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# A stochastic overlapping generation model with a continuum of agents

Emmanuelle Augeraud-Véron\*      Delphine David\*

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## Abstract

We consider a stochastic overlapping generations model for a continuum of individuals with finite lives in presence of a financial market. In this paper, agent's heterogeneity is given by the dates of birth of the households, on the contrary to standard models, in which each agent has his own aversion coefficient on his utility function. By means of the martingale arguments, we compute the agent's optimal consumption and portfolio. A characterization of interest rate trajectories is given by mixed-type functional differential equations and the stability of these trajectories is studied.

## 1 Introduction

We consider an OverLapping Generations (OLG) framework for a continuum of individuals in the presence of a financial market. In non-OLG models, as in [13] for example, agent's heterogeneity appears through the different relative risk aversion coefficients, the individuals commonly live on a finite time interval and the economy contains a finite number of agents. On the contrary, in OLG models, agent's heterogeneity appears through the demographic structure, the individuals are characterized by their birth date and a continuum of agents is considered. Furthermore, the economy lives on an infinite time horizon and the number of agents is unknown.

The main goal of this paper is the study of interest-rates and prices behaviors on the different markets. Our approach is a stochastic extension of the general equilibrium for deterministic OLG models developed in particular by Polemarchakis and Demichelis [19] and by d'Albis and Augeraud-Véron [6]. In these cases, the intertemporal equilibrium is the solution of a mixed type functional differential equation (MFDE). We

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show that the stochastic extension preserves this MFDE. Some results on mixed type and retarded functional differential equations are presented in Section 2.

In Section 3, we compute the optimal consumption/investment strategy of an household. This type of problem has been widely studied by authors in the last few decades. In two papers, Merton [17, 18] introduced a model with constant coefficients and solved the associated Hamilton-Jacobi-Bellman equation. Bismut [3] obtained formulas for optimal consumption using his stochastic duality theory (Bismut [2]). We also refer to Lehoczky, Sethi and Shreve [15], Karatzas, Lehoczky and Shreve [10] and Cox and Huang [5] for similar results.

In Section 4, we present our market equations and give conditions for existence of steady states. These conditions recover in particular the ones of d'Albis and Augeraud-Véron for the deterministic case. The multiple steady states we obtain are possible under conditions relying on endowment distribution similar to those proposed by Kim [14]. In standard stochastic equilibrium model, as in [11], [12] and [9], the endowment of agents is an exogeneous process and individuals wish to hedge the variability in their endowment processes by trading with one another. This justifies the introduction of a financial market. Applying this setting to our stochastic overlapping generations model leads to very complex equations. Consequently, deterministic wages are considered instead of stochastic endowment. For this reason, this model can't be seen as an equilibrium model.

Section 5 is dedicated to the local dynamics of the interest rates around the steady-states we have obtained. This stability analysis gives us the behaviors of the trajectories and uses the theory of MFDE, in particular the recent results of Mallet-Paret and Verduyn-Lunel [16].

## 2 Preliminaries

In this section, we present the theory of MFDE we use in Section 5. Results for advanced and retarded equations come from Mallet-Paret and Verduyn Lunel [16].

Let us consider the following linear functional differential equation of mixed type :

$$\dot{x}(t) = \int_{-1}^1 x(t + \theta) d\eta(\theta) \quad (2.1)$$

where  $d\eta(\theta)$  is a finite Lebesgue-Stieltjes measure on  $[-1,1]$ .

Replacing  $x(t) = e^{\lambda t}$ ,  $\lambda \in \mathbb{C}$  in (2.1), we can define the characteristic function as :

$$\Delta(\lambda) = \lambda - \int_{-1}^1 e^{\lambda\theta} d\eta(\theta). \quad (2.2)$$

In their paper, Mallet-Paret and Verduyn Lunel obtain a factorization of the characteristic function (2.2) :

$$\Psi(\lambda)\Delta(\lambda) = \Delta_-(\lambda)\Delta_+(\lambda) \quad (2.3)$$

where  $\Delta_-$  and  $\Delta_+$  are the characteristic functions of certain retarded and advanced equations, namely

$$\Delta_-(\lambda) = \lambda - \int_{-1}^0 e^{\lambda\theta} d\eta_-(\theta) \text{ and } \Delta_+(\lambda) = \lambda - \int_0^1 e^{\lambda\theta} d\eta_+(\theta)$$

and where  $\Psi$  is a polynomial of degree 1 which is needed as a correction factor to balance the growth rates on both sides of equation (2.3).

By means of this result, we can obtain a solution of (2.1) by solving the retarded functional differential equation (RFDE) :

$$\dot{x}(t) = \int_{-1}^0 x(t + \theta) d\eta_-(\theta). \quad (2.4)$$

The theory of RFDE gives the following result :

**Proposition 2.1** *If  $\Delta_-(\lambda)$  has no zeros in the right half plane  $\{\lambda | \operatorname{Re}(\lambda) \geq 0\}$ , all solutions of the RFDE (2.4) converge to zero exponentially as  $t \rightarrow +\infty$ .*

For more informations on RFDE, we refer the reader to Bellman and Cooke [1], Hale [7] and Hale and Verduyn Lunel [8].

### 3 The model

We consider a standard complete financial market with two assets. One of them is a non risky asset with price per unit  $B(t)$  governed by the equation :

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1,$$

where  $r(t)$  is the interest rate. The second asset is a risky one with price process  $(S(t), t \geq 0)$  defined by the following stochastic differential equation :

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(0) = x,$$

where  $W$  is a standard, one-dimensional Brownian motion on  $\mathbb{R}$ , defined on a probability space  $(\Omega, \mathcal{F}, P)$  equipped with the filtration  $(\mathcal{F}_t)$  generated by the Brownian motion  $W$  and augmented. In the all paper, we assume that  $r(\cdot)$ ,  $\mu(\cdot)$  and  $\sigma(\cdot)$  are deterministic functions. The economy is a pure exchange one with a single non-perishable good and we assume that the quantity produced in the economy is normalized to 1.

## Individual setting

We consider an agent born at time  $s \in \mathbb{R}^+$  and living for a time interval of a unit length. During his lifetime, he receives a wage  $w(s, t)$  defined by

$$w(s, t) = \begin{cases} w & \text{if } t \in [s + \alpha, s + \beta] \\ 0 & \text{otherwise} \end{cases}$$

with  $0 \leq \alpha < \beta \leq 1$  and  $w$  a constant. At each time  $t \in [s, s + 1]$ , he chooses the proportion of wealth  $\pi(s, t)$  he invests in the risky asset and his consumption  $c(s, t)$ .

**Definition 3.1** *The consumption process of an individual born at time  $s$  is an  $\{\mathcal{F}_t\}$ -progressively measurable, nonnegative process  $c(s, \cdot)$ , satisfying,*

$$\int_s^{s+1} c(s, t) dt < \infty, \quad \forall s \in \mathbb{R}^+,$$

and

$$\int_{t-1}^t c(s, t) ds < \infty, \quad \forall t \in \mathbb{R}^+,$$

almost surely.

Denoting by  $X(s, t)$  his wealth at time  $t$ , his budget constraint reads :

$$\begin{aligned} dX(s, t) &= [(r(t) + \pi(s, t)(\mu(t) - r(t))X(s, t) - c(s, t) + w(s, t)]dt \\ &\quad + \sigma(t)\pi(s, t)X(s, t)dW(t), \\ X(s, s) &= 0. \end{aligned} \tag{3.1}$$

## Utility functions

Denote  $U(s)$  the intertemporal utility of an individual born at time  $s \geq 0$  and given by :

$$U(s) = \mathbb{E} \left[ \int_s^{s+1} e^{-\rho(t-s)} u(c(s, t)) dt \right] \tag{3.2}$$

where  $u(\cdot)$  is a utility function and  $\rho \geq 0$  is the discount rate.

**Definition 3.2** *A utility function is a concave, nondecreasing, upper semicontinuous function  $u : \mathbb{R}_+^* \rightarrow \mathbb{R}$  satisfying :*

- the half-line  $\text{dom}(u) \triangleq \{x \in \mathbb{R}_+^* ; u(x) > -\infty\}$  is a nonempty subset of  $[0, \infty)$ ;
- $u'$  is continuous, positive, and strictly decreasing on the interior of  $\text{dom}(u)$ , and

$$u'(\infty) \triangleq \lim_{x \rightarrow \infty} u'(x) = 0.$$

- the Inada condition,

$$u'(0) \triangleq \lim_{x \rightarrow 0} u'(x) = \infty.$$

The strictly decreasing, continuous function  $u' : (0, \infty) \xrightarrow{\text{onto}} (0, u'(0))$  has a strictly decreasing, continuous inverse  $I : (0, u'(0)) \xrightarrow{\text{onto}} (0, \infty)$ . We set  $I(y) = 0$  for  $U'(0) \leq y \leq \infty$ .

**Definition 3.3** Let  $u$  be a utility function. The convex dual of  $u$  defined on  $\mathbb{R}_+^*$  is the function

$$\tilde{u}(y) \triangleq \sup_{x \geq 0} \{u(x) - xy\}.$$

The convex dual  $\tilde{u}$  is in fact the Legendre-Fenchel transform of  $u$  except for the minus sign. It satisfies the following property which is used for the computation of the optimal consumption.

**Property 3.4** The function  $\tilde{u}$  satisfies

$$\tilde{u}(y) = u(I(y)) - yI(y), \quad \text{if } y > 0. \quad (3.3)$$

*Proof.* The proof of this property can be found in [13], Chapter 3, Lemma 4.3.  $\square$

## The optimization problem

The individual program is to maximize (3.2) subject to (3.1). In the sequel, we will consider the particular case of  $u(c(s, t)) = \frac{c(s, t)^{1-\frac{1}{\gamma}}}{1 - \frac{1}{\gamma}}$  where  $0 < \gamma < 1$  stands for the elasticity of intertemporal substitution. Using the martingale approach, it is well-known that we can transform our constraint into the following one :

$$\mathbb{E} \left[ \int_s^{s+1} \frac{H(u)}{H(s)} c(s, u) du \right] = \mathbb{E} \left[ \int_s^{s+1} \frac{H(u)}{H(s)} w(s, u) du \right]$$

where  $H(t) = \exp \left\{ - \int_0^t r(\tau) + \frac{1}{2} \theta(\tau)^2 d\tau - \int_0^t \theta(\tau) dW(\tau) \right\}$  and the Sharpe ratio is  $\theta(t) = (\mu(t) - r(t))/\sigma(t)$ , see Karatzas and Shreve [13], Chapter 3, for more details.

**Proposition 3.5** The optimal consumption and portfolio process pair  $(c^*, \pi^*)$  is given by

$$c^*(s, t) = e^{\rho(t-s)} \left( \frac{H(t)}{H(s)} \right)^{-\gamma} \frac{\int_s^{s+1} \frac{B(s)}{B(u)} w(s, u) du}{\int_s^{s+1} \left( \frac{B(s)}{B(u)} \right)^{1-\gamma} e^{-\frac{\gamma}{2}(1-\gamma) \int_s^u \theta(\tau)^2 d\tau + \gamma \rho(u-s)} du},$$

$$\pi^*(s, t) = \frac{\Psi(s, t) H(s)}{\sigma(t) a(s, t) H(t)} + \frac{\theta(t)}{\sigma(t)},$$

where  $\Psi(s, t)$  is the integrand in the stochastic integral representation  $M(s, t) = \int_s^t \Psi(s, u) dW(u)$  of the martingale

$$M(s, t) \triangleq \mathbb{E} \left[ \int_s^{s+1} \frac{H(u)}{H(s)} (c(s, u) - w(s, u)) du \middle| \mathcal{F}_t \right].$$

*Proof.* The proof can be found in [13], Chapter 3, Section 3.6 and 3.7. □

## 4 Prices and interest rates behaviors

In this section, we give a characterization of the behaviors of prices and interest rate considering equalities on the different markets. These equalities can be seen as simplifications of equilibrium equations in the sense that we represent the demand/supply equalities in expectation.

### Market equations

First of all, we introduce the equalities on the different markets.

**Definition 4.1** *A vector  $(r^*(t), \theta^*(t))$  is a solution of the market equations if :*

1. *The pair  $(c^*, \pi^*)$  is optimal for each individual.*
2. *The markets clear :*

$$\mathbb{E} \left[ \int_{t-1}^t c^*(s, t) ds \right] = 1, \tag{4.1}$$

$$\mathbb{E} \left[ \int_{t-1}^t \pi^*(s, t) a(s, t) ds \right] = 0, \tag{4.2}$$

$$\int_{t-1}^t w(s, t) ds = 1. \tag{4.3}$$

This definition can be seen as a restriction for our model. However, this permits us to obtain explicit solutions for prices, otherwise this is not the case. These equations give the result bellow.

**Theorem 4.2** *The vector  $(r^*(t), \theta^*(t))$  is a solution of market equations if and only if, for all  $t \in \mathbb{R}^+$  :*

$$\int_{t-1}^t e^{-\gamma\rho(t-s)} \left( \frac{B(s)}{B(t)} \right)^{-\gamma} \frac{e^{\frac{\gamma}{2}(1+\gamma) \int_s^t \theta^*(\tau)^2 d\tau} \int_s^{s+1} \frac{B(s)}{B(u)} w(s, u) du}{\int_s^{s+1} \left( \frac{B(s)}{B(u)} \right)^{1-\gamma} e^{-\frac{\gamma}{2}(1-\gamma) \int_s^u \theta^*(\tau)^2 d\tau - \gamma\rho(u-s)} du} ds = 1 \tag{4.4}$$

and

$$\begin{aligned} & \frac{\theta(t)}{\sigma(t)} \left[ \int_{t-1}^t \int_t^{s+1} \frac{B(t)}{B(u)} w(s, u) du ds - (1-\gamma) \int_{t-1}^t \left( \frac{B(s)}{B(t)} \right)^{-\gamma} e^{\frac{\gamma}{2}(1+\gamma) \int_s^t \theta^*(\tau)^2 d\tau} \right. \\ & \left. \times \int_s^{s+1} \frac{B(s)}{B(u)} w(s, u) du \frac{\int_t^{s+1} \left( \frac{B(t)}{B(u)} \right)^{1-\gamma} e^{-\frac{\gamma}{2}(1-\gamma) \int_t^u \theta^*(\tau)^2 d\tau - \gamma \rho(u-s)} du}{\int_s^{s+1} \left( \frac{B(s)}{B(u)} \right)^{1-\gamma} e^{-\frac{\gamma}{2}(1-\gamma) \int_s^u \theta^*(\tau)^2 d\tau - \gamma \rho(u-s)} du} ds \right] = 0 \end{aligned} \quad (4.5)$$

where  $B(t) = \exp \left\{ \int_0^t r^*(s) ds \right\}$ .

Moreover, equation (4.3) gives  $w = \frac{1}{\beta - \alpha}$ .

*Proof.* The first part of this result is obtained by computing the expectations and using the martingale property of  $\exp\{\int_s^t \theta^*(\tau) dW(\tau) - \frac{1}{2} \int_s^t \theta^*(\tau)^2 d\tau\}$ . For the second part we use the definition of  $w(s, t)$  which is  $w(s, t) = w \mathbf{1}_{[s+\alpha, s+\beta]}(t)$  with  $0 \leq \alpha < \beta \leq 0$ . □

## Steady states

The goal is now to determine the conditions of existence and multiplicity of steady states for our market equations.

**Definition 4.3** A time-independent vector  $(r(t), \theta(t)) = (r^*, \theta^*)$  satisfying equations (4.4) and (4.5) is called a steady state.

The first result is the following :

**Lemma 4.4** For all steady state vectors  $(r^*, \theta^*)$  we have,

$$\frac{\int_0^1 e^{\gamma r^* \tau + \rho \tau + \frac{\gamma}{2}(1+\gamma)\theta^{*2}\tau} d\tau \int_0^1 e^{-r^* \tau + (\nu - \theta^* \kappa)\tau} d\tau}{\int_0^1 e^{-(1-\gamma)r^* \tau - \frac{\gamma}{2}(1-\gamma)\theta^{*2}\tau + \rho \tau} d\tau} = \int_0^1 e^{\nu \tau} d\tau \quad (4.6)$$

and at least one these equations are verified :

- $\theta^* = 0$ ,
- $\int_{\alpha}^{\beta} e^{r^* u} \int_u^{\beta} e^{-r^* \tau} d\tau ds + \int_0^{\alpha} e^{r^* u} \int_{\alpha}^{\beta} e^{-r^* \tau} d\tau ds - (1-\gamma) \int_0^1 \left( e^{r^* u + \gamma \theta^{*2} u} \int_{\alpha}^{\beta} e^{-r^* \tau} d\tau \right. \\ \left. \times \frac{\int_u^1 e^{-(1-\gamma)r^* \tau - \frac{\gamma}{2}(1-\gamma)\theta^{*2}\tau - \gamma \rho \tau} d\tau}{\int_0^1 e^{-(1-\gamma)r^* \tau - \frac{\gamma}{2}(1-\gamma)\theta^{*2}\tau - \gamma \rho \tau} d\tau} \right) ds = 0.$



One important remark is that we obtain steady states which are the same as the solutions which can be found in non-OLG equilibrium models.

In the sequel we focus on the case  $\theta^* = 0$  and study solutions to equation (4.6). To simplify the notation, we drop the star indices and define

$$\Phi(r; \alpha, \beta, \gamma, \rho) = \frac{\int_0^1 e^{\gamma rs + \gamma \rho s} ds \int_\alpha^\beta e^{-rs} ds}{(\beta - \alpha) \int_0^1 e^{-(1-\gamma)rs + \gamma \rho s} ds}$$

which is obtained by replacing  $\theta$  by 0 in equation (4.6). When there is no ambiguity for the parameter values, we denote  $\Phi(r; \alpha, \beta, \gamma, \rho) = \Phi(r)$ .

**Property 4.5** *A steady state is an  $r$  that satisfies  $\Phi(r) = 1$  if  $r \neq 0$  and  $\Phi'(r) = 0$  if  $r = 0$ .*

*Proof.* For  $r \neq 0$ , the property is an immediate implication of lemma 4.4. For  $r = 0$ , we just observe that  $\Phi(0) = 1$  and use l'Hôpital's Rule. □

To obtain  $r = 0$  as a solution, we have to make some restrictions on the parameters :

**Proposition 4.6**  *$r = 0$  is a steady state if and only if*

$$\begin{cases} \alpha + \beta = 2 \left( \frac{e^{\gamma \rho (\gamma \rho - 1) + 1}}{\gamma \rho (e^{\gamma \rho} - 1)} \right) & \text{if } \rho \neq 0 \\ \alpha + \beta = 1 & \text{if } \rho = 0 \end{cases}$$

*Proof.* The first derivative of function  $\Phi$  yields :

$$\Phi'(r) = \Phi(r) \left[ \gamma \frac{\int_0^1 s e^{\gamma rs + \gamma \rho s} ds}{\int_0^1 e^{\gamma \rho s} ds} - \frac{\int_\alpha^\beta s e^{-rs} ds}{\int_\alpha^\beta e^{-rs} ds} + (1 - \gamma) \frac{\int_0^1 s e^{-(1-\gamma)rs + \gamma \rho s} ds}{\int_0^1 e^{-(1-\gamma)rs + \gamma \rho s} ds} \right].$$

Replacing  $r = 0$  in  $\Phi'(r)$ , we conclude directly. □

The two following propositions present the existence and multiplicity of the steady states for equation (4.6).

**Proposition 4.7** *There exists  $(\alpha, \beta, \rho, \gamma)$  such that there is no steady-state.*

*Proof.* We show that there is no  $r$  such that  $\Phi(r) = 1$  if  $\alpha = 0$  and  $\beta \in (0, 1 - \gamma)$  with  $\rho = 0$ . Define  $\Psi(\beta) = \Phi(r; 0, \beta, \gamma, 0)$ .  $\Psi$  is such that

$$\Psi(\beta) = \frac{\int_0^1 e^{\gamma rs} ds \int_0^\beta e^{-rs} ds}{\beta \int_0^1 e^{-(1-\gamma)rs} ds}.$$

Then

$$\Psi'(\beta) = \left[ \frac{r}{e^{r\beta} - 1} - \frac{1}{\beta} \right] \Psi(\beta).$$

Consequently, if  $r > 0$  we verify that  $\Psi'(\beta) < 0$  and  $\lim_{\beta \rightarrow 1-\gamma} \Psi(\beta) > 1$ . Conversely, if  $r < 0$ ,  $\Psi'(\beta) > 0$  and  $\lim_{\beta \rightarrow 0} \Psi(\beta) > 1$ . This concludes the proof.

Notice that it can equally be shown that there is no solution  $r$  such that  $\Phi(r) = 1$  if  $\beta = 1$  and  $\alpha \in [\gamma, 1)$ .  $\square$

**Proposition 4.8** *There exists  $(\alpha, \beta, \rho, \gamma)$  such that there are multiple steady-state.*

*Proof.* Suppose  $\beta = 1 - \alpha$  and  $\rho = 0$ . With Proposition 4.6,  $r = 0$  is a steady state. We show that there are at least two other solutions such that  $\Phi(r) = 1$  if  $\gamma \in (\alpha, 2\alpha(1 - \alpha))$ . We have

$$\lim_{r \rightarrow -\infty} \frac{\Phi'(r)}{\Phi(r)} = \gamma - \alpha \text{ and } \lim_{r \rightarrow +\infty} \frac{\Phi'(r)}{\Phi(r)} = \alpha - \gamma.$$

Consequently, for  $\gamma > \alpha$

$$\lim_{r \rightarrow \pm\infty} \Phi(r) = +\infty.$$

Moreover,  $\Phi'(0) = 0$ , and since

$$\Phi''(0) = \frac{1}{6}(\gamma - 2\alpha(1 - \alpha)),$$

then  $\Phi''(0) < 0$ , for  $\gamma < 2\alpha(1 - \alpha)$ .  $\square$

## 5 Stability Analysis

In this section we are interested in the local dynamics around the steady-state  $r^* = 0$ . Let us define  $x(t)$  such that  $r(t) = r^* + \varepsilon x(t)$ . We take  $\rho = 0$ , then from Proposition 4.6,  $\beta = 1 - \alpha$ .

**Lemma 5.1** *Function  $x(\cdot)$  satisfies the following MFDE :*

$$\int_{t-1}^t \left[ (1 - 2\alpha) \left( \gamma \int_s^t x(u) du + (1 - \gamma) \int_s^{s+1} \int_s^u x(\tau) d\tau du \right) - \int_{s+\alpha}^{s+1-\alpha} \int_s^u x(\tau) d\tau du \right] ds = 0 \quad (5.1)$$

*Proof.* The result is obtained replacing  $r(t)$  by  $r^* + \varepsilon x(t)$  in equation (5.1) and by doing a Taylor expansion in the neighbourhood of  $\varepsilon = 0$ .  $\square$

In the sequel, to simplify, we take  $\alpha = 0$ . In this case, the new equation is :

$$\int_{t-1}^t \left[ \int_s^t x(u) du - \int_s^{s+1} \int_s^u x(\tau) d\tau du \right] ds = 0. \quad (5.2)$$

We now apply the results presented in Section 2 for our equation (5.2).

## Stability of Equation (5.1)

The characteristic function associated to equation (5.2) is :

$$\Delta(\lambda) = \frac{-\lambda^2 + e^\lambda - 2 + e^{-\lambda}}{\lambda^3}.$$

First, note that the roots of this function are symmetric with respect to abscissa and ordinate axes.

Defining  $\Lambda_U = \{\lambda \in \mathbb{C}; \Delta(\lambda) = 0 \text{ and } \text{Re}(\lambda) > 0\}$  and  $\Lambda_S = \{\lambda \in \mathbb{C}; \Delta(\lambda) = 0 \text{ and } \text{Re}(\lambda) < 0\}$ , the following lemma holds :

**Lemma 5.2** *If  $\lambda \in \Lambda_U$ ,  $-\lambda \in \Lambda_S$ .*

We use the factorization result of Mallet-Paret and Verduyn Lunel and remark that Proposition 2.1 can be applied. So we obtain this fundamental theorem :

**Theorem 5.3** *Let  $x(t)$  with  $t \in [-1, 0]$ . There exists a unique, bounded and continuous trajectory  $x(t)$ ,  $t \in \mathbb{R}^+$ , solution to*

$$\frac{d^2x(t)}{dt^2} = x(t+1) - 2x(t) + x(t-1). \quad (5.3)$$

*Proof.* This result is obtained by differentiating three times the first equation and using the exponential dichotomies result of Mallet-Paret and Verduyn-Lunel [16].  $\square$

## Comments

According to our stability analysis, the mixed-type functional differential equation theory tells us that there are oscillations around the steady states for the trajectory  $x(t)$ . Our steady state was for  $\theta^* = 0$ , so we have  $r^* = \mu^*$ . Denoting  $\mu(t)$  as  $\mu^* + \epsilon y(t)$ , the dynamics of  $y(t)$  are associated to the ones of  $x(t)$  and we obtain the same oscillations for  $y(t)$  as for  $x(t)$ . These oscillations decrease in magnitude and eventually disappear.

## 6 Conclusion

We considered a stochastic overlapping generations model in which the shock in the economy comes only from the risky asset on the financial market. In this approach, we obtained steady states which were very similar to the equilibrium solutions for non-OLG models. We added asymptotic results for an exchange economy without any terminal time and for heterogeneous agents. In this paper, we focused on the steady-state  $r^* = 0$  and  $\theta^* = 0$ . The dynamics of  $r(\cdot)$  were given by a functional differential equation of mixed type, due to the anticipation behavior in the individual program and to the demographic modelling. The study of these dynamics provided replacement echoes, which are usual in models with delays as in Boucekkine, Germain and Licandro [4].

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