{\mathbb{C}} = (\mathfrak{g}_0)^{\mathbb{C}}$ and $\overline{\mathfrak{g}_j^{\mathbb{C}}} = \mathfrak{g}_{-j^{\mathbb{C}}}$, $\forall j \in \mathbb{Z}/k'\mathbb{Z}$. Then we obtain the following eigenspace decomposition:

$$\mathfrak{g}^{\mathbb{C}} = \left(\mathfrak{g}_{-k}^{\mathbb{C}} \oplus \ldots \oplus \mathfrak{g}_{-1}^{\mathbb{C}}\right) \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \left(\mathfrak{g}_1^{\mathbb{C}} \oplus \ldots \oplus \mathfrak{g}_k^{\mathbb{C}}\right), \tag{25}$$

which provides in particular the following reductive decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$$

with $\mathfrak{m} = \oplus_{j=1}^{k} \mathfrak{m}_j$ and $\mathfrak{m}_j$ is the real subspace whose the complexified is $\mathfrak{m}_j^{\mathbb{C}} = \mathfrak{g}_{-j}^{\mathbb{C}} \oplus \mathfrak{g}_j^{\mathbb{C}}$. According to our convention, we suppose that $(\mathfrak{g}, \mathfrak{g}_0)$ is effective.
Then, as in the even case, integrating our setting and choosing G_0 such that $(G^\tau)^0 \subset G_0 \subset G^\tau$, we consider $N = G/G_0$ which is a locally $(2k+1)$-symmetric space and in particular a reductive homogeneous space. Moreover, the decomposition (25) gives rises to a splitting $TN^{\mathbb{C}} = T^{1,0}N \oplus T^{0,1}N$ defined by

$$\begin{array}{ccccc} TN^{\mathbb{C}} & = & \left(\oplus_{j=1}^{k}[\mathfrak{g}_{-j}^{\mathbb{C}}]\right) & \oplus & \left(\oplus_{j=1}^{k}[\mathfrak{g}_j^{\mathbb{C}}]\right) \\ & = & T^{1,0}N & \oplus & T^{0,1}N \end{array} \tag{26}$$

This splitting defines a canonical complex structure on G/G_0, that we will denote by $\underline{J}$.
Moreover, we have the following caracterisation

$$\mathfrak{g}_0 = \{\xi \in \mathfrak{g} | [\mathrm{ad}_{\mathfrak{m}}\xi, \tau_{|\mathfrak{m}}] = 0\}.$$

Let us suppose that $N = G/G_0$ is Riemannian then the subgroup generated by $\mathrm{Ad}_{\mathfrak{m}}G_0$ and $\tau_{|\mathfrak{m}}$ is compact (because $\tau_{|\mathfrak{m}}\mathrm{Ad}_{\mathfrak{m}}g\,\tau_{|\mathfrak{m}}^{-1} = \mathrm{Ad}_{\mathfrak{m}}g$, $\forall g \in G_0$, and

$\tau_{\mathfrak{m}}$ is of finite order) and thus we can choose an $\mathrm{Ad}G_0$-invariant inner product in $\mathfrak{m}$ for which $\tau_{|\mathfrak{m}}$ is an isometry. We will always do this choice (when N is Riemannian).

2.2 Definitions and general properties of the m-th elliptic system.

2.2.1 Definitions

Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an order k' automorphism with $k' \in \mathbb{N}^*$ (if $k' = 1$ then $\tau = \mathrm{Id}$). We use the notations of 2.1. Let us begin by defining some useful notations.

Notation and convention Given $I \subset \mathbb{N}$, we denote by $\prod_{j\in I} \mathfrak{g}_j^{\mathbb{C}}$, the product $\prod_{j\in I} \mathfrak{g}_{j \bmod k'}^{\mathbb{C}}$. In the case $\sum_{j\in I} \mathfrak{g}_{j \bmod k'}^{\mathbb{C}}$ is a direct sum in $\mathfrak{g}^{\mathbb{C}}$, we will identify it with the previous product via the canonical isomorphism

$$(a_j)_{j\in I} \longmapsto \sum_{j\in I} a_j, \tag{27}$$

and we will denote these two subspaces by the same notation $\oplus_{j\in I}\mathfrak{g}_j^{\mathbb{C}}$.

Now, let us define the m-th elliptic integrable system associated to a k'-symmetric space, in the sense of Terng [32].

Definition 2.2 *Let L be a Riemann surface. The m-th $(\mathfrak{g},\tau)$-system (with the $(-)$-convention) on L is the equation for $(u_0,\dots,u_m)$, $(1,0)$-type 1-form on L with values in $\prod_{j=0}^m \mathfrak{g}_{-j}^{\mathbb{C}}$:*

$$\begin{cases} \bar\partial u_j + \displaystyle\sum_{i=0}^{m-j} [\bar u_i \wedge u_{i+j}] = 0 \qquad (S_j), \quad \text{if } 1 \le j \le m, \\ \bar\partial u_0 + \partial \bar u_0 + \displaystyle\sum_{j=0}^{m} [u_j \wedge \bar u_j] = 0 \quad (S_0) \end{cases} \tag{Syst}$$

It is equivalent to say that the 1-form

$$\alpha_\lambda = \sum_{j=0}^{m} \lambda^{-j} u_j + \lambda^j \bar u_j = \sum_{j=-m}^{m} \lambda^j \hat\alpha_j \tag{28}$$

satisfies the zero curvature equation:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{C}^*. \tag{29}$$

Definition 2.3 *Let L be a Riemann surface. The m-th (G,τ)-system (with the $(+)$-convention) on L is the equation (*Syst*) as in definition 2.2 but for*

$(u_0, \dots, u_m)$, *$(1,0)$-type 1-form on L with values in* $\prod_{j=0}^m \mathfrak{g}_j^{\mathbb{C}}$.[4]
It is equivalent to say that the 1-form

$$\alpha_\lambda = \sum_{j=0}^{m} \lambda^j u_j + \lambda^{-j} \bar{u}_j = \sum_{j=-m}^{m} \lambda^j \hat{\alpha}_j \tag{30}$$

satisfies the zero curvature equation (29).

Remark 2.1 The difference between the two conventions is that in the first one $\alpha'_\lambda = \sum_{j=0}^m \lambda^{-j} u_j$ involves negative powers of λ whereas in the second one α'_λ involves positive powers of λ (in other words $\hat{\alpha}''_{-j} = 0$, for $j \geq 1$ in the first one whereas $\hat{\alpha}''_j = 0$, for $j \geq 1$ in the second one). In fact the second system is the first system associated to τ^{-1} and vice versa.
The first convention is the traditional one: it was used for harmonic maps into symmetric space (see [11]) and by Hélein-Romon [17, 18, 19] for Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric space – first example of second elliptic integrable system associated to a 4-symmetric space. Then the tradition was perpetuated in [23, 24, 25]. Terng [32], herself, in her definition of the elliptic integrable system uses also this convention. However in [6], this is the second convention which is used.
The $(+)$-convention is in fact the most natural, as we will see, since it uses the automorphism τ whereas in the $(-)$-convention, this is the automorphim τ^{-1} which appears in the geometrical interpretation. But the $(+)$-convention leads to several changes like for example in the DPW method, we must use the Iwasawa decomposition $\Lambda G_\tau^{\mathbb{C}} = \Lambda G_\tau . \Lambda_{\mathcal{B}}^- G_\tau^{\mathbb{C}}$ instead of $\Lambda G_\tau^{\mathbb{C}} = \Lambda G_\tau . \Lambda_{\mathcal{B}}^+ G_\tau^{\mathbb{C}}$ and in particular the holomorphic potential involves positive power of λ instead of negative one as it is the case traditionally. We decided here to continue to perpetuate the tradition as in [25] and to use the first convention. So in the following when we will speak about the m-th elliptic integrable system, it will be according to the definition 2.2.

Notation Sometimes, when it will be necessary to do precision we will denote (Syst) either by (Syst$(m, \mathfrak{g}, \tau)$), (Syst(m, τ)) or simply by (Syst(m)) depending on the context and the needs.
For shortness we will also often say the $(m, \mathfrak{g}, \tau)$-system instead of the m-th $(\mathfrak{g}, \tau)$-system. We will also say the m-th elliptic (integrable) system associated to (the k'-symmetric space) G/G_0.
We will say that a family of 1-forms $(\alpha_\lambda)_{\lambda \in \mathbb{C}^*}$ (denoted by abuse of notation, simply by α_λ) is solution of the $(m, \mathfrak{g}, \tau)$-system (or of (Syst)) if it corresponds to some solution u of this system, according to (28). Therefore α_λ is solution of the $(m, \mathfrak{g}, \tau)$-system if and only if it can be written in the form (28), for some $(1,0)$-type 1-form u on L with values in $\prod_{j=0}^m \mathfrak{g}_{-j}^{\mathbb{C}}$, and satisfies the zero curvature equation (29).

[4] instead of $\prod_{j=0}^m \mathfrak{g}_{-j}^{\mathbb{C}}$.

Definition 2.4 *Let us set* $m_{k'} = \left[\frac{k'+1}{2}\right] = \begin{cases} k \text{ if } k' = 2k \\ k+1 \text{ if } k' = 2k+1 \end{cases}$ *if* $k' > 1$, *and* $m_1 = 0$. *Concerning the* m*-th* (G,τ)*-system, we will say that:*
– we are in the ***primitive case*** *(or that that the system is primitive) if* $0 \leq m < m_{k'}$,
– in the ***determined case*** *(or that that the system is determined) if* $m_{k'} \leq m \leq k'-1$,
– and in the ***underdetermined case*** *(or that that the system is underdetermined) if* $m > k'-1$.
Moreover, the determined system of minimal order $m_{k'}$ *will be called "the minimal determined system", and the one of maximal order* $k'-1$ *will be called "the maximal determined system".*

Let us consider the $\mathfrak{g}$-valued 1-form $\alpha := \alpha_{\lambda=1}$. Then we have $\alpha = \sum_{j=0}^{m} u_j + \bar{u}_j$ according to (28) which is equivalent to $\alpha' = \sum_{j=0}^{m} u_j$, since α is $\mathfrak{g}$-valued.

• In the primitive and determined cases ($m \leq k'-1$), $\sum_{j=0}^{m} \mathfrak{g}_{-j}^{\mathbb{C}}$ is a direct sum so that $u = (u_0, \ldots, u_m)$ and $\sum_{j=0}^{m} u_j = \alpha'$ can be identified via (27) and according to our convention. We will then write simply $u = \alpha'$. In particular we have

$$u_j = \alpha'_{-j} \quad \forall j, 0 \leq j \leq m$$

with $\alpha_j := [\alpha]_{\mathfrak{g}_j}$ $\forall j \in \mathbb{Z}/k'\mathbb{Z}$. Hence in the primitive and determined cases the m-th (G,τ)-system can be considered as a system on α. Consequently, we can recover α_λ from α and we will speak about the "extended Maurer Cartan form" α_λ which is then associated to α by

$$\alpha_\lambda = \sum_{j=1}^{m} \lambda^{-j} \alpha'_{-j} + \alpha_0 + \sum_{j=1}^{m} \lambda^j \alpha''_j$$

according to (28).

• In the underdetermined case, $\sum_{j=0}^{m} \mathfrak{g}_{-j}^{\mathbb{C}}$ is not a direct sum so that to a given α (coming from some solution α_λ of the m-th (G,τ)-system, according to $\alpha = \alpha_{\lambda=1}$) there are a priori many (other) corresponding solutions $u = (u_0, \ldots, u_m)$ since

$$\forall j \in \mathbb{Z}/k'\mathbb{Z}, \alpha'_{-j} = \sum_{i \equiv j[k']} u_i.$$

In fact, we will prove that there are effectively an infinity of other solutions satisfying the condition $\alpha_{\lambda=1} = \alpha$ (see 2.4 for a begining of explanation).

2.2.2 The geometric solution

The equation (29) (as well as (28)) is invariant by gauge transformations by the group $C^\infty(L, G_0)$:

$$U_0 \cdot \alpha_\lambda = \mathrm{Ad}U_0^{-1} \alpha_\lambda - dU_0 . U_0^{-1}.$$

where $U_0 \in C^\infty(L, G_0)$. This allows us to define a **geometric solution** of (Syst) as a map $f\colon L \to G/G_0$ which can be lifted (locally, i.e. in the neighbourhood of each point in L) to some $U\colon L \to G$ (defined locally, on the neighbourhood under consideration) such that $U^{-1}.dU = \alpha_{\lambda=1}$ for some solution α_λ of (Syst) (on the neighbourhood under consideration)[5].

Now, to simplify the exposition, let us suppose that L is simply connected (until the end of 2.2.2). Then $(\alpha_\lambda)_{\lambda\in\mathbb{C}^*} \mapsto \alpha = \alpha_{\lambda=1}$ is a surjective map from the set of solution of (Syst) to the set of Maurer-Cartan forms of lifts of geometric solutions. According the discussion at the end of subsection 2.2.1, this map is bijective in the primitive and determined case ($m \leq k' - 1$) and not injective in the underdetermined case ($m > k' - 1$). By quotienting by $C^\infty(L, G_0)$, we obtain a surjective map π_m with the same properties, taking values in the set of geometric solutions.

Let us precise all that. We suppose, until the end of this subsection, that the automorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}$ is fixed (so that the only data which varies in the $(m, \mathfrak{g}, \tau)$-system is the order m). First, let us give an explicit expression of the space $\mathcal{S}(m)$ of solutions α_λ of the system (Syst(m)), i.e. the solutions of the zero curvature equation (29), which satisfies the equality (28) for some $(1,0)$-type 1-form u on L with values in $\prod_{j=0}^m \mathfrak{g}^{\mathbb{C}}_{-j}$. To do that, we want to express the condition (to be written in the form) (28) as a condition on the loop $\alpha_\bullet\colon \lambda \in S^1 \mapsto \alpha_\lambda$:[6]

$$(28) \iff (\alpha_\bullet \in \Lambda_m\mathfrak{g}_\tau \quad \text{and} \quad \alpha'_\bullet \in \Lambda^-\mathfrak{g}^{\mathbb{C}}_\tau) \tag{31}$$

where

$$\begin{aligned}
\Lambda\mathfrak{g}_\tau &= \{\eta_\bullet \in H^1(S^1, \mathfrak{g}) |\, \eta_{\omega\lambda} = \tau(\eta_\lambda), \forall \lambda \in S^1\} \\
\Lambda_m\mathfrak{g}_\tau &= \{\eta_\bullet \in \Lambda\mathfrak{g}_\tau |\, \eta_\lambda = \sum_{|j|\leq m} \lambda^j \hat{\eta}_j\} \\
\Lambda^-\mathfrak{g}^{\mathbb{C}}_\tau &= \{\eta_\bullet \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau |\, \eta_\lambda = \sum_{j\leq 0} \lambda^j \hat{\eta}_j\}
\end{aligned}$$

and ω is a k'-th root of unity; so that

$$\mathcal{S}(m) = \{\alpha_\bullet \in \mathcal{C}(T^*L \otimes \Lambda_m\mathfrak{g}_\tau) |\, \alpha'_\bullet \in \Lambda^-\mathfrak{g}^{\mathbb{C}}_\tau \quad \text{and} \quad d\alpha_\bullet + \frac{1}{2}[\alpha_\bullet \wedge \alpha_\bullet] = 0\}. \tag{32}$$

Let us remark that the condition $\alpha'_\bullet \in \Lambda^-\mathfrak{g}^{\mathbb{C}}_\tau$ can be interpretated as a condition of $\mathbb{C}$-linearity. Indeed, the Banach vector space $\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0$ is naturally endowed with the complex structure defined by the following decomposition

$$(\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0)^{\mathbb{C}} = \Lambda\mathfrak{g}^{\mathbb{C}}_\tau/\mathfrak{g}^{\mathbb{C}}_0 = \Lambda^-_*\mathfrak{g}_\tau \oplus \Lambda^+_*\mathfrak{g}_\tau, \tag{33}$$

[5] Our study, in the present paper, is local so one can suppose (when it is necessary to do) either that L is implicitely simply connected or that all lifts and integrations are done locally. We consider that these considerations are implicit and will not precise these most of the time.

[6] Remark that $\alpha_\bullet$ determines $(\alpha_\lambda)_{\lambda\in\mathbb{C}^*}$, when this latter satisfies (28).

where $\Lambda^{\pm}_{*}\mathfrak{g}_\tau = \{\eta_\bullet \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau |\, \eta_\lambda = \sum_{j\gtrless 0} \lambda^j \hat{\eta}_j\}$. Then the condition $\alpha'_\bullet \in \Lambda^- \mathfrak{g}^{\mathbb{C}}_\tau$ means that $[\alpha'_\bullet]_*\colon TL \to (\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0)^{\mathbb{C}}$ is $\mathbb{C}$-linear, where $[\ ]_*$ denotes the component in $\Lambda_*\mathfrak{g}_\tau = \{\eta_\bullet \in \Lambda\mathfrak{g}_\tau | \eta_\lambda = \sum_{j\neq 0} \lambda^j \hat{\eta}_j\} \cong \Lambda\mathfrak{g}_\tau/\mathfrak{g}_0$.

Now let us integrate our setting. Firstly, let us define the twisted loop group

$$\Lambda G_\tau = \{U_\bullet \in H^1(S^1, G) | U_{\omega\lambda} = \tau\,(U_\lambda)\}.$$

Then, let us set

$$\begin{aligned}
\mathcal{E}^m &= \{U_\bullet\colon L \to \Lambda G_\tau | U_\lambda(0) = 1, \forall \lambda \in S^1; \alpha_\lambda := U_\lambda^{-1}.dU_\lambda \text{ is a solution of } (\mathrm{Syst}(m))\} \\
\mathcal{E}^m_1 &= \{U\colon L \to G | \exists U_\bullet \in \mathcal{E}^m, U = U_1\} \\
\mathcal{G}^m &= \{f\colon L \to G/G_0 \text{ geom. sol. of } (\mathrm{Syst}(m)), f(0) = 1.G_0\} \\
\mathcal{G}^m_\bullet &= \{f_\bullet = \pi_{G/G_0} \circ U_\bullet,\, U_\bullet \in \mathcal{E}^m\}
\end{aligned}$$

The space of geometric solutions is obviously obtained from the space of *extended geometric solutions* $\mathcal{G}^m_\bullet$ by $\mathcal{G}^m = \mathcal{G}^m_1$. Moreover $\mathcal{S}(m) \simeq \mathcal{E}^m$ is determined by $\mathcal{G}^m_\bullet$ because of the gauge invariance: $\mathcal{E}(m).\mathcal{K} \subset \mathcal{E}(m)$ where $\mathcal{K} = C^\infty_*(L, G_0) = \{U \in C^\infty(L, G_0) | U(0) = 1\}$ so that we can write $\mathcal{G}^m_\bullet = \mathcal{E}(m)/\mathcal{K}$. Consequently, we have also $\mathcal{G}^m = \mathcal{E}^m_1/\mathcal{K}$.
Finally, we obtain the following diagram

$$\begin{array}{ccccccc}
\mathcal{S}(m) & \xrightarrow[\cong]{\text{int}} & \mathcal{E}^m & \xrightarrow{\pi_{\mathcal{K}}} & \mathcal{E}^m/\mathcal{K} & = & \mathcal{G}^m_\bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_m} \\
\mathcal{S}(m)_1 & \xrightarrow[\cong]{\text{int}} & \mathcal{E}^m_1 & \longrightarrow & \mathcal{E}^m_1/\mathcal{K} & = & \mathcal{G}^m
\end{array}.$$

Then π_m is bijective for $m \leq k' - 1$ and not injective for $m > k' - 1$. Therefore, in the primitive and determined case, we can consider that $(\mathrm{Syst}(m))$ is a system on the map f corresponding to α (since the Maurer-Cartan equation for α is always contained in $(\mathrm{Syst}(m))$ according to (29), and thus the existence of f is always guaranteed). This system on f is an elliptic PDE on f of order ≤ 2. In particular, we are led to the following definition:

Definition 2.5 *Given a $\mathfrak{g}$-valued Maurer-Cartan 1-form α on L, we define **the geometric map corresponding to** α, as $f = \pi_{G/G_0} \circ U$, U integrating α: $U^{-1}.dU = \alpha$, $U(0) = 1$.*

Let us summarize:

Proposition 2.1 *The natural map $\pi_m\colon \mathcal{G}^m_\bullet \to \mathcal{G}^m$ from the set of extended geometric solutions of the $(m, \mathfrak{g}, \tau)$-system into the set of geometric solutions is surjective. Moreover, it is bijective in the primitive and determined cases ($m \leq k' - 1$) and not injective in the underdetermined case ($m > k' - 1$). Moreover, in the primitive and determined cases, the $(m, \mathfrak{g}, \tau)$-system - on the (family of) 1-form α_λ - is in fact a system on the 1-form $\alpha := \alpha_{\lambda=1}$, itself equivalent to an elliptic PDE of order ≤ 2 on the corresponding geometric map $f\colon L \to G/G_0$.*

Furthermore, let us interpret the $\mathbb{C}$-linearity of $[\alpha'_\bullet]_*: TL \to (\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0)^{\mathbb{C}}$ in terms of the corresponding geometric solution $f_\bullet: L \to \Lambda G_\tau/G_0$, defined by $f_\bullet = \pi_{G/G_0} \circ U_\bullet$ where $U_\bullet$ integrates $\alpha_\bullet$. Firstly, the complex structure defined in $\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0$ by (33) is $\mathrm{Ad}G_0$-invariant so that it defines a ΛG_τ-invariant complex structure on the homogeneous space $\Lambda G_\tau/G_0$. Therefore the $\mathbb{C}$-linearity of $[\alpha'_\bullet]_*$ means exactly that $f_\bullet: L \to \Lambda G_\tau/G_0$ is holomorphic. Now, let us interpret the condition $\alpha_\bullet \in \Lambda_m\mathfrak{g}_\tau$ in terms of the map $f_\bullet$. Let us consider the following $\mathrm{Ad}G_0$-invariant decomposition

$$\Lambda\mathfrak{g}_\tau/\mathfrak{g}_0 = \Lambda_{m*}\mathfrak{g}_\tau \oplus \Lambda_{>m}\mathfrak{g}_\tau$$

where $\Lambda_{m*}\mathfrak{g}_\tau = \Lambda_m\mathfrak{g}_\tau \cap \Lambda_*\mathfrak{g}_\tau$ and $\Lambda_{>m}\mathfrak{g}_\tau = \{\eta_\bullet \in \Lambda\mathfrak{g}_\tau |\, \eta_\lambda = \sum_{|j|>m} \lambda^j \hat{\eta}_j\}$, which gives rise respectively to some ΛG_τ-invariant splitting

$$T(\Lambda G_\tau/G_0) = \mathcal{H}^\Lambda_m \oplus \mathcal{V}^\Lambda_m.$$

Then $\mathcal{H}^\Lambda_m$ and $\mathcal{V}^\Lambda_m$ inherit respectively the qualificatifs horizontal and vertical subbundle respectively. Therefore, in the same spirit as [11] (remark 2.5 and proposition 2.6), the equation (32) gives us the following familiar twistorial caracterisation

Proposition 2.2 *A map $f_\bullet: L \to \Lambda G_\tau/G_0$ is an extended geometric solution of the $(m, \mathfrak{g}, \tau)$-system if and only if it is holomorphic and horizontal.*

2.2.3 The increasing sequence of spaces of solutions: $(\mathcal{S}(m))_{m\in\mathbb{N}}$

Again, we suppose in all 2.2.3 that the automorphism τ is fixed and that L is simply connected. Then according to the realisation of (Syst(m)) in the forms (29) and (28), we see that any solution of (Syst(m)) is solution of (Syst(m')) for $m \leq m'$ (take $u_j = 0$ for $m < j \leq m'$). More precisely, (Syst(m)) is a reduction of (Syst(m')): (Syst(m)) is obtained from (Syst(m')) by putting $u_j = 0$, $m < j \leq m'$, in (Syst(m')). In particular, $\mathcal{S}(m) \subset \mathcal{S}(m')$ for $m \leq m'$; so that any solution in the primitive case ($m < m_{k'}$) is solution of any determined system ($m_{k'} \leq m \leq k'-1$), and any solution of a determined system is solution of any underdetermined system ($m > k'-1$). Besides we see that $\pi_{m'|\mathcal{G}^m_\bullet} = \pi_m$ if $m \leq m'$. In particular, $\pi_{m'}(\mathcal{G}^m_\bullet) = \mathcal{G}^m$. Thus we can set $\mathcal{S}(\infty) = \cup_{m\in\mathbb{N}}\mathcal{S}(m)$ and $\mathcal{G}^\infty = \cup_{m\in\mathbb{N}}\mathcal{G}^m$, then we can define a surjective map $\pi_\infty: \mathcal{E}^\infty/\mathcal{K} \to \mathcal{G}^\infty$ such that $\pi_\infty|_{\mathcal{E}^m/\mathcal{K}}$ is a bijection onto $\mathcal{G}^m$ for each $m \leq k'-1$.
Moreover any geometric solution $f \in \mathcal{G}^\infty$ has an order m which is the smaller m' such that $f \in \mathcal{G}^{m'}$. Then for any solution $\alpha_\lambda \in \mathcal{S}(\infty)$ giving rise to f, we have $\alpha_\lambda \in \mathcal{S}(m)$ (m is the maximal power on λ of α_λ, which does not depend on the choice of α_λ since these are all equivalent modulo the gauge group $\mathcal{K}$). Thus we have $\pi^{-1}_\infty(\mathcal{G}^m) = \mathcal{G}^m_\bullet$ or equivalently $\pi_\infty(\mathcal{G}^{m+1}_\bullet \setminus \mathcal{G}^m_\bullet) = \mathcal{G}^{m+1} \setminus \mathcal{G}^m$.

Remark 2.2 We can call $\mathcal{S}(\infty)$ the $(\mathfrak{g}, \tau)$-system, and then we can speak about its subsystem of order m, namely $\mathcal{S}(m)$. In particular, we have the following caracterization:

$$\mathcal{S}(\infty) = \{\alpha_\bullet \in \mathcal{C}(T^*L \otimes \Lambda_{(\infty)}\mathfrak{g}_\tau) |\, \alpha'_\bullet \in \Lambda^-\mathfrak{g}_\tau \quad \text{and} \quad d\alpha_\bullet + \frac{1}{2}[\alpha_\bullet \wedge \alpha_\bullet] = 0\}$$

where $\Lambda_{(\infty)}\mathfrak{g}_\tau = \cup_{m\in\mathbb{N}}\Lambda_m\mathfrak{g}_\tau$.

The primitive and determined cases ($m \leq k'-1$) Now, let us apply the previous discussion (about the increasing sequence $(\mathcal{S}(m))_{m\in\mathbb{N}}$) to the study of the determined case. Let us recall that in this case, we can consider that the system (Syst(m)) deals only with $\mathfrak{g}$-valued 1-forms α.

Proposition 2.3 *The solutions of a determined system* (Syst(m)), $m_{k'} \leq m \leq k'-1$, *are exactly the solutions of the maximal determined system, i.e.* (Syst($k'-1$)), *which satisfy the* holomorphicity conditions*:*

$$\alpha''_{-j} = 0,\ m - m_{k'} + 1 \leq j \leq m_{k'}.$$

Moreover, the solutions of a primitive system (Syst(m)), $1 \leq m \leq m_{k'} - 1$, *are the solutions of the minimal determined system, i.e.* (Syst($m_{k'}$)), *which satisfy the* horizontality conditions*:*

$$\alpha_k = \alpha_{\pm(k-1)} = \ldots \alpha_{\pm(m+1)} = 0.$$

The non injectivity of π_m in the underdetermined case Now, let us turn ourself to the underdetermined case.

Proposition 2.4 *In the underdetermined case, $m > k' - 1$, the map* $\mathrm{ev}_1 \colon \alpha_\bullet \in \mathcal{S}(m) \to \alpha \in \mathcal{S}(m)_1$ *is not injective.*

Sketch of Proof. Since in the underdetermined case, we have $m \geq k'$, then $\mathcal{S}(m) \supset \mathcal{S}(k')$ and thus it suffices to prove the non-injectivity on $\mathcal{S}(k')$. Let α_λ be a solution of (Syst(k')) and α its value at $\lambda = 1$. Then let us consider the fibre $\mathrm{ev}_1^{-1}(\alpha)$. Let us denote by $\tilde{\alpha}_\lambda$ the current element in $\mathrm{ev}_1^{-1}(\alpha)$, varying arbitrary, to differentiate it from the fixed element α_λ. Then let us consider the corresponding $(1,0)$-type 1-form $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_{k'-1}, \tilde{u}_{k'})$ taking values in $\prod_{j=0}^{k'} \mathfrak{g}^{\mathbb{C}}_{-j} = \left(\oplus_{j=0}^{k'-1}\mathfrak{g}^{\mathbb{C}}_{-j}\right) \times \mathfrak{g}_0$ and then let us set $u = (\underline{\tilde{u}}, \tilde{u}_{k'})$. Then we have by definition $\underline{\tilde{u}} + \tilde{u}_{k'} = \alpha'$ so that $\tilde{u}_0 = \alpha'_0 - \tilde{u}_{k'}$ and $\tilde{u}_j = \alpha'_{-j}$. Therefore, the variable $\tilde{u}$ can be parametrized, in $\mathrm{ev}_1^{-1}(\alpha)$, by $\tilde{u}_{k'}$. Furthermore, injecting these two previous equations in the system (Syst(k')) satisfied by $\tilde{u}$, this one becomes a system on $\tilde{u}_{k'}$ with some parameters depending on α, which by a straightforward computation can be written in the form

$$\begin{cases} \bar{\partial}\tilde{u}_{k'} + [\alpha''_0 \wedge \tilde{u}_{k'}] - [\overline{\tilde{u}}_{k'} \wedge \tilde{u}_{k'}] = 0 \quad \text{(a)} \\ \beta + [\gamma \wedge (\tilde{u}_{k'} + \overline{\tilde{u}}_{k'})] = 0 \quad \text{(b)} \end{cases}$$

Hence, $\mathrm{ev}_1^{-1}(\alpha)$ is (parametrized by) the solutions of (a) which lie in the affine space defined by (b). Therefore since we already know a solution of the system (a)-(b), namely the component $u_{k'}$ of the fixed solution α_λ, we can now apply the implicit functions theorem to the following initial value problem

$$\begin{cases} \bar{\partial}\tilde{u}_{k'} + [\alpha''_0 \wedge \tilde{u}_{k'}] - [\overline{\tilde{u}}_{k'} \wedge \tilde{u}_{k'}] = 0 \\ \tilde{u}_{k'}(z_0) = x_0 \in \{\xi_0 \in\ g_0 |\, [\gamma(z_0) \wedge (\xi_0 + \bar{\xi}_0)] = 0\} \end{cases}$$

restricted to the affine space of functions, defined by (b).

2.2.4 The decreasing sequence $(\mathrm{Syst}(m,\tau^p))_{p/k'}$

Any solution of the m-th $(\mathfrak{g},\tau)$-system is solution of the m-th $\mathfrak{g}$-system (take $\tau = \mathrm{Id}$, i.e. $u = (u_0,\dots,u_m) \in (\mathfrak{g}^{\mathbb{C}})^{m+1}$). More precisely, the m-th $(\mathfrak{g},\tau)$-system is the restriction to $\oplus_{j=0}^{m}\mathfrak{g}_{-j}(\tau)$ of the m-th $\mathfrak{g}$-system.
More generally, for any $p \in \mathbb{N}^*$ such that p divides k', the m-th $(\mathfrak{g},\tau)$-system is the m-th $(\mathfrak{g},\tau^p)$-system restricted to $\oplus_{j=0}^{m}\mathfrak{g}_{-j}(\tau)$, or equivalently - in terms of $\alpha_\lambda \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau^p}$ - restricted to $\Lambda\mathfrak{g}^{\mathbb{C}}_{\tau}$.

2.3 The minimal determined case

We study here the elliptic system $(\mathrm{Syst}(m))$ in the minimal determined case and by the way its subcase the primitive case. Let us recall again that in this case, we can consider that the system $(\mathrm{Syst}(m))$ deals only with Maurer-cartan forms α and consequently also with geometric maps f. Then we have to translate the equations on α into geometric conditions on f. This is what we will begin to do now.
The minimal determined case splits into two cases.

2.3.1 The even minimal determined case: $k' = 2k$ and $m = k$

Let us recall the following decomposition

$$\mathfrak{g}^{\mathbb{C}} = \left(\mathfrak{g}_{-(k-1)} \oplus \dots \oplus \mathfrak{g}_{-1}\right) \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus (\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}) \oplus \mathfrak{g}_k^{\mathbb{C}}.$$

The system $(\mathrm{Syst}(k,\tau))$ can be written

$$\begin{cases} \alpha''_{-j} = 0,\ 1 \le j \le k-1 & (H_j) \\ d\alpha + \dfrac{1}{2}[\alpha\wedge\alpha] = 0 & (\mathrm{MC}) \\ \bar\partial\alpha'_{-k} + [\alpha''_0 \wedge \alpha'_{-k}] = 0 & (S_k) \end{cases}. \tag{34}$$

More precisely the equations (S_j), $0 \le j \le k-1$, of $(\mathrm{Syst}(k,\tau))$ are respectively the projection on $\mathfrak{g}_{-j}, 0 \le j \le k-1$, of (MC) (owing to the holomorphicity conditions (H_j) given by proposition 2.3). Moreover, the projection of (MC) on $\mathfrak{g}_k$ gives us

$$d\alpha_k + [\alpha_0 \wedge \alpha_k] = 0$$

which is the real part of (S_k). Hence the only new information (in addition to (MC) and (H)) given by the minimal determined elliptic integrable system in the even case is the imaginary part of (S_k):

$$d(*\alpha_k) + [\alpha_0 \wedge (*\alpha_k)] = 0 \quad (E_k)$$

which is as we will see the vertical part of a harmonic map equation. Hence $(\mathrm{Syst}(k,\tau))$ is equivalent to

$$\begin{cases} \alpha''_{-j} = 0,\ 1 \le j \le k-1 & (H_j) \\ d(*\alpha_k) + [\alpha_0 \wedge (*\alpha_k)] = 0 & (E_k) \\ d\alpha + \dfrac{1}{2}[\alpha\wedge\alpha] = 0 & (\mathrm{MC}) \end{cases}.$$

Besides (S_k) can be written in the form:

$$[\bar{\partial}(\mathrm{Ad}U(\alpha'_k))]_{\mathfrak{g}_k} = 0.$$

where U integrates α. In term of $f\colon L \to G/G_0$, the projection of U, this last equation means (as we will see in section 4)

$$(\nabla^v)''\partial^v f = 0$$

where ∇^v is the vertical part of the Levi-Civita connection ∇ on the Riemanni-anhomogeneous space G/G_0, the vertical and horizontal spaces are defined by $\mathcal{V} = [\mathfrak{g}_k]$ and $\mathcal{H} = [\mathfrak{m}]$ since we can do the splitting: $T(G/G_0) = [\mathfrak{m}] \oplus [\mathfrak{g}_k]$. Then the equation (E_k) is equivalent to (see section 4)

$$d^{\nabla^v}(*d^v f) = 0 \Longleftrightarrow \tau^v(f) := \mathrm{Tr}_b(\nabla^v d^v f) = 0$$

(for any hermitian metric b on the Riemann surface L). It is easy to see that the equation (E_k) is a vertically harmonic map equation ($\tau^v(f) = 0$) for the canonical connection $\nabla^0 = d + \alpha_0$. In fact, we will see that in the vertical subbundle $\mathcal{V}$ we have $\nabla^v = \nabla^0$ (see section 4).

The primitive case The m-primitive case is obtained by putting $\alpha_k = \alpha_{\pm(k-1)} = \ldots \alpha_{\pm(m+1)} = 0$ in the minimal determined case (34). In particular $\alpha_k = 0$ and (S_k) is trivial so that the only additionnal conditions on the geometric map $f\colon L \to G/G_0$ (whose existence is guaranted by (MC)) are the equations (H_j): $\alpha''_{-j} = \alpha'_j = 0, 1 \leq j \leq m$, and $\alpha_k = \alpha_{\pm(k-1)} = \ldots \alpha_{\pm(m+1)} = 0$ (which both, let us recall it, come from $\alpha' = u \in \oplus_{j=0}^m \mathfrak{g}_{-j}$).

Proposition 2.5 *Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an order $2k$ automorphism, and an integer $m < k$ then the m-th elliptic integrable system* $(\mathrm{Syst}(m,\tau))$ *means that the geometric map $f\colon L \to G/G_0$ satisfies*

$$\partial f \in \oplus_{j=1}^m [\mathfrak{g}_{-j}] \subset T(G/G_0)^{\mathbb{C}}.$$

Proof. Let f be the geometric map corresponding to the Maurer-Cartan form α, that we integrate by $U\colon L \to G$, then we have $\partial f = \mathrm{Ad}U(\alpha'_{\mathfrak{n}})$ and α is solution of $(\mathrm{Syst}(m,\tau))$ if and only if

$$\alpha'_{\mathfrak{n}} = \alpha'_{-1} + \ldots + \alpha'_{-m} \in \oplus_{j=1}^m \mathfrak{g}_{-j} \Longleftrightarrow \partial f \in \oplus_{j=1}^m [\mathfrak{g}_{-j}].$$

This completes the proof. □

Remark 2.3 In particular, in the primitive case f is horizontal ($\alpha_k = 0$). Therefore (S_k) is trivial and (owing to the holomorphicity conditions $(H_j), 1 \leq j \leq k-1$) the free curvature equation (29) is equivalent to (MC) in the primitive case.

Definition 2.6 *We will call m-primitive map (into the locally $(2k)$-symmetric space G/G_0) a geometric solution of the system* $(\mathrm{Syst}(m,\tau))$ *for $m < k$.*

Geometric interpretation of the equations (H_j)**.** For $m < k$, let F^m be the f-structure on $N = G/G_0$ defined by the following (eigenspace) decomposition:

$$\begin{array}{rcccccc} TN^{\mathbb{C}} & = & \left(\oplus_{j=1}^{m}[\mathfrak{g}_{-j}]\right) & \oplus & \left(\oplus_{|j|>m}[\mathfrak{g}_j]\right) & \oplus & \left(\oplus_{j=1}^{m}[\mathfrak{g}_j]\right) \\ & = & T^+N & \oplus & T^0N & \oplus & T^-N \end{array} .$$

Then according to proposition 2.5 we have

Theorem 2.1 *A map $f\colon L \to G/G_0$ is m-primitive if and only if it is F^m-holomorphic.*

Remark 2.4 The equations (H_j): $\alpha''_{-j} = 0$, $1 \leq j \leq m$, on a Maurer-Cartan 1-form α means that the corresponding geometric map $f\colon L \to G/G_0$ satisfies $\mathrm{pr}_m \circ (df \circ j_L) = F^m \circ df$ where j_L is the complex structure in L, and $\mathrm{pr}_m\colon TN \to \oplus_{j=1}^{m}[\mathfrak{m}_j]$ is the projection on $\oplus_{j=1}^{m}[\mathfrak{m}_j]$ along $\left(\oplus_{j=m+1}^{k-1}[\mathfrak{m}_j]\right) \oplus [\mathfrak{g}_k]$. This means that the projection $\mathrm{pr}_m \circ df\colon TL \to \oplus_{j=1}^{m}[\mathfrak{m}_j]$ is a morphism of complex vector bundle. Let us denote $C^\infty_m(L, G/G_0) = \{f \in C^\infty(L, G/G_0) |\, df \in \oplus_{j=1}^{m}[\mathfrak{m}_j]\}$. Then we have the following equivalences between the Maurer-Cartan 1-form α and its geometric map:

$$\begin{array}{rcl} \alpha \in \mathfrak{g}_0 \oplus (\oplus_{j=1}^{m}\mathfrak{m}_j) & \iff & f \in C^\infty_m(L, G/G_0) \\ (H_j):\ \alpha''_{-j} = 0,\ 1 \leq j \leq m & \iff & \mathrm{pr}_m \circ (df \circ j_L) = F^m \circ df \end{array} \tag{35}$$

Then additionning these two equivalences, we recover the equivalence: "α solves $(\mathrm{Syst}(m, \tau))$" $\iff$ "f is F^m-holomorphic". Moreover, the equations $\alpha''_{-j} = 0$, $1 \leq j \leq k-1$, mean that f is horizontally holomorphic.

Theorem 2.2 *Let α be a $\mathfrak{g}$-valued 1-form on L and f its geometric map. The following statements are equivalent:*

(i) $\alpha''_{-j} = 0$, $1 \leq j \leq k-1$

(ii) *f is horizontally holomorphic: $(df \circ j_L)^{\mathcal{H}} = F^{k-1} \circ df$, $\mathcal{H} = [\mathfrak{m}]$ being the horizontal space and ${F^{k-1}}_{|\mathcal{H}}$ defining a complex structure on $\mathcal{H}$.*

So that we can conclude: **the even minimal determined system** $(\mathrm{Syst}(k, \tau))$ **means that the geometric map f is horizontally holomorphic and vertically harmonic.**

Remark 2.5 We can express what precedes in terms of the projection map $\bar{\pi}_{G/G_0}\colon \alpha \to f$ defined as follows. Let $\mathcal{MC}$ be the set of $\mathfrak{g}$-valued Maurer-Cartan 1-form on L and for $m < k$, $\mathcal{MC}^m$ the subset of Maurer-Cartan 1-form taking values in $\mathfrak{g}_0 \oplus (\oplus_{j=1}^{m}\mathfrak{m}_j)$, then $\bar{\pi}_{G/G_0}\colon \mathcal{MC} \to C^\infty(L, G/G_0)$ is defined by:

$$\bar{\pi}_{G/G_0}\colon \alpha \in \mathcal{MC} \xrightarrow{\ \mathrm{int}\ } U \in C^\infty_*(L, G) \xrightarrow{\ \pi_{G/G_0}\ } f = \pi_{G/G_0} \circ U \in C^\infty(L, G/G_0).$$

The preceding results can be summarized as follows: for any $m < k$

$$\bar{\pi}_{G/G_0}(\mathcal{MC}^m) = C^\infty_m(L, G/G_0) \quad \text{and} \quad \bar{\pi}_{G/G_0}(\mathcal{S}(m)) = \mathrm{Hol}((L, j), (G/G_0, F^m)),$$

the set of F^m-holomorphic maps; and the equations (H_j), $1 \leq j \leq m$, in $\mathcal{MC}$ are transformed by $\bar{\pi}_{G/G_0}$ into the equation $\mathrm{pr}_m \circ (df \circ j_L) = F^m \circ df$ in $C^\infty(L, G/G_0)$.

2.3.2 The minimal determined odd case

The order of the automorphism τ is odd $k' = 2k+1$, and $m = k+1$. Let us recall the following decomposition

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}) \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus (\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k)\,.$$

The equations (S_j), $0 \leq j \leq k-1$, of $(\mathrm{Syst}(k+1,\tau))$ are respectively the projection on $\mathfrak{g}_{-j}$, $0 \leq j \leq k-1$, of the Maurer-Cartan equation (MC) (owing to the holomorphicity conditions given by proposition 2.3). Hence the elliptic system $(\mathrm{Syst}(k+1,\tau))$ can be written:

$$\left\{\begin{array}{ll} \alpha''_{-j} = 0,\ 1 \leq j \leq k-1 & (H_j) \\ \bar{\partial}\alpha'_k + [\alpha''_0 \wedge \alpha'_k] = 0 & (S_{k+1}) \\ \bar{\partial}\alpha'_{-k} + [\alpha''_0 \wedge \alpha'_{-k}] + [\alpha''_1 \wedge \alpha'_k] = 0 & (S_k) \\ d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0 & (\mathrm{MC}) \end{array}\right. . \tag{36}$$

Then we see that the projection on $\mathfrak{g}_{-k}$ of (MC):

$$d\alpha_{-k} + [\alpha_0 \wedge \alpha_{-k}] + [\alpha_1 \wedge \alpha_k] = 0 \tag{37}$$

is nothing but $(S_k) + (\overline{S_{k+1}})$.

Now we have to distinguish two cases.

• Let us suppose that $k \geq 2$, then we have

$$(S_k) \vee (S_{k+1}) \equiv (S_k) + (S_{k+1}) \Longleftrightarrow \left[\bar{\partial}(\mathrm{Ad}U(\alpha'_{\mathfrak{m}_k}))\right]_{\mathfrak{m}_k} = 0$$

where U integrates α. For the last equivalence, just do the computation:

$$\begin{array}{rcl} \left[\bar{\partial}(\mathrm{Ad}U(\alpha'_{\mathfrak{m}_k}))\right]_{\mathfrak{m}_k} & = & \bar{\partial}\alpha'_{\mathfrak{m}_k} + [\alpha'' \wedge \alpha'_{\mathfrak{m}_k}]_{\mathfrak{m}_k} \\ & = & \bar{\partial}\alpha'_{\mathfrak{m}_k} + [\alpha''_0 \wedge \alpha'_{\mathfrak{m}_k}] + [\alpha''_1 \wedge \alpha'_k] + [\alpha''_{-1} \wedge \alpha'_{-k}] \\ & = & (S_k) + (S_{k+1}) \end{array}$$

since $\alpha''_{-1} = 0$. Hence we obtain that

$$(\mathrm{Syst}) \Longleftrightarrow \left\{\begin{array}{ll} \alpha''_{-j} = 0,\ 1 \leq j \leq k-1 & (H_j) \\ \bar{\partial}\alpha'_k + [\alpha''_0 \wedge \alpha'_k] = 0 & (S_{k+1}) \\ d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0 & (\mathrm{MC}) \end{array}\right. \Longleftrightarrow$$

$$\left\{\begin{array}{ll} \alpha''_{-j} = 0,\ 1 \leq j \leq k-1 & (H_j) \\ \bar{\partial}\alpha'_{-k} + [\alpha''_0 \wedge \alpha'_{-k}] + [\alpha''_1 \wedge \alpha'_k] = 0 & (S_k) \\ d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0 & (\mathrm{MC}) \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} \alpha''_{-j} = 0,\ 1 \leq j \leq k-1 & (H_j) \\ \left[\bar{\partial}(\mathrm{Ad}U(\alpha'_{\mathfrak{m}_k}))\right]_{\mathfrak{m}_k} = 0 & (S_{\mathfrak{m}_k}) \\ d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0 & (\mathrm{MC}) \end{array}\right.$$

In terms of the geometric map $f\colon L \to G/G_0$, we have according to remark 1.3 (see also section 5) the following geometric interpretation:

$$(S_{\mathfrak{m}_k}) \Longleftrightarrow ((\nabla^1)''\partial^v f)^v = 0,$$

where the splitting $TN = \mathcal{H} \oplus \mathcal{V}$ is defined by $\mathcal{H} = [\mathfrak{m}']$, $\mathcal{V} = [\mathfrak{m}_k]$ and $\mathfrak{m}' = \oplus_{j=1}^{k-1}\mathfrak{m}_j$. Moreover since $\mathrm{Re}(S_{\mathfrak{m}_k})$ is

$$d\alpha_{\mathfrak{m}_k} + [\alpha_0 \wedge \alpha_{\mathfrak{m}_k}] + [\alpha_{\mathfrak{m}_1} \wedge \alpha_{\mathfrak{m}_k}]_{\mathfrak{m}_k} = 0$$

which is nothing but $[\mathrm{MC}]_{\mathfrak{m}_k}$ (in the presence of (H)), the projection of (MC) on $\mathfrak{m}_k$, then the only new information (in addition to (MC) and (H)) given by the determined elliptic integrable system in the odd case is the imaginary part of $(S_{\mathfrak{m}_k})$ which means that f is vertically harmonic (with respect to ∇^1):

$$\mathrm{Im}(S_{\mathfrak{m}_k}) : d{*}\alpha_{\mathfrak{m}_k} + [\alpha_0 \wedge {*}\alpha_{\mathfrak{m}_k}] + [\alpha_{\mathfrak{m}_1} \wedge {*}\alpha_{\mathfrak{m}_k}]_{\mathfrak{m}_k} = 0 \Longleftrightarrow \tau_1^v(f) := \mathrm{Tr}((\nabla^1)^v d^v f) = 0.$$

• Let us suppose that $k = 1$, then we have

$$(S_1) \Longleftrightarrow \left[\bar\partial(\mathrm{Ad}U(\alpha'_{\mathfrak{m}}))\right]_{\mathfrak{m}}^{(1,0)} = 0$$

where U integrates α and $[\ ]^{(1,0)}$ denotes the $(1,0)$-component with respect to the canonical complex structure in N defined by the decomposition (26), i.e. in our case $TN^{\mathbb{C}} = [\mathfrak{g}_{-1}] \oplus [\mathfrak{g}_1]$. Indeed we have

$$\begin{aligned}
\left[\bar\partial(\mathrm{Ad}U(\alpha'_{\mathfrak{m}}))\right]_{\mathfrak{m}} &= \bar\partial\alpha'_{\mathfrak{m}} + [\alpha'' \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} \\
&= \bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_0 \wedge \alpha'_{\mathfrak{m}}] + [\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} \\
&= \bar\partial\alpha'_{-1} + [\alpha''_0 \wedge \alpha'_{-1}] + [\alpha''_1 \wedge \alpha'_1] \\
&+ \bar\partial\alpha'_1 + [\alpha''_0 \wedge \alpha'_1] + [\alpha''_{-1} \wedge \alpha'_{-1}]
\end{aligned}$$

the up term being the $(1,0)$-component and the down one, the $(0,1)$-component. Then recalling that $(S_1) + \overline{(S_2)}$ is the projection on $\mathfrak{g}_{-1}$ of (MC), we obtain that

$$(\mathrm{Syst}) \Leftrightarrow \left\{ \begin{array}{l} (S_2) \\ (\mathrm{MC}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (S_1) \\ (\mathrm{MC}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} \left[\bar\partial(\mathrm{Ad}U(\alpha'_{\mathfrak{m}}))\right]_{\mathfrak{m}}^{(1,0)} = 0 & (S_{\mathfrak{m}}^{(1,0)}) \\ d\alpha + \dfrac{1}{2}[\alpha \wedge \alpha] = 0 & (\mathrm{MC}) \end{array} \right.$$

and the only new information (in addition to (MC)) given by the minimal determined elliptic integrable system in this case is $(S_{\mathfrak{m}}^{(1,0)})$.
In terms of the geometric map, $f\colon L \to G/G_0$, we have according to remark 1.3 (see also section 5) the following geometric interpretation:

$$(S_{\mathfrak{m}}^{(1,0)}) \Longleftrightarrow \left[(\nabla^1)''\partial f\right]^{(1,0)} = 0,$$

we will say that f is holomorphically harmonic (see section 5 for a precise definition).

The primitive case. The m-primitive case is obtained by putting $\alpha_j = 0$, $m+1 \leq |j| \leq k-1$, and $\alpha''_{-k} = \alpha'_k = 0$ in the minimal determined case (36). As in the even case we obtain:

Proposition 2.6 *Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an order $2k+1$ automorphism, and an interger $m \leq k$ then the m-elliptic integrable system* $(\mathrm{Syst}(m,\tau))$ *means that the geometric map $f\colon L \to G/G_0$ satisfies:*

$$\partial f \in \oplus_{j=1}^{m}[\mathfrak{g}_{-j}] \subset T(G/G_0)^{\mathbb{C}}.$$

Geometric interpretation of the equations (H_j). Let F be the canonical complex structure on $N = G/G_0$ (see (26)) and set $F^m := \mathrm{pr}_m \circ F = F \circ \mathrm{pr}_m$ for $m \leq k$, where $\mathrm{pr}_m\colon TN \to \oplus_{j=1}^{m}[\mathfrak{m}_j]$ is the projection on $\oplus_{j=1}^{m}[\mathfrak{m}_j]$ along $\oplus_{j\geq m+1}^{k}[\mathfrak{m}_j]$ (remark that $\mathrm{pr}_k = \mathrm{Id}$). Then F^m is a f-structure on N (remark that $F^k = F$ is a complex structure). Then we have:

Theorem 2.3 *A map $f\colon L \to G/G_0$ is m-primitive if and only if it is F^m-holomorphic. In particular, f is k-primitive if and only if it is holomorphic (with respect to the canonical complex structure on G/G_0), and thus any m-primitive map is in particular a holomorphic curve in G/G_0. More precisely, m-primitive maps are exactly the integral holomorphic curves of the complex Pfaff system $\oplus_{j=1}^{m}[\mathfrak{m}_j] \subset TN$.*

Remark 2.6 The equivalences (35) hold also in the odd case. However for $m = k$, the first equivalence of (35) is trivial: $\alpha \in \mathfrak{g} \Longleftrightarrow f \in C^\infty(L, G/G_0)$. There is no restriction (in the form "α takes values in a subspace of $\mathfrak{g}$") in the highest primitive case.

Theorem 2.4 *Let α be a $\mathfrak{g}$-valued 1-form on L and f its geometric map. The following statements are equivalent:*

(i) $\alpha''_{-j} = 0$, $1 \leq j \leq k-1$

(ii) *f is horizontally holomorphic: $(df \circ j_L)^{\mathcal{H}} = F^{k-1} \circ df$, where $\mathcal{H} = [\mathfrak{m}']$ is the horizontal space, and $F^{k-1}{}_{|\mathcal{H}} = F_{|\mathcal{H}}$ defines a complex structure on $\mathcal{H}$.*

We can conclude that **the odd minimal determined system** $(\mathrm{Syst}(k+1,\tau))$ **means that the geometric map f is horizontally holomorphic and vertically harmonic if $k \geq 2$, and if $k = 1$, it means that f is holomorphically harmonic.**

2.4 The underdetermined case

Here we study the underdetermined case.

Theorem 2.5 *Let us consider an underdetermined system* $(\mathrm{Syst}(m, \mathfrak{g}, \tau))$, $m \geq k'$. *Let us write*

$$m = qk' + r, \quad 0 \leq r \leq k' - 1$$

the Euclidean division of m *by* k'. *Then let us consider the automorphism in* $\mathfrak{g}^{q+1}$ *defined by*

$$\tilde{\tau}(a_0, a_1, \ldots, a_q) \in \mathfrak{g}^{q+1} \longmapsto (a_1, \ldots, a_q, \tau(a_0)) \in \mathfrak{g}^{q+1}.$$

Then $\tilde{\tau}$ *is of order* $(q+1)k'$. *Moreover the* m*-th system associated to* $(\mathfrak{g}, \tau)$ *is in fact equivalent to the* m*-th system associated to* $(\mathfrak{g}^{q+1}, \tilde{\tau})$. *More precisely, denoting by* $\tilde{\omega}$ *a* $(q+1)k'$*-th primitive root of unity, then the map*

$$\alpha_\lambda \longmapsto (\alpha_\lambda, \alpha_{\tilde{\omega}}, \ldots, \alpha_{\tilde{\omega}^p \lambda})$$

is a bijection from the set of solutions of the ***underdetermined*** $(m, \mathfrak{g}, \tau)$*-system into the set of solutions of the* ***determined*** $(m, \mathfrak{g}^{q+1}, \tilde{\tau})$*-system.*

2.5 Examples

2.5.1 The trivial case: the 0-th elliptic system associated to a Lie group.

We consider the determined system $(\mathrm{Syst}(m, \tau))$ with $\tau = \mathrm{Id}$ and (thus) $k' = 1$ so that $m'_k = m_1 = 0 = k' - 1$. Then the determined system $(\mathrm{Syst}(0, \mathrm{Id}))$ is nothing but the Maurer-Cartan equation for $\mathfrak{g}$-valued 1-form α (i.e. in other words the "equation" for the trivial geometric map $f \colon L \to G/G$).

2.5.2 Even determined case

The first elliptic system associated to a symmetric space [11]. We consider the even determined system $(\mathrm{Syst}(k, \tau))$, with $k = 1$ and τ an involution. Then the horizontal subbundle is trivial $\mathcal{H} = [\mathfrak{m}] = \{0\}$ and $TN = [\mathcal{V}] = [\mathfrak{g}_1]$ so that the horizontal holomorphicity is trivial and vertical harmonicity means harmonicity. Hence the first elliptic system associated to a symmetric space, $(\mathrm{Syst}(1, \tau))$, is the equation for harmonic maps $f \colon L \to G/G_0$.

The second elliptic system associated to a 4-symmetric space ([25, 6]). Here τ is an order four automorphism and (thus) $k = 2$. Then we consider the even determined system $(\mathrm{Syst}(2, \tau))$. ...
Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces, ([19, 17, 18]). ρ-harmonic surfaces in $\mathbb{O}$, [23]. Surfaces with holomorphic mean curvature vector in 4-dimensional spaces form, surfaces with anti-holomorphic mean curvature vector in $\mathbb{C}P^2$, [6].

2.5.3 Primitive case

τ order k', $m = 1$. Affine Toda fields , [7]. Non-superminimal (weakly) conformal harmonic maps into S^n; (weakly) conformal non-isotropic harmonic maps into $\mathbb{C}P^n$; [5].

2.5.4 Odd determined case

τ of order 3, $m = 2$. Holomorphically harmonic maps (see 5.1.2).

2.5.5 Underdetermined case

First elliptic integrable system associated to a Lie group ([33, 32]). (Syst(m, τ)) with $m = 1$ and $\tau = \mathrm{Id}$, $k' = 1$ and $m_1 = 0 < m$, thus underdetermined system.

Second elliptic integrable system associated to the symmetric space S^n. Constrained Willmore surfaces in S^n...[8].

3 Finite order isometries and Twistor spaces

3.1 Isometries of order $2k$ with no eigenvalues $= \pm 1$

Let E be an Euclidean space and let us define (for $p \in \mathbb{N}^*$)

$$\mathcal{U}_p(E) = \{A \in SO(E),\ A^p = \mathrm{Id},\ A^k \neq \mathrm{Id} \text{ if } 1 \leq k < p\}$$
$$\mathcal{U}_p^*(E) = \{A \in \mathcal{U}_p(E) | 1 \notin \mathrm{Spect}(A)\}, \quad \mathcal{U}_p^{**}(E) = \{A \in \mathcal{U}_p(E) | \pm 1 \notin \mathrm{Spect}(A)\}.$$

Then for $k \in \mathbb{N}^*$ we set

$$\mathcal{Z}_{2k}(E) = \mathcal{U}_{2k}^{**}(E) \quad \text{and} \quad \mathcal{Z}_{2k+1}(E) = \mathcal{U}_{2k+1}^{*}(E) = \mathcal{U}_{2k+1}^{**}(E).$$

We will be interested here in the study of $\mathcal{Z}_{2k}(E)$. Then for each $A \in \mathcal{Z}_{2k}(E)$, we have the following eigenspace decomposition:

$$E^{\mathbb{C}} = \oplus_{j=1}^{k-1} \left(E_A(\omega_{2k}^j) \oplus E_A(\omega_{2k}^{-j}) \right)$$

with $E_A(\lambda) = \ker(A - \lambda \mathrm{Id})$ and $\omega_{2k} = e^{i\pi/k}$.
Let us set $\mathfrak{m}_j^{\mathbb{C}} = E_A(\omega_{2k}^j) \oplus E_A(\omega_{2k}^{-j})$ for $j \geq 0$. Then we have $\dim_{\mathbb{R}} \mathfrak{m}_j = \frac{1}{2}\left(\dim_{\mathbb{R}} E_A(\omega_{2k}^j) + \dim_{\mathbb{R}} E_A(\omega_{2k}^{-j})\right) = \dim_{\mathbb{R}} E_A(\omega_{2k}^j) = 2\dim_{\mathbb{C}} E_A(\omega_{2k}^j)$. Hence $\dim_{\mathbb{R}} \mathfrak{m}_j$ is even and hence we will suppose now that $E = \mathbb{R}^{2n}$ (in all section 3.1).

Example 3.1 We have $\mathcal{Z}_2(E) = \emptyset$, and $\mathcal{Z}_4(E) = \Sigma(E)$ the set of almost complex structure in E.

Situation in the plan Here $E = \mathbb{R}^2$, and any element of $\mathcal{Z}_{2k}(E)$ is written $A = R(\frac{l\pi}{k})$, with $(l, 2k) = 1$ ($R(\theta)$ being the rotation of angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$) i.e. A is a primitive $(2k)$-th root of the unity. Hence $\mathrm{card}(\mathcal{Z}_{2k}(\mathbb{R}^2)) = \phi(2k)$, ϕ being the Euler characteristic.

3.1.1 The set of connected components in the general case

Theorem 3.1 *$\pi_0(\mathcal{Z}_{2k}(\mathbb{R}^{2n}))$, the set of connected components of $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$ is (in one to one correspondance with):*

$$X_{2k} := \left\{ (\varepsilon, (p_1, \ldots, p_{k-1})) \in \mathbb{Z}_2 \times \mathbb{N}^{k-1} \;\middle|\; \sum_{j=1}^{k-1} p_j = n \text{ and } \exists j, (j, 2k) = 1 | p_j \neq 0 \right\}$$

Proof. Let $A \in \mathcal{Z}_{2k}(\mathbb{R}^{2n})$, then $A_{|\mathfrak{m}_j^{\mathbb{C}}} = \omega_{2k}^j \mathrm{Id}_{E_A(\omega_{2k}^j)} \oplus \omega_{2k}^{-j} \mathrm{Id}_{E_A(\omega_{2k}^{-j})}$. We choose an orientation on each $\mathfrak{m}_j$ (such that the induced orientation on $\oplus_1^{k-1} \mathfrak{m}_j$ is the one of $\mathbb{R}^{2n}$). Then there exist oriented plans P_j^l such that $\mathfrak{m}_j = \oplus_{l=1}^{p_j} P_j^l$ (sum of non oriented spaces), where $p_j = \frac{\dim \mathfrak{m}_j}{2}$, and

$$A_{|\mathfrak{m}_j} = \oplus_{l=1}^{p_j} R_{P_j^l}(\theta_j)$$

where $R_{P_j^l}(\theta_j)$ is the rotation on P_j^l of angle $\theta_j = \frac{j\pi}{k}$. Let ε_j be the orientation[7] of $\oplus_{l=1}^{p_j} P_j^l$ (sum of oriented spaces) in $\mathfrak{m}_j$. Now let us consider the map

$$f\colon A \in \mathcal{Z}_{2k}(\mathbb{R}^{2n}) \mapsto (\Pi_{j=1}^{k-1}\varepsilon_j, (p_j)_{1\leq j\leq k-1}) \in X_{2k}.$$

Then it is a continuous[8] surjection and $f^{-1}(\{(\varepsilon, \underline{p})\})$ is an $SO(2n)$-orbit in the action of $SO(2n)$ on $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$. This completes the proof. □

Remark 3.1

Remark 3.2

Each connected component is a $SO(2n)$-orbit and thus is compact, and consequently closed. Hence its complementary which is an finite union of closed subset is closed: each connected component is an open and closed submanifold of $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$ (which is itself a compact submanifold in $SO(2n)$).

Definition 3.1 *We will denote by $\mathcal{Z}_{2k}^{\alpha}(\mathbb{R}^{2n})$ (and sometimes only by $\mathcal{Z}_{2k}^{\alpha}$) the connected component $f^{-1}(\{\alpha\})$, for $\alpha = (\varepsilon, \underline{p}) \in X_{2k}$. We define*

$$\begin{aligned}
\mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n}) &= \left\{A \in \mathcal{Z}_{2k}(\mathbb{R}^{2n}) | A^k = -\mathrm{Id}\right\} = \bigsqcup_{\{\alpha|\forall j, p_{2j}=0\}} \mathcal{Z}_{2k}^{\alpha} \\
\mathcal{Z}_{2k}^{*}(\mathbb{R}^{2n}) &= \left\{A \in \mathcal{Z}_{2k}(\mathbb{R}^{2n}) | A^k \neq -\mathrm{Id}\right\} = \bigsqcup_{\{\alpha|\exists j, p_{2j}\neq 0\}} \mathcal{Z}_{2k}^{\alpha}
\end{aligned}$$

$\mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n})$ is the union of **order k components** in $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$, and $\mathcal{Z}_{2k}^{*}(\mathbb{R}^{2n})$ is the union of **order $2k$ components** (see below for the meaning of this appellation).
In the following we will denote by $\mathcal{Z}_{2k}^{a}(\mathbb{R}^{2n})$, for $a \in \{0, *\}$ any of the two spaces $\mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n})$ and $\mathcal{Z}_{2k}^{*}(\mathbb{R}^{2n})$, and r the order of these two spaces i.e. $r = \begin{cases} 2k & \text{in } \mathcal{Z}_{2k}^{*}(\mathbb{R}^{2n}) \\ k & \text{in } \mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n}) \end{cases}$. r is in fact the order of $\mathrm{Ad}J$, for $J \in \mathcal{Z}_{2k}(\mathbb{R}^{2n})$ (see below).
Let us compute the tangent space of $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$: $\forall J \in \mathcal{Z}_{2k}(\mathbb{R}^{2n})$,

$$T_J\mathcal{Z}_{2k}(\mathbb{R}^{2n}) = \left\{ A \in J.\mathfrak{so}(2n) \,\middle|\, \sum_{p+l=2k-1} J^p A J^l = 0 \right\} \tag{38}$$

and for $J \in \mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n})$, we have in addition

$$T_J\mathcal{Z}_{2k}(\mathbb{R}^{2n}) = T_J\mathcal{Z}_{2k}^{0}(\mathbb{R}^{2n}) = \left\{ A \in J.\mathfrak{so}(2n) \,\middle|\, \sum_{p+l=k-1} J^p A J^l = 0 \right\} \tag{39}$$

[7] see remark 3.1
[8] see remark 3.2

It could seem strange that the two expressions (38) and (39) are equal for $J \in \mathcal{Z}^0_{2k}(\mathbb{R}^{2n})$, but as we will see below, it comes from the fact that the "even" eigenspaces of $\mathrm{Ad}J$ vanish, for $J \in \mathcal{Z}^0_{2k}(\mathbb{R}^{2n})$, which leads to this last equality (which is in general an inclusion "$\supset$")

Example 3.2 If $k = 2$, then $X_{2k} = \{\pm 1\} = \mathbb{Z}_2$ and $\mathcal{Z}_4(\mathbb{R}^{2n}) = \mathcal{Z}^0_4(\mathbb{R}^{2n}) = \Sigma(\mathbb{R}^{2n}) = \{J \in SO(\mathbb{R}^{2n}) | J^2 = -\mathrm{Id}\} = \Sigma^+(\mathbb{R}^{2n}) \bigsqcup \Sigma^-(\mathbb{R}^{2n})$ (resp. the positive and negative components of $\Sigma(\mathbb{R}^{2n})$), whereas $\mathcal{Z}^*_4(\mathbb{R}^{2n}) = \emptyset$.

3.1.2 Study of $\mathrm{Ad}J$, for $J \in \mathcal{Z}^a_{2k}(\mathbb{R}^{2n})$

Let $J \in \mathcal{Z}^a_{2k}(\mathbb{R}^{2n})$. $\mathrm{Ad}J$ is then an order r automorphism of $\mathrm{End}(\mathbb{R}^{2n})$ (since $(\mathrm{Ad}J)^p = \mathrm{Id} \Leftrightarrow J^p = \pm \mathrm{Id}$) thus we have the following eigenspaces decomposition:

$$\mathrm{End}(\mathbb{R}^{2n})^{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \ker(\mathrm{Ad}J - \omega_r^j \mathrm{Id})$$

with $\omega_r = e^{2i\pi/r}$. Let us set

$$\mathcal{A}_j(J) = \ker(\mathrm{Ad}J - \omega_r^j \mathrm{Id}).$$

Then $\mathcal{A}_0(J) = \mathrm{Com}(J) := \{A \in \mathrm{End}(\mathbb{R}^{2n}) | [A, J] = 0\}$ and for $j \neq 0$ we have: $\forall A \in \mathcal{A}_j(J)(j \neq 0)$,

$$\sum_{l+p=r-1} J^l A J^p = \sum_{l+p=r-1} (\omega_r^j)^l A J^{p+l} = \left[\sum_0^{l-1} (\omega_r^j)^l\right] J^{r-1} = 0.$$

Hence

$$\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z} \setminus \{0\}} \mathcal{A}_j(J) \subset \ker\left(\sum_{l+p=r-1} L(J^l) \circ R(J^p)\right)$$

with obvious notation. This inclusion is in fact an equality. Indeed, let $A \in \mathrm{End}(\mathbb{R}^{2n})$, then $A = \sum_{j=0}^{r-1} A_j$, with $A_j \in \mathcal{A}_j(J)$, thus $\sum_{j=0}^{r-1} J^l A J^{r-1-l} = rA_0 J^{r-1} + 0 = rA_0 J^{r-1}$ which vanishes if and only if $A_0 = 0$. This proves the equality:

$$\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z} \setminus \{0\}} \mathcal{A}_j(J) = \ker\left(\sum_{l+p=r-1} L(J^l) \circ R(J^p)\right).$$

Now, let us restrict ourself to $J.\mathfrak{so}(2n)$, resp. to $\mathfrak{so}(2n)$, (which does not change the order of $\mathrm{Ad}J_{|J.\mathfrak{so}(2n)}$, resp. $\mathrm{Ad}J_{|\mathfrak{so}(2n)}$) and set[9]

$$\mathcal{B}_j(J) = \mathcal{A}_j(J) \cap (J.\mathfrak{so}(2n))^{\mathbb{C}}, \quad \text{resp.} \quad \mathfrak{so}_j(J) = \mathcal{A}_j(J) \cap \mathfrak{so}(2n)^{\mathbb{C}}.$$

[9] $\mathcal{B}_j(J)$ is stable by $\mathrm{Ad}J$ and we have $\mathcal{B}_j(J) = J.\mathfrak{so}_j(J) = \mathfrak{so}_j(J).J$. Besides we have more generally $J.\mathcal{A}_j(J) = \mathcal{A}_j(J).J = \mathcal{A}_j(J)$.

Then we have, according to (38)-(39),

$$T_J\mathcal{Z}^a_{2k}(\mathbb{R}^{2n}) = \left(\oplus_{j=1}^{r-1}\mathcal{B}_j(J)\right) \cap \mathrm{End}(\mathbb{R}^{2n}). \tag{40}$$

The inner automorphism[10] $T = \mathrm{Int}J_{|SO(2n)}$ gives rise to the r-symmetric space $SO(2n)/\mathbb{U}_0(J)$, where $\mathbb{U}_0(J) = SO(2n)^T = \mathrm{Com}(J)\cap SO(2n)$, which is nothing but the connected component $\mathcal{Z}^\alpha_{2k}$ of J (which is also the orbit $SO(2n)\cdot J = \mathrm{Int}(SO(2n))(J)$):

$$\mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n}) = SO(2n)/\mathbb{U}_0(J).$$

Consider now

$$\mathbb{U}_{j-1}(J) := \mathrm{Com}(J^j)\cap SO(2n) = SO(2n)^{T^j},$$

Then T is an order j automorphism[11] on $\mathbb{U}_{j-1}(J)$ and gives rises to the j-symmetric space $\mathbb{U}_{j-1}(J)/\mathbb{U}_0(J)$ which is in fact equal to

$$\mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j) := \{J' \in \mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n}) | (J')^j = J^j\}.$$

Indeed let $J' \in \mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n})$, then there exists $g \in SO(2n)$ such that $J' = gJg^{-1}$, then $(J')^j = J^j$ if and only if $gJ^jg^{-1} = J^j$ i.e. $g \in \mathbb{U}_{j-1}(J)$, which proves that $\mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j) = \mathrm{Int}(\mathbb{U}_{j-1}(J))(J)$ i.e.

$$\mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j) = \mathbb{U}_{j-1}(J)/\mathbb{U}_0(J).$$

Remark 3.3 Obviously, in this equation J can be replaced by any $J' \in \mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j)$.

Example 3.3 If $k = 2$, then we have $\mathcal{Z}^\alpha_{4,2}(\mathbb{R}^{2n}, J^2) = \mathcal{Z}^\alpha_{4,2}(\mathbb{R}^{2n}, -\mathrm{Id}) = \mathcal{Z}^\alpha_4(\mathbb{R}^{2n}) = \Sigma^\alpha(\mathbb{R}^{2n}) = SO(2n)/U(n)$, and the other values of j are trivial $\mathcal{Z}^\alpha_{4,\pm1}(\mathbb{R}^{2n}, J^{\pm1}) = \{J\}$.

Remark 3.4 Sometimes, we will need to precise clearly what is the eignevalues of the eigenspaces $\mathcal{A}_i(J)$ and $\mathfrak{so}_i(J)$, then we will simply use the notation

$$\mathcal{A}_{(\omega)}(J) = \ker(\mathrm{Ad}J - \omega\mathrm{Id})$$

and idem for $\mathfrak{so}_{(\omega)}(J)$ and $\mathcal{B}_{(\omega)}(J)$.
Besides, sometimes for the homogeneity of the equations, we will extend the notations $\mathcal{A}_i(J)$ for real index and set for $t \in \mathbb{R}$

$$\mathcal{A}_t(J) = \ker(\mathrm{Ad}J - \omega_r^t\mathrm{Id}).$$

[10] The conjugaison by J is denoted by $\mathrm{Int}J\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ when the domain of defintion is a Lie subgroup and by $\mathrm{Ad}J\colon \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$ when it is a Lie subalgebra.

[11] We confuse $j \in \mathbb{Z}_r$ and its representant in $\{0, \dots, r-1\}$.

3.1.3 Study of $\mathrm{Ad}J^j$

Let $j \in \mathbb{Z}^*$. Then we have

$$\mathrm{Ad}J^j = (\mathrm{Ad}J)^j = \oplus_{l=0}^{r-1} \left(\omega_r^l\right)^j \mathrm{Id}_{\mathcal{A}_l}(J),$$

ω_r^j is of order $p = \dfrac{r}{(r,j)}$, i.e. it is in $\hat{U}_p = \{z \in S^1 | z^p = 1\} = \exp\left((\mathbb{Z}/p\mathbb{Z}) \cdot \dfrac{2i\pi}{p}\right)$.
Hence

$$\mathrm{Ad}J^j = \bigoplus_{q=0}^{(r,j)-1} \left[\oplus_{l=0}^{p-1} \left(\omega_r^j\right)^l \mathrm{Id}_{\mathcal{A}_{qp+l}}(J)\right]$$

hence writing ($\mathrm{Ad}J^j$ is of order p):

$$\mathrm{Ad}J^j = \oplus_{l=0}^{p-1} \omega_p^l \,\mathrm{Id}_{\mathcal{A}_l}(J^j)$$

we obtain that

$$\mathcal{A}_l(J^j) = \oplus_{q=0}^{(r,j)-1} \mathcal{A}_{qp+l'}(J) \tag{41}$$

where $l' = (j')^{-1} l$ in the ring $\mathbb{Z}/p\mathbb{Z}$, and $j' = \left[\dfrac{j}{(j,r)}\right]_{\mathrm{mod}\ p}$ (j' is inversible in the ring $\mathbb{Z}/p\mathbb{Z}$, since $(j', p) = 1$ by definition of (r, j)).
In particular,

$$\mathrm{Com}(J^j) = \mathcal{A}_0(J^j) = \oplus_{q=0}^{(r,j)-1} \mathcal{A}_{qp}(J). \tag{42}$$

More particulary,

$$\mathrm{Com}(J^k) = \begin{cases} \oplus_{q=0}^{k-1} \mathcal{A}_q(J) = \mathrm{End}(\mathbb{R}^{2n})^{\mathbb{C}} & \text{if } r = k \\ \oplus_{q=0}^{k-1} \mathcal{A}_{2q}(J) & \text{if } r = 2k \end{cases}$$

and

$$\mathrm{Com}(J^2) = \left\{ \begin{array}{ll} \mathcal{A}_0(J) \oplus \mathcal{A}_k(J) & \text{if } r = 2k \\ \mathcal{A}_0(J) \oplus \mathcal{A}_{\frac{k}{2}}(J) & \text{if } r = k \in 2\mathbb{Z} \end{array} \right\} = \mathcal{A}_0(J) \oplus \mathcal{A}_{\frac{r}{2}}(J) \quad \text{if } r \text{ is even}$$
$$\mathcal{A}_0(J) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } r = k \text{ is odd}$$

.

We can rewrite all what precedes in $J.\mathfrak{so}(2n)$ (resp. in $\mathfrak{so}(2n)$) by replacing $\mathcal{A}_l$ by $\mathcal{B}_l$ (resp. $\mathfrak{so}_l$). In particular we have, according to (42),

$$\mathfrak{u}_{j-1}(J) := \mathrm{Lie}\left(\mathbb{U}_{j-1}(J)\right) = \mathfrak{so}_0(J^j) \cap \mathfrak{so}(2n) = \left(\oplus_{q=0}^{(r,j)-1} \mathfrak{so}_{qp}(J)\right) \cap \mathfrak{so}(2n),$$

this[12] is the eigenspace decomposition of the order j automorphism obtained by restricting $T = \mathrm{Ad}J$ to $\mathfrak{u}_{j-1}(J)$. Moreover we have

$$T_J \mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j) = \left(\bigoplus_{q=1}^{(r,j)-1} \mathcal{B}_{qp}(J)\right) \cap \mathrm{End}(\mathbb{R}^{2n}). \tag{43}$$

[12] We mean $\mathfrak{u}_{j-1}(J)^{\mathbb{C}} = \oplus_{q=0}^{(r,j)-1} \mathfrak{so}_{qp}(J)$

Indeed, $g \in \mathbb{U}_{j-1}(J) \mapsto gJg^{-1} \in \mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j)$ is a surjective submersion whose the (surjective) derivative at $g = 1$,

$$A \in \mathfrak{u}_{j-1}(J) \mapsto [A, J] = \sum_{q=0}^{(r,j)-1} [A_{qp}, J] = \sum_{q=0}^{(r,j)-1} (1-\omega_r^{qp}) A_{qp} J \in T_J \mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j)$$

has $\left(\oplus_{q=1}^{(r,j)-1} \mathcal{B}_{qp}(J)\right) \cap \mathrm{End}(\mathbb{R}^{2n})$ as image, which proves the equality (43).
More simply by differentiating the definition equation of $\mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j)$ we obtain

$$T_J \mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j) = \left\{ A \in J.\mathfrak{so}(2n) \,\middle|\, \sum_{p+l=j-1} J^p A J^l = 0 \right\} \tag{44}$$

In particular, let us apply (43) for $j = 2$:

$$T_J \mathcal{Z}^{\alpha}_{2k,2}(\mathbb{R}^{2n}, J^2) = \begin{cases} \mathcal{B}_{\frac{r}{2}}(J) = \{A \in J.\mathfrak{so}(2n) | AJ + JA = 0\} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

This can be recovered from (44) by remarking that if r is odd then -1 is not a r-th root of unity (and thus not an eigenvalue of $\mathrm{Ad}J$).

Remark 3.5 If $(j, 2k) = 1$ (so that $(j, r) = 1$ also) then $J^j \in \mathcal{Z}_{2k}(\mathbb{R}^{2n})$ and T^j is of order r and we have, according to (41)

$$\mathcal{A}_l(J^j) = \mathcal{A}_{[j]_r^{-1} \cdot l}(J), \quad \forall l \in \mathbb{Z}/r\mathbb{Z},$$

in others words $\mathcal{A}_l(J) = \mathcal{A}_{j \cdot l}(J^j)$, $\forall l \in \mathbb{Z}/r\mathbb{Z}$. In particular

$$\mathbb{U}_{j-1}(J) = \mathcal{A}_0(J^j) \cap SO(2n) = \mathcal{A}_0(J) \cap SO(2n) = \mathbb{U}_0(J).$$

Hence

$$\mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^j) = \{J\}.$$

More generally, we have, according to (42), since $(jl, r) = (l, r)$,

$$\mathrm{Com}((J^j)^l) = \mathrm{Com}(J^{jl}) = \oplus_{q=0}^{(l,r)-1} \mathcal{A}_{qp}(J) = \mathrm{Com}(J^l)$$

with $p = \dfrac{r}{(l, r)}$, and thus

$$\mathrm{Com}((J^j)^l) = \mathrm{Com}(J^l) \quad \forall l \in \mathbb{Z}/r\mathbb{Z}.$$

In particular, $\mathbb{U}_{l-1}(J^j) = \mathbb{U}_{l-1}(J)$ $\forall l \in \mathbb{Z}/r\mathbb{Z}$ and thus

$$\mathcal{Z}^{\alpha}_{2k,l}(\mathbb{R}^{2n}, (J^j)^l) = \mathcal{Z}^{[j]_{2k}^{-1} \cdot \alpha}_{2k,l}(\mathbb{R}^{2n}, J^l)$$

where $[j]_{2k}^{-1} \cdot \alpha$ is the action of $[j]_{2k}^{-1}$ on $\alpha \in X_{2k}$, the action of $l \in (\mathbb{Z}/r\mathbb{Z})^*$ on X_{2k} being defined by the bijective map

$$J \in \mathcal{Z}_{2k}(\mathbb{R}^{2n}) \mapsto J^l \in \left(\mathcal{Z}_{2k}(\mathbb{R}^{2n})\right)^l = \mathcal{Z}_{2k}(\mathbb{R}^{2n})$$

which sends a connected component onto another one

$$\left(\mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n})\right)^{l} =: \mathcal{Z}^{l\cdot\alpha}_{2k}(\mathbb{R}^{2n}).$$

In particular, for $j = 1$, we have

$$\mathbb{U}_{l-1}(J^{-1}) = \mathbb{U}_{l-1}(J) \quad \forall l \in \mathbb{Z}/r\mathbb{Z}$$

and thus

$$\mathcal{Z}^{\alpha}_{2k,l}(\mathbb{R}^{2n}, (J^{-1})^{l}) = \mathcal{Z}^{-\alpha}_{2k,l}(\mathbb{R}^{2n}, J^{l})$$

where $-(\varepsilon, \underline{p}) = ((-1)^{n}\varepsilon, \underline{p})$ in X_{2k}. Hence $\mathcal{Z}^{\alpha}_{2k,l}(\mathbb{R}^{2n}, (J^{-1})^{l}) = \mathcal{Z}^{\alpha}_{2k,l}(\mathbb{R}^{2n}, J^{l})$ if and only if n is even (i.e. J and J^{-1} are in the same connected component).

3.2 Isometries of order $2k+1$ with no eignevalue $= 1$

We can do exactly the same study for $\mathcal{Z}_{2k+1}(E)$ as we did for $\mathcal{Z}_{2k}(E)$, with however the following simplification: all the connected components have the same order $r = 2k+1$ and we do not have to distinguish two types of orbits as previously.

3.3 The effect of the power maps on the finite order isometries

Let $J \in \mathcal{U}_{k'}$ then $J^{j} \in \mathcal{U}_{p}$ with $p = \frac{k'}{(k',j)}$. Moreover it is easy to see (from the diagonalisation) that the power map

$$J \mapsto J^{j}$$

is surjective from $\mathcal{U}_{k'}$ onto $\mathcal{U}_{p}$ (since $z \in \hat{U}_{k'} \mapsto z^{j} \in \hat{U}_{p}$ is surjective). Besides, since each connected component in $\mathcal{U}_{k'}$ (and in $\mathcal{U}_{p}$) is a $SO(2n)$-orbit, then the power map $J \mapsto J^{j}$ sends one component in $\mathcal{U}_{k'}$ onto another one in $\mathcal{U}_{p}$ so that it induces a map:

$$\alpha \in \pi_0(\mathcal{U}_{k'}) \longrightarrow j \cdot \alpha \in \pi_0(\mathcal{U}_{p})$$

such that

$$(\mathcal{U}^{\alpha}_{k'})^{j} =: \mathcal{U}^{j\cdot\alpha}_{p}, \quad \forall \alpha \in \pi_0(\mathcal{U}_{k'}).$$

Remark 3.6 In general we have $(\mathcal{Z}_{2k})^{j} \nsubseteq \mathcal{Z}_{p}$. For example, for $j = 2$, we have

$$(\mathcal{Z}_{2k}(\mathbb{R}^{2n}))^{2} = \begin{cases} \mathcal{U}^{*}_{k} & \text{if k is even} \\ \mathcal{Z}_{k} & \text{if k is odd} \end{cases}$$

Besides, given $J \in \mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n})$, then $\mathcal{Z}^{\alpha}_{2k,j}(\mathbb{R}^{2n}, J^{j})$ is the inverse image of J^{j} by the map

$$J' \in \mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n}) \longmapsto (J')^{j} \in (\mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n}))^{j} = \mathcal{U}^{j\cdot\alpha}_{p}(\mathbb{R}^{2n}).$$

Since $(J')^j$ is constant in $\mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j)$, we can denote it by J_j and then

$$\mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J^j) = \mathcal{Z}^\alpha_{2k,j}(\mathbb{R}^{2n}, J_j).$$

■

Furthermore, we have also for any $J \in \mathcal{U}^\alpha_{k'}$, $\mathbb{U}_0(J^j) = \mathbb{U}_{j-1}(J)$ so that

$$\mathcal{U}^{j\cdot\alpha}_{k'} = SO(2n)/\mathbb{U}_0(J^j) = SO(2n)/\mathbb{U}_{j-1}(J)$$

so that

$$(\mathcal{U}^\alpha_{k'})^j = SO(2n)/\mathbb{U}_{j-1}(J) \tag{45}$$

which we can recover directly by taking the power j in the equality $\mathcal{U}^\alpha_{k'} = \{gJg^{-1}, g \in SO(2n)\}$.

Convention: for each $\alpha \in \pi_0(\mathcal{Z}_{2k})$, we will choose (and fix) a canonical representant in $\mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n})$. For example, let $(\epsilon_1, \ldots, \epsilon_{2n})$ be the canonical basis in $\mathbb{R}^{2n}$, and

$$\begin{aligned} e_{2l+1} &= \frac{\epsilon_{2l+1} + i\epsilon_{2l+2}}{\sqrt{2}},\ 0 \leq l \leq n-1, \\ e_{2l} &= \overline{e}_{2l-1},\ 1 \leq l \leq n. \end{aligned}$$

Then $e = (e_1, \ldots, e_{2n})$ is a hermitian basis in $\mathbb{C}^{2n}$ and we can take J^α_0 such that

$$\mathcal{M}at_e(J^\alpha_0) = \mathrm{Diag}\left(\begin{pmatrix} e^{i\theta_j}\mathrm{Id}_{p_j} & 0 \\ 0 & e^{-i\theta_j}\mathrm{Id}_{p_j} \end{pmatrix}, 1 \leq j \leq n-1\right)$$

where $\underline{p} = (p_1, \ldots, p_{k-1})$ is determined by $\alpha = (\varepsilon, \underline{p}) \in \pi_0(\mathcal{Z}_{2k})$ (see section 3.1.1), $\theta_j = \dfrac{j\pi}{k}$, and $\mathcal{M}at_e(\cdot)$ means "the matrice in the basis e of ".

3.4 The Twistor spaces of a Riemannian manifolds and its reductions

Let M be an oriented (even dimensional) Riemannian manifolds and let us consider the bundle of order $2k$ isometries $\mathcal{U}_{2k}(M)$ as well as its subbundles $\mathcal{U}^*_{2k}(M)$ and $\mathcal{Z}_{2k}(\mathbb{R}^{2n})$. Let us fix $\alpha \in \pi_0(\mathcal{Z}_{2k}(\mathbb{R}^{2n}))$ and consider the component $\mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n})$. Then denoting by $\mathcal{SO}(M)$ the $SO(2n)$-bundle of positively oriented orthonormal frames on M, we have

$$\mathcal{Z}^\alpha_{2k}(\mathbb{R}^{2n}) = \mathcal{SO}(M)/\mathbb{U}_0(J^\alpha_0)$$

(see section 4.3 for more precisions about this equality). We want to ask the following question: does $\mathcal{SO}(M)$ admit $\mathbb{U}_{j-1}(J^\alpha_0)$-reduction for $1 \leq j \leq r$. We know (according to [27]) that $\mathcal{SO}(M)$ admits an $\mathbb{U}_{j-1}(J^\alpha_0)$-reduction if and only if the associated bundle $\mathcal{SO}(M)/\mathbb{U}_{j-1}(J^\alpha_0)$ $(= \mathcal{SO}(M) \times_{SO(2n)}$

$SO(2n)/\mathbb{U}_{j-1}(J_0^\alpha)$) admits a global section $J_j\colon M \to \mathcal{SO}(M)/\mathbb{U}_{j-1}(J_0^\alpha)$. Besides $\mathbb{U}_{j-1}(J_0^\alpha) = \mathbb{U}_0((J_0^\alpha)^j)$ and according to (45),

$$\mathcal{SO}(M)/\mathbb{U}_{j-1}(J_0^\alpha) = (\mathcal{Z}_{2k}^\alpha(\mathbb{R}^{2n}))^j = \mathcal{U}_p^{j\cdot\alpha}(M)$$

with $p = \dfrac{2k}{(2k,j)}$. Hence J_j (when it exists) is a global section of $(\mathcal{Z}_{2k}^\alpha(M))^j$ and then the $\mathbb{U}_{j-1}(J_0^\alpha)$-reduction is given in terms of J_j by:

$$\mathfrak{U}_{j-1}^\alpha(M) := \{e = (e_1, \dots, e_{2n}) \in \mathcal{SO}(M) |\, \mathcal{M}at_e(J_j) = (J_0^\alpha)^j\}.$$

Then we have

$$\mathfrak{U}_{j-1}^\alpha(M)/\mathbb{U}_0(J_0^\alpha) = \mathcal{Z}_{2k,j}^\alpha(M, J_j).$$

In particular, since $(J_0^\alpha)^r = \pm\mathrm{Id}$, we have $\mathbb{U}_{r-1}(J_0^\alpha) = SO(2n)$ and $\mathcal{SO}(M)$ has always an (unique and trivial) $SO(2n)$-reduction for which $J_r = \pm\mathrm{Id}_{TM}$ and thus $\mathfrak{U}_{r-1}^\alpha(M) = \mathfrak{U}_0^\alpha(M) = \mathcal{SO}(M)$ and $\mathcal{Z}_{2k,r}^\alpha(M, J_r) = \mathcal{Z}_{2k}^\alpha(M)$.

Example 3.4 For $k = 2$, and thus $r = 2$, $J = -\mathrm{Id}$ defines the trivial reduction; and J_1(when it exists) defines on M an (almost) complex structure and $\mathfrak{U}_0^\alpha(M)$ is then the subbundle of hermitian frames on M (with respect to this complex structure).

3.5 Return to an order $2k$ automorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}$.

We give ourself the same ingredients as in section 2.1 and we use the same notations. In particular, we suppose that the subgroup H is chosen such that $(G^\sigma)^0 \subset H \subset G^\sigma$. In addition to that we suppose G/H Riemannian.

3.5.1 Case $r = k$

Suppose that we have $\tau_{|\mathfrak{m}}^k = -\mathrm{Id}$ i.e. $\tau_{|\mathfrak{m}} \in \mathcal{Z}_{2k}^0(\mathfrak{m})$. Then $\mathfrak{g}_{2j} = 0$ for all $2j \in \mathbb{Z}/(2k)\mathbb{Z} \setminus \{0, k\}$. Hence we have

$$[\mathfrak{g}_p, \mathfrak{g}_l] = \{0\} \qquad \text{if } p + l \neq 0, k.$$

Indeed, if p or l is even then the corresponding eigenspace vanishes. If p and l are odd then $[\mathfrak{g}_p, \mathfrak{g}_l] \subset \mathfrak{g}_{p+l}$ and $p+l$ is even, thus $\mathfrak{g}_{p+l} = \{0\}$ except if $p+l = 0$ or k. Consequently, we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and thus G/H is a (locally) symmetric space. Let us distinguish the following two cases.

k is odd. Then $[\mathfrak{g}_k, \mathfrak{g}_j] \subset \mathfrak{g}_{k+j} = \{0\}$ for all j odd $\neq 0, k$. Hence $[\mathfrak{g}_k, \mathfrak{m}] = \{0\}$ i.e. $\mathrm{ad}_{\mathfrak{m}}\mathfrak{g}_k = 0$ so that $\mathfrak{g}_k = 0$ and thus this case is trivial because $H = G_0$ up to covering and thus the fibre H/G_0 is trivial (i.e. a discret set). Moreover we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} = \mathfrak{g}_0$ and $G/H = G/G_0$ (up to covering) is the (locally) symmetric space associated to the involution τ^k.

k **is even.** Then the symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the eigenspace decomposition of τ^k, and G/H is the (locally) symmetic space corresponding to this involution τ^k.

In conclusion, if $r = k$, then G/H is the (locally) symmetic space corresponding to τ^k.

Example 3.5 For $2k = 4$, we always have $r = k = 2$ (since $\tau^2_{|\mathfrak{m}} = -\mathrm{Id}$) and G/H is the symmetric space corresponding to $\sigma = \tau^2$.

3.5.2 Action of $\mathrm{Ad}\tau_{|\mathfrak{m}}$ on $\mathrm{ad}\mathfrak{g}_j$

We have $\tau \circ \mathrm{ad}X \circ \tau^{-1} = \mathrm{ad}\tau(X)$, $\forall X \in \mathfrak{g}$. In particular,

$$\forall X_j \in \mathfrak{g}_j,\ \tau \circ \mathrm{ad}X_j \circ \tau^{-1} = \omega_{2k}^j \mathrm{ad}X_j \quad (j \in \mathbb{Z}/(2k)\mathbb{Z}).$$

Hence by taking the restriction to $\mathfrak{m}$ and projecting on $\mathfrak{m}$:

$$\tau_{|\mathfrak{m}} \circ [\mathrm{ad}_{\mathfrak{m}} X_j]_{\mathfrak{m}} \circ \tau_{|\mathfrak{m}}^{-1} = \omega_{2k}^j [\mathrm{ad}_{\mathfrak{m}} X_j]_{\mathfrak{m}}$$

so that

$$[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_j]_{\mathfrak{m}} \subset \mathcal{A}_j(\tau_{|\mathfrak{m}})\ ^{13} \quad \forall j \in \mathbb{Z}/(2k)\mathbb{Z}.$$

If $r = k$ then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, hence $[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_j]_{\mathfrak{m}} = 0$, for all $j \in \mathbb{Z}/(2k)\mathbb{Z} \setminus \{0, k\}$.
Let us recall that we always have ($r = 2k$ or k)

$$[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_0]_{\mathfrak{m}} = \mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_0 \subset \mathfrak{so}_0(\tau_{|\mathfrak{m}}) = \mathrm{Com}(\tau_{|\mathfrak{m}}) \cap \mathfrak{so}(\mathfrak{m})$$

$$[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_k]_{\mathfrak{m}} = \mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_k \subset \begin{cases} \mathfrak{so}_{\frac{r}{2}}(\tau_{|\mathfrak{m}}) = \mathrm{Ant}(\tau_{|\mathfrak{m}}) \cap \mathfrak{so}(\mathfrak{m}) & \text{if } r = k \text{ is even} \\ 0 & \text{if } r = k \text{ is odd (trivial case)} \end{cases}$$

where $\mathrm{Ant}(\tau_{|\mathfrak{m}}) = \{A \in \mathrm{End}(\mathfrak{m}) | A\,\tau_{|\mathfrak{m}} + \tau_{|\mathfrak{m}} A = 0\}$.

Remark 3.7 There is no raison to have $[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_j]_{\mathfrak{m}} \subset \mathfrak{so}_j(\tau_{|\mathfrak{m}})$ (if the metric is not naturally reductive) and no raison also to have $[\mathrm{ad}_{\mathfrak{m}} \mathfrak{g}_j]_{\mathfrak{m}} \subset \mathcal{B}_j(\tau_{|\mathfrak{m}})$.

3.6 The canonical section in $(\mathcal{Z}_{2k}(G/H))^2$, the canonical embedding, and the Twistor lifts

Once more, we give ourself the same ingredients and notations as in section 2.1. We suppose that $(G^\sigma)^0 \subset H \subset G^\sigma$ and that G/H is Riemannian. We denote by $p_0 := 1.H$ the reference point in G/H. According to the definition of H, we have

[13] see remark 3.7

Lemma 3.1 *Let J_0 be the element in $\mathcal{Z}_{2k}^{\alpha_0}(T_{p_0}M)$ corresponding* [14] *to $\tau_{|\mathfrak{m}}^{-1}$ (or more generally to $\tau_{|\mathfrak{m}}^{j}$ with $(j,2k)=1$) under the identification $T_{p_0}M=\mathfrak{m}$. Then we have $\forall g\in H$, $gJ_0^2g^{-1}=J_0^2$. Hence there exists a unique section*

$$J_2\colon G/H\mapsto(\mathcal{Z}_{2k}^{\alpha_0})^2=\mathcal{U}_k^{\alpha_0^2}(G/H)$$

defined by

$$g.p_0\in G/H\mapsto gJ_0^2g^{-1}\in(\mathcal{Z}_{2k}^{\alpha_0})^2.$$

Proceeding as in [25], Theorem 3, we obtain:

Theorem 3.2 *Let $\tau\colon\mathfrak{g}\to\mathfrak{g}$ be an order $2k$ automorphism and $M=G/H$ a (locally) Riemannian k-symmetric space corresponding to $\sigma=\tau^2$. Let us make G acting on $\mathcal{Z}_{2k}(M)$: $g\cdot J=gJg^{-1}$. Let $J_0\in\mathcal{Z}_{2k}^{\alpha_0}(T_{p_0}M)$ be the finite order isometry corresponding to $\tau_{|\mathfrak{m}}^{-1}$ under the identification $T_{p_0}M=\mathfrak{m}$. Then the orbit of J_0 under the action of G is an immersed submanifold in $\mathcal{Z}_{2k}^{\alpha_0}(M)$. Denoting by G_0 the stabilizer of J_0, then $G_0=G^\tau\cap H$ and thus $N=G/G_0$ is a locally $2k$-symmetric bundle over M and the natural map:*

$$\begin{array}{rccc}\mathfrak{I}_{J_0}\colon & G/G_0 & \longrightarrow & \mathcal{Z}_{2k,2}^{\alpha_0}(G/H,J_2)\\ & g.G_0 & \longmapsto & gJ_0g^{-1}\end{array}$$

is an injective immersion and a morphism of bundle. Moreover, if the image of G in $\mathrm{Is}(M)$ (the group of isometry of M) is closed, then $\mathfrak{I}_{J_0}$ is an embedding.

Remark 3.8 Say something about the choice $J_0=\tau_{|\mathfrak{m}}^{j}$ with $(j,2k)=1$ etc...

Notation For a geometric map $f\colon L\to G/G_0$, we will denote by J the corresponding map $\mathfrak{I}_{J_0}\circ f\colon L\to Z_{2k,2}(G/H,J_2)$ under the previous inclusion $G/G_0\hookrightarrow Z_{2k,2}(G/H,J_2)$.

3.6.1 The Twistor lifts

Definition 3.2 *An isometry $A\in SO(\mathbb{R}^{2n})$ will be called an $e^{i\theta}$-structure if $\mathrm{Spect}(A)=\{e^{i\theta},e^{-i\theta}\}$. An isometry $A\in SO(\mathbb{R}^{2n})$ will be called a $2k$-structure if $A\in\mathcal{Z}_{2k}(\mathbb{R}^{2n})$.*

Definition 3.3 *Let (L,i) be a complex manifold (of dimension $d\geq 1$), M an oriented Riemannian manifold and $u\colon L\to M$ a immersion. Then an element $J\colon L\to u^*(\mathcal{Z}_{2k}(M))$ is an admissible twistor lift of u if one of the following equivalent statements holds:*

(i) *Let E_j be the orthogonal projection of the tangent subbundle $u_*(TL)$ on the subbundle $\mathfrak{m}_j(J)$ (with obvious notations). Then for all $j\in\{1,\dots,k-1\}$, J stabilizes E_j and $J_{|E_j}$ is a ω_{2k}^j-structure and $\mathrm{pr}_{E_j}\circ i=J\circ\mathrm{pr}_{E_j}$.*

[14] α_0 denotes off course the connected component of J_0 in $\mathcal{Z}_{2k}(T_{p_0}M)$.

(ii) *J stabilizes E_j and induces on it an ω_{2k}^j-structure, i induces on E_j a well defined map which is nothing but the ω_{2k}^j-structure $J_{|E_j}$.*

(iii) *$[\partial u]_{\mathfrak{m}_j(J)^{\mathbb{C}}} \in \mathfrak{g}_j(J)$ for $1 \leq j \leq k-1$ (still with obvious notations).*

(iv) *Let $\underline{\mathrm{J}}$ be the complex structure defined on $u^*(TM)$ by the decomposition*

$$u^*(TM)^{\mathbb{C}} = \underset{(1,0)}{\left[\oplus_{j=1}^{k-1}\mathfrak{g}_j(J)\right]} \oplus \underset{(0,1)}{\left[\oplus_{j=1}^{k-1}\mathfrak{g}_{-j}(J)\right]}$$

then u is $\underline{\mathrm{J}}$-holomorphic: $*du := du \circ i = \underline{\mathrm{J}} \circ dX$

In particular, if (L, i) is a Riemann surface, then we can add that the existence of an admissible twistor lift J of u implies in particular that u is a conformal immersion.

Theorem 3.3 *In the situation described in theorem 3.2, let α be a $\mathfrak{g}$-valued Maurer-Cartan 1-form on a Riemann surface L and $f\colon L \to G/G_0$ its geometric map and $J = \mathfrak{I}_{J_0} \circ f$. The the following statements are equivalent:*

(i) $\alpha''_{-j} = 0$, $1 \leq j \leq k-1$

(ii) *$J\colon L \to \mathcal{Z}_{2k,2}(G/H, J_2)$ is an admisible twistor lift.*

4 Vertically Harmonic maps and Harmonic sections of submersions

We will recall here some definitions and properties about vertical harmonicity and refer to [35, 36] for details and proofs (section 4.1 and 4.2). Then we will apply these latter to the study of the examples we are interested in (and that we have already introduced and begun to study in 4.1) : homogeneous spaces and Twistor spaces (section 4.3). Finally, we will conclude with a geometric interpretation of the even determined elliptic integrable system in terms of vertically harmonic twistor lifts (section 4.4).

4.1 Definitions, general properties and examples

4.1.1 The vertical energy fonctional

Let (M, g), (N, h) be Riemannian manifolds and $\pi\colon N \to M$ a submersion. We can do the splitting $TN = \mathcal{V} \oplus \mathcal{H}$, where the vertical and horizontal subbundles are defined by $\mathcal{V} = \ker d\pi$ and $\mathcal{H} = (\ker d\pi)^{\perp} = \mathcal{V}^{\perp}$.
For any map $u\colon M \to N$, we denote by $d^v u = (du)^v$ the vertical component of du. Following [35], this allows us to define the vertical energy density of u, $e^v(u) = \dfrac{1}{2}|d^v u|^2$, and the associated vertical energy fonctional:

$$E^v(u) = \frac{1}{2}\int_M |d^v u|^2 d\mathrm{vol}_g.$$

Let us define the vertical tension field of $u\colon M \to N$ by

$$\tau^v(u) = \mathrm{Tr}(\nabla^v d^v u)$$

where ∇^v denotes the vertical component of the Levi-Civita connection (of N) in TN, and Tr_g the trace with respect to g. Then we have

Theorem 4.1 *[35] The map $u\colon M \to N$ is a critical point of E^v with respect to vertical variations if and only if $\tau^v(u) = 0$. In particular, if u is a section, i.e. $\pi \circ u = \mathrm{Id}_M$, then it is a critical point of E^v with respect to variations through sections if and only if $\tau^v(u) = 0$.*

We define a map $u\colon M \to N$ to be vertically harmonic if $\tau^v(u) = 0$, and if u is a section we will say that it is a harmonic section.

4.1.2 Examples

Example 4.1 Let $\pi\colon N \to M$ be like above. Let (L, b) be a Riemannian manifold and $f\colon L \to N$ a map. Then we can consider the projection $u = \pi \circ f\colon L \to M$ and the manifold

$$u^*N = \{(z, n) \in L \times N,\, n \in \pi^{-1}(\{u(z)\})\}.$$

Then we have the submersion $u^*\pi\colon (z,n) \in u^*N \mapsto z \in L$. Furthermore, u^*N can be endowed canonically with a Riemannian metric: take the metric induced by the product metric

$$|(dz, dn)|^2 = |dz|^2 + |dn|^2$$

in $L \times N \supset u^*N$. Then we will say that $f\colon L \to N$ is vertically harmonic if and only if

$$\mathrm{Tr}_b(\nabla^v d^v f) = 0.$$

When u is an isometry and π a Riemannian submersion **this is equivalent to say that the corresponding section $\tilde{f}\colon L \to u^*N$ is a harmonic section** (see the Appendix, theorem 6.1).

Example 4.2 Let $\mathrm{p}\colon (E, \nabla, \langle\cdot,\cdot\rangle) \mapsto (M, g)$ be a Riemannian vector bundle of rank $2n$ (in particular $\langle\cdot,\cdot\rangle$ is ∇-parallel). Then we consider the bundle of orhogonal almost complex structure: $N_\Sigma = \Sigma(E) = \{(x, J_x), J_x \in \Sigma(E_x)\}$, where $\Sigma(E_x) = \{J \in \mathfrak{so}(E_x) | J^2 = -\mathrm{Id}\}$. We have a fibration $\pi_\Sigma\colon N_\Sigma \to M$. The vertical space is given by: $\forall J \in N_\Sigma$,

$$\mathcal{V}_J := T_J\Sigma(E_x) = \{A \in so(E_x) | AJ + JA = 0\}$$

where $x = \pi_\Sigma(J)$.
The metric connection ∇ gives us a splitting : $T\Sigma(E) = \mathcal{V}^\Sigma \oplus \mathcal{H}^\Sigma$. Indeed we have the following splitting (coming from ∇)

$$T\mathfrak{so}(E) = \mathrm{p}^*\mathfrak{so}(E) \oplus \mathcal{H} \tag{46}$$

where $\mathrm{p}\colon \mathfrak{so}(E) \to M$ is the natural fibration[15]. Then for any (local) section $J\colon U \subset M \to \Sigma(E)$, we have

$$0 = \nabla J^2 = (\nabla J)J + J(\nabla J)$$

so that $\nabla J \in \mathcal{V}^\Sigma$ and thus in the decomposition (46): $[dJ]_{so(E)} = \nabla J \in \mathcal{V}^\Sigma$ and thus $[dJ]_{\mathcal{H}} = dJ - \nabla J \in TN_\Sigma$ which allows us to conclude that

$$T\Sigma(E) = \mathcal{V}^\Sigma \oplus \mathcal{H}_{|\Sigma(E)}.$$

Then we can endow N_Σ with the metric

$$h = \pi^*g + \langle\ ,\ \rangle_{\mathcal{V}^\Sigma} \tag{47}$$

where $\langle\ ,\ \rangle_{\mathcal{V}^\Sigma}$ is the fibre metric in $\mathcal{V}^\Sigma$ induced by the metric in $\mathfrak{so}(E)$:

$$\langle A, B\rangle = \mathrm{Tr}(A^t.B). \tag{48}$$

With this metric we have obviously $\mathcal{H}^\Sigma = \mathcal{V}^{\Sigma\perp}$.

[15] we denote by the same letter the fibration $\mathrm{p}\colon E \to M$ and all its "tensorial extensions": $\mathrm{p}\colon \mathrm{End}(E) \to M$, $\mathrm{p}\colon \mathfrak{so}(E) \to M$, etc..

Furthermore, let us remark that $T\Sigma(E)$ is a subbundle of $T\mathfrak{so}(E)_{|\Sigma(E)}$ and that we have

$$T\mathfrak{so}(E)_{|\Sigma(E)} = \pi_\Sigma^*\mathfrak{so}(E) \oplus \mathcal{H}_{|\Sigma(E)} = \mathfrak{so}_-(\pi_\Sigma^*E) \oplus \mathfrak{so}_+(\pi_\Sigma^*E) \oplus \mathcal{H}_{|\Sigma(E)} \quad (49)$$
$$= T\Sigma(E) \oplus \mathfrak{so}_+(\pi_\Sigma^*E) \quad (50)$$

with[16]

$$\mathfrak{so}_+(\pi_\Sigma^*E)_J = \mathfrak{so}_+(E_x, J) := \{A \in \mathfrak{so}(E_x) |\, [A, J] = 0\}$$
$$\mathfrak{so}_-(\pi_\Sigma^*E)_J = \mathfrak{so}_-(E_x, J) := \{A \in \mathfrak{so}(E_x) |\, AJ + JA = 0\} = \mathcal{V}_J^\Sigma$$

for all $J \in \Sigma(E)$ (and where $x = \pi_\Sigma(J)$). In other words, π_Σ^*E is canonically endowed with a complex structure: $\mathcal{I}_J = J$, $\forall J \in N_\Sigma$, and this complex structure defines the two spaces $\mathfrak{so}_\pm(\pi_\Sigma^*E)$ by

$$\mathfrak{so}_\pm(\pi_\Sigma^*E) = \mathfrak{so}_\pm(\pi_\Sigma^*E, \mathcal{I}).$$

Now given a section $J \in \mathcal{C}(\pi_\Sigma)$, then we consider the vertical part of the rough Laplacian $\nabla^*\nabla J$, in the decomposition (49): $(\nabla^*\nabla J)^{\mathcal{V}^\Sigma} = \frac{1}{2}J[J, \nabla^*\nabla J]$. We will see in section 4.3.2 that this is in fact exactly the vertical tension field of J in N_Σ:

$$\tau^v(J) = \frac{1}{2}J[J, \nabla^*\nabla J].$$

In particular, we recover the definition of vertical harmonicity used in [25] and [6].

Example 4.3 Let $\mathrm{p}\colon (E, \nabla, \langle\cdot,\cdot\rangle) \mapsto (M, g)$ be a Riemannian vector bundle of rank $2n$. Then we consider more generally the bundle of order $2k$ isometries $\mathcal{U}_{2k}(E)$ as well as its subbundles $\mathcal{U}_{2k}^*(E)$ and $\mathcal{Z}_{2k}(E)$. Let us fix $\alpha \in \pi_0(\mathcal{Z}_{2k}(\mathbb{R}^{2n}))$ and consider the component $\mathcal{Z}_{2k}^\alpha(E) := N_\mathcal{Z}$. We have a natural fibration $\pi_\mathcal{Z}\colon \mathcal{Z}_{2k}^\alpha(E) \to M$. The vertical space is given by

$$\forall J \in N_\mathcal{Z}, \quad \mathcal{V}_J^\mathcal{Z} = T\mathcal{Z}_{2k}^\alpha(E_x) = \left(\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}\setminus\{0\}} \mathcal{B}_j(E_x, J)\right) \bigcap \mathrm{End}(E_x) = J.\mathfrak{so}_*(E_x, J) \quad (51)$$

according to section 3.1.2 (more particulary equation (40)) and where

$$\mathfrak{so}_*(E_x, J) := \left(\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}\setminus\{0\}} \mathfrak{so}_j(E_x, J)\right) \bigcap \mathfrak{so}(E_x).$$

The metric connection ∇ gives us a splitting: $T\mathcal{Z}_{2k}^\alpha(E) = \mathcal{V}^\mathcal{Z} \oplus \mathcal{H}^\mathcal{Z}$. Indeed we have the following splitting (coming from ∇)

$$TSO(E) = \mathcal{V}^{SO(E)} \oplus \mathcal{H} \quad (52)$$

[16] using the notations defined in section 3.1.2 (i.e. the definition of $\mathfrak{so}_j(J)$ for $j \in \mathbb{Z}/2\mathbb{Z}$).

where $\mathcal{V}_J^{SO(E)} = T_J SO(E_x) = J.\mathfrak{so}(E_x)$ (since $0 = \nabla(J^tJ) = (\nabla J)^t J + J^t(\nabla J)$ $\Longrightarrow \nabla J \in T_J SO(E_x)$). Then for all (local) section $J\colon U \subset M \to N_{\mathcal{Z}}$, we have

$$0 = \nabla J^{2k} = \sum_{p+l=2k-1} J^p(\nabla J)J^l$$

so that according to (38), $\nabla J \in \mathcal{V}^{\mathcal{Z}}$ and thus in the decomposition (52), we have $[dJ]_{\mathcal{V}^{\mathfrak{so}(E)}} \in \mathcal{V}^{\mathcal{Z}}$ and hence $[dJ]_{\mathcal{H}} = dJ - [dJ]_{\mathcal{V}^{\mathfrak{so}(E)}} \in TN_{\mathcal{Z}}$ which leads to

$$T\mathcal{Z}_{2k}^{\alpha}(E) = \mathcal{V}^{\mathcal{Z}} \oplus \mathcal{H}_{|\mathcal{Z}_{2k}^{\alpha}(E)}. \tag{53}$$

Then we can endow $N_{\mathcal{Z}}$ with the metric defined as in (47) and where the fibre $\langle\ ,\ \rangle_{\mathcal{V}^{\mathcal{Z}}}$ is induced by the trace metric (48), for which we have $\mathcal{H}^{\mathcal{Z}} = \mathcal{V}^{\mathcal{Z}\perp}$.
Furthermore let us remark that $T\mathcal{Z}_{2k}^{\alpha}(E)$ is a subbundle of $TSO(E)_{|\mathcal{Z}_{2k}^{\alpha}(E)}$ and that we have

$$\begin{aligned} TSO(E)_{|\mathcal{Z}_{2k}^{\alpha}(E)} &= \mathcal{B}_0(\pi_{\mathcal{Z}}^*E) \oplus \mathcal{B}_*(\pi_{\mathcal{Z}}^*E) \oplus \mathcal{H}_{|\mathcal{Z}_{2k}^{\alpha}(E)} \\ &= \mathcal{B}_0(\pi_{\mathcal{Z}}^*E) \oplus T\mathcal{Z}_{2k}^{\alpha}(E) \end{aligned}$$

where [17]

$$\begin{aligned} \mathcal{B}_0(\pi_{\mathcal{Z}}^*E)_J &= \mathcal{B}_0(E_x, J) \quad \text{and} \\ \mathcal{B}_*(\pi_{\mathcal{Z}}^*E)_J &= \mathcal{B}_*(E_x, J) := \left(\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}\setminus\{0\}} \mathcal{B}_j(E_x, J) \right) \bigcap \mathrm{End}(E_x) = \mathcal{V}_J^{\mathcal{Z}} \end{aligned}$$

for all $J \in \mathcal{Z}_{2k}^{\alpha}(E)$. In other words, $\pi_{\mathcal{Z}}^*E$ is canonically endowed with a $2k$-structure: $\mathcal{I}_J = J$, $\forall J \in N_{\mathcal{Z}}$, and this $2k$-structure defines the spaces $\mathcal{B}_j(\pi_{\mathcal{Z}}^*E) := \mathcal{B}_j(\pi_{\mathcal{Z}}^*E, \mathcal{I})$.
Now let us precise the relation between $SO(E)$ and $\mathfrak{so}(E)$ and in particular the relation $T_J SO(E_x) = J.\mathfrak{so}(E_x)$. For $J \in SO(E)$, let

$$L_J\colon A \in \mathrm{End}(E_x) \longmapsto J.A \in \mathrm{End}(E_x)$$

be the left multiplication by J in $\mathrm{End}(E_x)$, with $x = \mathrm{p}(J)$. Let us still denote by $\mathcal{I}$, the tautological section of $\mathrm{p}^*SO(E)$ defined by $\mathcal{I}_J = J$, $\forall J \in SO(E)$, and whose restriction to $N_{\mathcal{Z}}$ is our canonical $2k$-structure $\mathcal{I}$ on $\pi_{\mathcal{Z}}^*E$. Then let $L_{\mathcal{I}}\colon SO(E) \longrightarrow \mathrm{Aut}(\mathrm{End}(\mathrm{p}^*E))$ be the section of the bundle of linear automorphism of the vector bundle $\mathrm{End}(\mathrm{p}^*E)$ defined by

$$L_{\mathcal{I}}\colon J \in SO(E) \longmapsto L_J \in \mathrm{Aut}(\mathrm{End}(E_{\mathrm{p}(J)}))$$

or more concretely

$$L_{\mathcal{I}}\colon (J, A) \in \mathrm{End}(\mathrm{p}^*E) \longmapsto (J, J.A) \in \mathrm{End}(\mathrm{p}^*E).$$

[17] still with the notation defined in section 3.1.2

Then we have

$$\mathcal{V}^{SO(E)} = L_{\mathcal{I}}(\mathfrak{so}(\mathrm{p}^*E)) \quad \text{and} \quad \mathcal{B}_j(\pi_{\mathcal{Z}}^*E) = L_{\mathcal{I}}(\mathfrak{so}_j(\pi_{\mathcal{Z}}^*E))$$

which we will denote more simply by

$$\mathcal{V}^{SO(E)} = \mathcal{I}.\mathfrak{so}(\mathrm{p}^*E) \quad \text{and} \quad \mathcal{B}_j(\pi_{\mathcal{Z}}^*E) = \mathcal{I}.\mathfrak{so}_j(\pi_{\mathcal{Z}}^*E).$$

Example 4.4 Let us consider the previous example and let us suppose that there exists a (global) section J_j of $(\mathcal{Z}_{2k}(E))^j = \mathcal{U}_{p'}^{j.\alpha}(E)$ for some $j \in \mathbb{Z}$ and $p' = \dfrac{2k}{(2k,j)}$. Let us consider the subbundle

$$N_{\mathcal{Z}}^j := \mathcal{Z}_{2k,j}^{\alpha}(E, J_j) = \{J \in \mathcal{Z}_{2k}^{\alpha}(E) |\, J^j = J_j\}$$

for which we have the natural fibration $\pi_{\mathcal{Z}}^j \colon \mathcal{Z}_{2k,j}^{\alpha}(E, J_j) \to M$. The vertical space is given by

$$\forall J \in N_{\mathcal{Z}}^j, \quad \mathcal{V}_J^{\mathcal{Z},j} = T_J\mathcal{Z}_{2k,j}^{\alpha}(E_x, J_j) = \oplus_{q=1}^{(r,j)-1}\mathcal{B}_{qp}(E_x, J) = J.\mathfrak{u}_{j-1}^*(E_x, J)$$

according to (43)[18], where

$$\mathfrak{u}_{j-1}^*(E_x, J) = \oplus_{q=1}^{(r,j)-1}\mathfrak{so}_{pq}(E_x, J) = \mathfrak{u}_{j-1}(E_x, J)/\mathfrak{u}_0(E_x, J) = \mathfrak{so}_0(E_x, J)/\mathfrak{so}_0(E_x, J).$$

Furthermore, differentiating the definition equation of $\mathcal{Z}_{2k,j}(E, J_j)$: $J^j = J_j$, we obtain: for all (local) section J of $\pi_{\mathcal{Z}}^j$,

$$\nabla J^j = \sum_{l+q=j-1} J^l \nabla J\, J^q = \nabla J_j \tag{54}$$

so that

$$\nabla J \in \mathcal{V}^{\mathcal{Z},j} \Longleftrightarrow \nabla J_j = 0,$$

therefore in general, we have $\nabla J \notin \mathcal{V}^{\mathcal{Z},j}$. We will simply set

$$\mathcal{H}^{\mathcal{Z},j} = \mathcal{V}^{\mathcal{Z},j^{\perp}} \cap T\mathcal{Z}_{2k,j}^{\alpha}(E, J_j).$$

Then the splitting

$$T\mathcal{Z}_{2k,j}^{\alpha}(E, J_j) = \mathcal{V}^{\mathcal{Z},j} \oplus \mathcal{H}^{\mathcal{Z},j}$$

do not correspond to the splitting (52) or equivalently to (53), in general. In other words, the connection in $\pi_{\mathcal{Z}}$ defined by the horizontal distribution $\mathcal{H}_{|N_{\mathcal{Z}}}$ is not reducible to a connection in $\pi_{\mathcal{Z}}^j$ (which could only be $\mathcal{H}^{\mathcal{Z},j}$): it happens if and only if $\mathcal{H}$ is tangent to $N_{\mathcal{Z}}^j$. Besides we have two different ways to decompose

[18] and with the notations of section 3.1.3, in particular $p = \dfrac{r}{(r,j)}$.

the orthogonal of $\mathcal{V}^{\mathcal{Z},j}$ in $TN_{\mathcal{Z}}$, using the decompositions $TN_{\mathcal{Z}} = \mathcal{V}^{\mathcal{Z}} \oplus \mathcal{H}^{\mathcal{Z}}$ or $TN_{\mathcal{Z}_{|N^j_{\mathcal{Z}}}} = TN^j_{\mathcal{Z}} \oplus {TN^j_{\mathcal{Z}}}^{\perp}$:

$$\begin{aligned} TN_{\mathcal{Z}_{|N^j_{\mathcal{Z}}}} &= \mathcal{V}^{\mathcal{Z},j} \oplus \mathcal{V}^{\mathcal{Z},j^{\perp}} \\ &= \mathcal{V}^{\mathcal{Z},j} \oplus \mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}} \oplus \mathcal{H}_{|N^j_{\mathcal{Z}}} \\ &= \underbrace{\mathcal{V}^{\mathcal{Z},j} \oplus \mathcal{H}^j_{\mathcal{Z}}}_{TN^j_{\mathcal{Z}}} \oplus {TN^j_{\mathcal{Z}}}^{\perp} \end{aligned}$$

In particular, we have for any (local) section $J\colon U \subset M \to N^j_{\mathcal{Z}}$

$$[dJ]_{\mathcal{V}^{\mathcal{Z},j}} = [\nabla J]_{\mathcal{V}^{\mathcal{Z},j}} = \mathrm{pr}^{\mathcal{V}^{\mathcal{Z}}}_{\mathcal{V}^{\mathcal{Z},j}}(\nabla J)$$

where $[\ ]_{\mathcal{V}^{\mathcal{Z},j}}\colon TN_{\mathcal{Z}} \to \mathcal{V}^{\mathcal{Z},j}$ and $\mathrm{pr}^{\mathcal{V}^{\mathcal{Z}}}_{\mathcal{V}^{\mathcal{Z},j}}\colon \mathcal{V}_{\mathcal{Z}} \to \mathcal{V}^{\mathcal{Z},j}$ are resp. the orthogonal projections. Moreover, let us decompose $T\mathcal{SO}(E)_{|\mathcal{Z}^{\alpha}_{2k,j}(E)}$ into an orthogonal sum making appear the vertical subbundle $\mathcal{V}^{\mathcal{Z},j}$ of $N^j_{\mathcal{Z}}$:

$$\begin{aligned} T\mathcal{SO}(E)_{|N^j_{\mathcal{Z}}} &= \mathcal{B}_0({\pi^j_{\mathcal{Z}}}^* E, \mathcal{I}) \oplus \quad \mathcal{B}_*({\pi^j_{\mathcal{Z}}}^* E, \mathcal{I}) \quad \oplus \quad \mathcal{H}_{|N^j_{\mathcal{Z}}} \\ &= \underbrace{\mathcal{B}_0({\pi^j_{\mathcal{Z}}}^* E, \mathcal{I}) \oplus \mathcal{V}^{\mathcal{Z},j}}_{\mathcal{I}.\mathfrak{so}_*({\pi^j_{\mathcal{Z}}}^* E, \mathcal{I}^j)} \oplus \underbrace{\mathcal{I}.\mathfrak{so}_*({\pi^j_{\mathcal{Z}}}^* E, \mathcal{I}^j)}_{\mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}}} \oplus \mathcal{H}_{|N^j_{\mathcal{Z}}} \ . \end{aligned}$$

Now let us see how we can determine $\mathcal{H}^{\mathcal{Z},j}$ from the section J_j. First we remark that $\mathcal{H}^{\mathcal{Z},j} \cap \mathcal{V}^{\mathcal{Z}} = \{0\}$ (indeed $\ker d\pi_{\mathcal{Z}} \cap \mathcal{H}^{\mathcal{Z},j} \subset \ker d\pi_{\mathcal{Z}} \cap TN^j_{\mathcal{Z}} = \mathcal{V}^{\mathcal{Z},j}$ and of course $\mathcal{V}^{\mathcal{Z},j} \cap \mathcal{H}^{\mathcal{Z},j} = \{0\}$). Therefore $\mathcal{H}^{\mathcal{Z},j}$ is a vector subbundle of $(\mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}}) \oplus \mathcal{H}$ which satisfies $(\mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}}) \cap \mathcal{H}^{\mathcal{Z},j} = \{0\}$. Thus $\mathcal{H}^{\mathcal{Z},j}$ is the graph of some linear map[19] $\Gamma\colon \mathcal{H} \to \mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}}$, [20]

$$\mathrm{Id} + \Gamma\colon W \in \mathcal{H} \mapsto W + \Gamma(W) \in \mathcal{H} \oplus (\mathcal{V}^{\mathcal{Z},j^{\perp}} \cap \mathcal{V}^{\mathcal{Z}})$$

has $\mathcal{H}^{\mathcal{Z},j}$ as image.
Let us concentrate ourself on (54). ∇J is in $\mathcal{V}^{\mathcal{Z}}$ so that we can write it $\nabla J = \sum_{i=1}^{r-1} JA_i$ with $A_i \in \mathfrak{so}_i(E, J)$, according to (51). Then we have $\forall i \in \{1, \dots, r-1\}$,

$$\sum_{l+q=j-1} J^q(JA_i)J^l = \sum_{l=0}^{j-1} J^{j-1}\omega_r^l JA_i = \frac{1-(\omega_r^i)^j}{1-\omega_r^i} J^j A_i$$

so that

$$\sum_{l+q=j-1} J^q(\nabla J)J^l = J^j \left(\sum_{\substack{i=1 \\ i \notin p.\mathbb{Z}}}^{r-1} \frac{1-(\omega_r^i)^j}{1-\omega_r^i} A_i \right)$$

[19] i.e. a morphism of vector bundle

[20] In the following reasoning, we will forget the index "$|N^j_{\mathcal{Z}}$" in $\mathcal{H}_{|N^j_{\mathcal{Z}}}$ to do not weigh down the equations. The right notation will reapear in the final equation.

where as usual $p = \dfrac{r}{(r,j)}$ is the order of ω_r^j. In particular, we remark that (with obvious notation)[21]

$$\sum_{l+q=j-1} L(J^l) \circ R(J^q)^{-1} \colon \mathcal{B}_*(E,J) \longmapsto \mathcal{B}_*(E,J_j) \tag{55}$$

is a surjective map with kernel[22]

$$\bigoplus_{i \in p.\mathbb{Z}_r \setminus \{0\}} \mathcal{B}_i(E,J) = J^* \mathcal{V}^{\mathcal{Z},j},$$

so that it induces an isomorphism from

$$J.\mathfrak{so}_*(E,J_j) = J^*(\mathcal{V}^{\mathcal{Z},j^\perp} \cap \mathcal{V}^{\mathcal{Z}})$$

onto $\mathcal{B}_*(E,J_j)$. Let us denote by $P^j(J)$ the surjective map (55) and by $P^j(J)^{-1}$ the inverse map of the isomorphism induced on $J.\mathfrak{so}_*(E,J_j)$. Then we have

$$P^j(J)(\nabla J) = \nabla J_j$$

so that $[\nabla J]^{\mathcal{V}^{\mathcal{Z},j^\perp} \cap \mathcal{V}^{\mathcal{Z}}} = P^j(J)^{-1}(\nabla J_j)$, but we have $\nabla J = [dJ]^{\mathcal{V}^{\mathcal{Z}}}$, and therefore

$$[dJ]^{\mathcal{V}^{\mathcal{Z},j^\perp} \cap \mathcal{V}^{\mathcal{Z}}} = P^j(J)^{-1}(\nabla J_j). \tag{56}$$

On the other hand, $d\pi^{\mathcal{Z}} \circ dJ = \mathrm{Id}_{TM}$ so that $d\pi^{\mathcal{Z}} \circ [dJ]^{\mathcal{H}} = \mathrm{Id}_{TM}$, which with (56) allows to conclude that

$$\Gamma = P^j(J)^{-1} \circ (\nabla J_j) \circ d\pi^{\mathcal{Z}}_{|\mathcal{H}}$$

that is to say, for all $W \in \mathcal{H}_{|N^j_{\mathcal{Z}}}$

$$\Gamma(W) = P^j(J_0)^{-1} \cdot \nabla_{(\pi^{\mathcal{Z}})_* W} J_j$$

where $W = (J_0, W_{J_0})$, $J_0 \in N^j_{\mathcal{Z}}$, $W_{J_0} \in \mathcal{H}_{J_0}$.

4.1.3 Ψ-torsion, Ψ-difference tensor, and curvature of a Pfaff system

Ψ-torsion, Ψ-difference. Let us consider a vector bundle morphism

$$\begin{array}{ccc} TM & \xrightarrow{\Psi} & (E,\nabla) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & N \end{array},$$

∇ being a connections on the vector bundle E. Then the Ψ-torsion of ∇ is the ψ^*E-valued 2-form on M,

$$T^\Psi(X,Y) = \nabla_X(\Psi Y) - \nabla_Y(\Psi X) - \Psi[X,Y] = d^\nabla \Psi(X,Y) \quad \forall X, Y \in \mathcal{C}(TM).$$

Let us give now some examples.

[21] remark that $\mathcal{B}_*(E,J_j) = J^j.\mathfrak{so}_*(E,J^j) = J^j.\left(\left(\oplus_{i \in \mathbb{Z}_r \setminus p.\mathbb{Z}_r} \mathfrak{so}_i(E,J)\right) \cap \mathfrak{so}(E,J)\right)$

[22] Obviously, since J is a local section, everything is local here and E must be replaced by $E_U := \mathrm{p}^{-1}(U)$, but we do not want to weigh down the notations.

Example 4.5 Let N be a manifold and suppose that we have a splitting $TN = \mathcal{V} \oplus \mathcal{H}$ and suppose also that the vertical bundle $\mathcal{V}$ is endowed with a covariant derivative ∇^c and let $\phi\colon TN \to \mathcal{V}$ be the projection (morphism) on $\mathcal{V}$ along $\mathcal{H}$, then we can speak about the ϕ-torsion of ∇^c, $T^\phi = d^{\nabla^c}\phi$.

$$\begin{array}{ccc} TN & \xrightarrow{\phi} & (\mathcal{V}, \nabla^c) \\ \downarrow & & \downarrow \\ N & \xrightarrow{\mathrm{Id}} & N \end{array},$$

Example 4.6 Let $s\colon M \to (N, \nabla)$ be a map from a manifold M into an affine manifold (N, ∇) and suppose that we have a splitting $TN = \mathcal{V} \oplus \mathcal{H}$, then let us consider the morphism of bundle

$$\begin{array}{ccc} TM & \xrightarrow{d^v s} & (\mathcal{V}, \nabla^v) \\ \downarrow & & \downarrow \\ M & \xrightarrow{s} & N \end{array}$$

where ∇^v is the vertical part of the affine connection∇.

Then the *vertical s-torsion* of N is $T^s := T^{d^v s} = d^{\nabla^v} d^v s$.

Example 4.7 In particular let us take $s = \mathrm{Id}_N$ (in the previous example) and thus $d^v s = \phi$ the projection on $\mathcal{V}$ and then the ϕ-torsion of ∇^v (or Id_N-torsion of N) is the *vertical torsion* in $\mathcal{V}$: $T^v = d^{\nabla^v}\phi$.
Now for any map $s\colon M \to N$ we have

$$T^s = s^* T^v.$$

We will say that s is *vertically torsion free* if $T^s = 0$. ■

Now, we define the ψ-difference and the ψ-equivalence.

Definition 4.1 *Let $E \to N$ be a vector bundle and let us suppose that we have a morphism of bundle $\psi\colon TN \to E$ (over Id_N). Let us consider the ψ-torsion of a given connection ∇ in E (if ψ is an isomorphism then $T^\psi = \psi \circ T$, where T is the torsion of the linear connection $\psi^{-1} \circ \nabla \circ \psi$ on N). Given another connection ∇' in E, the ψ-**difference tensor** S^ψ for the pair (∇, ∇') is defined by*

$$S^\psi(X, Y) = \nabla_X(\psi Y) - \nabla'_X(\psi Y) = (\nabla - \nabla')_X(\psi Y).$$

*Then S^ψ is symmetric precisely when ∇ and ∇' have the same ψ-torsion. On the other hand, if S^ψ is skew-symmetric we will way (following [36]) that ∇ and ∇' are ψ-**equivalent**: it means that these have the same ψ-geodesics, a ψ-geodesic of ∇ being a path $y(t)$ in N solution of the equation*

$$\nabla_{y'(t)}(\psi y'(t)) = 0$$

(if ψ is an isomorphism then ψ-geodesics are precisely the geodesics of $\psi^{-1} \circ \nabla \circ \psi$).

Curvature of a Pfaff system

Definition 4.2 *Let $\mathcal{P}$ be a Pfaff system on the manifold N. Then for any local sections of $\mathcal{P}$, $X, Y\colon (N,a) \to \mathcal{P}$, defined in the neighbourhood of $a \in N$, the image $([X,Y]_a)_\mathcal{V}$ of $[X,Y]_a$ by the canonical projection $T_aN \to \mathcal{V}_a = T_aN/\mathcal{P}_a$, depends only on the values X_a, Y_a at $a \in N$, of the vector fields X, Y. We define the curvature of $\mathcal{P}$ as the tensor $R \in \Lambda^2\mathcal{P}^* \otimes \mathcal{V}$,*

$$R_a(X_a, Y_a) := -([X,Y]_a)_\mathcal{V}.$$

Definition 4.3 *Let N be a manifold endowed with a Pfaff system $\mathcal{V}$ ("vertical subbundle") and let us suppose that $\mathcal{V}$ admits a connection i.e. a complement $\mathcal{H}$ ("horizontal subbundle"): $TN = \mathcal{V} \oplus \mathcal{H}$. Then $\mathcal{V}$ is identified to $TN/\mathcal{P}$ so that the curvature of the connection $\mathcal{H}$ becomes the tensor $R \in \Lambda^2\mathcal{H}^* \otimes \mathcal{V}$ defined by*

$$R(X,Y) = -[X,Y]_\mathcal{V} \quad \forall X, Y \in \mathcal{C}(\mathcal{H})$$

the subscripts "$\mathcal{V}$" designing the $\mathcal{V}$-component along $\mathcal{H}$.

Convention We will often extend R to the corresponding horizontal 2-form on N, still denoted by R: $R \in \Lambda^2 T^*N \otimes \mathcal{V}$ such that $R(X,Y) = 0$ if X or $Y \in \mathcal{V}$.

Theorem 4.2 *Let $\pi\colon Q \to Q/H = M$ be a H-principal bundle endowed with a connection 1-form $\omega\colon TQ \to \mathfrak{h}$. Let $\mathcal{H} = \ker\omega \subset TQ$ be the corresponding horizontal subbundle. Let be $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ the curvature 2-form. Then we have*

$$R^\mathcal{H}_q(X,Y) = q.\Omega_q(X,Y) \quad \forall q \in Q, \forall X, Y \in \mathcal{H}_q$$

$R^\mathcal{H}$ being the curvature of the connection $\mathcal{H}$. In other words, we have

$$R^\mathcal{H} = \Omega^*$$

where $\Omega^*_q = q.\Omega_q$.

4.2 Harmonic sections of homogeneous fibre bundles

In this section, we study fibre bundles $\pi\colon N \to M$ for which the fibre is a homogeneous space H/K. To do that, we follow the exposition of [36] (subsection 4.2.1 and 4.2.2) and then we add a generalisation of the results (of [36]) to non section maps in the end of 4.2.2, and finally we study the homogeneous fibre bundle reductions in 4.2.3.

4.2.1 Definitions and Geometric properties

Let $\pi_M\colon Q \to M$ be a principal H-bundle, with H a Lie group. Let K be a Lie subgroup of H and $N = Q/K$. Then the map $\pi_N\colon Q \to N$ is a principal K-bundle and we have $\pi_M = \pi \circ \pi_N$ where $\pi\colon N \to M$ is a fibre bundle with fibre H/K, which is naturally isomorphic to the associated bundle $Q \times_H H/K$. We assume the following hypothesis

(i) H/K is reductive: $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathrm{Ad}K(\mathfrak{p}) \subset \mathfrak{p}$, where $\mathfrak{h}$ and $\mathfrak{k}$ are respectively the Lie algebras of H and K.

(ii) M is endowed with a Riemannian metric g

(iii) H/K is Riemannian: there exists a H-invariant Riemannian metric on H/K (equivalently an $\mathrm{Ad}K$-invariant (positive definite) inner product on $\mathfrak{p}$). Equivalently $\mathrm{Ad}_{\mathfrak{p}}K$ is compact.

(iv) The principal H-bundle $\pi_M \colon Q \to M$ is endowed with a connection. We denote by ω the corresponding $\mathfrak{h}$-valued connection form on Q.

Then the splitting $TQ = \mathcal{V}_0 \oplus \mathcal{H}_0$ defined by ω ($\mathcal{V}_0 = \ker d\pi_M$, $\mathcal{H}_0 = \ker \omega$) gives rise by $d\pi_N$, to the following decomposition $TN = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} = \ker d\pi = d\pi_N(\mathcal{V}_0)$ and $\mathcal{H} = d\pi_N(\mathcal{H}_0)$. Let $\mathfrak{p}_Q := Q \times_K \mathfrak{p} \to N$ be the vector bundle associated to $\pi_N \colon Q \to N$ with fibre $\mathfrak{p}$. Let us denote by $[q, a] \in \mathfrak{p}_Q$ the element defined by $(q, a) \in Q \times \mathfrak{p}$. Then we have the following vector bundle isomorphism

$$\begin{array}{rccc} I \colon & \mathcal{V} & \longrightarrow & \mathfrak{p}_Q \\ & d\pi_N(q.a) & \longmapsto & [q, a] \end{array}$$

where $q \in Q$, $a \in \mathfrak{p}$ and as usual $q.a = \dfrac{d}{dt}_{|t=0} q.\exp(ta) \in T_qQ$. Decomposing $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{p}}$ following $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, then since H/K is reductive, $\omega_{\mathfrak{p}}$ is a K-equivariant ($\omega_{\mathfrak{p}}(X.h) = \mathrm{Ad}h\, \omega_{\mathfrak{p}}(X)$) and π_N-horizontal ($\omega_{\mathfrak{p}|\mathcal{V}_0} = 0$) $\mathfrak{p}$-valued 1-form on Q and hence projects to a $\mathfrak{p}$-valued 1-form ϕ on N:

$$\phi(d\pi_N(X)) = [q, \omega_{\mathfrak{p}}(X)].$$

Then we have

$$\phi_{|\mathcal{V}} = I \quad \text{and} \quad \ker \phi = \mathcal{H}.$$

We can now construct a Riemannian metric h on N:

$$h = \pi^* g + \langle \phi, \phi \rangle \tag{57}$$

where $\langle\ ,\ \rangle$ is the fibre metric induced on $\mathfrak{p}_Q$ by the inner product on $\mathfrak{p}$.

In the same way, let Φ be the $\mathfrak{p}_Q$- valued 2-form on N defined by the component $\Omega_{\mathfrak{p}}$ of the curvature form Ω of ω. Since $\Omega_{\mathfrak{p}}$ is π_M-horizontal ($\Omega(X, Y) = 0$ if $X \in \mathcal{V}_0$ or $Y \in \mathcal{V}_0$), then Φ is π-horizontal: $\Phi(X, Y) = 0$, if $X \in \mathcal{V}$ or $Y \in \mathcal{V}$.

Remark 4.1 In [36], $\mathfrak{p}_Q$ is called the *canonical bundle*, I the *canonical isomorphism*, ϕ the *homogeneous connection form*, and Φ the *homogeneous curvature form*.

The 1-form $\omega_{\mathfrak{k}}$ (which is a connection form in π_N because H/K is reductive) defines a connection in π_N called the *canonical connection*. This connection induces a covariant derivative ∇^c in the associated bundle $\mathfrak{p}_Q$, with respect to which the fibre metric is parallel. ∇^c defines a exterior derivative d^c on the

space of $\mathfrak{p}_Q$-valued differential forms on N. This allows us to define the *canonical torsion* T^c which is nothing but the ϕ-torsion of ∇^c (see section 4.1.3)

$$T^c(A,B) = d^c\phi(A,B) = \nabla^c_A(\phi B) - \nabla^c_B(\phi A) - \phi[A,B], \quad \forall A,B \in \mathcal{C}(TN) \quad (58)$$

Let $\mathfrak{h}_Q := Q \times_H \mathfrak{h} \to M$ be the vector bundle associated to π_M with fibre $\mathfrak{h}$, and in the same way $\mathfrak{k}_Q := Q \times_K \mathfrak{k} \to N$ the bundle associated to π_N with fibre $\mathfrak{h}$. Then we have

$$\pi^*\mathfrak{h}_Q = \mathfrak{k}_Q \oplus \mathfrak{p}_Q.$$

The Lie bracket of $\mathfrak{h}$ induces a bracket on the fibres of $\mathfrak{h}_Q$, and those of $\pi^*\mathfrak{h}_Q$, which we continue to denote by $[\ ,\ ]$, and we denote also its $\mathfrak{p}_Q$-component (when there is no risk of confusion) by $[\ ,\ ]_\mathfrak{p}$ (otherwise we denote it by $[\ ,\ ]_{\mathfrak{p}_Q}$). Taking the $\mathfrak{p}$-component of the structure equation $d\omega = \Omega - \frac{1}{2}[\omega \wedge \omega]$ and then projecting on N, we obtain *the homogeneous structure equation*:

$$T^c = \Phi - \frac{1}{2}[\phi \wedge \phi]_\mathfrak{p} \quad (59)$$

and thus

$$\begin{aligned} T^c_{|\mathcal{V}\times\mathcal{V}} &= -[I\cdot, I\cdot]_\mathfrak{p}, \quad T^c_{|\mathcal{V}\times\mathcal{H}} = 0 \\ T^c_{|\mathcal{H}\times\mathcal{H}} &= \Phi_{|\mathcal{H}\times\mathcal{H}}. \end{aligned}$$

In particular, T^c is horizontal if and only if H/K is a (locally) symmetric space, and in this case

$$T^c = \Phi \quad (60)$$

Remark 4.2 According to (59) and (58), for all $X, Y \in \mathcal{H}$, (extended to vector fields in N denoted by the same letters), we have

$$\Phi(X,Y) = T^c(X,Y) = -\phi([X,Y])$$

so that

$$\Phi = R^\mathcal{H},$$

according to definition 4.3. The homogeneous curvature form is nothing but the curvature of the connection $\mathcal{H}$. ∎

Now, let U be the $\mathfrak{p}_Q$-valued symmetric bilinear form defined on $\mathfrak{p}_Q$ by:

$$\langle \mathrm{U}(a,b), c\rangle = \langle [c,a]_\mathfrak{p}, b\rangle + \langle a, [c,b]_\mathfrak{p}\rangle \quad (61)$$

where $\langle\ ,\ \rangle$ is the fibre metric, and $a, b, c \in \mathfrak{p}_Q$. Let us set

$$\mathrm{B} = \mathrm{U} + [\ ,\ ]_\mathfrak{p}$$

which is a $\mathfrak{p}_Q$-valued bilinear form on $\mathfrak{p}_Q$, whose the symmetric and skew symmetric components are respectively U and $[\ ,\ ]_\mathfrak{p}$. U vanishes if and only if H/K is naturally reductive and B if and only if H/K is (locally) symmetric . Then denoting by ∇^N the Levi-Civita connection on N, we have:

Theorem 4.3 *[36] Let us consider the difference tensor:*

$$S(A,B) = \phi(\nabla^N_A B) - \nabla^c_A(\phi B)$$

then we have

$$2S = \phi^*\mathrm{U} - T^c = \phi^*\mathrm{B} - \Phi.$$

Consequently, $\forall V \in \mathcal{C}(\mathcal{V})$

$$I(\nabla^v_A V) = \nabla^c_A(IV) + \frac{1}{2}\mathrm{B}(\phi A, IV). \tag{62}$$

In particular, if H/K *is a (locally) symmetric space, we have*

$$I\nabla^v_A V = \nabla^c_A(IV).$$

Remark 4.3 If H/K is a symmetric space, under the canonical identification $I\colon \mathcal{V} \xrightarrow{\simeq} \mathfrak{p}_Q$, we have $\nabla^v = \nabla^c$ on $\mathcal{V}$. More generally the difference between ∇^v and ∇^c looks like to the difference between the Levi-Civita and canonical connections of a reductive Riemannian homogeneous space (see section 1.6). Moreover, ∇^v is ϕ-equivalent to ∇^c when H/K is naturally reductive, according to (62).

Let ∇^ω be the covariant derivative in the vector bundle $\mathfrak{h}_Q$ (associated to π_M), defined by the connection form ω. Let us decompose (the π-pullback of) ∇^ω following the decomposition $\pi^*\mathfrak{h}_Q = \mathfrak{k}_Q \oplus \mathfrak{p}_Q$, and the $\mathfrak{p}_Q$-component gives us a connection $\nabla^{\mathfrak{p}}$ in $\mathfrak{p}_Q$.

Theorem 4.4 *For all* $V \in \mathcal{C}(\mathfrak{p}_Q)$*,*

$$\nabla^{\mathfrak{p}} V = \nabla^\omega V - [\phi, \alpha]_{\mathfrak{h}}$$

and

$$\nabla^c V = \nabla^{\mathfrak{p}} V - [\phi, V]_{\mathfrak{p}} = \nabla^\omega V - [\phi, V].$$

Consequently, $\nabla^{\mathfrak{p}}$ *and* ∇^c *are* ϕ*-equivalent (since their* ϕ*-difference is* $[\phi,\phi]_{\mathfrak{p}}$*). In particular* $\nabla^c = \nabla^{\mathfrak{p}}$ *if* H/K *is a (locally) symmetric space.*

Example 4.8 Let us consider the situation described by example 4.1 and suppose that $u\colon L \to M$ is an isometry. Then if $\pi\colon N \to M$ is a homogeneous fibre bundle like above then this is also the case for $u^*\pi\colon u^*N \to L$.
Indeed let us set

$$u^*Q = \{(z,q) \in L \times Q,\ q \in \pi_M^{-1}(\{u(z)\})\} = \bigsqcup_{z\in L} \{z\} \times f(z).H$$

then $u^*\pi_M\colon (z,q) \in u^*Q \mapsto z \in L$ is a principal H-bundle over L. Then we have $u^*N = u^*Q/K$, and $u^*\pi\colon u^*N \to L$ is a fibre bundle with fibre H/K.
Finally we have to define a connection on $u^*\pi_M\colon u^*Q \to L$. Let us extend the

connection ω, to a connection on $\mathrm{Id}_L \times \pi_M \colon L \times Q \to L \times M$ by $\tilde{\omega}_{(z,q)}(dz+dq) = \omega_q(dq)$ and then let us set

$$u^*\omega := \tilde{\omega}_{|T(u^*Q)}.$$

In the same way, the homogeneous connection and curvature forms on u^*N are given respectively by

$$u^*\phi := \tilde{\phi}_{|T(u^*N)} \quad \text{and} \quad u^*\Phi := \tilde{\Phi}_{|T(u^*N)\oplus T(u^*N)}.$$

The canonical torsion T^c on u^*N is also given by $u^*T^c := \tilde{T}^c_{|T(u^*N)\oplus T(u^*N)}$.

4.2.2 Vertical harmonicity equation

We know that the structure group H of $\pi_M \colon Q \to M$ is reducible to K (i.e. there exists an K-bundle $\pi'_M \colon Q' \to M$) if and only if the associated bundle $\pi \colon N \to M$ admits a (global) section $s \colon M \to N$ (see [27]) so that there is a one to one correspondance between the K-reductions of π_M and the space of sections $\mathcal{C}(\pi)$.
Let $\omega' = \omega_{\mathfrak{h}|TQ'}$. Then ω' a connection in π'_M, and ω is reducible if and only if $\omega_{|TQ'} = \omega'$ (see [27]). The reducibility of ω can be characterized as follows.

Proposition 4.1 *The following statements are equivalent:*

(i) *s is horizontal;*

(ii) *$s^*\phi = 0$;*

(iii) *s is an isometric immersion;*

(iv) *ω is reducible*

Now we have the following expression of the tension field for sections $s \colon M \to N$.

Theorem 4.5 *[36] For all $s \in \mathcal{C}(\pi)$,*

$$I(\tau^v(s)) = -d^*(s^*\phi) + \frac{1}{2}\mathrm{Tr}(s^*\phi^*\mathrm{U})$$

where d^* is the coderivative for $s^*\mathfrak{p}_Q$-valued differential forms on M relative to the s-pullback of any connection in $\mathfrak{p}_Q$ which is ϕ-equivalent to ∇^c. In particular, if H/K is naturally reductive then s is an harmonic section if and only if $s^*\phi$ is coclosed.

Remark 4.4 If H/K is naturally reductive, to compute the vertical tension field $\tau^v(s) = \mathrm{Tr}(\nabla^v d^v s)$, we can use instead of ∇^v any connection in $\mathcal{V} \cong \mathfrak{p}_Q$ which is ϕ-equivalent to ∇^c.

From the homogeneous structure equation (59), we obtain

$$s^*\Phi = d^c(s^*\phi) + \frac{1}{2}[s^*\phi \wedge s^*\phi]_{\mathfrak{p}},$$

hence every horizontal section is flat (i.e. $s^*\Phi = 0$).
Let us introduce the following 3-covariant tensor $\langle s^*\phi \otimes s^*\Phi\rangle$ on M:

$$\langle s^*\phi \otimes s^*\Phi\rangle(X,Y,Z) = \langle s^*\phi(X), s^*\Phi(Y,Z)\rangle,$$

then we have

Theorem 4.6 *[36] For all $s \in \mathcal{C}(\pi)$ we have*

(i) $\phi(\nabla ds) = \nabla^c(s^*\phi) + \frac{1}{2}s^*\phi^*\mathrm{B} - \frac{1}{2}s^*\Phi$.
In particular, if s is vertically geodesic then s is a harmonic section.

(ii) $2g(\pi_*\nabla ds(X,Y),Z) = \langle s^*\phi \otimes s^*\Phi\rangle(X,Y,Z) + \langle s^*\phi \otimes s^*\Phi\rangle(Y,X,Z)$.
Therefore s is horizontally geodesic if and only if $\langle s^*\phi \otimes s^*\Phi\rangle$ is a 3-form on M. In particular, if s is flat then s is horizontally geodesic.

Theorem 4.7 *[36]*

(i) *The symmetric and skew symmetric components of $\Pi^v s := \nabla^v d^v s$ are given by:*

$$I(\Pi^v s) = \phi \circ \nabla ds + \frac{1}{2}s^*\Phi.$$

(ii) *The section s is superflat if and only if s is flat and totally geodesic. In particular, if s is flat then s is totally geodesic if and only if s is super-flat.*

(iii) *Moreover $\tau^v(s)$ is the vertical component of the tension field $\tau(s)$. So if s is an harmonic map, then it is certainly a harmonic section.*

Theorem 4.8 *[36] An harmonic section s is a harmonic map if and only if $\langle s^*\phi, s^*\Phi\rangle = 0$ where*

$$\langle s^*\phi, s^*\Phi\rangle(X) = \sum_i \langle s^*\phi \otimes s^*\Phi\rangle(E_i, E_i, X)$$

for any orthonormal tangent frame (E_i) of M.
In particular, if s is flat ($s^*\Phi = 0$) then s is a harmonic map if and only if s is a harmonic section.

Remark 4.5 Let us consider the situation described by examples 4.1 and 4.8. Then if $f^*\Phi = 0$, $f\colon L \to N$ is vertically harmonic if and only if $\tilde{f}\colon L \to u^*N$ is an harmonic section if and only if $\tilde{f}\colon L \to u^*N$ is an harmonic map. But it does not imply that $f\colon L \to N$ is harmonic! (See the Appendix.) Indeed in the previous theorem it is essential that s be a section: $\pi \circ s = \mathrm{Id}$.

In fact the previous theorems can be easily generalized for non section map . The proofs in [36] holds without any change for theorems 4.5, 4.6-(i), 4.7-(i, iii), while for theorems 4.6-(ii), 4.7-(ii), 4.8: follow the proof of [36], just replace the starting equation $\pi \circ s = \mathrm{Id}$ by $\pi \circ s = u$. Then we obtain

Theorem 4.9 *For all $s \in C^\infty(M, N)$, we have*

(i) $I(\tau^v(s)) = -d^*(s^*\phi) + \frac{1}{2}\mathrm{Tr}(s^*\phi^*\mathrm{U})$

(ii) $\phi(\nabla ds) = \nabla^c(s^*\phi) + \frac{1}{2}s^*\phi^*\mathrm{B} - \frac{1}{2}s^*\Phi.$

In particular if s is vertically geodesic then s is a harmonic section.

(iii) $I(\Pi^v s) = \phi \circ \nabla ds + \frac{1}{2}s^*\Phi.$

The map s is superflat if and only if s is flat and vertically geodesic. Moreover $\tau^v(s)$ is the vertical component of the tension field $\tau(s)$. So if s is an harmonic map, then it is certainly vertically harmonic.

(iv) $2g(\pi_*\nabla ds(X, Y), u_*Z) = \langle s^*\phi \otimes s^*\Phi\rangle(X, Y, Z) + \langle s^*\phi \otimes s^*\Phi\rangle(Y, X, Z) + 2g\left(\nabla du(X, Y), u_*Z\right).$

Let us suppose now that u is an immersion, then this equation determines the horizontal part of ∇ds. In particular, if s is flat then s is horizontally geodesic if and only if u is totally geodesic; and s is totally geodesic if and only if s is superflat and u is totally geodesic.

(v) *A vertically harmonic map s is a harmonic map if and only if*

$$g(\tau(u), \cdot) + \langle s^*\phi, s^*\Phi\rangle = 0.$$

In particular if s is flat, then s is a harmonic map if and only if s is vertically harmonic and $u = \pi \circ s$ is harmonic.

We could also deduce this generalisation from the previous theorems 4.5-4.8 themself. Indeed we can apply these to the section $\tilde{s} \in \mathcal{C}(u^*N)$ corresponding to s and use theorems 6.1 and 6.2 in the Appendix, but we must suppose in addition that u is an isometry.
Let us go further in the generalisation and consider maps $f \in \mathcal{C}^\infty(L, N)$ with(L, b) a Riemannian manifold (see examples 4.1 and 4.8). Then the proofs in [36] holds fo theorems 4.5, 4.6-(i), 4.7-(i, iii), whereas theorems 4.6-(ii), 4.7-(ii), 4.8 are no longer valid. Indeed the equation in theorem 4.9-(iii) holds, but it gives us only $[\pi_*\nabla df]_{u_*TL}$, the component of $[\pi_*\nabla df]$ in the tangent bundle u_*TL. So if we want $[\pi_*\nabla df]_{(u_*TL)^\perp}$ we must introduce the 3-linear form $\langle f^*\phi \otimes f^*\Phi_{(1,\cdot)}\rangle \in \mathcal{C}(T^*L \otimes T^*L \otimes f^*\mathcal{H})$ defined by :

$$\langle f^*\phi \otimes f^*\Phi_{(1,\cdot)}\rangle(a, b, Z) = \langle f^*\phi(a), \Phi(f_*a, Z)\rangle.$$

Then we have

Theorem 4.10 *For all $f \in \mathcal{C}^\infty(L,N)$, we have*

(i) $g(\pi_*\nabla df(a,b),\pi^*Z) = \langle f^*\phi \odot f^*\Phi_{(1,\cdot)}\rangle(a,b,Z) + g(\nabla du(a,b),\pi_*Z)$.

In particular, if f is strongly flat i.e. ($f^*\Phi_{(1,\cdot)} = \Phi_f(df,\cdot) = 0$) then:
- f is horizontaly geodesic if and only if u is totally geodesic.
- f is totally geodesic if and only if f is superflat and u is totally geodesic.

(ii) *A vertically harmonic map f is a harmonic map if and only if*

$$g(\tau(u),\cdot) + \langle f^*\phi, f^*\Phi_{(1,\cdot)}\rangle = 0$$

where $\langle f^*\phi, f^*\Phi_{(1,\cdot)}\rangle(X) = \sum_i \langle f^*\phi \otimes f^*\Phi_{(1,\cdot)}\rangle(e_i,e_i,X)$ for any tangent frame (e_i) of L. In particular if f is strongly flat ($f^*\Phi_{(1,\cdot)} = 0$) then f is a harmonic map if and only if f is vertically harmonic and u is harmonic.

4.2.3 Reductions of homogeneous fibre bundles

Let us suppose now that the structure group H of $\pi_M\colon Q \to M$ is reducible to a (closed) subgroup $H^{\mathrm{v}} \supset K$. That is to say, there exists a principal H^{v}-subbundle $\pi^{\mathrm{v}}_M\colon Q^{\mathrm{v}} \to M$. Let us suppose in addition to that, that H^{v}/K is reductive: $\mathfrak{h}^{\mathrm{v}} = \mathfrak{k} \oplus \mathfrak{p}^{\mathrm{v}}$ and $\mathrm{Ad}k(\mathfrak{p}^{\mathrm{v}}) = \mathfrak{p}^{\mathrm{v}}$, $\forall k \in K$. The restriction to $\mathfrak{p}^{\mathrm{v}}$ of the $\mathrm{Ad}K$-invariant inner product on $\mathfrak{p}$ defines a H^{v}-invariant metric on H^{v}/K which is nothing but the metric induced by the H-invariant metric on H/K, so that the inclusion $H^{\mathrm{v}}/K \to H/K$ is an isometric embedding. Let $\mathfrak{p}' = (\mathfrak{p}^{\mathrm{v}})^\perp$ in $\mathfrak{p}$ and let us suppose that $\mathfrak{p}'$ is $\mathrm{Ad}H^{\mathrm{v}}$-invariant, so that $\mathfrak{h} = \mathfrak{h}^{\mathrm{v}} \oplus \mathfrak{p}'$ is a reductive decomposition and H/H^{v} is reductive. Conversely if H/H^{v} is reductive: $\mathfrak{h} = \mathfrak{h}^{\mathrm{v}} \oplus \mathfrak{p}'$ with $\mathfrak{p}'$ $\mathrm{Ad}H^{\mathrm{v}}$-invariant, then we can always complete any $\mathrm{Ad}H^{\mathrm{v}}$-inner product in $\mathfrak{p}'$ by an $\mathrm{Ad}K$-invariant inner product $\mathfrak{p} = \mathfrak{p}^{\mathrm{v}} \oplus \mathfrak{p}'$:

$$\langle\ ,\ \rangle_{\mathfrak{p}} = \langle\ ,\ \rangle_{\mathfrak{p}^{\mathrm{v}}} + \langle\ ,\ \rangle_{\mathfrak{p}'}$$

for which $\mathfrak{p}' = (p^{\mathrm{v}})^\perp$ in $\mathfrak{p}$.
In the following we suppose that H/H^{v} is reductive andthat the inner product in $\mathfrak{p}$ is chosen as described above.
Now let us turn toward the connection 1-form ω. Its restriction $\omega^{\mathrm{v}} := \omega_{\mathfrak{h}^{\mathrm{v}}|TQ^{\mathrm{v}}}$ defines a connection on $\pi^{\mathrm{v}}_M\colon \to M$. We endow Q^{v} with ω^{v} and $(Q^{\mathrm{v}}, H^{\mathrm{v}}, K, \omega^{\mathrm{v}})$ is then a homogeneous fibre bundle as defined in the begining of 4.2.
Moreover ω is reducible (to ω^{v}) in Q^{v} if and only if one of the following equivalent statements holds ([27])

- $\forall q \in Q^{\mathrm{v}}$, $(\mathcal{H}_0)_q$ is tangent to Q^{v}.
- $\omega_{|TQ^{\mathrm{v}}} = \omega^{\mathrm{v}}$ (i.e. $\omega_{|TQ^{\mathrm{v}}}$ is $\mathfrak{h}^{\mathrm{v}}$-valued).
- The canonical cross section s^{v} of the associated bundle $E^{\mathrm{v}} := Q/H^{\mathrm{v}} = Q \times_H (H/H^{\mathrm{v}})$, which defines the H^{v}-reduction Q^{v} is horizontal.

The vertical bundle (in TQ), $\mathcal{V}_0$, splits as follows

$$\mathcal{V}_0 = \mathcal{V}'_0 \oplus \mathcal{V}^{\mathrm{v}}_0$$

where $(\mathcal{V}_0^{\mathrm{v}})_q = q.\mathfrak{h}^{\mathrm{v}} = T_q(q.H^{\mathrm{v}})$ and $(\mathcal{V}'_0)_q = q.\mathfrak{p}'$, and quotienting by $\mathfrak{k}$, i.e. by applying $d\pi_N$ we obtain the following decomposition of $\mathcal{V}$:

$$\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}^{\mathrm{v}}$$

with $\mathcal{V}' = d\pi_N(\mathcal{V}'_0)$ and $\mathcal{V}^{\mathrm{v}} = d\pi_N(\mathcal{V}_0^{\mathrm{v}})$.
Then the canonical isomorphism $I\colon \mathcal{V} \to \mathfrak{p}_Q$ sends the previous decomposition onto the following $\mathfrak{p}_Q = \mathfrak{p}'_Q \oplus \mathfrak{p}_Q^{\mathrm{v}}$ (i.e. $\mathcal{V}'$, $\mathcal{V}^{\mathrm{v}}$ are sent resp. onto $\mathfrak{p}'_Q$ and $\mathfrak{p}_Q^{\mathrm{v}}$).
Then the vertical space in TN^{v} is $\mathcal{V}^{\mathrm{v}}_{|N^{\mathrm{v}}}$ that we will also denote by $\mathcal{V}^{\mathrm{v}}$ when there is no possibilities of confusion. The splitting of TN^{v} by ω^{v} is then

$$TN^{\mathrm{v}} = \mathcal{V}_{|N^{\mathrm{v}}} \oplus \mathcal{H}^{\mathrm{v}}_{|N^{\mathrm{v}}}$$

where $\mathcal{H}^{\mathrm{v}} = d\pi_N(\mathcal{H}_0^{\mathrm{v}})$ and $\mathcal{H}_0^{\mathrm{v}} = \ker \omega_{\mathfrak{h}^{\mathrm{v}}}$. Let us remark that ω is reducible if and only if $\mathcal{H}^{\mathrm{v}}_{|N^{\mathrm{v}}} = \mathcal{H}_{|N^{\mathrm{v}}}$.
The canonical bundle on N^{v}, $\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}} = Q^{\mathrm{v}} \times_K \mathfrak{p}^{\mathrm{v}} \to N^{\mathrm{v}}$ is the restriction to N^{v} of $\mathfrak{p}^{\mathrm{v}}_Q \to N$, and the canonical isomorphism $I^{\mathrm{v}}\colon \mathcal{V}^{\mathrm{v}}_{|N^{\mathrm{v}}} \to \mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}$ is the restriction to $\mathcal{V}^{\mathrm{v}}_{|N^{\mathrm{v}}}$ of $I\colon \mathcal{V} \to \mathfrak{p}_Q$.
Since $\omega = \omega_{\mathfrak{p}^{\mathrm{v}}} + \omega_{\mathfrak{p}'}$, the homogeneous connection form on N^{v}, ϕ^{v} (the $\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}$-valued 1-form on N^{v} defined by $\omega_{\mathfrak{p}^{\mathrm{v}}}$) is the restriction to N^{v} of the $\mathfrak{p}^{\mathrm{v}}_Q$-component of ϕ:

$$\phi^{\mathrm{v}} = [\phi]_{\mathfrak{p}^{\mathrm{v}}_Q|N^{\mathrm{v}}} = [\phi_{|N^{\mathrm{v}}}]_{\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}}.$$

The homogeneous curvature form Φ^{v} (defined by $\Omega^{\mathrm{v}}_{\mathfrak{p}^{\mathrm{v}}}$, with $\Omega^{\mathrm{v}} = d\omega^{\mathrm{v}} + \frac{1}{2}[\omega^{\mathrm{v}} \wedge \omega^{\mathrm{v}}]$) is given by

$$\Phi^{\mathrm{v}} = [\Phi]_{\mathfrak{p}^{\mathrm{v}}_Q|N^{\mathrm{v}}} = [\Phi_{|N^{\mathrm{v}}}]_{\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}}.$$

Furthermore, $\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}$ is ∇^c-parallel: the covariant derivative on $\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}$ defined by $\omega_{\mathfrak{k}|TQ^{\mathrm{v}}}$ is the restriction of ∇^c to $\mathfrak{p}^{\mathrm{v}}_{Q^{\mathrm{v}}}$. In other words ∇^c commutes with the projection on $\mathfrak{p}^{\mathrm{v}}_Q$. The canonical torsion on N^{v} is given by

$$(T^c)^{\mathrm{v}} = d^c\phi^{\mathrm{v}} = [T^c]_{\mathfrak{p}^{\mathrm{v}}_Q|N^{\mathrm{v}}}.$$

Let us denote by $\tau^{\mathrm{v}}(s)$ the vertical tension field of $s \in \mathcal{C}^\infty(M, N^{\mathrm{v}})$. According to theorem 4.7, we have $\tau^v(s) = [\tau_N(s)]_{\mathcal{V}}$ for all $s \in \mathcal{C}^\infty(M, N)$, and $\tau^{\mathrm{v}}(s) = [\tau_{N^{\mathrm{v}}}(s)]_{\mathcal{V}^{\mathrm{v}}}$ for all $s \in \mathcal{C}^\infty(M, N^{\mathrm{v}})$. But if ω is reducible in Q^{v}, then the inclusion $N^{\mathrm{v}} \to N$ is an isometry and hence the Levi-Civita connection in N^{v} is the orthogonal projection in TN^{v} of the Levi-Civita connection in N. Thus we have $\tau_{N^{\mathrm{v}}}(s) = [\tau_N(s)]_{TN^{\mathrm{v}}}$ for all $s \in \mathcal{C}^\infty(M, N^{\mathrm{v}})$, so that $\tau^{\mathrm{v}}(s) = [[\tau_N(s)]_{TN^{\mathrm{v}}}]_{\mathcal{V}^{\mathrm{v}}} = [\tau_N(s)]_{\mathcal{V}^{\mathrm{v}}}$. Therefore we obtain

Theorem 4.11 *Let us suppose that ω is reducible in Q^{v}. Let $s \in \mathcal{C}^\infty(M, N^{\mathrm{v}})$, then s is vertically harmonic in N ($\tau^v(s) = 0$) if and only if it is vertically harmonic in N^{v} ($\tau^{\mathrm{v}}(s) = 0$) and $[\tau_N(s)]^{\mathcal{V}'} = 0$. In particular if s is vertically harmonic in N then it is also vertically harmonic in N^{v}.*

4.3 Examples of Homogeneous fibre bundles

In this section, we give examples and applications for the theory developped in the previous sections whose we use here the same notations.

4.3.1 Homogeneous spaces fibration

Let us take $Q = G$ a Lie group, and $K \subset H \subset G$ subgroups of G, (H, K) satisfying the hypothesis in the begining of section 4.2.1. Let us suppose that $M = G/H$ is reductive and Riemannian: that is to say if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the reductive decomposition, then $\mathrm{Ad}_{\mathfrak{m}}H$ is compact and we choose an $\mathrm{Ad}H$-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{m}}$ in $\mathfrak{m}$. For ω, we take the canonical connection on $\pi_M \colon G \to G/H$ which is given, let us recall it, by $\omega = \theta_{\mathfrak{h}}$ where θ is the Maurer-Cartan form in G (see section 1.3). Then the corresponding decomposition $TQ = \mathcal{V}_0 \oplus \mathcal{H}_0$ is given by

$$T_gG = g.\mathfrak{g} = \underbrace{g.\mathfrak{h}}_{\mathcal{V}_0} \oplus \underbrace{g.\mathfrak{m}}_{\mathcal{H}_0}.$$

Since $\mathfrak{n} := \mathfrak{p} \oplus \mathfrak{m}$ is $\mathrm{Ad}K$- invariant, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ is a reductive decomposition and $N = G/K$ is reductive. Let us recall that we have the canonical identification $G \times_K \mathfrak{g} \cong N \times \mathfrak{g}$ given by (7), which gives us an identification $\mathfrak{n}_G = G \times_K \mathfrak{n} \cong [\mathfrak{n}]$. Then under this last identification and under the one given by the Maurer-Cartan form of G/K, $\beta \colon TN \xrightarrow{\cong} [\mathfrak{n}]$ (see section 1.2), the splitting $TN = \mathcal{V} \oplus \mathcal{H}$ is

$$TN = [\mathfrak{p}] \oplus [\mathfrak{m}],$$

the canonical isomorphism $I \colon \mathcal{V} \to \mathfrak{p}_G$ is then the identity, and $\phi \colon TN \to \mathfrak{p}_G$ the projection on $[\mathfrak{p}]$ along $[\mathfrak{m}]$. The metric h on G/K is then defined by the $\mathrm{Ad}K$-invariant inner product:

$$\langle\ ,\ \rangle_{\mathfrak{n}} = \langle\ ,\ \rangle_{\mathfrak{p}} + \langle\ ,\ \rangle_{\mathfrak{m}}.$$

Furthermore, $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] = d\theta_{\mathfrak{h}} + \frac{1}{2}[\theta_{\mathfrak{h}} \wedge \theta_{\mathfrak{h}}]$ and thus

$$\Omega_{\mathfrak{p}} = d\theta_{\mathfrak{p}} + [\theta_{\mathfrak{k}} \wedge \theta_{\mathfrak{p}}] + \frac{1}{2}[\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}]_{\mathfrak{p}}.$$

Since $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$, then (projecting on $\mathfrak{h}$) we have

$$d\theta_{\mathfrak{h}} + \frac{1}{2}[\theta_{\mathfrak{h}} \wedge \theta_{\mathfrak{h}}] + \frac{1}{2}[\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}}]_{\mathfrak{h}} = 0$$

thus

$$\Omega = -\frac{1}{2}[\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}}]_{\mathfrak{h}}$$

so that

$$\Omega_{\mathfrak{p}} = -\frac{1}{2}[\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}}]_{\mathfrak{p}} \tag{63}$$

therefore

$$\Phi = -\frac{1}{2}[\psi \wedge \psi]_{\mathfrak{p}} \tag{64}$$

where $\psi\colon TN \to \mathcal{H} = [\mathfrak{m}]$ is the projection on $\mathcal{H}$ along $\mathcal{V} = [\mathfrak{p}]$.
The covariant derivative ∇^c, which lifts into $d + \theta_{\mathfrak{k}}$ in G, is nothing but the canonical affine connection ∇^0 in $N = G/K$ restricted to $[\mathfrak{p}] \subset TN$ (see section 1.4 and 1.5).
The canonical torsion T^c, which lifts in G into

$$d\theta_{\mathfrak{p}} + [\theta_{\mathfrak{k}} \wedge \theta_{\mathfrak{p}}] = -\frac{1}{2}[\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}}]_{\mathfrak{p}} - \frac{1}{2}[\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}]_{\mathfrak{p}} \tag{65}$$

is given by [23]

$$\begin{aligned} T^c &= -\frac{1}{2}[\psi \wedge \psi]_{\mathfrak{p}} - \frac{1}{2}[\phi \wedge \phi]_{\mathfrak{p}} \\ &= [\ ,\ ]_{\mathfrak{n}} + [\phi \wedge \psi]_{\mathfrak{p}}. \end{aligned}$$

The associated bundle

$$\mathfrak{h}_G = G \times_H \mathfrak{h} \cong [\mathfrak{h}]^M := \{(g.p_0, \mathrm{Ad}g(a)), g \in G, a \in \mathfrak{h}\} \subset M \times \mathfrak{g}$$

can be embedded into $\mathfrak{so}(TM)$ by[24]

$$\xi = \mathrm{Ad}g(a) \in \mathrm{Ad}g(\mathfrak{h}) = [\mathfrak{h}]^M_{g.p_0} \longmapsto \mathrm{ad}\xi_{|\mathrm{Ad}g(\mathfrak{m})} = \mathrm{Ad}g \circ \mathrm{ad}_{\mathfrak{m}} a \circ \mathrm{Ad}g^{-1} \in \mathfrak{so}(\mathrm{Ad}g(\mathfrak{m})). \tag{66}$$

In the same way, $\mathfrak{k}_G = G \times_K \mathfrak{k} \cong [\mathfrak{k}]^N$ embedds in $\mathfrak{so}(N)$. Moreover let us remark that we have

$$\pi^*\mathfrak{h}_G = G \times_K \mathfrak{h} = [\mathfrak{h}]^N = \{(g.n_0, \mathrm{Ad}g(a)), g \in G, a \in \mathfrak{h}\},$$

and that $\pi^*\mathfrak{h}_G$ embedds into $\mathfrak{so}(\pi^*TM)$.
As concerns the covariant derivative ∇^ω, defined in $\mathfrak{h}_G$, it lifts into $d + \theta_{\mathfrak{h}}$ in G and under the embedding (66), it is nothing but the restriction to the subbundle $\mathfrak{so}(TM)$ of the endomorphism connection on M (i.e. the tensor product connection in $T^*M \otimes T^*M$) defined by the canonical affine connection in M, $\overset{M}{\nabla^0}$. Indeed under the embedding (66), ∇^ω lifts to the derivative $d + \mathrm{ad}_{\mathfrak{m}}\theta_{\mathfrak{h}}$ and equation (18) allows to conclude.
Therefore $\nabla^{\mathfrak{p}}$ is given by its lift [25]

$$[(d + \theta_{\mathfrak{h}})_{|\mathfrak{p}}]_{\mathfrak{p}} = d + \mathrm{ad}_{\mathfrak{p}}\theta_{\mathfrak{k}} + [\theta_{\mathfrak{p}}, \cdot_{|\mathfrak{p}}]_{\mathfrak{p}} \tag{67}$$

that is to say $\nabla^{\mathfrak{p}}$ is the $[\mathfrak{p}]$-component of the affine connection ∇^1 in G/K (see section 1.6) restricted to $[\mathfrak{p}] \subset TN$. (Indeed we have $[\theta_{\mathfrak{n}}, \cdot_{|\mathfrak{p}}]_{\mathfrak{p}} = [\theta_{\mathfrak{p}}, \cdot_{|\mathfrak{p}}]_{\mathfrak{p}} +$

[23] In particular, according to (64), we recover, for this example, the Homogeneous structure equation (59).
[24] Or in other words $[g, a] \in \mathfrak{h}_G = G \times_H \mathfrak{h} \mapsto [g, \mathrm{ad}_{\mathfrak{m}} a] \in G \times_H \mathfrak{so}(\mathfrak{m}) \cong \mathfrak{so}(TM)$.
[25] $[\ ]_{\mathfrak{p}}$ denotes as usual the $\mathfrak{p}$-component.

$[\theta_{\mathfrak{m}}, \cdot_{|\mathfrak{p}}]_{\mathfrak{p}}$, but $[\mathfrak{m}, \mathfrak{p}] \subset [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$ by reductivity.)
Moreover the Levi-Civita connection in N is given by (see section 1.6)

$$\overset{N}{\nabla} = \overset{\text{met}}{\nabla^{\frac{1}{2}}} = \nabla^0 + \frac{1}{2}\mathrm{B}^N$$

where $\mathrm{B}^N = [\ ,\ ]_{[\mathfrak{n}]} + \mathrm{U}^N$ and U^N is defined by equation (15). Then we have by taking the projection on the vertical subbundle $[\mathfrak{p}]$:

$$\phi(\nabla^N_A V) = \nabla^0_A(\phi V) + \frac{1}{2}\phi \circ \mathrm{B}^N(A, V)$$

so that we can conclude according to theorem (4.3) that

$$\phi \circ \mathrm{B}^N = \phi^* \mathrm{B} - \Phi$$

which can be verified directly using the expressions of B^N, B and Φ.

If H/K is (locally) symmetric. In this case, we have $T^c = \Phi$ (see (59), or (65) and (63)). Moreover, according to (67), $\nabla^{\mathfrak{p}}$ lifts to $d + \theta_{\mathfrak{k}}$, so that we recover that $\nabla^{\mathfrak{p}} = \nabla^c$ in this case. Now, let us apply the equality $\nabla^v = \nabla^c$ in $\mathcal{V}$ (theorem 4.3)[26].
Let $f\colon (L, b) \to N$ be a map then we have

$$\tau^v(f) = \mathrm{Tr}_b(\nabla^v d^v f) = *d^{\nabla^v} * d^v u = *\mathrm{Ad}F\left(d(*\alpha_{\mathfrak{p}}) + [\alpha_{\mathfrak{k}} \wedge (*\alpha_{\mathfrak{p}})]\right).f$$

where F lifts f in G and $\alpha = F^{-1}dF$. Then f is vertically harmonic if and only if

$$d(*\alpha_{\mathfrak{p}}) + [\alpha_{\mathfrak{k}} \wedge (*\alpha_{\mathfrak{p}})] = 0$$

Moreover f is flat ($f^*\Phi = 0$) if and only if it is vertically torsion free ($f^*T^c = 0$) if and only if

$$[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{p}} = 0 \Longleftrightarrow d\alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{p}}] = 0.$$

G/K is a (locally) $2k$-symmetric space Let us suppose that there exists an order $2k$ automorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}$ such that $K = G_0$ with G_0 such that $(G^\tau)^0 \subset G_0 \subset G^\tau$, and $(G^\sigma)^0 \subset H \subset G^\sigma$ with $\sigma = \tau^2$ (see section 2.1). Then H/K is (locally) symmetric (see section 2.1). We have the following identities (with the notation of section 2.1)

$$\mathfrak{m} = \oplus_{j=1}^{k-1}\mathfrak{m}_j \quad \text{and} \quad \mathfrak{k} = \mathfrak{g}_0, \mathfrak{p} = \mathfrak{g}_k.$$

Then we have

$$\Omega_{\mathfrak{p}} = \frac{1}{2}[\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}}]_{\mathfrak{p}} = -\frac{1}{2}\sum_{\substack{i+j=k \\ i,j \in \mathbb{Z}_{2k}\setminus\{0,k\}}} [\theta_j \wedge \theta_i],$$

so that in particular

[26] We can also use directely theorem 4.5.

Proposition 4.2 Let (L, j) be a Riemann surface. If $f\colon (L, j) \to N$ satisfies the equations $\alpha''_{-j} = 0$, $1 \leq j \leq k-1$ then we have $f^*\Phi = 0$. In other words if $f\colon (L, j) \to N$ is horizontally holomorphic then it it is flat, that is to say f is vertically torsion free or equivalently

$$d\alpha_k + [\alpha_0 \wedge \alpha_k] = 0 \tag{68}$$

Theorem 4.12 *In the even determined elliptic integrable system* $(\mathrm{Syst}(k, \tau))$, *the last equation* (S_k) *is equivalent to*

$$\begin{cases} (\mathrm{Re}(S_k)) \equiv d\alpha_k + [\alpha_0 \wedge \alpha_k] = 0 \iff f \text{ is vertically torsion free i.e. } f \text{ is flat,} \\ (\mathrm{Im}(S_k)) \equiv d(*\alpha_k) + [\alpha_0 \wedge (*\alpha_k)] = 0 \iff f \text{ is vertically harmonic.} \end{cases}$$

In conclusion the even determined elliptic system $(\mathrm{Syst}(k, \tau))$ *means that the geometric map* f *is horizontally holomorphic (which implies that* f *is flat) and vertically harmonic.*

Remark 4.6 The vertical torsion free equation (68) is the projection on $\mathfrak{p}$ of the Maurer-Cartan equation provided that we assume the horizontal holomorphicity $\alpha''_{-j} = 0, 1 \leq j \leq k-1$. In the same way, the equations (S_j), $0 \leq j \leq k-1$, of the elliptic system $(\mathrm{Syst}(k, \tau))$, are the projections on the different spaces $\mathfrak{g}_{-j}$, of the Maurer-Cartan equation, provided that we assume the horizontal holomorphicity. In section 2, this hyphothesis was (sometimes) implicitely assumed implicitely by definition of α: $\alpha' = u$.

Use of the canonical $2k$-structure $\mathfrak{I}_{J_0}$. Furthermore the morphism of bundle (over M) $\mathfrak{I}_{J_0}\colon N \to \mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2) \subset \mathcal{Z}^{\alpha_0}_{2k}(M)$ defines a $2k$-structure on π^*TM (still denoted by $\mathfrak{I}_{J_0}$), which according to (23) allows to precise the subbundles $\mathfrak{k}_G$ and $\mathfrak{p}_G$ (under the embedding $\pi^*\mathfrak{h}_G \hookrightarrow \mathfrak{so}(\pi^*TM)$)

$$\mathfrak{k}_G = \{A \in \pi^*\mathfrak{h}_G | [A, \mathfrak{I}_{J_0}] = 0\} := \mathfrak{so}_{(+1)}(\pi^*TM, \mathfrak{I}_{J_0}) \cap \pi^*\mathfrak{h}_G \tag{69}$$

$$\mathfrak{p}_G = \{A \in \pi^*\mathfrak{h}_G | A\mathfrak{I}_{J_0} + \mathfrak{I}_{J_0}A\} := \mathfrak{so}_{(-1)}(\pi^*TM, \mathfrak{I}_{J_0}) \cap \pi^*\mathfrak{h}_G \tag{70}$$

Remark 4.7 The embedding $\mathfrak{h}_G \hookrightarrow \mathfrak{so}(TM)$ is the H-equivariant extention of the map $a \in \mathfrak{h} \mapsto \mathrm{ad}_{\mathfrak{m}}a \in \mathfrak{so}(\mathfrak{m}) \cong \mathfrak{so}(T_{p_0}M)$, and in the same way $\mathfrak{I}_{J_0}$ is the H-equivariant extention of the map $h.G_0 \in H/G_0 \mapsto hJ_0h^{-1} \in \mathcal{Z}(T_{p_0}M, J_0)$, so that the equations (69) are obtained by H-equivariance from (23).

Let us now express the homogeneous fibre bundle tools ϕ, Φ, and ∇^c in terms of the embedding $\mathfrak{I}_{J_0}$. To do not weigh the notation we will forget the index J_0 in $\mathfrak{I}_{J_0}$, in the following theorem.

Theorem 4.13 *If* $A, B \in TN$, $F \in \mathcal{C}(\mathfrak{p}_G)$ *then*

(i) $\phi A = -\frac{1}{2}\mathfrak{I}^{-1}\overset{M}{\nabla}{}^0_A\mathfrak{I}$

(ii) $\Phi(A,B) = \frac{1}{2}\mathfrak{I}^{-1}[\mathfrak{I}, \pi^* \overset{M}{R^{\nabla^0}}(A,B)]$ *where* $\overset{M}{R^{\nabla^0}}$ *is the curvature of* $\overset{M}{\nabla^0}$.

(iii) $\nabla^c_A F = \overset{N}{\nabla^0_A} F = \frac{1}{2}\mathfrak{I}^{-1}[\mathfrak{I}, \overset{M}{\nabla^0_A} F]$.

Theorem 4.14 *Let* $s \in \mathcal{C}(\pi)$ *and* $J = s^*\mathfrak{I}_{J_0} \in \mathcal{C}(\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2))$ *be the corresponding* $2k$*-structure. Then*

(i) $I(d^v s) = -\frac{1}{2} J^{-1} \overset{M}{\nabla^0} J$. *Thus* s *is horizontal if and only if* J *is* $\overset{M}{\nabla^0}$*-parallel.*

(ii) $I(\Pi^v(s)) = -\frac{1}{4}[J^{-1}, \overset{M}{(\nabla^0)^2} J]$.

Thus s *is superflat if and only if* $\overset{M}{(\nabla^0)^2} J$ *commutes with* J.

(iii) $I(\tau^v(s)) = \frac{1}{4}[J^{-1}, \overset{M}{(\nabla^0)^*} \overset{M}{\nabla^0} J]$.

Thus s *is a harmonic section if and only if* $\overset{M}{(\nabla^0)^*} \overset{M}{\nabla^0} J$ *commutes with* J.

(iv) $s^*\Phi = \frac{1}{2} J^{-1}[J, \overset{M}{R^{\nabla^0}}]$.

These properties hold also for maps $f \in \mathcal{C}^\infty(L, N)$, (L, b) *being a Riemannian manifold: (i),(ii),(iii) without any change and (iv) becoming* $f^*\Phi = \frac{1}{2} J^{-1}[J, u^* \overset{M}{R^{\nabla^0}}]$, *with* $u = \pi \circ f$.

Corollary 4.1 *Let* (L, j) *be a Riemann surface,* $f \colon L \to N$ *a map and* $J = f^*\mathfrak{I}_{J_0}$ *the corresponding map into* $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$. *Then* f *is a geometric solution of the even determined system* $(\mathrm{Syst}(k, \tau))$ *if and only if*

(i) J *is an admissible twistor lift (*$\Leftrightarrow$ f *is horizontally holomorphic).*

(ii) J *is vertically harmonic*[27]*:* $\left[\overset{M}{(\nabla^0)^*} \overset{M}{\nabla^0} J, J\right] = 0$ *(*$\Leftrightarrow$ f *is vertically harmonic).*

Moreover the first condition implies that $\left[u^* \overset{M}{R^{\nabla^0}}, J\right] = 0$ *i.e. that* J *is a flat section in* $(\mathrm{End}(u^*TM), u^* \overset{M}{\nabla^0})$ *(*$\Leftrightarrow$ f *is flat).*
Furthermore f *is a primitive geometric solution (i.e. there exists* $m \leq k$ *such that* f *is* m*-primitive, which is equivalent to say that* f *is* k*-primitive) if and only if*

(i) J *is an admissible twistor lift*

(ii) J *is parallel:* $\overset{M}{\nabla^0} J = 0$ *(*$\Leftrightarrow$ f *is horizontal).*

[27] See section 4.3.4 for the definition of vertical harmonicity in $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$.

4.3.2 The twistor bundle of almost complex structures $\Sigma(E)$

We give ourself the same ingredients as in example 4.2. Let us suppose that the vector bundle E is oriented. Then the bundle of positive (resp. negative) orthogonal almost complex structure on E (i.e. the component $\Sigma^\varepsilon(E)$ of $\Sigma(E)$ with $\varepsilon = \pm 1$), $\pi_\Sigma \colon \Sigma^\varepsilon(E) \to M$ is a homogeneous fibre bundle. Indeed, we take $Q = \mathcal{SO}(E)$ the $SO(2n)$-bundle of positively oriented orthonormal frames of E, $H = SO(2n)$ and $K = U(n)$ (embedded in $SO(2n)$ via $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$). K is the subgroup of $SO(2n)$ which commutes with $J_0^\varepsilon = \varepsilon \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$. The involution $T = \mathrm{Int} J_0^\varepsilon$ in $SO(2n)$ gives rise to the symmetric space $H/K = \Sigma^\varepsilon(\mathbb{R}^{2n})$, and to the following symmetric decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\begin{aligned} \mathfrak{k} &= \{A \in \mathfrak{so}(2n) | [A, J_0^\varepsilon] = 0\} \\ \mathfrak{p} &= \{A \in \mathfrak{so}(2n) | AJ_0^\varepsilon + J_0^\varepsilon A = 0\}. \end{aligned}$$

Concerning ω, we take the $\mathfrak{so}(2n)$-valued connection 1-form on Q corresponding to the covariant derivative ∇ in E: if $e = (e_1, \ldots, e_{2n})$ is a (local) moving frame of E (i.e. a section of Q) then

$$\nabla(e_1, \ldots, e_{2n}) = (e_1, \ldots, e_{2n})\omega(e; de).$$

Now, let us consider the isomorphism of bundle:

$$\mathcal{J} \colon e.U(n) \in \mathcal{SO}(E)/U(n) \overset{\cong}{\longmapsto} J \in \Sigma^\varepsilon(E) |\, \mathcal{M}at_{e.U(n)}(J) = J_0^\varepsilon.$$

The isomorphism $\mathcal{J}$ defines a bijection between the set of section of $\pi \colon N \to M$[28] and the set of complex structure of E (sections of π_Σ): $s \in \mathcal{C}(N) \to J = \mathcal{J} \circ s \in \mathcal{C}(\Sigma^+(E))$.
The existence of a (positive) complex structure J in E – i.e. a section of $\pi_\Sigma \colon \Sigma^+(E) \to M$– is equivalent to the existence of an $U(n)$-reduction of the principal bundle $\mathcal{SO}(E) \to M$: J defines a Hermitian structure on E and then the $U(n)$-subbundle of unitary frames for this Hermitian structure, and vice versa.
The isomorphism of bundle over M, $\mathcal{J} \colon N \to \Sigma^+(E)$ defines tautologically a canonical complex structure on $\pi^*E \to N$ (which we still denote by $\mathcal{J}$)[29] $\mathcal{J} \colon N \to \Sigma^+(\pi^*E)$. Under this identification, let us precise the subbundles $\mathfrak{p}_Q$ and $\mathfrak{k}_Q$. First, we have $\mathfrak{h}_Q = \mathfrak{so}(E)$[30], the bundle of skew-symmetric endomorphism of E and then[31]

$$\begin{aligned} (\mathfrak{k}_Q)_y &= \{F \in \mathfrak{so}(E_{\pi(y)}) | [F, \mathcal{J}(y)] = 0\} =: \mathfrak{so}_+(E_{\pi(y)}, \mathcal{J}(y)) =: \mathfrak{so}_+(\pi^*E)_y \\ (\mathfrak{p}_Q)_y &= \{F \in \mathfrak{so}(E_{\pi(y)}) | F\mathcal{J}(y) + \mathcal{J}(y)F = 0\} =: \mathfrak{so}_-(E_{\pi(y)}, \mathcal{J}(y)) =: \mathfrak{so}_-(\pi^*E)_y. \end{aligned}$$

[28] In all the section 4.3, as it was the case in all the section 4.2, $N := Q/K$.
[29] and which is in fact nothing but $\mathcal{J}^*\mathcal{I} = \mathcal{I} \circ \mathcal{J}$, see example 4.2.
[30] See remark (4.8) (and more precisely equation (71)) for the identification map.
[31] Since π^*E is canonically endowed with the complex structure $\mathcal{J}$, we need not to precise this latter in the notation $\mathfrak{so}_\pm(\pi^*E)$, whereas E_x, for $x \in M$, could be endowed with any element $J_x \in \Sigma^+(E_x)$, this is why we must precise it in the notation $\mathfrak{so}_\pm(E_x, J_x)$.

Then the decomposition following $\pi^*\mathfrak{h}_Q = \mathfrak{k}_Q \oplus \mathfrak{p}_Q$ of any element $F \in \pi^*\mathfrak{h}_Q = \mathfrak{so}(\pi^*E)$ is given by

$$F = \frac{1}{2}\mathcal{J}\{F, \mathcal{J}\} + \frac{1}{2}\mathcal{J}[F, \mathcal{J}]$$

where $\{\ ,\ \}$ is the anticommutator.

Remark 4.8 The canonical complex structure $\mathcal{J}$ is a section of the associated bundle over N: $\Sigma^+(\pi^*E) = \pi^*(Q \times_H \Sigma^\varepsilon(\mathbb{R}^{2n})) = \pi^*Q \times_H \Sigma^\varepsilon(\mathbb{R}^{2n})$, so that it can be lifted to a H-equivariant map $\tilde{\mathcal{J}}\colon \pi^*Q \to \Sigma^\varepsilon(\mathbb{R}^{2n}) \subset \mathfrak{h}$, which is given by

$$\tilde{\mathcal{J}}\colon (e.K, e.h^{-1}) \in \pi^*Q \longmapsto hJ_0^\varepsilon h^{-1} \in \Sigma^\varepsilon(\mathbb{R}^{2n}).$$

Remark that the restriction of $\tilde{\mathcal{J}}$ to $Q \subset \pi^*Q$ is the constant map J_0^ε (the inclusion $Q \subset \pi^*Q$ is given by $e \mapsto (e.U(n), e)$), and that $\tilde{\mathcal{J}}\colon \pi^*Q \to \Sigma^\varepsilon(\mathbb{R}^{2n})$ is the H-equivariant extension of the K-equivariant constant map J_0^ε on Q. $\tilde{\mathcal{J}}$ can also be given in term of $\mathcal{J}$ by

$$\tilde{\mathcal{J}}\colon (y; e) \in \pi^*Q \longmapsto \mathcal{M}at_e(\mathcal{J}(y)) \in \Sigma^\varepsilon(\mathbb{R}^{2n}) \subset \mathfrak{h}.$$

Furthermore, we have a canonical identification $N = Q \times_H H/K$ (via $[e, h.K] \mapsto (e.h).K$) and the identification depending on J_0^ε: $H/K = \Sigma^\varepsilon(\mathbb{R}^{2n})$ (via $h.K \mapsto hJ_0h^{-1}$) so that $N = Q \times_H \Sigma^\varepsilon(\mathbb{R}^{2n})$ (via $e.K \mapsto [e, J_0]$). Then under this last identification, $\mathcal{J}$ is the restriction to N of the canonical identification

$$\begin{array}{rcl} \mathfrak{h}_Q := Q \times_H \mathfrak{h} & \stackrel{\cong}{\longrightarrow} & \mathfrak{so}(E) \\ [e, a] & \longmapsto & A|\ \mathcal{M}at_e(A) = a. \end{array} \tag{71}$$

Therefore

$$\mathfrak{h}_Q = \mathcal{J}^*\mathfrak{so}_+(\pi_\Sigma^*E) \quad \text{and} \quad \mathfrak{p}_Q = \mathcal{J}^*\mathfrak{so}_-(\pi_\Sigma^*E),$$

with the notations of example 4.2.

Let us now express the homogeneous connection ϕ, the curvature foms Φ and the canonical connection ∇^c in terms of $\mathcal{J}$ (following [36]).

Theorem 4.15 *[36] If $A, B \in TN$, $F \in \mathcal{C}(\mathfrak{p}_Q)$ then:*

(i) $\phi A = \frac{1}{2}\mathcal{J}.\nabla_A\mathcal{J}$

(ii) $\Phi(A, B) = \frac{1}{2}\mathcal{J}[\pi^*R(A, B), \mathcal{J}]$, *where R is the curvature operator of the ∇.*

(iii) $\nabla^c F = \frac{1}{2}\mathcal{J}[\nabla_A F, \mathcal{J}]$

Theorem 4.16 *[36] Let $s \in \mathcal{C}(\pi)$ and $J = s^*\mathcal{J}$ be the corresponding complex structure, and $\nabla^*\nabla = -\mathrm{Tr}\nabla^2$, the rough Laplacian of E. Then*

(i) $I(d^v s) = \frac{1}{2}J.\nabla J = \frac{1}{4}[J, \nabla J]$. *Thus s is horizontal if and only if J is parallel.*

(ii) $I(\Pi^v(s)) = \dfrac{1}{4}[J, \nabla^2 J]$. *Thus s is superflat if and only if $\nabla^2 J$ commutes with J.*

(iii) $I(\tau^v(s)) = -\dfrac{1}{4}[J, \nabla^* \nabla J]$. *Thus s is a harmonic section if and only if $\nabla^* \nabla J$ commutes with J.*

(iv) $s^*\Phi = \dfrac{1}{2} J[R, J]$.

From theorem 4.15-(i) (or theorem 4.16-(i)) it follows that $d\mathcal{J}$ sends the decomposition $TN = \mathcal{V} \oplus \mathcal{H}$ onto the decomposition $T\Sigma^\varepsilon(E) = \mathcal{V}^\Sigma \oplus \mathcal{H}^\Sigma$ coming from ∇ (see example 4.2) so that we can consider $\pi_\Sigma \colon \Sigma^\varepsilon(E) \to M$ as a homogeneous fibre bundle over M with structure group $H = SO(2n)$ and $K = U(n)$. Besides, since the vertical and horizontal subbundles corresponds via $\mathcal{J}$, then we can conclude according to (47) and (57) that $\mathcal{J}$ **is an isometry**.

Moreover, we see that s is vertically harmonic in N if and only if the rough Laplacian $\nabla^* \nabla J$ of J in $\mathfrak{so}(E)$ is vertical (i.e. in $\mathcal{V}_J^\Sigma$, see example 4.2) so that we recover the definition of vertically harmonic twistor lifts used in [25] and [6]. More precisely, via the isometry $\mathcal{J}$, **the vertical tension field of** s – which is, let us recall it, defined using the Levi-Civita connection in N which corresponds via the isometry $\mathcal{J}$ to the Levi-Civita connection in $\Sigma^+(E)$– **is exactly the vertical part in $\pi_\Sigma^* \mathfrak{so}(E)$ of the rough laplacian of** J:

$$d\mathcal{J}(\tau^v(s)) = \nabla_{\tau^v(s)} \mathcal{J} = -2\mathcal{J} \circ \phi(\tau^v(s)) = \frac{1}{2} J[J, \nabla^* \nabla J]$$

according to theorem 4.15-(i) and theorem 4.16-(iii). Concretely, **to compute the vertical tension field in** $\Sigma^+(E)$, instead of using the (abstract) Levi-Civita connection, it is enough to **take the vertical part of the rough Laplacian** (which uses the concrete metric connection ∇).

4.3.3 The twistor bundle $\mathcal{Z}_{2k}(E)$ of a Riemannian vector bundle

We give ourself the same ingredients and notations as in example 4.3. Let us suppose that the vector bundle E is oriented. Then the bundle $\pi_{\mathcal{Z}} \colon \mathcal{Z}_{2k}^\alpha(E) \to M$ is a homogeneous fibre bundle. Indeed, we take $Q = \mathcal{SO}(E)$, $H = SO(2n)$ and $K = \mathbb{U}_0(J_0^\alpha)$. Let us recall that the order r automorphism $T = \mathrm{Int} J_0^\alpha$ in $SO(2n)$ gives rise to the r-symmetric space $H/K = \mathcal{Z}_{2k}^\alpha(\mathbb{R}^{2n})$, and to the following reductive decompostion $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{k} = \mathfrak{so}_0(J_0^\alpha) \quad \text{and} \quad \mathfrak{p} = \mathfrak{so}_*(J_0^\alpha) := \left(\bigoplus_{j \in \mathbb{Z}/r\mathbb{Z} \setminus \{0\}} \mathfrak{so}_j(J_0^\alpha) \right) \bigcap \mathfrak{so}(2n).$$

Concerning ω, we take the same as in the previous example. Now let us consider the isomorphism of bundle:

$$\mathcal{J} \colon e.\mathbb{U}_0(J_0^\alpha) \in \mathcal{SO}(E)/\mathbb{U}_0(J_0^\alpha) \xmapsto{\cong} J \in \mathcal{Z}_{2k}^\alpha(E) |\, \mathcal{M}at_e(J) = J_0^\alpha. \tag{72}$$

The isomorphism $\mathcal{J}$ defines a bijection between the sections of $\pi\colon N \to M$ and the set of sections of $\pi_{\mathcal{Z}}\colon \mathcal{Z}^{\alpha}_{2k}(E) \to M$, $s \in \mathcal{C}(N) \mapsto J = \mathcal{J} \circ s \in \mathcal{C}(\mathcal{Z}^{\alpha}_{2k}(E))$.
The isomorphism of bundle over M, $\mathcal{J}\colon N \to \mathcal{Z}^{\alpha}_{2k}(E)$ defines tautologically a canonical $2k$-structure[32] on $\pi^*E \to N$ (still denoted by $\mathcal{J}$), $\mathcal{J}\colon N \to \mathcal{Z}^{\alpha}_{2k}(\pi^*E)$. Under this consideration, we therefore have $\mathfrak{h} = \mathfrak{so}(E)$ and for all $y \in N$,

$$\begin{aligned}\mathfrak{k}_Q &= \mathfrak{so}_0(\pi^*E, \mathcal{J}) \\ \mathfrak{p}_Q &= \mathfrak{so}_*(\pi^*E, \mathcal{J}) = \left(\bigoplus_{j\in\mathbb{Z}/r\mathbb{Z}\setminus\{0\}} \mathfrak{so}_j(\pi^*E, \mathcal{J})\right) \bigcap \mathfrak{so}(\pi^*E).\end{aligned}$$

Since π^*E is canonically endowed with $\mathcal{J}$, we will not precise it and use the notation $\mathfrak{so}_j(\pi^*E) := \mathfrak{so}_j(\pi^*E, \mathcal{J})$.
Let us consider the surjective morphism of vector bundle

$$\begin{array}{rccl} \mathrm{ad}\mathcal{J}\colon & \pi^*\mathfrak{h}_Q = \mathfrak{so}(\pi^*E) & \longrightarrow & \mathcal{B}_*(\pi^*E) = \mathcal{J}.\mathfrak{so}_*(\pi^*E) = \mathcal{J}.\mathfrak{p}_Q \\ & (J, A) & \longmapsto & \mathrm{ad}J(A) = [J, A] = J\sum_{j=1}^{r}(1-\omega_r^j)A_j \end{array}$$

where $A_j = [A]_{\mathfrak{so}_j(E_x)}$ is the $\mathfrak{so}_j(E_x)$-component of $A \in \mathfrak{so}(E_x)$. The kernel of $\mathrm{ad}\mathcal{J}$ is $\mathfrak{k}_Q = \mathfrak{so}_0(\pi^*E)$ so that $\mathrm{ad}\mathcal{J}$ induces an isomorphism from $\mathfrak{p}_Q$ onto $\mathcal{J}.\mathfrak{p}_Q$. We will set

$$(\mathrm{ad}\mathcal{J})^{-1} = (\mathrm{ad}\mathcal{J}_{|\mathcal{J}.\mathfrak{p}_Q})^{-1} \oplus 0_{\mathcal{J}.\mathfrak{k}_Q}$$

so that

$$(\mathrm{ad}\mathcal{J})^{-1} \circ \mathrm{ad}\mathcal{J} = \mathrm{pr}_{\mathfrak{p}_Q}, \text{ the projection on } \mathfrak{p}_Q \text{ along } \mathfrak{k}_Q, \text{ and} \quad (73)$$

$$\mathrm{ad}\mathcal{J} \circ (\mathrm{ad}\mathcal{J})^{-1} = \mathrm{pr}_{\mathcal{J}.\mathfrak{p}_Q}, \text{ the projection on } \mathcal{J}.\mathfrak{p}_Q \text{ along } \mathcal{J}.\mathfrak{k}_Q. \quad (74)$$

Let us remark that $\mathcal{J}.\mathfrak{p}_Q = \mathcal{J}^*\mathcal{V}^{\mathcal{Z}}$ is the (pullback by $\mathcal{J}$ of the) vertical space of $\pi_{\mathcal{Z}}$ (see example 4.3). More precisely the $\mathcal{J}$-pullback of the decomposition $\mathcal{V}^{SO(E)}{}_{|N_{\mathcal{Z}}} = \mathcal{B}_0(\pi_{\mathcal{Z}}^*E) \oplus \mathcal{B}_*(\pi_{\mathcal{Z}}^*E)$ (see example 4.3) is the decomposition $\mathcal{J}.\mathfrak{so}(E) = \mathcal{J}.\mathfrak{k}_Q \oplus \mathcal{J}.\mathfrak{p}_Q$.
Let us now express the homogeneous fibre bundle tools ϕ, Φ and $\nabla^{\mathfrak{p}}$ in terms of $\mathcal{J}$.

Theorem 4.17 *If $A, B \in TN$, $F \in \mathcal{C}(\mathfrak{p}_Q)$ then*

(i) $\nabla\mathcal{J} = -\mathrm{ad}\mathcal{J} \circ \phi$ *thus* $\phi A = -(\mathrm{ad}\mathcal{J})^{-1}\nabla_A\mathcal{J}$

(ii) $\Phi(A, B) = (\mathrm{ad}\mathcal{J})^{-1}[\mathcal{J}, \pi^*R(A, B)]$

(iii) $\nabla^{\mathfrak{p}}_A F = (\mathrm{ad}\mathcal{J})^{-1}[\mathcal{J}, \nabla_A F]$

Theorem 4.18 *Let $s \in \mathcal{C}(\pi)$ and $J = s^*\mathcal{J}$ be the corresponding $2k$-structure. Then*

(i) $I(d^v s) = -(\mathrm{ad}J)^{-1}\nabla J$. *Thus s is horizontal if and only if J is parallel.*

[32] and which is in fact nothing but $\mathcal{J}^*\mathcal{I} = \mathcal{I} \circ \mathcal{J}$, see example 4.3.

(ii) $I(\Pi^v(s)) = -(\mathrm{ad}J)^{-1}\nabla^2 J + \frac{1}{2}(\mathrm{ad}J)^{-1}[\nabla J \odot (\mathrm{ad}J)^{-1}\nabla J]$.

Thus s is superflat if and only if $\nabla^2 J - \frac{1}{2}[\nabla J \odot (\mathrm{ad}J)^{-1}\nabla J]$ commutes with J.

(iii) $I(\tau^v(s)) = +(\mathrm{ad}J)^{-1}\nabla^*\nabla J + \ (\mathrm{ad}J)^{-1}\mathrm{Tr}\left([\nabla J, (\mathrm{ad}J)^{-1}\nabla J]\right)$.

Thus s is a harmonic section if and only if $\nabla^*\nabla J + \ \mathrm{Tr}\left([\nabla J, (\mathrm{ad}J)^{-1}\nabla J]\right)$ commutes with J.

(iv) $s^*\Phi = (\mathrm{ad}J)^{-1}[J, R]$.

As above, from theorem 4.17-(i), we conclude that $d\mathcal{J}$ sends the decomposition $TN = \mathcal{V} \oplus \mathcal{H}$ onto the decomposition $TN_{\mathcal{Z}} = \mathcal{V}^{\mathcal{Z}} \oplus \mathcal{H}^{\mathcal{Z}}$ coming from ∇ (see example 4.3) so that we can consider $\pi_{\mathcal{Z}} : \mathcal{Z}^{\alpha}_{2k}(E) \to M$ as a homogeneous fibre bundle over M with structure groups $H = SO(2n)$ and $K = \mathbb{U}_0(J_0^{\alpha})$. We will call this structure *the homogeneous fibre bundle structure defined in $N_{\mathcal{Z}}$ by ∇ (or by the Riemannian vector bundle (E, ∇)).*
Besides, since the vertical and horizontal subbundles corresponds via $\mathcal{J}$, then we can conclude according to (47) and (57) that $\mathcal{J}$ **is an isometry**.
Moreover, the vertical tension field of J in $N_{\mathcal{Z}} = \mathcal{Z}^{\alpha}_{2k}$ is given by

$$\begin{aligned} d\mathcal{J}(\tau^v(s)) &= \nabla_{\tau^v(s)}\mathcal{J} = -(\mathrm{ad}\mathcal{J}) \circ \phi(\tau^v(s)) \\ &= -(\mathrm{ad}\mathcal{J}) \circ (\mathrm{ad}\mathcal{J})^{-1}(\nabla^*\nabla J + \ \mathrm{Tr}\left([\nabla J, (\mathrm{ad}J)^{-1}\nabla J]\right)) \\ &= -\left[\nabla^*\nabla J + \ \mathrm{Tr}\left([\nabla J, (\mathrm{ad}J)^{-1}\nabla J]\right)\right]_{\mathcal{V}^{\mathcal{Z}}} \end{aligned}$$

By taking $k = 2$ in the two preceding theorems, we recover of course the results of the previous section: just remark that in this case, $\mathrm{ad}\mathcal{J} = 0_{\mathfrak{k}_Q} \oplus 2L_{\mathcal{J}|\mathfrak{p}_Q}$, and that ∇J anticommutes with J.

Remark 4.9 Let us consider the canonical identification

$$\begin{array}{rcl} H_Q : \ = Q \times_H H & \stackrel{\cong}{\longrightarrow} & SO(E) \\ [e, h] & \longmapsto & A |\ \mathcal{M}at_e(A) = h. \end{array} \tag{75}$$

then $\mathcal{J}$ is the restriction to $N \cong Q \times_H \mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n})$ (via $e.K \mapsto [e, J_0^{\alpha}]$) of (75).
More generally, for $j \in \mathbb{Z}$, we can consider $\mathcal{J}_j$ the restriction of (75) to $Q/\mathbb{U}_{j-1}(J_0^{\alpha}) = Q \times_{SO(2n)} (\mathcal{Z}_{2k}(\mathbb{R}^{2n}))^j$ (via $e.\mathbb{U}_{j-1}(J_0^{\alpha}) \mapsto [e, (J_0^{\alpha})^j]$):

$$\mathcal{J}_j : e.\mathbb{U}_{j-1}(J_0^{\alpha}) \in \mathcal{SO}(E)/\mathbb{U}_{j-1}(J_0^{\alpha}) \stackrel{\cong}{\longmapsto} J \in (\mathcal{Z}^{\alpha}_{2k}(E))^j |\ \mathcal{M}at_e(J) = (J_0^{\alpha})^j. \tag{76}$$

Remark 4.10 The previous study could have been done (without any change) for any component $\mathcal{U}^{\alpha}_{2k}(E)$. In particular, by replacing J_0^{α} by $(J_0^{\alpha})^j$ in what precedes, we get the isomorphism (76)

$$\mathcal{J}_j : \mathcal{SO}(E)/\mathbb{U}_{j-1}(J_0^{\alpha}) \stackrel{\cong}{\longmapsto} (\mathcal{Z}^{\alpha}_{2k}(E))^j = \mathcal{U}_p^{j\cdot\alpha}(E),$$

where $p = \dfrac{2k}{(2k,j)}$, and by applying theorem 4.18, we see that a cross section $s_j \colon M \to \mathcal{SO}(E)/\mathbb{U}_{j-1}(J_0^\alpha)$ is horizontal if and only if the corresponding section $J_j = \mathcal{J}_j \circ s_j \colon M \to (\mathcal{Z}_{2k}^\alpha(E))^j$ is parallel: $\nabla J_j = 0$.

4.3.4 The Twistor subbundle $\mathcal{Z}_{2k,j}^\alpha(E)$

We continue here the study of example 4.4, $\pi_{\mathcal{Z}}^j \colon \mathcal{Z}_{2k,j}^\alpha(E, J_j) \to M$, and prove that it defines a homogeneous bundle fibre bundle. Let us recall that we have a bijection between the set of (global) sections J_j in $(\mathcal{Z}_{2k}^\alpha(E))^j = \mathcal{U}_p^{j\cdot\alpha}(E)$ and the set of $\mathbb{U}_{j-1}(J_0^\alpha)$-reductions $\pi^j \colon Q^j \to M$ of $\mathcal{SO}(E)$, which is given by

$$Q^j = \mathfrak{U}_{j-1}^\alpha(E) := \{e \in \mathcal{SO}(E) | \mathcal{M}at_e(J_j) = (J_0^\alpha)^j\}. \tag{77}$$

Let us consider such a reduction Q^j (defined by some J_j). Then $\pi^j \colon Q^j \to M$ is a principal bundle with structural group $H^j = \mathbb{U}_{j-1}(J_0^\alpha)$ and we take for the second structural group $K = \mathbb{U}_0(J_0^\alpha)$ as in the previous example. Let us recall that the order j automorphism $T = \mathrm{Int} J_0^\alpha{}_{|\mathbb{U}_{j-1}(J_0^\alpha)}$ gives rise to the j-symmetric space $H^j/K = \mathcal{Z}_{2k,j}^\alpha(\mathbb{R}^{2n}, (J_0^\alpha)^j)$, and to the following reductive decomposition $\mathfrak{h}^j = \mathfrak{k} \oplus \mathfrak{p}^j$ where

$$\begin{aligned} \mathfrak{k} &= \mathfrak{so}_0(J_0^\alpha) \quad \text{and} \\ \mathfrak{p}^j &= \mathfrak{u}_{j-1}^*(J_0^\alpha) = \oplus_{q=1}^{(r,j)-1} \mathfrak{so}_{pq}(J_0^\alpha) = \mathfrak{u}_{j-1}(J_0^\alpha)/\mathfrak{u}_0(J_0^\alpha) = \mathfrak{so}_0((J_0^\alpha)^j)/\mathfrak{so}_0(J_0^\alpha) \cong T_{J_0^\alpha}\mathcal{Z}_{2k,j}^\alpha(\mathbb{R}^{2n}, (J_0^\alpha)^j), \end{aligned}$$

the last identification is given by

$$A \in \oplus_{q=1}^{(r,j)-1} \mathfrak{so}_{pq}(J_0^\alpha) \longmapsto A \cdot J_0^\alpha = [A, J_0^\alpha] \in \oplus_{q=1}^{(r,j)-1} \mathcal{B}_{pq}(J_0^\alpha)$$

(see section 3.1.3).
For the connection form on Q^j we take

$$\omega^j := \omega_{\mathfrak{h}_j | TQ^j}.$$

We set as usual $N^j = Q^j/K$ which is a Homogeneous fibre bundle over M. Moreover the isomorphism of bundle (and isometry) $\mathcal{J} \colon N \to \mathcal{Z}_{2k}^\alpha(E)$ satisfies

$$\mathcal{J}(N^j) = \mathcal{Z}_{2k,j}^\alpha(E, J_j)$$

by definition of $\mathcal{J}$ and Q^j (see (72) and (77)), so that it induces an isomorphism of bundle from N^j onto $N_{\mathcal{Z}}^j$.
Let us denote by $TN^j = \mathcal{V}^j \oplus \mathcal{H}^j$ the splitting in terms of vertical and horizontal subbundles given by ω^j. Then denoting by s_j the cross section in the associated bundle $Q/H^j = \mathcal{SO}(E)/\mathbb{U}_{j-1}(J_0^\alpha)$ defining the H^j-reduction Q^j (i.e. $\mathcal{J}_j \circ s_j = J_j$)[33], according to section 4.2.3, we have the following equivalences

$$\begin{array}{ccccc} \omega \text{ is reducible in } Q^j \text{ (to } \omega^j) & \overset{4.2.3}{\Longleftrightarrow} & s_j \text{ is horizontal} & \overset{4.2.3}{\Longleftrightarrow} & \mathcal{H}^j = \mathcal{H}_{|N^j} \\ & & \overset{\text{Rmk 4.10}}{\Updownarrow} & & \Updownarrow \\ & & \nabla J_j = 0 & \overset{\text{ex. 4.4}}{\Longleftrightarrow} & \mathcal{H}^{\mathcal{Z},j} = \mathcal{H}^{\mathcal{Z}}{}_{|N_{\mathcal{Z}}^j} \end{array}$$

[33] See remark 4.9 and 4.10.

Example 4.9 *Let $M = G/H$ be the k-symmetric space correponding to some $2k$-symmetric space G/G_0 (see section 2.1.1), and take $(E, \nabla) = (TM, \overset{M}{\nabla^0})$, $j = 2$ and J_2 given by lemma 3.1. Then we have*

$$\overset{M}{\nabla^0} J_2 = 0.$$

Indeed $\overset{M}{\nabla^0} J_2$ lifts in G into

$$(d + \theta_{\mathfrak{h}})J_0^2 = dJ_0 + [\theta_{\mathfrak{h}}, J_0^2] = 0$$

(see lemma 3.1). Therefore we can conclude that in this case ω is reducible in Q^2 (to ω^2).

If ω is not reducible in Q^j (to ω^j), then according to (47) and (57), $N^j \hookrightarrow N$ and $N^j_{\mathcal{Z}} \hookrightarrow N_{\mathcal{Z}}$ are not isometries, and thus we can not say directly that $\mathcal{J}$ induces an isometry from N^j onto $N^j_{\mathcal{Z}}$, even if as we will see below it is effectively the case. As above, the result of (47) and (57), and $d\mathcal{J}(\mathcal{V}^j) = \mathcal{V}^{\mathcal{Z},j}$, is that: $\mathcal{J} \colon N^j \to N^j_{\mathcal{Z}}$ is an isometry if and only if $d\mathcal{J}(\mathcal{H}^j) = \mathcal{H}^{\mathcal{Z},j}$.

Now let us come back to the connection form $\omega^j \colon TQ^j \to \mathfrak{h}^j \subset \mathfrak{so}(2n)$. It defines a metric covariant derivative $\nabla^{[j]}$ in the associated vector bundle E. Then we have

$$\nabla^{[j]} J^j = 0.$$

Indeed J_j lifts into the H^j-equivariant (constant) map $\tilde{\mathcal{J}}_j \colon e \in Q^j \to (J_0^\alpha)^j \in (\mathcal{Z}_{2k}^\alpha(\mathbb{R}^{2n}))^j \subset \mathfrak{gl}_{2n}(\mathbb{R}^{2n})$ and $\nabla^{[j]} J_j$ lifts into

$$\bar{D}^j \tilde{J}^j = d\tilde{J}_j + [\omega^j, \tilde{J}_j] = 0 + 0 = 0,$$

since by definition $\mathfrak{h}^j = \mathfrak{u}_{j-1}(J_0^\alpha) = \mathfrak{so}_0((J_0^\alpha)^j)$ commutes with $(J_0^\alpha)^j$.

Remark 4.11 We can do the things more concretely by using a (local) moving frame e in Q^j: $\nabla^{[j]}$ is then caracterized by

$$\nabla^{[j]}(e_1, \ldots, e_{2n}) = (e_1, \ldots, e_{2n}).\omega^j(e; de) \tag{78}$$

Then by definition of Q^j we have

$$J_j e = e.(J_0^\alpha)^j \tag{79}$$

so that

$$(\nabla^{[j]} J_j)e + J_j(\nabla^{[j]} e) = e.\omega^j(J_0^\alpha)^j$$

then using (79) and (78), we obtain

$$(\nabla^{[j]} J_j)e = e.\omega^j(J_0^\alpha)^j - J_j(e.\omega^j) = e.(\omega^j(J_0^\alpha)^j - (J_0^\alpha)^j\omega^j) = 0$$

since ω^j takes values in $\mathfrak{h}_j = \mathfrak{so}_0((J_0^\alpha)^j)$.

In fact we can caracterize $\nabla^{[j]}$ in the following more general way, which in particular generalizes a well-known result of Rawnsley [31] about complex structures on vector bundles.

Theorem 4.19 *Let E be a Riemannian vector bundle as above. Let $p' \in \mathbb{N}^*$ and $J \in \mathcal{C}(\mathcal{U}_{p'}(E))$, then* $\mathrm{Ad}J$ *defines an automorphism of the linear bundle* $\mathrm{End}(E)$ *(over* Id_M*), i.e. a section of* $\mathrm{End}(\mathrm{End}(E))$*. Then the metric covariant derivative ∇ in E admits an unique decomposition in the form:* [34]

$$\nabla = \overset{J}{\nabla}{}^0 + \sum_{i=1}^{r_{p'}-1} A_i \tag{80}$$

where $\overset{J}{\nabla}{}^0$ *is a metric covariant derivative for which*

$$\overset{J}{\nabla}{}^0 J = 0$$

and $A_i \in \mathcal{C}(T^*M \otimes \mathfrak{so}_i(E,J))$, i.e. $JA_iJ^{-1} = \omega_r^i A_i$ and $A_i \in \mathfrak{so}(E)^{\mathbb{C}}$.
$\overset{J}{\nabla}{}^0$ *will be called the J-commuting component of ∇, $A_* = \sum_{i=1}^{r_{p'}-1} A_i \in \mathcal{C}(T^*M \otimes \mathfrak{so}_*(E,J))$ the $\mathfrak{so}_*(E,J)$-component of ∇, and A_i the $\mathfrak{so}_i(E,J)$-component of ∇.*

Proof. *Unicity.* Let us suppose that (80) exists then we have

$$\nabla J = \sum_{i=1}^{r_{p'}-1} [A_i, J]$$

so that

$$\sum_{i=1}^{r_{p'}-1} A_i = -(\mathrm{ad}J)^{-1}(\nabla J)$$

(see section 4.3.2) which proves the unicity of $(A_i)_{1\leq i\leq r_{p'}-1}$ (these are determined by ∇ and J, more precisely these are the components of $-(\mathrm{ad}J)^{-1}(\nabla J)$).
Now $\overset{J}{\nabla}{}^0 = \nabla - \sum_{i=1}^{r_{p'}-1} A_i$ is also unique.
Existence. Let ∇^0 be any metric covariant derivative commuting with J, that is to say ∇^0 corresponds to a connection on the principal bundle of Hermitian frames on $(E, \langle\ ,\ \rangle, J)$ (such a connection always exists, see [27]). Then consider

$$A = \nabla - \nabla^0 \in \mathcal{C}(T^*M \otimes \mathfrak{so}(E))$$

and let $A = \sum_{i=0}^{r_{p'}-1} A_i$ be the decomposition of A following $\mathfrak{so}(E,J)^{\mathbb{C}} = \oplus_{i=0}^{r_{p'}-1} \mathfrak{so}_i(E,J)$. Let us set

$$\overset{J}{\nabla}{}^0 = \nabla^0 + A_0$$

[34] As usual $r_{p'}$ is the order of $\mathrm{Ad}J$, i.e. $r_{p'} = p'$ if p' is odd, and if p' is even then $r = p'$ if $J^{\frac{p'}{2}} \neq -\mathrm{Id}$ and $r_{p'} = \dfrac{p'}{2}$ if $J^{\frac{p'}{2}} = -\mathrm{Id}$.

then $\overset{J}{\nabla}{}^0$ is a J-commuting metric covariant derivative in E and we have

$$\nabla = \overset{J}{\nabla}{}^0 + \sum_{i=1}^{r_{p'}-1} A_i$$

which proves the existence. □

Applying this theorem to J_j, we obtain the following.

Corollary 4.2 *$\nabla^{[j]}$ is the J_j-commuting component of ∇.*

Proof. The H-equivariant lift of ∇ is the covariant derivative on Q:[35]

$$d + \omega = (d + \omega_0) + \sum_{i=1}^{r_{p'}-1} \omega_i \tag{81}$$

where $\omega_i = [\omega]_{\mathfrak{so}_i((J_0^\alpha)^j)}$, and in particular $\omega_0 = \omega^j$. Then restricting (81) to Q^j, and projecting on M, we obtain the decomposition (80) of ∇:

$$\nabla = \nabla^{[j]} + \sum_{i=1}^{r_{p'}-1} A_i$$

that is to say $d+\omega_0$ is the H^j-equivariant lift of $\overset{J_j}{\nabla}{}^0$, which is thus equal to $\nabla^{[j]}$, and ω_i is the H^j-equivariant lift of the $\mathfrak{so}_i(E, J_j)$-valued 1-form on M, A_i. This completes the proof.

Remark 4.12 Moreover ω is reducible in Q^j ($\nabla J_j = 0$) if and only if $\nabla^{[j]} = \nabla$.

Remark 4.13 Under the hypothesis of theorem 4.19 we have

$$\forall F \in \mathcal{C}(\mathcal{A}_0(E, J)), \quad \overset{J}{\nabla}{}^0 F = \mathrm{pr}_{\mathcal{A}_0(E,J)} \circ \nabla F$$

where $\mathrm{pr}_{\mathcal{A}_0(E,J)} \colon \mathrm{End}(E) \to \mathcal{A}_0(E, J)$ is the orthogonal projection (i.e. along $\mathcal{A}_*(E, J)$) so that in particular

$$\forall F \in \mathcal{C}(\mathfrak{so}_0(E, J)), \quad \overset{J}{\nabla}{}^0 F = \mathrm{pr}_{\mathfrak{so}_0(E,J)} \circ \nabla F$$

where $\mathrm{pr}_{\mathfrak{so}_0(E,J)} \colon \mathfrak{so}(E) \to \mathfrak{so}_0(E, J)$ is the orthogonal projection. Indeed,

$$\nabla F = \overset{J}{\nabla}{}^0 F + \sum_{i=1}^{r_{p'}-1} [A_i, F]$$

[35] Remark that here $r_{p'}$ is the order of $\mathrm{Ad}(J_0^\alpha)^j$, so that $r_{p'} = \dfrac{r}{(r, j)} = p$.

and J commutes with $\overset{J}{\nabla}{}^0$ and F so with $\overset{J}{\nabla}{}^0 F : (\overset{J}{\nabla}{}^0 F).J = \overset{J}{\nabla}{}^0(F.J) - F\overset{J}{\nabla}{}^0 J = \overset{J}{\nabla}{}^0(J.F) = J\overset{J}{\nabla}{}^0 F$. Moreover $[A_i, F] \in [\mathcal{A}_i(J), \mathcal{A}_0(J)] \subset \mathcal{A}_i(J)$, so that we can conclude. ∎

The canonical $2k$-structure in π^*E, $\mathcal{J} \colon N \to \pi^*E$ induces by restriction a $2k$-structure in ${\pi^j}^*E$, still denoted by $\mathcal{J} \colon N^j \to {\pi^j}^*E$.
Now, let us precise the subbundles $\mathfrak{p}^j_{Q^j}$ and $\mathfrak{k}_{Q^j}$. First, we have $\mathfrak{h}^j_{Q^j} = \mathfrak{u}_0(E, J_j)$[36] and then

$$\begin{aligned} \mathfrak{k}_{Q^j} &= \mathfrak{k}_{Q_{|Q^j}} = \mathfrak{so}_0({\pi^j}^*E, \mathcal{J}) \\ \mathfrak{p}^j_{Q^j} &= \mathfrak{u}^*_{j-1}(E, \mathcal{J}) = \Big(\oplus_{i \in p.\mathbb{Z}_r \setminus \{0\}} \mathfrak{so}_i({\pi^j}^*E, \mathcal{J})\Big) \bigcap \mathfrak{so}({\pi^j}^*E). \end{aligned}$$

The morphism of vector bundle $\mathrm{ad}\mathcal{J} \colon \mathfrak{so}(\pi^*E) \to \mathcal{J}.\mathfrak{p}_Q$ induces a surjective morphism from ${\pi^j}^*\mathfrak{h}^j_{Q^j} = \mathfrak{u}_{j-1}({\pi^j}^*E, \mathcal{J})$ onto $\mathcal{J}.\mathfrak{p}^j_{Q^j}$, with kernel $\mathfrak{k}_{Q^j}$:

$$\begin{array}{rccl} \mathrm{ad}\mathcal{J}\colon & {\pi^j}^*\mathfrak{h}^j_{Q^j} = \mathfrak{u}_{j-1}({\pi^j}^*E, \mathcal{J}) & \longrightarrow & \mathcal{J}.\mathfrak{u}^*_{j-1}({\pi^j}^*E, \mathcal{J}) = \mathcal{J}.\mathfrak{p}^j_{Q^j} \\ & (J, A) & \longmapsto & \mathrm{ad}J(A) = [J, A] = J \sum_{i=1}^{(r,j)-1} (1 - \omega_r^{ip}) A_{ip} \end{array}$$

where $A_i = [A]_{\mathfrak{so}_i(E_x)}$.
As above, now we express the homogeneous fibre bundle tools ϕ^j, Φ^j and $\nabla^{\mathfrak{p}^j}$ in terms of $\mathcal{J}$.

Theorem 4.20 *If $A, B \in TN^j$, $F \in \mathcal{C}(\mathfrak{p}^j_{Q^j})$ then*

(i) $\nabla^{[j]}\mathcal{J} = -\mathrm{ad}\mathcal{J} \circ \phi^j$ *thus* $\phi^j A = -(\mathrm{ad}\mathcal{J})^{-1}\nabla^{[j]}_A \mathcal{J}$

(ii) $\Phi^j(A, B) = (\mathrm{ad}\mathcal{J})^{-1}\left[\mathcal{J}, {\pi^j}^* R^{\nabla^{[j]}}(A, B)\right]$ *where $R^{\nabla^{[j]}}$ is the curvature of $\nabla^{[j]}$.*

(iii) $\nabla^{\mathfrak{p}^j}_A F = (\mathrm{ad}\mathcal{J})^{-1}\left[\mathcal{J}, \nabla^{[j]}_A F\right]$

In the following theorem, we use the notation of 4.2.3. In particular, we denote by " $\cdot^{\mathrm{v}}$ " instead of " $\cdot^{v}$ " the vertical component in $\mathcal{V}^j \subset TN^j$.

Theorem 4.21 *Let $s \in \mathcal{C}(\pi^j)$ and $J = s^*\mathcal{J}$ be the corresponding $2k$-structure. Then*

(i) $I(d^{\mathrm{v}}s) = -(\mathrm{ad}J)^{-1}\nabla^{[j]}J$. *Thus s is horizontal if and only if J is $\nabla^{[j]}$-parallel.*

[36] The restriction of the identification (71), $\mathfrak{h}_Q \cong \mathfrak{so}(E)$ to $\mathfrak{h}^j_{Q^j}$ gives rises to an identification $\mathfrak{h}^j_{Q^j} := Q^j \times_{H^j} \mathfrak{h}^j \cong \mathfrak{so}_0(E, J_j)$.

(ii) $I(\Pi^{\mathrm{v}}(s)) = -(\mathrm{ad}J)^{-1}(\nabla^{[j]})^2 J + \frac{1}{2}(\mathrm{ad}J)^{-1}\left[\nabla^{[j]}J \odot (\mathrm{ad}J)^{-1}\nabla^{[j]}J\right]$.

Thus s is superflat if and only if $(\nabla^{[j]})^2 J - \frac{1}{2}\left[\nabla^{[j]}J \odot (\mathrm{ad}J)^{-1}\nabla^{[j]}J\right]$ *commutes with J.*

(iii) $I(\tau^{\mathrm{v}}(s)) = +(\mathrm{ad}J)^{-1}{\nabla^{[j]}}^*\nabla^{[j]}J + (\mathrm{ad}J)^{-1}\mathrm{Tr}\left(\left[\nabla^{[j]}J, (\mathrm{ad}J)^{-1}\nabla^{[j]}J\right]\right)$.

Thus s is a harmonic section if and only if ${\nabla^{[j]}}^*\nabla^{[j]}J + \mathrm{Tr}\left(\left[\nabla^{[j]}J, (\mathrm{ad}J)^{-1}\nabla^{[j]}J\right]\right)$ *commutes with J.*

(iv) $s^*\Phi^j = (\mathrm{ad}J)^{-1}\left[J, R^{\nabla^{[j]}}\right]$.

As above, from theorem 4.20-(i), we conclude that $d\mathcal{J}$ sends the decomposition $TN^j = \mathcal{V}^j \oplus \mathcal{H}^j$ onto the decomposition $TN^j_{\mathcal{Z}} = \mathcal{V}^{\mathcal{Z},j} \oplus \mathcal{H}^{\mathcal{Z},j}$ (see example 4.4) so that we can consider $\pi^j_{\mathcal{Z}} \colon N^j_{\mathcal{Z}}(E) \to M$ as a homogeneous fibre bundle over M with structure groups $H^j = \mathbb{U}_{j-1}(J_0^\alpha)$ and $K = \mathbb{U}_0(J_0^\alpha)$. We will call this structure *the homogeneous fibre bundle structure defined by (the J_j-commuting part of)* ∇.
Besides, since the vertical and horizontal subbundles corresponds via $\mathcal{J}$, then we can conclude according to (47) and (57) that $\mathcal{J} \colon N^j \to N^j_{\mathcal{Z}}$ is an isometry.
Moreover, the vertical tension field of J in $N^j_{\mathcal{Z}} = \mathcal{Z}^\alpha_{2k,2}(E, J_j)$ is given by

$$d\mathcal{J}(\tau^{\mathrm{v}}(s)) = -\left[{\nabla^{[j]}}^*\nabla^{[j]}J + \mathrm{Tr}\left(\left[\nabla^{[j]}J, (\mathrm{ad}J)^{-1}\nabla^{[j]}J\right]\right)\right]_{\mathcal{V}^{\mathcal{Z},j}}$$

Remark 4.14 According to 4.2.3, the canonical connection in $\mathfrak{p}^j_{Q^j} \to N^j$ is the restriction of the canonical connection in $\mathfrak{p}_Q \to N$, to $\mathfrak{p}^j_{Q^j}$.

Remark 4.15 If we endow E with $\nabla^{[j]}$ and apply the theorems 4.17 and 4.18 (with the Riemannian vector bundle $(E, \nabla^{[j]})$), then by restriction to N^j, we obtain theorems 4.20 and 4.21, which is not surprising since in this case ω is reducible in Q^j and then everything corresponds in the reduction $N^j \hookrightarrow N$.
In particular, superflatness and vertical harmonicity (for sections in N^j) are the same in N^j and N. This is what happens in particular in example 4.9.

The particular case of $\mathcal{Z}^\alpha_{2k,2}(E, J_2)$ According to theorem 3.2, we will be especially interested by this subcase in our interpretation of the elliptic integrable system. In this subcase the fibre $H^2/K = \mathcal{Z}_{2k,2}(\mathbb{R}^{2n}, (J_0^\alpha)^2)$ is symmetric so that we obtain simplifications (coming in particular from the facts that $\nabla^c = \nabla^{\mathfrak{p}}$ and that any section $J \in \mathcal{C}(\pi^2_{\mathcal{Z}})$ anticommutes with $\nabla^{[2]}J$) in theorems 4.20 and 4.21 which then take the same forms as theorems 4.15 and 4.16 about the twistor bundle $\Sigma^\varepsilon(E)$, just by doing the change $\nabla \longleftrightarrow \nabla^{[2]}$. Therefore the case $\mathcal{Z}^\alpha_{2k,2}(E, J_2)$ is very similar to that of $\Sigma^\varepsilon(E)$.
Before writing the simplified theorems for $j = 2$, let us do some useful observations.

First, we have[37]

$$\mathfrak{k}_{Q^2} = \mathfrak{so}_0(\pi^{2^*}E,\mathcal{J}) = \{A \in \mathfrak{so}(\pi^{2^*}E) |\, [A,\mathcal{J}] = 0\} = \mathfrak{so}_{(+1)}(\pi^{2^*}E,\mathcal{J}) \quad (82)$$

$$\mathfrak{p}^2_{Q^2} = \mathfrak{so}_{\frac{r}{2}}(\pi^{2^*}E,\mathcal{J}) = \{A \in \mathfrak{so}(\pi^{2^*}E) |\, A.\mathcal{J} + \mathcal{J}.A = 0\} = \mathfrak{so}_{(-1)}(\pi^{2^*}E,\mathcal{J}). \quad (83)$$

Then $\mathrm{ad}\mathcal{J}$ induces a surjective morphism from $\pi^{2^*}\mathfrak{h}^2_{Q^2} = \mathfrak{u}_1(\pi^{2^*}E,\mathcal{J})$ onto $\mathcal{J}.\mathfrak{p}^2_{Q^2} = \mathcal{B}_{\frac{r}{2}}(\pi^{2^*}E,\mathcal{J})$ with kernel $\mathfrak{k}_{Q^2}$

$$\begin{array}{rccl} \mathrm{ad}\mathcal{J}: & \mathfrak{u}_1(\pi^{2^*}E,\mathcal{J}) = \mathfrak{so}_{(+1)}(\pi^{2^*}E,\mathcal{J}) \oplus \mathfrak{so}_{(-1)}(\pi^{2^*}E,\mathcal{J}) & \longrightarrow & \mathcal{B}_{(-1)}(\pi^{2^*}E,\mathcal{J}) = \mathcal{J}.\mathfrak{so}_{(-1)}(\pi^{2^*}E,\mathcal{J}) \\ & (J, A_0 + A_1) & \longmapsto & \mathrm{ad}J(A) = [J,A] = 2JA_1 \end{array}$$

where we denote A_0+A_1 the decomposition following $\mathfrak{so}_{(+1)}(\pi^{2^*}E,\mathcal{J})\oplus\mathfrak{so}_{(-1)}(\pi^{2^*}E,\mathcal{J})$ instead of $A_0 + A_{\frac{r}{2}}$.

Theorem 4.22 *If $A, B \in TN^2$, $F \in \mathcal{C}(\mathfrak{p}^2_{Q^2})$ then*

(i) $\phi^2 A = -\frac{1}{2}\mathcal{J}^{-1}\nabla^{[2]}\mathcal{J}$

(ii) $\Phi^2(A,B) = \frac{1}{2}\mathcal{J}^{-1}\left[\mathcal{J}, \pi^{2^*}R^{\nabla^{[2]}}(A,B)\right]$ *where $R^{\nabla^{[2]}}$ is the curvature of $\nabla^{[2]}$.*

(iii) $\nabla^c_A F = \frac{1}{2}\mathcal{J}^{-1}\left[\mathcal{J}, \nabla^{[2]}_A F\right]$ [38]

Theorem 4.23 *Let $s \in \mathcal{C}(\pi^2)$ and $J = s^*\mathcal{J}$ be the corresponding $2k$-structure. Then*

(i) $I(d^{\mathrm{v}}s) = -\frac{1}{2}J^{-1}\nabla^{[2]}_A J$. *Thus s is horizontal if and only if J is $\nabla^{[2]}$-parallel.*

(ii) $I(\Pi^{\mathrm{v}}(s)) = -(\mathrm{ad}J)^{-1}\left(\nabla^{[2]}\right)^2 J = -\frac{1}{4}\left[J^{-1}, \left(\nabla^{[2]}\right)^2 J\right]$.

Thus s is superflat if and only if $\left(\nabla^{[2]}\right)^2 J$ commutes with J.

(iii) $I(\tau^{\mathrm{v}}(s)) = (\mathrm{ad}J)^{-1}\nabla^{[2]^*}\nabla^{[2]}J = \frac{1}{4}\left[J^{-1}, \nabla^{[2]^*}\nabla^{[2]}J\right]$.

Thus s is a harmonic section if and only if $\nabla^{[2]^*}\nabla^{[2]}J$ commutes with J.

(iv) $s^*\Phi^2 = \frac{1}{2}J^{-1}\left[J, R^{\nabla^{[2]}}\right]$.

Let us add that the vertical tension field $N^2_{\mathcal{Z}}$ is given

$$d\mathcal{J}(\tau^{\mathrm{v}}(s)) = -\left[\nabla^{[2]^*}\nabla^{[2]}J\right]_{\mathcal{V}^{\mathcal{Z},2}} = -\frac{1}{2}J\left[J^{-1}, \nabla^{[2]^*}\nabla^{[2]}J\right] \quad (84)$$

[37] with notation defined in remark 3.4.

[38] see remark 4.14

4.4 Geometric interpretation of the even determined system

4.4.1 The injective morphism of homogeneous fibre bundle $\mathfrak{I}_{J_0}: G/G_0 \hookrightarrow \mathcal{Z}^{\alpha_0}_{2k,2}(G/H, J_2)$.

Here, we want to ask ourself if the inclusion $\mathfrak{I}_{J_0}: G/G_0 \hookrightarrow \mathcal{Z}^{\alpha_0}_{2k,2}(G/H, J_2)$ given by theorem 3.2 conserves the homogeneous fibre bundle structure, in particular: the vertical harmonicity is it conserved. We use the notations of 4.3.1 and 4.3.4 (with $E = TM$, ∇ a metric connection on M and $j = 2$)[39]. First, we see that $\mathfrak{I}_{J_0}$ is obtained by "passage to the quotient" from the following injective morphism of bundle (which is an embedding if G is closed in $\mathrm{Is}(M)$):

$$\begin{array}{rccc} \mathfrak{I}_{e_0}: & G & \hookrightarrow & Q^2 = \mathcal{U}_1^{\alpha_0}(G/H, J_2) \subset \mathcal{SO}(M) \\ & g & \longmapsto & g \cdot e_0 \end{array} \tag{85}$$

where $e_0 \in \mathcal{SO}(T_{p_0}M)$ is such that $\mathcal{M}at(J_0) = J_0^{\alpha_0}$, and $J_0 = \tau^{-1}_{|\mathfrak{m}}$. In other words $G \to M$ is a reduction of $\mathcal{U}_1^{\alpha_0}(G/H, J_2) \to M$ itself a reduction of $\mathcal{SO}(M) \to M$.
Further quotienting in (85) by $\mathbb{U}_0(J_0^\alpha)$ the target space and then by G_0 the domain, we obtain (by definition of $G_0 = G^\tau \cap H$, see theorem 3.2) the injective morphism of bundle

$$\mathfrak{I}_{\overline{e_0}}: g.G_0 \longmapsto (g \cdot e_0)\mathbb{U}_0(J_0^\alpha) \in \mathcal{U}_1^{\alpha_0}(M, J_2)/\mathbb{U}_0(J_0^\alpha) \subset \mathcal{SO}(M)/\mathbb{U}_0(J_0^\alpha)$$

where $\overline{e_0} = e_0\mathbb{U}_0(J_0^\alpha) \in N^2$, and finally composing with $\mathcal{J}$ (in the target space) we obtain the map $\mathfrak{I}_{J_0}$:

$$g.G_0 \longmapsto g \cdot (e_0\mathbb{U}_0(J_0^\alpha)) \overset{\mathcal{J}}{\longmapsto} J = gJ_0g^{-1} \in \mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2).$$

Since $\mathfrak{I}_{J_0}$ (resp. $\mathfrak{I}_{\overline{e}_0}$) is an injective morphism of bundle (and an immersion) $d\mathfrak{I}_{J_0}$ (resp. $d\mathfrak{I}_{\overline{e}_0}$) induces an injective morphism of bundle from the vertical subbundle $\mathcal{V}^{G/G_0} = [\mathfrak{g}_k]$ into the vertical subbundle $\mathcal{V}^{\mathcal{Z},2}$ (resp. $\mathcal{V}^2$).
$\mathfrak{I}_{J_0}$ is the restriction to G/G_0 of the inclusion map $\mathfrak{I}: \mathrm{End}(G/H) \to M \times \mathrm{End}(\mathfrak{g})$ (see 1.7). Indeed, we have the inclusion depending on J_0: $g.G_0 \in G/G_0 \mapsto [g, J_0] \in G \times_H \mathrm{End}(\mathfrak{m}) = \mathrm{End}(G/H)$ which under the inclusion $\mathfrak{I}$ gives $g.G_0 \in G/G_0 \mapsto (g.x_0, \mathrm{Ad}g \circ \tau^{-1}_{|\mathfrak{m}} \circ \mathrm{Ad}g^{-1}) \in M \times \mathrm{End}(\mathfrak{g})$ which is in nothing but $\mathfrak{I}_{J_0}$ (as usual under the identification $TM = [\mathfrak{m}]$). Then under the inclusion $\mathfrak{h}_G \subset \mathfrak{so}(TM)$, we have $\mathfrak{h}_G \subset \mathfrak{h}^2_{Q^2} = \mathfrak{u}_0(TM, J_2)$. Indeed, under the linear isotropy representation of H in $T_{x_0}M$, we have $H \subset \mathbb{U}_0(T_{x_0}M, J_0^2) = \mathbb{U}_1(T_{x_0}M, J_0)$ so that $\mathfrak{h} \subset \mathfrak{u}_0(T_{p_0}M, J_0^2)$ and thus $\mathfrak{h}_G := G \times_H \mathfrak{h} \subset \mathfrak{u}_0(TM, J_2)$. Moreover let us remark that $\pi^2 \circ \mathfrak{I}_{\overline{e}_0} = \pi$ so that $\pi^*\mathfrak{h}_G \subset \pi^{2*}\mathfrak{h}^2_{Q^2}$ over $\mathfrak{I}_{\overline{e}_0}: N \to N^2$ (i.e. the inclusion is a morphism of bundle over $\mathfrak{I}_{\overline{e}_0}$).
Furthermore, since $\mathrm{Ad}J_0$ leaves invariant $\mathfrak{h} \subset \mathfrak{u}_1(T_{x_0}M, J_0)$, the restriction to $\mathfrak{h}$ of the symmetric decomposition

$$\mathfrak{u}_1(T_{x_0}M, J_0) = \mathfrak{so}_{(+1)}(T_{x_0}M, J_0) \oplus \mathfrak{so}_{(-1)}(T_{x_0}M, J_0)$$

[39] That is to say the notations for $\mathcal{Z}^{\alpha_0}_{2k,2}(G/H, J_2)$ will have the subscript "2" and these of G/G_0 will not have subscript according to 4.3.1 and 4.3.4

gives rise to the decomposition $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_k$ according to (23), so that the symmetric decomposition given by $\mathrm{Ad}\mathcal{J}$ on ${\pi^2}^*\mathfrak{h}^2_{Q^2} = \mathfrak{u}_1({\pi^2}^*TM, \mathcal{J})$, that is to say

$$\mathfrak{u}_1(TM, \mathcal{J}) = \mathfrak{so}_{(+1)}({\pi^2}^*TM, \mathcal{J}) \oplus \mathfrak{so}_{(-1)}({\pi^2}^*TM, \mathcal{J})$$

gives rise in the $\mathrm{Ad}\mathcal{J}$-invariant subspace $\pi^*\mathfrak{h}_G \subset {\pi^2}^*\mathfrak{h}^2_{Q^2}$ to the symmetric decomposition of $\mathrm{Ad}\mathfrak{I}_{J_0}$ (restricted to $\pi^*\mathfrak{h}_G \subset \mathfrak{so}(\pi^*TM)$)

$$\pi^*\mathfrak{h}_G = \mathfrak{k}_G \oplus \mathfrak{p}_G$$

according to (69). In other words, the decomposition given by (69) injects into the decomposition given by (82) via the inclusion $\pi^*\mathfrak{h}_G \subset {\pi^2}^*\mathfrak{h}^2_{Q^2}$.

Now let us interpret theorems 4.13 and 4.14 using the homogeneous fibre bundle structure in $\mathcal{Z}^{\alpha_0}_{2k}(M, J_2)$ defined by the Riemannian vector bundle $(E, \nabla) = (TM, \overset{M}{\nabla^0})$ (in the sense of 4.3.4). We continue to use the same conventions for the notations in N and N^2 (no subscript for N and subscript 2 for N^2 and $N^{\mathcal{Z},2}$). Recall that we have $\mathfrak{I}_{J_0} = \mathcal{J} \circ \mathfrak{I}_{\overline{e}_0} = \mathfrak{I}^*_{\overline{e}_0}\mathcal{J}$ and that $(\overset{M}{\nabla^0})^{[2]} = \overset{M}{\nabla^0}$. Then according to theorems 4.22 and 4.23, theorems 4.13 and 4.14 implies

Theorem 4.24 *We have the following identities*

(i) $\phi = \mathfrak{I}^*_{\overline{e}_0}\phi^2$

(ii) $\Phi = \mathfrak{I}^*_{\overline{e}_0}\Phi^2$

(iii) $\nabla^c = \overset{N}{\nabla^0}|[\mathfrak{p}] = \mathfrak{I}^*_{\overline{e}_0}\nabla^{c,2}$, *where* $\nabla^{c,2}$ *is the canonical connection in* $\mathfrak{p}^2_{Q^2}$.

Theorem 4.25 *Let* $s \in \mathcal{C}(\pi)$ *and identify it (temporarily) with* $s^*\mathfrak{I}_{\overline{e}_0} \in \mathcal{C}(\pi^2)$. *Then under the inclusion* $\mathfrak{I}_{\overline{e}_0} : N \to N^2$, *we have:*

(i) $d^v s = d^{\mathrm{v},2}s$

(ii) $\Pi^v s = \Pi^{\mathrm{v},2}$

(iii) $\tau^v s = \tau^{\mathrm{v},2}s$

(iv) $s^*\Phi = s^*\Phi^2$

These properties holds also, without any change, for maps $f \in \mathcal{C}(L, N)$, (L, b) *being a Riemannian manifold.*

Let us remark that since the connection form ω, on $Q = \mathcal{SO}(TM)$ defined by $\overset{M}{\nabla^0}$ is reducible in Q^2, then in the previous theorems all the "quantities" in N^2 (right handside) can also be computed in $\mathcal{SO}(TM)/\mathbb{U}_0(J_0^{\alpha_0}) \cong \mathcal{Z}^{\alpha_0}_{2k}(M)$, since "everything is reducible" in this case (see remark 4.15).

Now, let us compute the vertical tension field of $J\colon L \to N^2_{\mathcal{Z}}$ for the homogenous fibre bundle structure defined in $N^2_{\mathcal{Z}}$ by $\overset{M}{\nabla^0}$: according to (84) we have

$$\tau^{\mathrm{v},2}(J) = -\frac{1}{2} J \left[J^{-1}, (\overset{M}{\nabla^0})^* \overset{M}{\nabla^0} J \right]. \tag{86}$$

Then suppose that $J \in \mathfrak{I}_{J_0}(G/G_0)$ i.e. $J = s^* \mathfrak{I}_{J_0}$ for a certain $s \in \mathcal{C}^\infty(L, N)$, then according to theorem 4.25 (and $\mathfrak{I}_{J_0} = \mathcal{J} \circ \mathfrak{I}_{\overline{e}_0}$) we have

$$d\mathfrak{I}_{J_0}(\tau^v(s)) = d\mathcal{J}(\tau^{\mathrm{v},2}(\bar{s})) = \tau^{\mathrm{v},2}(J)$$

where $\bar{s} = \mathcal{J}^{-1} \circ J$ i.e. $J = \bar{s}^* \mathcal{J}$.
The tension fields (and thus vertical harmonicity) correspond via the different inclusions and identifications, in particular via $\mathfrak{I}_{J_0}\colon N \to N^2_{\mathcal{Z}}$.
In fact in what precedes we can replace the canonical connection in M, $\overset{M}{\nabla^0}$, by (the J_2 commuting part of) the Levi-Civita connection in M.

Theorem 4.26 *The canonical affine connection on M is the J_2-commuting component of the Levi-Civita connection $\overset{M}{\nabla}$ on M:*

$$\overset{M}{\nabla^0} = \overset{J_2}{\nabla^0} = \overset{M}{\nabla^{[2]}}.$$

Corollary 4.3 *The homogeneous fibre bundle structures in $N^2_{\mathcal{Z}}$ defined by the canonical affine coonection $\overset{M}{\nabla^0}$ and by (the J_2-commuting part of) the Levi-Civita connection $\overset{M}{\nabla}$, in M, are the same. Therefore theorems 4.13, 4.14 and corollary 4.1 still hold if we replace $\overset{M}{\nabla^0}$ by $\overset{M}{\nabla^{[2]}}$. Moreover theorems 4.24 and 4.25 hold with the homogeneous fibre bundle structure defined in N^2 by the (J_2-commuting part of) the Levi-Civita connection $\overset{M}{\nabla}$.*

Let us conclude this subsection by some additionnnal equalities.

Theorem 4.27 *The canonical affine connection on M is the J_2-commuting component of the connections $\overset{\mathrm{met}}{\nabla^t}$ on M:*

$$\overset{M}{\nabla^0} = \left(\overset{\mathrm{met}}{\nabla^t} \right)^{[2]}.$$

Theorem 4.28 *Let $J_1 \in \mathcal{C}(\mathcal{U}^*_{2k}(N))$ be the section defined by $\tau^{-1}_{|\mathfrak{n}}$ with, let us recall it, $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m} = \mathfrak{g}_k \oplus \mathfrak{m}$, then*

(i) *The J_1-commuting component of the (π-pullback of the) canonical affine connection in M, $\pi^*\overset{M}{\nabla}{}^0$, is $\overset{N}{\nabla}{}^0$ the canonical affine connection in N. This latter is also the J_1-commuting component of the (π-pullback of the) Levi-Civita connection, and more generally of the connections $\overset{\text{met,M}}{\nabla^t}$.*

(ii) *The J_1-commuting component of the Levi-Civita connection in N, $\overset{N}{\nabla}$, is $\overset{N}{\nabla}{}^0$.*

(iii) *More generally, the J_1-commuting component of $\overset{\text{met,N}}{\nabla^t}$ is $\overset{N}{\nabla}{}^0$.*

(iv) *Let $s \in \mathcal{C}(\pi)$ and $J = s^*\mathfrak{I}_{J_0}$ the corresponding $2k$-structure on M, then $s^*\overset{N}{\nabla}{}^0$ is the J_1-commuting component of $\overset{M}{\nabla}{}^0$, and also the J_1-commuting component of (the s-pulback of) the Levi-Civita connection on M, $s^*\overset{M}{\nabla}$; and more generally of (the s-pulback of) the the connections $\overset{\text{met,N}}{\nabla^t}$.*

We recover in particular [6] from (iv).

4.4.2 Conclusion

Now we can conclude:

Theorem 4.29 *Let (L, j) be a Riemann surface, $f\colon L \to N = G/G_0$ be a map and $J = f^*\mathfrak{I}_{J_0}$ the corresponding map into $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$. Then f is a geometric solution of the even determined system* $(\mathrm{Syst}(k, \tau))$ *if and only if*

(i) *J is an admissible twistor lift ($\Leftrightarrow$ f is horizontally holomorphic)*

(ii) *J is vertically harmonic in $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$ endowed with its homogeneous fibre bundle structure defined by the Levi-Civita connection, ∇, in M:*

$$\left[\nabla^{[2]^*}\nabla^{[2]}J, J\right] = 0,$$

where $\nabla^{[2]}$ is the J_2-commuting component of ∇. ($\Leftrightarrow$ f is vertically harmonic in G/G_0).

Moreover the first condition implies that J is flat in $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$:

$$J^*\Phi^{\mathcal{Z},2} = \left[u^*R^{\nabla^{[2]}}, J\right] = 0,$$

where $\Phi^{\mathcal{Z},2}$ is the homogeneous curvature form in $\mathcal{Z}^{\alpha_0}_{2k,2}(M, J_2)$, which means also that J is a flat section in $\mathrm{End}(u^*TM, u^*\nabla^{[2]})$. *($\Leftrightarrow$ f is flat in the homogeneous fibre bundle $N \to M$).*
Furthermore f is a primitive geometric solution (i.e. there exists $m \leq k$ such that f is m-primitive, which is equivalent to say that f is k-primitive) if and only if

(i) *J is an admissible twistor lift*

(ii) *J is parallel: $\nabla^{[2]}J = 0$ ($\Leftrightarrow$ f is horizontal).*

Besides in these characterizations, in the points (ii) the Levi-Civita connection can be replaced by any G-invariant metric connection ∇' whose the J_2-commuting component $\nabla'^{[2]}$ leaves invariant $\mathfrak{h}_G \subset \mathfrak{so}(TM)$. This is the case in particular for the connections

$$\overset{\text{met}}{\nabla^t} = \overset{M}{\nabla^0} + t\left([\ ,\ ]_{[\mathfrak{m}]} + \bar{\mathrm{U}}^M\right), \quad 0 \leq t \leq 1,$$

for which the J_2-commuting component is the canonical connection on M: $\overset{M}{\nabla^0}$.

5 Affine (vertically) harmonic maps

5.1 Affine harmonic maps and holomorphically harmonic maps

A map $u\colon M \to N$ between two Riemannian manifolds (M, g) and (N, h) is harmonic if it extremizes the energy functional

$$E(u) = \frac{1}{2}\int_D |du|^2 d\mathrm{vol}_g$$

for all compact subdomains $D \subset M$, where $|du|^2 = \mathrm{Tr}_g(u^*h)$. The associated Euler-Lagrange equation is $\tau(u) := \mathrm{Tr}_g(\nabla du) = 0$, where ∇ is the connection on $T^*M \otimes u^*TN$ induced by the Levi-Civita connections of M and N.

Now, we generalise this definition for maps from a Riemannian manifold into an affine manifold. We present to different ways to do that. The first one is the natural one (see also [21]) and concerns general affine manifolds whereas the second one concerns maps from Riemann surfaces into affine almost complex manifolds.

5.1.1 Affine harmonic maps: general properties

Definition 5.1 *Let $s\colon (M, g) \to (N, \nabla)$ be a smooth map from a Riemannian manifold (M, g) into an affine manifold (N, ∇). We set*

$$\tau(s) = \mathrm{Tr}_g(\nabla ds) = -\nabla^* ds = *d^\nabla * ds$$

and we say that s is affine harmonic with respect to ∇ or ∇-harmonic if $\tau(s) = 0$.

Now, let us consider the case where (M, g) is a Riemannian surface surface i.e. a Riemann surface (L, j) with a Hermitian metric b. Then the action of the Hodge operator $*$ of L, is independent of the metric b on 1-forms ($*\alpha = \alpha \circ j$), whereas in 2-forms (resp. 0-forms) it is multiplied by the factor λ^2 (resp. $\lambda^{-2} > 0$) when the metric is multiplied by the factor $\lambda \in C^\infty(L, \mathbb{R}^*_+)$. Hence the tension field $\tau(f) = *d^\nabla(*df)$ is multiplied by λ^2, under this last transformation. In particular the affine harmonicity for maps $f\colon (L, j) \to (N, \nabla)$ does not depend on the hermitian metric L but only on the conformal structure of (L, j). Thus we have:

Theorem 5.1 *Let (L, j) be a Riemann surface and $f\colon (L, j) \to (N, \nabla)$ a smooth map. Let $TL^{\mathbb{C}} = T'L \oplus T''L$ be the decomposition into the $(1, 0)$ and $(0, 1)$-parts, and $d = \partial + \bar\partial$ and $\nabla^{f^*(TN)} = \nabla' + \nabla''$ the corresponding splittings. Then we have*

$$2\,\bar\partial^\nabla \partial f = d^\nabla df + i d^\nabla * df,$$

moreover $d^\nabla df = f^*T$, where T is the torsion of ∇ and $d^\nabla * df = \tau(f)\mathrm{vol}_b$ for any hermitian metric b in M. Therefore the following statements are equivalent:

(i) $\nabla'' \partial f = 0$

(ii) $\bar{\partial}^{\nabla} \partial f = 0$

(iii) $\nabla_{\frac{\partial}{\partial \bar{z}}} \left(\frac{\partial f}{\partial z} \right) = 0$, *for any holomorphic local coordinate* $z = x + iy$ *(i.e.* (x, y) *are conformal coordinates for any hermitian metric in* L*).*

(iv) f *is* ∇*-harmonic with respect to any hermitian metric in* L *and torsion free:* $f^*T = 0$ *(i.e.* $T(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$ *for any conformal coordinates* (x, y)*).*

We will say in this case that f *is strongly* ∇*-harmonic.*

Remark 5.1 *We remark that the imaginary part (resp. the real part) of equation (ii) (resp. equation (iii)) is the* ∇*-harmonic maps equation whereas its real part (resp. imaginary part) is the torsion free equation* $f^*T = 0$*.*
If $T = 0$ *or more generally* $f^*T = 0$*, then* f *is strongly* ∇*-harmonic if and only if it is* ∇*-harmonic.*

5.1.2 Holomorphically harmonic maps

In the case the target space N is endowed with an almost complex structure J then we have another way to generalise the definition of harmonicity to maps from a Riemann surface into N.

Definition 5.2 *Let* (L, j) *be a Riemann surface and* (N, ∇) *be an affine manifold endowed with a complex structure* J*. Let us denote* $TN^{\mathbb{C}} = T^{1,0}N \oplus T^{0,1}N$ *the corresponding decomposition of* $TN^{\mathbb{C}}$*. We will say that* $f \colon L \to N$ *is holomorphically harmonic if*

$$[\bar{\partial}^{\nabla} \partial f]^{1,0} = 0.$$

Proposition 5.1 *Let* (L, j) *be a Riemann surface and* (N, ∇) *be an affine manifold endowed with a complex structure* J*. Then* f *is holomorphically harmonic if and only if (for any hermitian metric* b *in* M*)*

$$T_b(f) + J\tau_b(f) = 0$$

where $T_b(f) = *(f^*T) = f^*T(e_1, e_2)$*, with* (e_1, e_2) *an orthonormal basis of* TL*, or equivalently*

$$\tau_b(f) \mathrm{vol}_b = J(f^*T).$$

Therefore f *is strongly harmonic if and only if it is torsion free and holomorphically harmonic. In particular, if* $T = 0$*, or more generally* $f^*T = 0$*, then* f *is holomorphically harmonic if and only if it is harmonic. Hence for torsion free connection* ∇ *harmonicity and holomorphic harmonicity are the same.*

Proof. Let $Z = X + iY \in TN^{\mathbb{C}}$ with $X, Y \in TN$, then since $T^{1,0}N$ and $T^{0,1}N$ are given respectively by $\{V \mp iJV, V \in TN\}$, we deduce that

$$[Z]^{1,0} = 0 \Leftrightarrow X + JY = 0 \quad \text{and} \quad [Z]^{0,1} = 0 \Leftrightarrow X - JY = 0.$$

Now, let us apply that to the $TN^{\mathbb{C}}$-valued 2-form $\bar{\partial}^{\nabla}\partial f$, we obtain

$$[\bar{\partial}^{\nabla}\partial f]^{1,0} = 0 \Longleftrightarrow d^{\nabla}df + Jd^{\nabla} * df = 0$$

according to theorem 5.1. This proves the first assertion. Then the assertion concerning strongly harmonicity follows from theorem 5.1(iv). This completes the proof. □

Let us remark that

Proposition 5.2 *In the same situation as in the previous proposition, let us suppose in addition that $\nabla J = 0$. Then if a map $f\colon L \to N$ is holomorphic i.e. $df \circ j_L = Jdf$, then f is anti-holomorphically harmonic (i.e. holomorphically harmonic with respect to $-J$).*

Proof. f is holomorphic if and only if $df(T^{1,0}L) \subset T^{1,0}N$ i.e. $[\partial f]^{0,1} = 0$. Then we have

$$[\bar{\partial}^{\nabla}\partial f]^{0,1} = \bar{\partial}^{\nabla}[\partial f]^{0,1} = 0$$

since ∇ commutes with J. This completes the proof. □

Holomorphic sections of complex vector bundles Now, we need to do some recalls about complex vector bundles that we will apply in the next paragraph to obtain an interpretation of the holomorphic harmonicity in terms of holomorphic 1-forms.

Let $E \to M$ be a real vector bundle (over a manifold M) endowed with a complex structure $J \in \mathrm{End}(E)$. Then any frame in the form $(e^1_x, \dots, e^r_x, Je^1_x, \dots, Je^r_x)$ at some point $x \in M$ can be extended to a local frame $(e^1, \dots, e^r, Je^1, \dots, Je^r)$ in the neighbourhood of x. Then there exists an open covering $(U_\alpha)_{\alpha\in I}$ of M and local trivialisations $\Phi_\alpha\colon (E_{|U_\alpha}, J) \to U_\alpha \times (\mathbb{C}^r, i\mathrm{Id})$ which are $\mathbb{C}$-linear isomorphisms ($\Phi_\alpha \circ J = i\Phi_\alpha$), or equivalently of which transition maps take values in the endomorphisms of $\mathbb{C}^r$: $\phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}\colon U_\alpha \cap U_\beta \to GL(\mathbb{C}^r)$. Therefore E is a complex vector bundle.

Remark 5.2 Let us set $\hat{\mathbb{C}} = \mathbb{R}[J]$, then $\hat{\mathbb{C}} = \mathbb{R}[J]$ is a vector bundle over M whose fibres are fields isomorphic to $\mathbb{C} = \mathbb{R}[i]$ and each fibre E_x of E is a $\hat{\mathbb{C}}_x$-vector space. Then $E^{\mathbb{C}}$ is endowed with two different structures of vector bundle: one over the field $\mathbb{C}$ (the tautological one defined by the complexification of E) and another one "over the distribution of field $\hat{\mathbb{C}}$" (i.e. the one defined by J). Therefore we have two different complex conjugaisons in $E^{\mathbb{C}}$, that we will call respectively the $\mathbb{C}$-conjugaison and the $\hat{\mathbb{C}}$-conjugaison.

Now, let us suppose that E is endowed with a complex connection ∇, i.e. a connection which commutes with J: $\nabla J = 0$. Then for all $X \in TM$, $\nabla_X\colon \mathcal{C}(E) \to \mathcal{C}(E)$ is $\mathbb{C}$-linear with respect to the complex vector space structure defined on $\mathcal{C}(E)$ by the complex vector bundle structure on E. Then we have two different ways to extend ∇ to $TM^{\mathbb{C}}$.

1. The canonical one: for any section $s \in \mathcal{C}(E^{\mathbb{C}})$, we extend ∇s by $\mathbb{C}$-linearity to a linear morphism from $TM^{\mathbb{C}}$ to $E^{\mathbb{C}}$,

$$\nabla_{iX} s = i\nabla_X s, \quad \forall X \in TM, s \in \mathcal{C}(E^{\mathbb{C}})$$

after, of course, having extended ∇ to a connection on $E^{\mathbb{C}}$ by setting $\nabla is = i\nabla s$, $\forall s \in \mathcal{C}(E)$. In conclusion, $\forall s \in \mathcal{C}(E^{\mathbb{C}})$, $\nabla s \in \mathcal{C}(T^*M^{\mathbb{C}} \otimes E^{\mathbb{C}})$.

2. By using the complex vector bundle structure of E defined by J: for any $s \in \mathcal{C}(E)$, we extend ∇s by $\mathbb{C}$-linearity to a linear morphism from $TM^{\mathbb{C}}$ to E:

$$\widehat{\nabla}_{iX} s = J\widehat{\nabla}_X s, \quad \forall X \in TM, s \in \mathcal{C}(E).$$

Let us remark that $\widehat{\nabla}$ depends on J, and since we use the complex vector space structure defined by J, one needs that ∇ and J commute. One the other side the simple canonical complex extention defined in 1 (that we still denote by ∇) is independant of J and one needs not to do any additionnal hypothesis. Remark that the extention 1 is nothing but the extention $\widehat{\nabla}$ defined by the complex structure $i\mathrm{Id}_{E^{\mathbb{C}}}$ on E (which commutes obviously with ∇).

Now let us suppose that M is an (almost) complex manifold with (almost) complex structure j_M. Then we have the splitting $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ defined by j_M which gives rise respectively to the following decompositions of ∇ and $\widehat{\nabla}$:

$$\begin{aligned} \widehat{\nabla} &= \nabla^{(1,0)} + \nabla^{(0,1)} \\ \nabla &= \nabla' + \nabla''. \end{aligned}$$

More generally, let $\eta \in \mathcal{C}(T^*M \otimes E)$ be a 1-form on M with values in E. Then we can extend it in two different ways by $\mathbb{C}$-linearity in $TM^{\mathbb{C}}$ by setting:

$$\begin{aligned} \eta^{\mathbb{C}}(X + iY) &= \eta(X) + i\eta(Y), \quad \forall X, Y \in TM \\ \hat{\eta}(X + iY) &= \eta(X) + J\eta(Y), \quad \forall X, Y \in TM. \end{aligned}$$

Remark that $\eta^{\mathbb{C}} \in \mathcal{C}(T^*M^{\mathbb{C}} \otimes E^{\mathbb{C}})$ whereas $\hat{\eta} \in \mathcal{C}(T^*M^{\mathbb{C}} \otimes E)$. As above we can decompose $\eta^{\mathbb{C}}$ and $\hat{\eta}$ according to the decomposition $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$:

$$\eta^{\mathbb{C}} = \eta' + \eta'' \tag{87}$$

$$\hat{\eta} = \eta^{(1,0)} + \eta^{(0,1)}. \tag{88}$$

Then we have the following relations

Lemma 5.1

$$\begin{array}{ll} [\eta']^{1,0} = \eta^{(1,0)} - iJ\eta^{(1,0)} & [\eta'']^{0,1} = \eta^{(1,0)} + iJ\eta^{(1,0)} \\ [\eta']^{0,1} = \eta^{(0,1)} + iJ\eta^{(0,1)} & [\eta'']^{1,0} = \eta^{(0,1)} - iJ\eta^{(0,1)} \end{array} \tag{89}$$

Proof. Let $Z = X - ij_M X \in T^{1,0}M$ with $X \in TM$. Then

$$\begin{aligned}[\eta(Z)]^{1,0} = [\eta(X) - i\eta(j_M X)]^{1,0} &= \eta(X) - iJ\eta(X) - i\left(\eta(j_M X) - iJ\eta(j_M X)\right) \\ &= \eta(X) - J\eta(j_M X) - iJ\left(\eta(X) - J\eta(j_M X)\right) \\ &= \eta^{(1,0)}(Z) - iJ\eta^{(1,0)}(Z).\end{aligned}$$

This gives us $[\eta']^{1,0}$. Then by taking the $\hat{\mathbb{C}}$-conjugate, we obtain $[\eta']^{0,1}$. Finally, the second column of (89) is obtained by $\mathbb{C}$-conjugaison from the first column. This completes the proof. □

We can apply what precedes to the flat differentiation d. Let (N, J) be an almost complex manifold and $s\colon M \to N$ a map. Then we consider the complex vector bundle $E = s^*TN$ over M. Then applying what precedes to the 1-form $\eta = ds$, we can consider the extensions $\widehat{ds}$ and $(ds)^{\mathbb{C}}$, which then allows us to define the following extension of d to $TM^{\mathbb{C}}$:

$$\hat{d}s = \widehat{ds} \quad \text{and} \quad d^{\mathbb{C}}s = (ds)^{\mathbb{C}},$$

and **by abuse of notation**[40] $d^{\mathbb{C}}$ **will be still denoted by** d. Then we can write the following decompositions

$$\hat{d} = \hat{\partial} + \overline{\hat{\partial}} \quad \text{and} \quad d = \partial + \bar{\partial}$$

according to the decomposition $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$.

Now let us come back to the general situation of a complex vector bundle E over an almost complex manifold (M, j_M), endowed with a complex connection ∇. Let us set

$$\mathrm{H}(M, E) = \{\eta \in T^*M \otimes E | \eta\, j_M = J\eta\}.$$

Then $\mathrm{H}(M, E)$ is a vector subbundle of the vector bundle $T^*M \otimes E$ and is naturally endowed with the complex structure defined by

$$I(\eta) = \eta\, j_M = J\eta, \quad \forall \eta \in T^*M \otimes E, \tag{90}$$

which makes $\mathrm{H}(M, E)$ being a complex vector bundle whose the set of sections is

$$\mathrm{Hom}((TM, j_M), (E, J)) = \{\eta \in \mathcal{C}(T^*M \otimes E) | \eta \circ j_M = J \circ \eta\}.$$

The sections of $\mathrm{H}(M, E)$ can also be caracterized by using the splittings (87-88):

Lemma 5.2 *We have the following equivalences for 1-forms $\eta \in \mathcal{C}(T^*M \otimes E)$:*

$$\eta \circ j_M = J \circ \eta \iff \eta' \in T^*M \otimes E^{1,0} \iff \eta^{(0,1)} = 0.$$

Then we deduce in particular

[40] and to be coherent with the notation used until now, in the paper.

Lemma 5.3 *Let $s \in \mathcal{C}(E)$, then we have the following equivalences:*

$$\nabla s \circ j_M = J \circ \nabla s \Longleftrightarrow \nabla^{(0,1)} s = 0 \Longleftrightarrow [\nabla' s]^{0,1} = 0 \Longleftrightarrow \nabla'(s - iJs) = 0.$$

We will say that s is a vertically holomorphic section.

In fact we can say more

Lemma 5.4 *Let us consider the splitting $TE = \mathcal{H} \oplus \mathcal{V}$ given by ∇, where $\mathcal{V} = \ker \pi = \pi^* E$ is the vertical subbundle and $\mathcal{H}$ the horizontal one. Then let us define an almost complex structure $\check{J}$ on the manifold E by setting*

$$\check{J} = ((d\pi)^* j_M)_{|\mathcal{H}} \oplus \pi^* J.$$

Then a section $s \in \mathcal{C}(E)$ is $\check{J}$-holomorphic if and only if it is vertically holomorphic.

Proof. It suffices to prove that any section $s \in \mathcal{C}(E)$ is horizontally holomorphic, i.e. satisfies the horizontal part of the equation $ds \circ j_M = J \circ ds$. We have $d\pi \circ (ds \circ j_M) = j_M$ since s is a section. In the other side we have $d\pi \circ (\check{J} \circ ds) = j_M \circ d\pi \circ ds = j_M$, by definition of $\check{J}$ and using the fact s is a section. In conclusion $d\pi \circ (ds \circ j_M) = d\pi \circ (J \circ ds)$. This completes the proof.□

In the following, we will say that a section of a complex vector bundle (E, J, ∇) is holomorphic if it is $\check{J}$-holomorphic.

Now, let us apply the two previous lemmas to the vector bundle $\mathrm{H}(M, E)$. First, let us endow M with an almost complex connection ∇^M (it means $\nabla^M J = 0$; such a connection always exists, see [27]). Then $T^*M \otimes E$ is naturally endowed with the connection $\overset{\otimes}{\nabla}$ defined by ∇^M and ∇. Further, we denote by $\overline{\nabla}$ the restriction to $\mathrm{H}(M, E)$ of $\overset{\otimes}{\nabla}$. Then we remark that $\overline{\nabla}$ commutes with the complex structure I (defined by (90)). Therefore, we can now apply the two previous lemmas to the complex vector bundle $\big(\mathrm{H}(M, E), I, \overline{\nabla}\big)$:

Proposition 5.3 *A section of $\mathrm{H}(M, E)$, $\eta \in \mathrm{Hom}((TM, j_M), (E, J))$, is holomorphic if and only if it satisfies one of the following equivalent statements*

(i) $\overline{\nabla}^{(0,1)} \eta = 0$

(ii) $\overline{\nabla}'' \eta' = 0$

(iii) $\left[\overline{\nabla}'' \eta\right]^{1,0}$.

Moreover if M is a Riemann surface[41]*, then it also equivalent to*

(iv) $\overline{\hat{\partial}}^{\widehat{\nabla}} \hat{\eta} = \overline{\hat{\partial}}^{\widehat{\nabla}} \eta^{(1,0)} = 0$, *or*

[41] and ∇^M the unique torsion free complex connection in M, which coincides also with the Levi-Civita connection of any Hermitian metric on M.

(v) $\bar{\partial}^{\nabla}\eta' = 0$

Moreover, if M is a complex manifold (i.e. j_M is integrable) then we choose for ∇^M the unique torsion free complex connection on M. Then we obtain the following result:

Proposition 5.4 *Let $l \in TM$ be a complex line in the tangent bundle of the complex manifold M. Then for any section $\eta \in \mathrm{Hom}((TM, j_M), (E, J))$ we have the following equality*

$$\overline{\nabla}^{(0,1)}\eta_{|l\times l} = d^{\nabla}\eta_{|l\times l}.$$

Moreover if η is holomorphic then $d^{\nabla}\eta = 0$. More particulary, if M is a Riemann surface then we have the following equivalence

$$\eta \text{ is holomorphic } \iff d^{\nabla}\eta = 0.$$

Remark 5.3 One could directly deduces the case of a Riemann surface by using proposition 5.3.
Indeed, the first way to do that is to write $d^{\nabla}\eta = d^{\nabla}\eta' + d^{\nabla}\eta''$. Then remark that η' and η'' takes values in $E^{1,0}$ and $E^{0,1}$ respectively, according to lemma 5.3. Therefore since $E^{1,0}$ and $E^{0,1}$ are ∇-parallel, we can say that $d^{\nabla}\eta'$ and $d^{\nabla}\eta''$ take values resp. in $E^{1,0}$ and $E^{0,1}$ resp., so that $d^{\nabla}\eta = 0 \Leftrightarrow d^{\nabla}\eta' = 0 \Leftrightarrow d^{\nabla}\eta'' = 0$. Then if M is a Riemann surface $d^{\nabla}\eta' = \bar{\partial}^{\nabla}\eta'$, and we conclude by using proposition 5.3.
The second way to do that is to use the $\hat{\mathbb{C}}$-linearity. Indeed, the extension to $TM^{\mathbb{C}}$ by $\hat{\mathbb{C}}$-linearity of $d^{\nabla}\eta$ is $\widehat{d^{\nabla}\eta} = d^{\widehat{\nabla}}\hat{\eta} = d^{\widehat{\nabla}}\eta^{(1,0)} = 0$, since $\eta^{(0,1)} = 0$ (see lemma 5.3). Then if M is a Riemann surface $d^{\widehat{\nabla}}\eta^{(1,0)} = \overline{\partial}^{\widehat{\nabla}}\eta^{(1,0)}$, and we conclude by using proposition 5.3-(iv).

Remark 5.4 Let us consider a 1-form $\beta \in \mathcal{C}(T^*M \otimes E)$, then we can associate to it

$$\eta = \beta - J\beta \circ j_M = \hat{\beta} \circ (\mathrm{Id} - ij_M) = \beta^{(1,0)} \circ (\mathrm{Id} - ij_M).$$

By definition $\eta \in \mathcal{C}(\mathrm{H}(M, E))$, i.e. $\eta \circ j_M = J \circ \eta$. Moreover, still suppposing that M is complex and that ∇^M is the unique torsion free complex connection on M, we have

$$\widehat{\overline{\nabla}}\hat{\eta} = \widehat{\overline{\nabla}}\beta^{(1,0)} \circ (\mathrm{Id} - ij_M) \tag{91}$$

because $(\mathrm{Id} - ij_M)$ is ∇^M-parallel.
Let us remark that since j_M and the multiplication by i coincide in $T^{1,0}M$, they define the same complex structure, which we will suppose $T^{1,0}M$ to be canonically endowed with. Then, since $\hat{\beta}$ is by definition a complex linear morphism from $TM^{\mathbb{C}}$ to E, $\beta^{(1,0)}$ is also a complex linear morphism from $T^{1,0}M$ to E. Hence $\beta^{(1,0)}$ is a section of the complex vector bundle $T^*_{1,0}M \otimes_{\mathbb{C}} E$. Therefore, from equation (91), we deduce that η is a holomorphic section of $\mathrm{H}(M, E)$ if

and only if $\beta^{1,0}$ is a holomorphic section of $T^*_{1,0}M \otimes_{\mathbb{C}} E$. In particular if M is a Riemann surface, we deduce from (91), that

$$\overline{\partial}^{\widehat{\nabla}} \hat{\eta} = \overline{\partial}^{\widehat{\nabla}} \beta^{(1,0)} \circ (\mathrm{Id} - i j_M).$$

Now, we come back to our complex vector bundle (E, J, ∇) and we recall a theorem ([26]) which caracterizes when $\check{J}$ is integrable.

Theorem 5.2 *Let $(E, J, \nabla) \to (M, j_M)$ be a complex vector over a complex manifold, with a complex connection ∇. Then we will say that a holomorphic structure $\mathcal{E}$ is compatible with ∇ (or that ∇ is adapted to $\mathcal{E}$) if it is induced by the almost complex structure $\check{J}$ (defined by lemma 5.4). In other words, a section $s \in \mathcal{C}(E)$ is holomorphic with respect to $\mathcal{E}$ if and only if*

$$\forall Z \in T^{1,0}M,\ \widehat{\nabla}_{\bar{Z}} s = 0.$$

An holomorphic structure $\mathcal{E}$ exists on E if and only if $\check{J}$ is integrable, and in this case $\mathcal{E}$ is unique. Moreover $\check{J}$ is integrable if and only if the $(0,2)$-component of the curvature operator[42] *R of ∇ vanishes.*

When M is of dimension 2, then the $(0,2)$-component of the curvature operator always vanishes so that E always admits a holomorphic structure compatible with ∇, that we will call, following [9], the Koszul-Malgrange holomorphic structure induced by ∇. In the following, we suppose that a complex vector bundle (E, J, ∇) over a Riemann surface is always endowed with its Koszul-Malgrange holomorphic structure.

Interpretation of the holomorphic harmonicity in terms of holomorphic 1-forms. Now we come back to the situation in the begining of 5.1.2. More precisely, we consider (N, J) an almost complex manifold, with ∇ an almost complex connection, (L, j_L) a Riemann surface and $f\colon L \to N$ a map. Then we apply what precedes to the complex vector bundle $E = (f^*TN, f^*\nabla, f^*J)$ over L (i.e. L plays the role of M and f the one of s with respect to the notation of the previous paragraphs). We obtain a first theorem:

Proposition 5.5 *Let $f\colon (L, j_M) \to (N, J)$ be a map from a Riemann surface into an almost complex manifold. Let us set*

$$\eta = df - Jdf \circ j_M.$$

Then η is a section of $\mathrm{H}(L, f^*TN)$, i.e. $\eta \circ j_M = J \circ \eta$. Moreover f is holomorphically harmonic if and only if η is a holomorphic section of the complex vector bundle $\mathrm{H}(L, f^*TN)$, i.e.

$$\bar{\partial}^{\nabla} \eta' = 0.$$

[42] i.e. the $(0,2)$-component of the extension $\hat{R}$ of R to $\Lambda^2 T^*M^{\mathbb{C}}$ by $\hat{\mathbb{C}}$-linearity.

Proof. We write

$$d^{\nabla}\eta = d^{\nabla}\left(df - Jdf \circ j_M)\right) = d^{\nabla}\left(df + *Jdf\right) = d^{\nabla}df + Jd^{\nabla} * df$$

so that we can conclude according to proposition 5.4 and proposition 5.1. This completes the proof. □

We can give a caracterisation which looks like very closely to the one which holds for harmonic maps ([9]):

Theorem 5.3 *A map $f\colon (L, j_M) \to (N, J)$ from a Riemann surface into an almost complex manifold, is holomorphically harmonic if and only if*

$$\overline{\hat{\partial}}^{\widehat{\nabla}}\hat{\partial}f = 0, \tag{92}$$

i.e. $\hat{\partial}f$ is a holomorphic section of $T^*_{1,0}L \otimes_{\mathbb{C}} f^*TN$.

Proof. Apply remark 5.4 to $\beta = df$ and then use proposition 5.5 to prove that $\hat{\partial}f$ is a holomorphic section and proposition 5.3-(iv) to prove the equation (92). □

5.2 The sigma model with a Wess-Zumino term

Here we present an interpretation of the holomorphic harmonicity in terms of the sigma model with a Wess-Zumino term.

5.2.1 The general case of an almost Hermitian manifold

Proposition 5.6 *Let (N, J, ∇) be an almost complex manifold with an almost connection ∇. Then J* anticommutes with the torsion *T of ∇ i.e.*

$$T(X, JY) = -JT(X, Y)$$

if and only if

$$T = \frac{1}{4}N_J \quad \textit{and} \quad T(JX, Y) = T(X, JY),$$

where N_J denotes the torsion of J i.e its Nijenhuis tensor.

From now until the end of this section 5.2, we consider (N, J) an almost complex manifold with an almost complex affine connection ∇ and a ∇-parallel Hermitian metric h. Therefore (N, J, h) is an almost Hermitian manifold with a Hermitian connection ∇.

Proposition 5.7 *Let (N, J, h) be an almost Hermitian manifold with a Hermitian connection ∇. Let us suppose that J anticommutes with the torsion T of ∇. Let us suppose also that the torsion of ∇ is totally skew-symmetric i.e.*

$$T^*(X, Y, Z) = \langle T(X, Y), Z\rangle$$

is a 3-form. Lastly, we suppose that the torsion is ∇-parallel, i.e. $\nabla T^* = 0$ which is equivalent to $\nabla T = 0$. Then the 3-form

$$H(X,Y,Z) = T^*(X,Y,JZ) = \langle T(X,Y), JZ\rangle$$

is closed $dH = 0$.

Theorem 5.4 *Let (N,J,h) be an almost Hermitian manifold with a Hermitian connection ∇. Then, under the hyphothesis of the previous proposition, the equation for holomorphically harmonic maps $f\colon L \to N$ is the equation of motion (i.e. Euler-Lagrange equation) for the sigma model in N with the Wess-Zumino term defined by the closed 3-form H. The action functional is given by*

$$S(f) = E(f) + S^{WZ}(f) = \frac{1}{2}\int_L |df|^2 d\mathrm{vol}_g + \int_B H,$$

where B is 3-submanifold (or indeed a 3-chain) in N whose boundary is $f(L)$.

5.2.2 The example of a 3-symmetric space

Let us suppose now that $N = G/G_0$ is a (locally) 3-symmetric space. We use the notations of subsection 2.1.2. In particular, N is endowed with its canonical almost complex structure $\underline{J}$ defined by (26).

Proposition 5.8 *The canonical connection ∇^0 in N commutes with the canonical almost complex structure $\underline{J}$*

$$\nabla^0 \underline{J} = 0.$$

Moreover, $\underline{J}$ anticommutes with the torsion T^0 of ∇^0. Lastly, if N is Riemannian, then ∇^0 is metric and (N,J,h) is almost Hermitian for any G-invariant metric h.[43]
Furthermore, the torsion of ∇^0 is totally skew-symmetric if and only if h is naturally reductive.

Now, we can conclude

Theorem 5.5 *Let us suppose that the (locally) 3-symmetric space $N = G/G_0$ is Riemannian and naturally reductive. Let h be a G-invariant naturally reductive metric on N. Then the equation for holomorphically harmonic maps $f\colon L \to N$ is the equation of motion (i.e. Euler-Lagrange equation) for the sigma model in N with the Wess-Zumino term defined by the closed 3-form H, corresponding to the canonical almost complex structure $\underline{J}$ and the canonical connection ∇^0.*

[43] chosen according to our convention explained in subsection 2.1.2: that is $\tau_{\mathfrak{m}}$ leaves invariant the inner product defining h.

5.3 Affine harmonic maps into reductive homogeneous spaces.

Let $N = G/K$ be a reductive homogeneous space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ a reductive decomposition of the Lie algebra $\mathfrak{g}$. We use the notations of section 1 (applied to $N = G/K$ instead of $M = G/H$).

Theorem 5.6 *Let (L, j) be a Riemann surface and $f\colon (L, j) \to N$ be a smooth map, let $F\colon L \to G$ be a (local) lift of u and $\alpha = F^{-1}.dF$. Then the following statements are equivalent:*

(i) *f is ∇^t-harmonic for one $t \in [0, 1]$.*

(ii) *f is ∇^t-harmonic for every $t \in [0, 1]$.*

(iii) $d * \alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{m}}] = 0$.

(iv) $\operatorname{Im}\left(\bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] + t[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}}\right) = 0$, $\forall t \in [0, 1]$.

In fact, the tension field $\tau^t(f)$ of f with respect to ∇^t is independent of $t \in [0, 1]$.

Theorem 5.7 *In the same situation as above, the following statements are equivalent:*

(i) *f is strongly ∇^t-harmonic for one $t \in [0, 1] \setminus \{\frac{1}{2}\}$.*

(ii) *f is strongly ∇^t-harmonic for every $t \in [0, 1] \setminus \{\frac{1}{2}\}$.*

(iii) $\bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] + t[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0$, $\forall t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$.

(iv) *f is ∇^t-harmonic for one $t \in [0, 1]$ and $[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0$.*

(iv) $d\alpha_\lambda + \dfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$, $\forall \lambda \in S^1$, *with* $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$.

Furthermore f is strongly $\nabla^{\frac{1}{2}}$-harmonic if and only if it is $\nabla^{\frac{1}{2}}$-harmonic: indeed $\nabla^{\frac{1}{2}}$ is torsion free.

Proof of theorem 5.6 The tension field $\tau^t(f)$ of f with respect to ∇^t is given by

$$\begin{aligned}\tau^t(f) = *d^{\nabla^t} * df &= *\mathrm{Ad}F\left(d * \alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{m}}] + t[\alpha_{\mathfrak{m}} \wedge *\alpha_{\mathfrak{m}}]_{\mathfrak{m}}\right)\\ &= *\mathrm{Ad}F\left(d * \alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{m}}]\right)\end{aligned}$$

(see section 1.4 (especially equation (9)) and section 1.6). This proves the equivalence between (i), (ii) and (iii). Then we conclude by remarking that $2\operatorname{Im}\left(\bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]\right) = d*\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{m}}]$ and that $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = \dfrac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}}$ is real. This completes the proof. □

Proof of theorem 5.7 We have for all $\in [0, 1]$

$$\bar\partial^{\nabla^t}\partial f = \mathrm{Ad}F\left(\bar\partial\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] + t[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}}\right) \tag{93}$$

so that the ∇^t-strongly harmonicity of f is written:

$$\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] + t[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} = 0. \qquad (S^t_{\mathfrak{m}})$$

Then the imaginary part of $\bar{\partial}^{\nabla^t}\partial f = 0$ gives us the ∇^t-harmonicity whereas the real part gives us

$$d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{m}}] + t[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \qquad (\mathrm{Re}(t))$$

which is nothing but the lift of the torsion free equation: $f^*T^t = 0$, where $T^t = T^{\nabla^t}$. Moreover the projection on $\mathfrak{m}$ of the Maurer-Cartan equation (on α) gives us the structure equation

$$d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{m}}] + \frac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \qquad [\mathrm{MC}]_{\mathfrak{m}}$$

which is nothing but $(\mathrm{Re}(\frac{1}{2}))$ (so that we recover that $T^{\frac{1}{2}} = 0$) but (since it can be written $((\mathrm{Re}(0)) + \frac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0)$ it is also the lift of (the f-pullback of) the equation expressing the canonical torsion T^0 in term of the Lie bracket (see theorem 1.4 or equation (14)) :

$$T^0 + [\ ,\ ]_{[\mathfrak{m}]} = 0. \qquad (94)$$

which combining with the fact that the left hand side of $(\mathrm{Re}(t))$ is the lift of f^*T^t, gives us back $T^t = (2t-1)[\ ,\ ]_{[\mathfrak{m}]}$ (see (14)).

Hence according to (93) and $[\mathrm{MC}]_{\mathfrak{m}}$ the strongly harmonicity for one $t \neq \frac{1}{2}$ is equivalent to the harmonicity (imaginary part) and $[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0$ (real part $(\mathrm{Re}(t))$ combining with $[\mathrm{MC}]_{\mathfrak{m}}$). We can also simply say that f is strongly harmonic if and only if f is harmonic and torsion free i.e. $[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0$ according to (14). This proves the equivalence between (i), (ii), and (iii). Now, let us decompose the curvature of α_λ, with respect to powers of λ:

$$\begin{aligned} d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] &= \lambda^{-1}\left(d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]\right) \\ &+ \left(d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{k}}\right) + \frac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad \lambda\left(d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]\right) \end{aligned}$$

hence using the fact that α_λ is real (i.e. $\mathfrak{g}$-valued)

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \Leftrightarrow \begin{cases} d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = 0 \quad (S^0_{\mathfrak{m}}) \\ [\mathrm{MC}]_{\mathfrak{k}} \\ [\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \end{cases} \Leftrightarrow .$$

Thus the zero curvature equation on α_λ is equivalent to the strongly ∇^0-harmonicity, i.e. the strongly ∇^t-harmonicity for all $t \in [0,1] \setminus \{\frac{1}{2}\}$. Finally the last assertion is obvious. This completes the proof. □

We are led naturally to the following definitions.

Definition 5.3 *We will say that $u\colon L \to G/K$ is torsion free if $u^*T^t = 0$ for $t \in [0,1] \setminus \{\frac{1}{2}\}$ (this equation does not depend on t).*

Definition 5.4 *In the situation described by theorem 5.7-(iv), we will say that the $\mathfrak{g}$-valued 1-form on L, α, is solution of the the first elliptic system associated to the reductive homogeneous space G/K, and that the corresponding geometric map f is a geometric solution of this system.*

Affine harmonic maps into symmetric spaces

Now, if we suppose in particular that N is (locally) symmetric, i.e. $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$, then all the connections ∇^t, $0 \leq t \leq 1$, coincide. Moreover, if N is also Riemannian then these are equal to the Levi-Civita connection. Therefore we obtain:

Corollary 5.1 *The first elliptic integrable system associated to a (locally) symmetric space $N = G/K$ is the equation for ∇^0-harmonic maps $f\colon L \to N$. If N is Riemannian this means that it is the equation for harmonic maps $f\colon L \to N$ (with respect to Levi-Civita in N).*

5.4 Affine (holomorphically) harmonic maps into 3-symmetric spaces

Let us suppose now that $N = G/G_0$ is a (locally) 3-symmetric space. We use the notations of section 2. N is endowed with its canonical almost complex structure $\underline{J}$ defined by (26). We continue here the study begun in 2.3.2 concerning the lowest order determined odd system.

Theorem 5.8 *Let (L,j) be a Riemann surface and $f\colon L \to N$ a smooth map. Let $F\colon L \to G$ be a (local) lift of f and $\alpha = F^{-1}.dF$. Then the following statement are equivalent*

(i) $\bar\partial\alpha'_{-1} + [\alpha''_0 \wedge \alpha'_{-1}] + [\alpha''_1 \wedge \alpha'_1] = 0 \qquad (S_1)$

(ii) $\bar\partial\alpha'_1 + [\alpha''_0 \wedge \alpha'_1] = 0 \qquad (S_2)$

(iii) *f is holomorphically ∇^1-harmonic:* $\left[(\nabla^1)''\partial f\right]^{(1,0)} = 0.$

(iv) *f is anti-holomorphically ∇^0-harmonic:* $\left[(\nabla^0)''\partial f\right]^{(0,1)} = 0.$

(v) *f is a geometric solution of the second elliptic integrable system associated to the (locally) 3-symmetric space G/G_0:*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in S^1,$$

where $\alpha_\lambda = \lambda^{-2}\alpha'_1 + \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1 + \lambda^2\alpha''_{-1}$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) have been proved in 2.3.2. To prove (i) $\Leftrightarrow$ (iii): just take the $(1,0)$-component in $TN^{\mathbb{C}}$ of (93) for $t=1$. Idem for (ii) $\Leftrightarrow$ (iv). This completes the proof. $\square$

Now, additionning theorems 5.8, 5.7, 5.1 and proposition 5.1, we obtain

Corollary 5.2 *The following statements are equivalent*

(i) *f is strongly ∇^t-harmonic for one $t \in [0,1] \setminus \{\frac{1}{2}\}$.*

(ii) *f is ∇^t-harmonic for one $t \in [0,1]$ and torsion free.*

(iii) *f is holomorphically ∇^t-harmonic for one $t \in [0,1]$ and torsion free.*

(iv) *f is a geometric solution of the first elliptic system associated to the reductive homogeneous space G/G_0.*

(v) *f is a geometric solution of the second elliptic system associated to the 3-symmetric space G/G_0, and moreover $\mathrm{Re}\,([\alpha_1 \wedge \alpha_1]) = 0$.*

(vi) *f is in the same time a geometric solution of the determined odd elliptic systems $(\mathrm{Syst}(2,\tau))$ and $(\mathrm{Syst}(2,\tau^{-1}))$.*

Now, let us apply theorem 5.5 to the equivalence (iv) $\Leftrightarrow$ (v) of theorem 5.8.

Theorem 5.9 *The second elliptic integrable system associated to the 3-symmetric space $N = G/G_0$ is the equation of motion for the sigma model in N with the Wess-Zumino term defined by the closed 3-form H, corresponding to the anti-canonical almost complex structure $-\underline{J}$ and the canonical connection ∇^0.*

5.5 Affine vertically harmonic maps: general properties

Here we generalise the definition of vertical harmonicity for maps from a Riemannian surface into an affine manifold.

Definition 5.5 *Let (N, ∇^N) be an affine manifold. Let us suppose that we have a splitting $TN = \mathcal{V} \oplus \mathcal{H}$. In other words N is endowed with a Pfaff system (the vertical subbundle $\mathcal{V}$) and with a connection on this Pfaff system. Let $s\colon (L,b) \to N$ be a smooth map from a Riemannian manifold (L,b) into N. Then we set*

$$\tau^v(s) = \mathrm{Tr}_b(\nabla^v d^v s) = *d^{\nabla^v} * d^v s,$$

where ∇ is the connection on $T^*L \otimes s^*TN$ induced by the Levi-Civita connection of L and ∇^N; and ∇^v is its vertical component. We will say that s is affine vertically harmonic with respect to ∇^N or ∇^N-vertically harmonic if $\tau^v(s) = 0$.

Theorem 5.10 *Let (L,j) be a Riemann surface and $s\colon (L,j) \to (N,\nabla^N)$ a smooth map. Then we have*

$$2\,(\nabla'')^v\,\partial^v u = d^{\nabla^v} d^v u + i d^{\nabla^v} * d^v u,$$

moreover $d^{\nabla^v} d^v u = u^* T^v$, *where* T^v *is the vertical torsion (see 4.1.3) and* $d^{\nabla^v} * d^v u = \tau^v(u)\mathrm{vol}_b$ *for any hermitian metric* b *in* L. *Therefore the following statements are equivalent:*

(i) $(\nabla'')^v \partial^v u = 0$

(ii) $\nabla^v_{\frac{\partial}{\partial \bar{z}}} \left(\frac{\partial^v u}{\partial z}\right) = 0$, *for any holomorphic local coordinate* $z = x + iy$ *(i.e.* (x, y) *are conformal coordinates for any hermitian metric in* L*).*

(iii) u *is* ∇^v*-vertically harmonic with respect to any hermitian metric in* L *and vertically torsion free:* $u^* T^v = 0$ *(i.e.* $T^v(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$ *for any* x, y *conformal coordinates).*

We will say in this case that u *is strongly* ∇^N*-vertically harmonic.*

5.6 Affine vertically harmonic maps into reductive homogeneous space

Let G be a Lie group, and $K \subset H \subset G$ subgroups of G such that $M = G/H$ and H/K are reductive. We use the notations of 4.3.1 (but we do not suppose a priori that the reductive homogeneous spaces are Riemannian).

Theorem 5.11 *Let* (L, j) *be a Riemannian surface and* $f : L \to N = G/K$ *be a smooth map,* $F : L \to G$ *a (local) lift of* F *and* $\alpha = F^{-1}.dF$. *Then the following statement are equivalent:*

(i) f *is* ∇^t*-vertically harmonic for one* $t \in [0, 1]$.

(ii) f *is* ∇^t*-vertically harmonic for all* $t \in [0, 1]$.

(iii) $d * \alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge * \alpha_{\mathfrak{p}}] = 0$.

(iv) $\mathrm{Im}\left(\bar{\partial}\alpha'_{\mathfrak{p}} + [\alpha''_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] + t[\alpha''_{\mathfrak{p}} \wedge \alpha'_{\mathfrak{p}}]_{\mathfrak{p}}\right) = 0$, $\forall t \in [0, 1]$.

The ∇^t*-vertical tension field* $\tau^{v,t}(u)$, *with respect of* ∇^t, *is independent of* $t \in [0, 1]$.

Proof. We have

$$\begin{aligned} \tau^v_t(f) = *d^{\nabla^t} * d^v f &= *\mathrm{Ad}F(d * \alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge * \alpha_{\mathfrak{p}}] + t[\alpha_{\mathfrak{n}} \wedge * \alpha_{\mathfrak{p}}]_{\mathfrak{p}}) \\ &= *\mathrm{Ad}F(d * \alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge * \alpha_{\mathfrak{p}}] + t[\alpha_{\mathfrak{p}} \wedge * \alpha_{\mathfrak{p}}]_{\mathfrak{p}}) \\ &\quad \text{since } [\mathfrak{m}, \mathfrak{p}] \subset [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m} \\ &= *\mathrm{Ad}F(d * \alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge * \alpha_{\mathfrak{p}}]) \end{aligned}$$

Moreover let us compute the complex second derivative:

$$(\nabla^t)^{v\,\prime\prime} \partial^v f = *\mathrm{Ad}F(\bar{\partial}\alpha'_{\mathfrak{p}} + [\alpha''_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] + [\alpha''_{\mathfrak{p}} \wedge \alpha'_{\mathfrak{p}}]_{\mathfrak{p}}).$$

Then remark that $[\alpha''_{\mathfrak{p}} \wedge \alpha'_{\mathfrak{p}}]_{\mathfrak{p}} = \frac{1}{2}[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}]_{\mathfrak{p}}$ is in the real subspace $\mathfrak{p}$. This completes the proof. □

Now, let $f\colon L \to N$ be an arbitrary map from a Riemann surface into N. Then the (f-pullback of the) vertical torsion with respect ot ∇^t is

$$\begin{aligned} f^*T^{t,v} &= \mathrm{Ad}F\left(d\alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{p}}] + t[\alpha \wedge \alpha_{\mathfrak{p}}]_{\mathfrak{p}}\right) \\ &= f^*\left(T^{0,v} + t[\phi \wedge \phi]_{[\mathfrak{p}]}\right) \end{aligned}$$

where $\phi\colon TN \to [\mathfrak{p}]$ is the projection on the vertical subbundle along the horizontal subbundle $[\mathfrak{m}]$. Therefore

$$T^{t,v} = T^{0,v} + t[\phi \wedge \phi]_{[\mathfrak{p}]}.$$

Moreover, recall that, according to section 4.3.1, the projection on $[\mathfrak{p}]$ of the Maurer-Cartan equation gives us the homogeneous structure equation (see equations (65), (64) and footnote 23)

$$T^{0,v} = \Phi - \frac{1}{2}[\phi \wedge \phi]_{[\mathfrak{p}]}$$

where $\Phi = -\frac{1}{2}[\psi \wedge \psi]_{[\mathfrak{p}]}$ is the homogeneous curvature form and $\psi\colon TN \to [\mathfrak{m}]$ is the projection on $[\mathfrak{m}]$ along $[\mathfrak{p}]$. Then we have

$$T^{t,v} = \Phi + \left(t - \frac{1}{2}\right)[\phi \wedge \phi]_{[\mathfrak{p}]}.$$

Therefore

Theorem 5.12 *Let us consider the same situation as in theorem 5.11.*

• *If f is flat then strongly ∇^t-vertical harmonicity and torsion freedom, for f, do not depend on t, if $t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$.*
Moreover $T^{\frac{1}{2},v} = \Phi$ so that (if f is flat) strongly vertical harmonicity and vertical harmonicity with respect to $\nabla^{\frac{1}{2}}$ are equivalent.

• *If H/K is locally symmetric, i.e. $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, then $\forall t \in [0,1]$, $T^{t,v} = \Phi$.*
In particular, the ∇^t-vertical torsion does not depend on $t \in [0,1]$, and thus neither strongly harmonicity does.

Corollary 5.3 *Let us suppose now that $N = G/K$ is a (locally) $2k$-symmetric space and that $M = G/H$ is the corresponding (locally) k-symmetric space. Then the even minimal determined system $(\mathrm{Syst}(k,\tau))$ associated to N means that the geometric map $f\colon L \to N$ is horizontally holomorphic and vertically harmonic with respect to any affine connection ∇^t, $0 \le t \le 1$. Moreover the horizontal holomorphicity implies the flatness of f and thus its vertical torsion freedom (with respect to any connection ∇^t, $0 \le t \le 1$). More precisely the (last) equation (S_k) of the system means*

$$(\nabla^t)^{v''}\partial^v f = 0$$

i.e. that f is strongly ∇^t-vertically harmonic, so that its real part means that f is vertically torsion free and its imaginary part that f is vertically harmonic.

The Riemannian case

Now, let us suppose that M is Riemannian, and then so is N. In other words, we are in the situation described by 4.3.1. Let us consider the metric connections in N:

$$\overset{\text{met}}{\nabla}{}^t = \nabla^0 + t\mathrm{B}^N, \qquad 0 \leq t \leq 1$$

with $\mathrm{B}^N = [\ ,\ ]_{[\mathfrak{n}]} + \mathrm{U}^N$ and U^N defined by equation (15).
For any $\mathrm{Ad}K$-invariant subspace $\mathfrak{l} \subset \mathfrak{n}$, we will denote by $\mathrm{U}^{\mathfrak{l}}\colon \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ the bilinear symmetric map defined by

$$\langle \mathrm{U}^{\mathfrak{l}}(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{l}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{l}}\rangle \quad \forall X, Y \in \mathfrak{l},$$

and by $\mathrm{U}^{[\mathfrak{l}]}$ its extension to the subbundle $[\mathfrak{l}] \subset TN$. Then we have in particular

$$\mathrm{U}^N = \mathrm{U}^{[\mathfrak{n}]} \quad \text{and} \quad \mathrm{U} = \mathrm{U}^{[\mathfrak{p}]}$$

where, let us recall it, U is defined by (61).

Now, let us project the definition equation of $\overset{\text{met}}{\nabla}{}^t$ in the vertical subbundle: we obtain $\forall V \in \mathcal{C}(TN)$,

$$\phi(\overset{\text{met}}{\nabla}{}^t V) = \nabla^0 \phi V + t\phi \circ \mathrm{B}^N.$$

Moreover, according to 4.3.1, we have $\phi \circ \mathrm{B}^N = \phi^* \mathrm{B} - \Phi$ so that

$$\phi(\overset{\text{met}}{\nabla}{}^t V) = \nabla^0 \phi V + t(\phi^* \mathrm{B} - \Phi)$$

and in particular $\forall V \in \mathcal{C}(\mathcal{V})$, $\forall A \in TN$,

$$\overset{\text{met}}{\nabla}{}^t_A V = \nabla^0_A V + t\left([\phi A, V]_{[\mathfrak{p}]} + \mathrm{U}^{[\mathfrak{p}]}(\phi A, V)\right).$$

Then according to theorem 4.5 and remark 4.4 it follows:

Theorem 5.13 • *If H/K is naturally reductive, then the connections defined by the restriction to $\mathcal{V}$ of $\overset{\text{met}}{\nabla}{}^{t,v}$, $0 \leq t \leq 1$, are all ϕ-equivalent. Therefore vertical harmonicity, with respect to $\overset{\text{met}}{\nabla}{}^t$, is the same for all $0 \leq t \leq 1$.*

• *If H/K is locally symmetric, then all the $\overset{\text{met}}{\nabla}{}^{t,v}$, $0 \leq t \leq 1$, coincide in $\mathcal{V}$. In particular strongly harmonicity is the same for all the connections $\overset{\text{met}}{\nabla}{}^t$, $0 \leq t \leq 1$.*

The vertical torsion of $\overset{\text{met}}{\nabla}{}^t$. We have seen in 1.6 that the torsion of $\overset{\text{met}}{\nabla}{}^t$ is the same as that of ∇^t. Now let us see what happens for the vertical torsion. The vertical torsion with respect to $\overset{\text{met}}{\nabla}{}^{t,v}$,

$$\overset{\text{met}}{T}{}^{t,v} = d^{\overset{\text{met}}{\nabla}{}^{t,v}}\phi$$

lifts into

$$d\theta_{\mathfrak{p}} + [\theta_{\mathfrak{k}} \wedge \theta_{\mathfrak{p}}] + t[\theta_{\mathfrak{n}} \wedge \theta_{\mathfrak{p}}]_{\mathfrak{p}} + t\left[\mathrm{U}^{\mathfrak{n}}(\theta_{\mathfrak{n}} \wedge \theta_{\mathfrak{p}})\right]_{\mathfrak{p}},$$

where as usually θ denotes the Maurer-Cartan form in G. We will prove that the last term in the right hand side vanishes. Indeed we have $\forall X, Y, Z \in \mathfrak{g}$,

$$\begin{array}{rcccc} \langle \mathrm{U}^{\mathfrak{n}}(X_{\mathfrak{n}}, Y_{\mathfrak{p}}), Z_{\mathfrak{p}} \rangle & = & \langle [Z_{\mathfrak{p}}, X_{\mathfrak{n}}]_{\mathfrak{n}}, Y_{\mathfrak{p}} \rangle & + & \langle X_{\mathfrak{n}}, [Z_{\mathfrak{p}}, Y_{\mathfrak{p}}]_{\mathfrak{n}} \rangle \\ & = & \langle [Z_{\mathfrak{p}}, X_{\mathfrak{p}}]_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle & + & \langle X_{\mathfrak{p}}, [Z_{\mathfrak{p}}, Y_{\mathfrak{p}}]_{\mathfrak{p}} \rangle \\ & & \text{since } [\mathfrak{p}, \mathfrak{m}] \subset [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, & & \text{since } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} \\ & = & \langle \mathrm{U}^{\mathfrak{p}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}), Z_{\mathfrak{p}} \rangle. & & \end{array} \tag{95}$$

Then $\left[\mathrm{U}^{\mathfrak{n}}(\theta_{\mathfrak{n}} \wedge \theta_{\mathfrak{p}})\right]_{\mathfrak{p}} = \mathrm{U}^{\mathfrak{p}}(\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}) = 0$, because $\mathrm{U}^{\mathfrak{p}}$ is symmetric. Therefore

$$\overset{\text{met}}{T}{}^{t,v} = T^{t,v} \tag{96}$$

Corollary 5.4 *In corollary 5.3, the affine connection ∇^t can be replaced by the metric connection $\overset{\text{met}}{\nabla}{}^t$.*

Remark 5.5 In particular for $t = \frac{1}{2}$, we recover theorem 4.12. Moreover, we see that in general the value $t = \frac{1}{2}$, i.e. the Levi-Civita connections, plays a special role according to theorem 5.12 and equation (96). Indeed for the Levi-Civita connection, we always have $\overset{\text{met}}{T}{}^{\frac{1}{2},v} = \Phi$, so that if f is flat, the strongly harmonicity and the vertical harmonicity are equivalent.

6 Appendix

6.1 Vertical harmonicity

Theorem 6.1 *Let us consider the situation described by example 4.1 and suppose that $\pi\colon N \to M$ is a Riemannian submersion and $u\colon L \to M$ is an isometry . Then $f\colon L \to N$ is vertically harmonic if and only if the corresponding section $\tilde{f}\colon L \to u^*N$ is a harmonic section. Furthermore $f\colon L \to N$ is harmonic if and only if $\tilde{f}\colon L \to u^*N$ is harmonic and $[\tau(f)]^{\mathcal{H}}_{u_*(TL)^\perp} = 0$ i.e. the component of the tension field in the subspace of $\mathcal{H}$ corresponding by the isometry $d\pi_{|\mathcal{H}}$ to the normal bundle $u_*(TL)^\perp$ in TM, vanishes, or equivalently $[d\pi(\tau(f))]_{u_*(TL)^\perp} = 0$.*

Proof. The Levi-Civita in u^*N is the orthogonal projection ofthe Levi-Civita connection in $L \times N$, on the tangent bundle $T(u^*N)$. Let us determine this orthonormal projection. First let us express clearly what is the tangent subbundle $T(u^*N)$ in $T(L \times N)$.

$$T_{(l,n)}(u^*N) = \{(\xi,\eta) \in T_{(l,n)}L \times N | du(\xi) = d\pi(\eta)\}.$$

Let us do some identifications. First an usual one: consider that TL is a subbundle of $TM_{|L}$ (and forget the "u_*" in $u_*(TL)$), secondly: we consider that $\pi^*TM = \mathcal{H}$, identifying these by the isometry $d\pi_{|\mathcal{H}}$, so that we will write $\mathcal{H}_{|\pi^{-1}(L)} = \pi^*TL \oplus \pi^*TL^\perp$, where $TL^\perp$ is the normal bundle of L in M. Moreover, for any $\eta \in TN_{|\pi^{-1}(L)}$ let us write its decomposition following $TN_{\pi^{-1}(L)} = \pi^*TL \oplus \pi^*TL^\perp \oplus \mathcal{V}_{\pi^{-1}(L)}$ as

$$\eta = \eta^{\mathcal{H}}_{TL} + \eta^{\mathcal{H}}_{TL^\perp} + \eta^{\mathcal{V}}.$$

Then under the previous identifications, we have

$$\begin{aligned} T_{(l,n)}(u^*N) &= \{(\xi,\eta) \in T_lL \times T_nN | \eta^{\mathcal{H}}_{TL} = \xi, \eta^{\mathcal{H}}_{TL^\perp} = 0\} \\ &= \{(\xi, \xi + \eta^{\mathcal{V}}), \xi \in T_lL, \eta^{\mathcal{V}} \in \mathcal{V}_n\}. \end{aligned}$$

This gives us a splitting $T(u^*N) = \mathcal{V}^{u^*N} \oplus \mathcal{H}^{u^*N}$ where $\forall (l,n) \in u^*N$,

$$\mathcal{V}^{u^*N}_{(l,n)} = \{0\} \times \mathcal{V}_n \quad \text{and} \quad \mathcal{H}^{u^*N}_{(l,n)} = T_lL \times \mathcal{H}_n \cap T_{(l,n)}(u^*N) = \{(\xi,\xi) \in T_lL \times T_lL\}.$$

Let us determine the orthogonal of the tangent space $T(u^*N)$:

$$\begin{aligned} &(\alpha,\beta) \in \big(T_{(l,n)}(u^*N)\big)^\perp \iff \\ &\quad \forall (\xi,\eta) \in T_{(l,n)}(u^*N), \quad 0 &&= \langle (\xi,\eta), (\alpha,\beta)\rangle \\ &&&= \langle \xi,\alpha\rangle + \langle \eta,\beta\rangle \\ &&&= \langle \xi,\alpha\rangle + \langle \xi,\beta^{\mathcal{H}}_{TL}\rangle + \langle 0,\beta^{\mathcal{H}}_{TL^\perp}\rangle + \langle \eta^{\mathcal{V}},\beta^{\mathcal{V}}\rangle \\ &&&= \langle \xi,\alpha + \beta^{\mathcal{H}}_{TL}\rangle + \langle \eta^{\mathcal{V}},\beta^{\mathcal{V}}\rangle \\ &&&\qquad \iff (\alpha + \beta^{\mathcal{H}}_{TL}, \beta^{\mathcal{V}}) = 0. \end{aligned}$$

Therefore

$$\big(T_{(l,n)}(u^*N)\big)^\perp = \{(-\beta^{\mathcal{H}}_{TL}, \beta), \beta \in \mathcal{H}_n\}.$$

Decomposing each $(a,b) \in T(L\times N)_{|u^*N}$ following the decomposition $T(L\times N)_{|u^*N} = T(u^*N)\oplus T(u^*N)^\perp$: $(a,b) = (\xi,\eta)+(\alpha,\beta)$, then we obtain

$$\left\{\begin{array}{l} a = \eta^{\mathcal{H}}_{TL} - \beta^{\mathcal{H}}_{TL} \\ b = (\eta^{\mathcal{H}}_{TL} + \beta^{\mathcal{H}}_{TL}) + \beta^{\mathcal{H}}_{TL} + \eta^{\mathcal{V}} \end{array}\right.$$

so that this decomposition is therefore given by

$$(a,b) = \left(\frac{a+b^{\mathcal{H}}_{TL}}{2}, a + b^{\mathcal{H}}_{TL} + b^{\mathcal{V}}\right) + \left(-\frac{(b^{\mathcal{H}}_{TL}-a)}{2}, \frac{(b^{\mathcal{H}}_{TL}-a)}{2} + b^{\mathcal{H}}_{TL^\perp}\right).$$

Now, let us come back to our fonction $f\colon L\to N$ and the corresponding section $\tilde{f}\colon (L,b)\to u^*N$. Then let us compute

$$\begin{aligned} \overset{u^*N}{\nabla^v} d^v\tilde{f} = \overset{u^*N}{\nabla^v}(dl,df)^{\mathcal{V}^{u^*N}} &= \overset{u^*N}{\nabla^v}(0,d^vf) = \left([\nabla(0,d^vf)]_{T(u^*N)}\right)^{\mathcal{V}^{u^*N}} \\ &= \left([(0,\nabla d^vf)]_{T(u^*N)}\right)^{\mathcal{V}^{u^*N}} \\ &= \left(\frac{1}{2}(\nabla d^vf)^{\mathcal{H}}_{TL}, \frac{1}{2}(\nabla d^vf)^{\mathcal{H}}_{TL} + \nabla^v d^v f\right)^{\mathcal{V}^{u^*N}} \\ &= (0,\nabla^v d^v f) \end{aligned}$$

Finally, we have proved

$$\overset{u^*N}{\nabla^v} d^v\tilde{f} = \nabla^v d^v f \tag{97}$$

and by taking the trace, we obtain the first assertion of the theorem.
Now, in the same way we obtain

$$\overset{u^*N}{\nabla} d\tilde{f} = \left(\frac{1}{2}(\nabla df)^{\mathcal{H}}_{TL}, \frac{1}{2}(\nabla df)^{\mathcal{H}}_{TL} + \nabla^v df\right) \tag{98}$$

so that $\tilde{f}\colon N\to u^*N$ is harmonic if and only if $[\tau(f)]^{\mathcal{H}}_{TL} = 0$ and $[\tau(f)]^{\mathcal{V}} = 0$. Therefore $f\colon L\to N$ is harmonic if and only if $\tilde{f}\colon N\to u^*N$ is harmonic and $[\tau(f)]^{\mathcal{H}}_{TL^\perp} = 0$. This completes the proof. □
From the proof of theorem 6.1 (more precisely from (97) and (98)), we obtain:

Theorem 6.2 *Let us consider the situation described by theorem 6.1. Then $f\colon L\to N$ is superflat if and only if the corresponding section $\tilde{f}\colon L\to u^*N$ is superflat. Furthermore $f\colon L\to N$ is totally geodesic if and only if $\tilde{f}\colon L\to u^*N$ is totally geodesic and $[\nabla df]^{\mathcal{H}}_{TL^\perp} = 0$ (i.e. $[d\pi(\nabla df)]_{TL^\perp} = 0$).*

Remark 6.1 The metric defined in example 4.1 in u^*N (and thus in theorems 6.1 and 6.2, i.e. the metric induced by the product metric, is given by

$$|(\xi,\eta)|^2 = 2|\xi|^2 + |\eta^{\mathcal{V}}|^2 \tag{99}$$

whereas, when $\pi\colon N \to M$ is a Homogeneous fibre bundle, the metric in u^*N, considered as an Homogeneous fibre bundle, is defined in 4.2 by equation (57) and is given by

$$|(\xi,\eta)|^2 = |\xi|^2 + |\eta^{\mathcal{V}}|^2. \tag{100}$$

However, theorems 6.1 and 6.2 hold, of course, also with the metric (100). Indeed, first remark that the theorems hold if we multiply the product metric in $L \times N$ by a constant factor. Then just apply these theorems with the same (M, g) (and thus the same (L, u^*g)), N endowed with the new metric $|\cdot|^2_{\mathcal{H}} + 2|\cdot|^2_{\mathcal{V}}$ (the old one being $|\cdot|^2_{\mathcal{H}} + |\cdot|^2_{\mathcal{V}}$) and endow $L \times N$ with $\frac{1}{2}$ times the product metric, then the induced metric on u^*N is (100): $\dfrac{1}{2}(|\xi|^2 + (|\xi|^2 + 2|\eta^{\mathcal{V}}|^2)) = |\xi|^2 + |\eta^{\mathcal{V}}|^2$.

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