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Optimal control of stochastic delayed systems with jumps

Delphine David*

November 15, 2008

Abstract

We consider the optimal control of stochastic delayed systems with jumps, in which both the state and controls can depend on the past history of the system, for a value function which depends on the initial path of the process. We derive the Hamilton-Jacobi-Bellman equation and the associated verification theorem and prove a necessary and a sufficient maximum principles for such problems. Explicit solutions are obtained for a mean-variance portfolio problem and for the optimal consumption and portfolio case in presence of a financial market with delay and jumps.

Key-words: Stochastic delayed systems with jumps, Hamilton-Jacobi-Bellman equation, Pontryagin-type maximum principle, sufficient maximum principle.

Mathematics Subject Classification (2000): 93E20, 49K25, 49L20, 91B28.

1 Introduction

This paper deals with stochastic optimal control when the state is driven by a Stochastic Differential Delay Equation with Jumps (SDDEJ). Such problems arise in finance if the price of a risky asset depends on its own past. However, in general, these problems are very difficult to solve because of their infinite dimension. Nonetheless, if the growth at time t depends on $X(t - \delta)$, $\delta > 0$, and on some sliding average of previous values, it is possible to obtain some explicit solutions. In the Brownian

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motion case, this type of setting was first used by Kolmanovskii and Maizenberg [8] for a linear delay system with a quadratic cost functional. In the same framework, Øksendal and Sulem [14] proved a sufficient maximum principle and applied their result to optimal consumption and to optimal portfolio problems separately. We also refer to Elsanousi, Øksendal and Sulem [4] for applications to optimal harvesting and to Elsanousi and Larssen [3] for optimal consumption results. All these articles are considering undelayed controls.

Many authors, see [1, 2, 20] and references therein, argued that Lévy processes are relevant to the modelling of asset prices in mathematical finance. In optimal control theory, the Hamilton-Jacobi-Bellman equation with jumps was first proved by Sheu [19]. A necessary maximum principle for processes with jumps has been given by Tang and Li [22]. The sufficient version of this principle was proved by Framstad, Øksendal and Sulem [6] in order to solve an optimal consumption and portfolio problem in a Lévy type Black-Scholes market.

Our objective is to consider an optimal control problem which includes both delays and jumps. Moreover, we modify the primary model settled by Kolmanovskii and Maizenberg [8] to allow one of the control processes to be delayed. This framework is particularly adapted to financial applications : It is natural to consider a delayed portfolio if we assume that a risky asset price is governed by a SDDEJ. In their paper, Gozzi and Marinelli [7] also studied a delayed control but considered a specific SDDE for advertising models that cannot be used in our setting.

Let $(B(t))_{t \in [0, T]}$ a Brownian motion and $\tilde{N}(dt, dk) = N(dt, dk) - \lambda(dk)dt$ a compensated Poisson random measure with finite Lévy measure λ . We denote by $u(\cdot)$ and $v(\cdot)$ the control processes. We assume that they take values in a given closed set $U \subset \mathbb{R}^2$. We consider the state $X(\cdot)$ driven by a SDDEJ of the form :

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t), u(t), v(t))dt + \sigma(t, X(t), Y(t), Z(t), u(t), v(t))dB(t) \\ \quad + \int_{\mathbb{R}} \eta(t^-, X(t^-), Y(t^-), Z(t^-), u(t^-), v(t^-), k) \tilde{N}(dt, dk), \quad t \in [0, T], \\ X(t) = \xi(t-s), \quad v(t) = \nu(t-s), \quad s - \delta \leq t \leq s, \quad \xi \in C([- \delta, 0]; \mathbb{R}) \end{cases} \quad (1.1)$$

where the continuous function $\xi \in C([- \delta, 0]; \mathbb{R})$ is the initial path of X , $\nu \in C([- \delta, 0]; \mathbb{R})$ the initial path of v and where

$$Y(t) = \int_{-\delta}^0 e^{\rho s} v(t+s) X(t+s) ds \quad (1.2)$$

and

$$Z(t) = v(t-\delta)X(t-\delta)$$

are some functionals of the path segments $X_t := \{X(t+s); s \in [-\delta, 0]\}$ of X and $v_t := \{v(t+s); s \in [-\delta, 0]\}$ of v . Moreover, $b : [0, T] \times \mathbb{R}^3 \times U \mapsto \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R}^3 \times U \mapsto \mathbb{R}$ and $\eta : [0, T] \times \mathbb{R}^3 \times U \times \mathbb{R} \mapsto \mathbb{R}$ are given continuous functions, $\rho \in \mathbb{R}$ is a given averaging parameter and $\delta > 0$ is a fixed delay.

We also define the performance function as

$$J(s, \xi; u, v) = \mathbb{E}^{s, \xi} \left[\int_s^T f(t, X(t), Y(t), u(t), v(t)) dt + g(X(T), Y(T)) \right] \quad (1.3)$$

with $f : [0, T] \times \mathbb{R}^2 \times U \mapsto \mathbb{R}$ and $g : \mathbb{R}^2 \mapsto \mathbb{R}$ some given lower bounded C^1 functions. $\mathbb{E}^{s, \xi}$ is the expectation given that the initial path of X is $\xi \in C([- \delta, 0]; \mathbb{R})$. Our goal is to find admissible controls $u(\cdot)$ and $v(\cdot)$ that maximize the performance function (1.3). In mathematical terms, we aim at solving the following problem :

Problem 1.1 *Find admissible controls $u^*(\cdot)$ and $v^*(\cdot)$ such that*

$$J(s, \xi; u^*, v^*) = \sup_{u, v} J(s, \xi; u, v).$$

To solve Problem 1.1, we define the associated value function :

$$\begin{cases} \tilde{V}(s, \xi) = \sup_{u, v} J(s, \xi; u, v), \quad (s, \xi) \in [0, T] \times C([- \delta, 0], \mathbb{R}) \\ \tilde{V}(T, \xi) = g(X(T), Y(T)). \end{cases} \quad (1.4)$$

In general, this function depends on the initial path in a complicated way. However, looking at the functional (1.3), one might expect that the value function depends on ξ only through the two functionals :

$$x = \xi(0), \quad y = \int_{-\delta}^0 e^{\rho\tau} \nu(\tau) \xi(\tau) d\tau.$$

Consequently, in the sequel, we will work with a new value function V which is, by hypothesis, only depending on x and y instead of ξ in the following way :

$$\tilde{V}(s, \xi) = V(s, x, y), \quad V : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}. \quad (1.5)$$

In Theorem 3.1 below, we show that under conditions (1.4) and (1.5), the value function associated with our problem verifies the Hamilton-Jacobi-Bellman equation (3.1). Conversely, we prove that if we can find a function which verifies the Hamilton-Jacobi-Bellman equation, then it is the value function for our problem (see Theorem 3.2).

Since the coefficients of (1.1) enter into the proof of the HJB equation, the solution may also depend on a third functional, namely

$$z = \xi(-\delta).$$

Thus we cannot *a priori* expect the HJB equation to have a solution independent of z . In Theorem 3.3, we derive necessary conditions on b , σ and η for which this condition is verified.

As the second main result of this paper, we show in Theorem 4.1 that if we find optimal controls for Problem 1.1, then the derivatives with respect to u and v of the Hamiltonian H defined by :

$$\begin{aligned} & H(t, X(t), Y(t), Z(t), u(t), v(t), p(t), q(t), r(t, \cdot)) \\ &= f(t, X(t), Y(t), u(t), v(t)) + b(t, X(t), Y(t), Z(t), u(t), v(t))p_1(t) \\ & \quad + (v(t)X(t) - e^{-\rho\delta}Z(t) - \rho Y(t))p_2(t) + \sigma(t, X(t), Y(t), Z(t), u(t), v(t))q_1(t) \\ & \quad + \int_{\mathbb{R}} \eta(t, X(t), Y(t), Z(t), u(t), v(t), k)r_1(t, k)\lambda(dk) \end{aligned}$$

for p, q, r some adjoint processes, are equal to 0. Moreover, we show that if the controls maximize the Hamiltonian for H and g concave, then they are also optimal controls

for Problem 1.1 (see Theorem 4.2). These theorems are named the necessary and the sufficient maximum principles.

We proceed as follows. In Section 2.2 we give some notation, the Itô formula and the dynamic programming principle. By means of these results we derive the Hamilton-Jacobi-Bellman equation and the associated verification theorem in Section 2.3. We also give conditions for which equality (1.5) is verified. Section 2.4 presents the necessary and the sufficient maximum principles. Finally, in Section 2.5, we consider two financial applications. The first one is a mean-variance portfolio problem. In this example, we search an optimal control π which minimizes the variance of the terminal wealth of an agent under a constant expectation condition. The wealth of the agent is given by :

$$\begin{cases} dX(t) = [(\mu(t) - b(t))\pi(t)X(t) + b(t)X(t) + \alpha(t)Y(t) + \beta Z(t)]dt + \sigma(t)\pi(t)X(t)dB(t) \\ \quad + \pi(t^-)X(t^-) \int_{\mathbb{R}} \eta(t^-, k) \tilde{N}(dt, dk) \\ X(s) = \xi(s), \quad \pi(s) = \nu(s), \quad s \in [-\delta, 0] \end{cases}$$

For this problem, we show that the optimal portfolio is expressed as

$$\pi^*(t) = \frac{\mu(t) - b(t) + \beta e^{\rho\delta}}{X^*(t)(\sigma^2(t) + \int_{\mathbb{R}} \eta(t, k)^2 \lambda(dk))} h(t, X^*(t), Y^*(t))$$

where $\mu(\cdot), b(\cdot), \sigma(\cdot), \eta(\cdot, \cdot)$ are parameters of the market and h is a function that we determine explicitly.

The second application is to the classical optimal consumption and portfolio problem. In this case, the goal is to find an optimal consumption rate $c(\cdot)$ and an optimal portfolio $\pi(\cdot)$ that maximize :

$$\mathbb{E} \left[\int_0^T e^{-\rho t} \frac{c(t)^\gamma}{\gamma} dt + \frac{1}{\gamma} (\theta_1 X(T) + \theta_2 Y(T))^\gamma \right]$$

under the wealth constraint

$$\begin{cases} dX(t) = [(\mu - b)\pi(t)X(t) + bX(t) + \alpha Y(t) + \beta Z(t) - c(t)]dt + \sigma\pi(t)X(t)dB(t) \\ X(s) = \xi(s), \quad \pi(s) = \nu(s), \quad s \in [-\delta, 0] \end{cases}$$

We show that the optimal controls are :

$$c^*(t) = [e^{\rho t} \gamma b h(t)]^{\frac{1}{\gamma-1}} (bX(t) + \alpha Y(t)) \quad \text{and} \quad \pi^*(t) = \frac{\mu - b + \beta}{X(t)\sigma^2 b(1 - \gamma)} (bX(t) + \alpha Y(t))$$

where $h(\cdot)$ is a deterministic function that we give explicitly.

2 Notation and preliminary results

We work on a product of probability spaces

$$(\Omega, \mathbb{P}) = (\Omega_B \times \Omega_M, \mathbb{P}_B \otimes \mathbb{P}_M)$$

on which are respectively defined a standard Brownian motion $(B(t))_{t \in \mathbb{R}^+}$ and a compound Poisson process $(M(t))_{t \in \mathbb{R}^+}$ such that

$$M(t) = \int_0^t \int_{\mathbb{R}} k N(ds, dk)$$

where $N(dt, dk)$ is a Poisson random measure with intensity measure $\lambda(dk)dt$. We define a pure jump Lévy process $\{\tilde{M}(t)\}_{t \in \mathbb{R}^+}$ such that

$$\tilde{M}(t) = \int_0^t \int_{\mathbb{R}} k \tilde{N}(ds, dk)$$

where $\tilde{N}(dt, dk) = N(dt, dk) - \lambda(dk)dt$ is the compensated Poisson random measure. For $s, t \geq 0$, we take $\mathcal{F}_{s,t} = \sigma\{(B(\tau), \tilde{M}(\tau)), s \leq \tau \leq t\}$.

We also define the processes $u(\cdot)$ and $v(\cdot)$ as the control processes. We assume that they have values in a given closed set $U \subset \mathbb{R}^2$ and that they are adapted and càdlàg (right continuous and with left limits). We also require that $u(\cdot)$ and $v(\cdot)$ are such that system (1.1) has a unique, strong solution. Such controls will be called admissible. Let us denote by \mathcal{A}_s the set of all these admissible controls $\{(u(t), v(t)), s \leq t \leq T\}$.

Let us denote by $C([-\delta, 0], \mathbb{R})$ the Banach space of all continuous paths $[-\delta, 0] \rightarrow \mathbb{R}$ given the supremum norm

$$\|\gamma\|_{C([-\delta, 0], \mathbb{R})} := \sup_{s \in [-\delta, 0]} |\gamma(s)|, \quad \gamma \in C([-\delta, 0], \mathbb{R}).$$

To ensure the existence and the uniqueness of a solution for system (1.1), we make the following assumptions (see for example [17]).

(A.1) Lipschitz continuity: there exists a constant L_1 such that

$$|b(t, \gamma, u, v) - b(t, \hat{\gamma}, u, v)| + |\sigma(t, \gamma, u, v) - \sigma(t, \hat{\gamma}, u, v)|$$

$$+ \int_{\mathbb{R}} |\eta(t, \gamma, u, v, k) - \eta(t, \hat{\gamma}, u, v, k)| \lambda(dk) < L_1 \|\gamma - \hat{\gamma}\|_{\mathbb{R}^3},$$

for all $t \in [0, T]$, $\gamma, \hat{\gamma} \in \mathbb{R}^3$, $(u, v) \in U$.

(A.2) Linear growth: there is a constant $L_2 > 0$ such that

$$|b(t, \gamma, u, v)| + |\sigma(t, \gamma, u, v)| + \int_{\mathbb{R}} |\eta(t, \gamma, u, v, k)| \lambda(dk) \leq L_2(1 + \|\gamma\|_{\mathbb{R}^3}),$$

for all $t \in [0, T]$, $\gamma \in \mathbb{R}^3$, $(u, v) \in U$.

(A.3) The maps $f : [0, T] \times \mathbb{R}^2 \times U \rightarrow \mathbb{R}$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ are uniformly continuous, and there exist constants $L_3, l > 0$ such that for all $t \in [0, T]$, $\gamma, \hat{\gamma} \in \mathbb{R}^2$, $(u, v) \in U$,

$$|f(t, \gamma, u, v) - f(t, \hat{\gamma}, u, v)| + |h(\gamma) - h(\hat{\gamma})| \leq L_3 \|\gamma - \hat{\gamma}\|_{\mathbb{R}^2},$$

$$|f(t, \gamma, u, v)| + |h(\gamma)| \leq L_3(1 + \|\gamma\|_{\mathbb{R}^2})^l.$$

Moreover, by extension of Theorem 11.2.3 in [13], under the following mild condition

:

$$\begin{aligned} \text{(C)} \quad & \mathbb{E}^{s, \xi} \left[\int_s^t \left| \left(\frac{\partial V}{\partial \tau}(\tau, X(\tau), Y(\tau)) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(\tau, X(\tau), Y(\tau)) \sigma^2(\tau, X(\tau), Y(\tau), Z(\tau), u(\tau), v(\tau)) \right. \right. \right. \\ & + \frac{\partial V}{\partial x}(\tau, X(\tau), Y(\tau)) b(\tau, X(\tau), Y(\tau), Z(\tau), u(\tau), v(\tau)) + (v(\tau) X(\tau) - e^{-\rho \delta} Z(\tau) - \rho Y(\tau)) \\ & \times \frac{\partial V}{\partial y}(\tau, X(\tau), Y(\tau)) + \int_{\mathbb{R}} \left(V(\tau, X(\tau) + \eta(\tau, X(\tau), Y(\tau), Z(\tau), u(\tau), v(\tau), k), Y(\tau)) \right. \\ & \left. \left. - V(\tau, X(\tau), Y(\tau)) - \frac{\partial V}{\partial x}(\tau, X(\tau), Y(\tau)) \eta(\tau, X(\tau), Y(\tau), Z(\tau), u(\tau), v(\tau), k) \right) \lambda(dk) \right| d\tau \\ & \left. + |V(t, X(t), Y(t))| \right] < \infty, \end{aligned}$$

it suffices to consider Markov controls, i.e. controls $u(t)$ and $v(t)$ of the form $u(t) = \tilde{u}(t, X(t^-), Y(t^-), Z(t^-))$ and $v(t) = \tilde{v}(t, X(t^-), Y(t^-), Z(t^-))$.

Therefore, from now on we will only consider Markov controls and we will, with a slight abuse of notation, write $u(t) = u(t, X(t^-), Y(t^-), Z(t^-))$ and $v(t) = v(t, X(t^-), Y(t^-), Z(t^-))$.

Next, we present two preliminary results we use in the rest of the paper.

Proposition 2.1 Itô's formula.

Let $X(\cdot)$ and $Y(\cdot)$ given by (1.1) and (1.2) respectively and $h \in C^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R})$, then we have :

$$\begin{aligned}
& dh(t, X(t), Y(t)) \\
&= \left(\frac{\partial h}{\partial t}(t, X(t), Y(t)) + \frac{1}{2} \frac{\partial^2 h}{\partial x^2}(t, X(t), Y(t)) \sigma^2(t, X(t), Y(t), Z(t), u(t), v(t)) \right. \\
&\quad + \frac{\partial h}{\partial x}(t, X(t), Y(t)) b(t, X(t), Y(t), Z(t), u(t), v(t)) \\
&\quad - \int_{\mathbb{R}} \frac{\partial h}{\partial x}(t, X(t), Y(t)) \eta(t, X(t), Y(t), Z(t), u(t), v(t), k) \lambda(dk) \\
&\quad \left. + (v(t)X(t) - e^{-\rho\delta} Z(t) - \rho Y(t)) \frac{\partial h}{\partial y}(t, X(t), Y(t)) \right) dt \\
&\quad + \left(\frac{\partial h}{\partial x}(t, X(t), Y(t)) \sigma(t, X(t), Y(t), Z(t), u(t), v(t)) \right) dB(t) \\
&\quad + \int_{\mathbb{R}} \left(h(t^-, X(t^-) + \eta(t^-, X(t^-), Y(t^-), Z(t^-), u(t^-), v(t^-), k), Y(t^-)) \right. \\
&\quad \left. - h(t^-, X(t^-), Y(t^-)) \right) N(dt, dk).
\end{aligned}$$

Proof. First, let us recall that,

$$Y(t) = \int_{-\delta}^0 e^{\rho s} v(t+s) X(t+s) ds,$$

which, by a change of variables, is rewritten as

$$Y(t) = \int_{t-\delta}^t e^{\rho(s-t)} v(s) X(s) ds,$$

which implies

$$dY(t) = (v(t)X(t) - e^{-\rho\delta} Z(t) - \rho Y(t)) dt.$$

We conclude the proof applying the standard Itô formula (see e.g. Protter [16], Chapter 2, Section 7) to $h(t, X(t), Y(t))$. \square

The second result of this section is the dynamic programming principle. This principle will be used to prove the Hamilton-Jacobi-Bellman equation in Section 3.

Proposition 2.2 Dynamic programming principle.

Let (A.1)-(A.3) hold. Then for any $(s, \xi) \in [0, T] \times C([-\delta, 0], \mathbb{R})$ and $s \leq t \leq T$, the

value function defined in (1.4) and (1.5) satisfies

$$V(s, x, y) = \sup_{(u, v) \in \mathcal{A}_s} \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + V(t, X(t), Y(t)) \right].$$

Proof. By definition of $J(s, \xi; u, v)$, we have

$$\begin{aligned} J(s, \xi; u, v) &= \mathbb{E}^{s, \xi} \left[\int_s^T f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + g(X(T), Y(T)) \right] \\ &= \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + \int_t^T f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau \right. \\ &\quad \left. + g(X(T), Y(T)) \right] \\ &= \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + \mathbb{E}^{s, \xi} \left[\int_t^T f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau \right. \right. \\ &\quad \left. \left. + g(X(T), Y(T)) \middle| \mathcal{F}_{s, t} \right] \right] \\ &= \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau \right. \\ &\quad \left. + \mathbb{E}^{t, X_t^s, \xi} \left[\int_t^T f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + g(X(T), Y(T)) \right] \right] \\ &= \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + J(t, X_t; u, v) \right]. \end{aligned}$$

Given any $\varepsilon_1 > 0$ there exist u_{ε_1} and v_{ε_1} such that $V(s, x, y) - \varepsilon_1 < J(s, \xi; u_{\varepsilon_1}, v_{\varepsilon_1})$, then

$$\begin{aligned} V(s, x, y) - \varepsilon_1 &< J(s, \xi; u_{\varepsilon_1}, v_{\varepsilon_1}) \\ &< \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X_{\varepsilon_1}^{s, x}(\tau), Y_{\varepsilon_1}^{s, y}(\tau), u_{\varepsilon_1}(\tau), v_{\varepsilon_1}(\tau)) d\tau + J(t, X_t; u_{\varepsilon_1}, v_{\varepsilon_1}) \right] \\ &< \sup_{(u, v) \in \mathcal{A}_s} \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + V(t, X(t), Y(t)) \right]. \end{aligned} \tag{2.1}$$

Let us now define the new controls \tilde{u} and \tilde{v} such that

$$\tilde{u}(\tau) = \begin{cases} u(\tau) & \text{if } \tau \in [s, t] \\ u_{\varepsilon_1}(\tau) & \text{if } \tau \in [t, T] \end{cases} \quad \text{and} \quad \tilde{v}(\tau) = \begin{cases} v(\tau) & \text{if } \tau \in [s, t] \\ v_{\varepsilon_1}(\tau) & \text{if } \tau \in [t, T] \end{cases}.$$

Then by definition of $V(s, x, y)$:

$$\begin{aligned}
V(s, x, y) &\geq J(s, \xi; \tilde{u}, \tilde{v}) \\
&\geq \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + J(t, X_t; u_{\varepsilon_1}, v_{\varepsilon_1}) \right] \\
&\geq \mathbb{E}^{s, \xi} \left[\int_s^t f(\tau, X(\tau), Y(\tau), u(\tau), v(\tau)) d\tau + V(t, X(t), Y(t)) \right] - \varepsilon_1.
\end{aligned} \tag{2.2}$$

We obtain the result from the inequalities (2.1) and (2.2). \square

3 Hamilton-Jacobi-Bellman equation and verification theorem

In this section, we derive the Hamilton-Jacobi-Bellman equation and we prove its associated verification theorem. We also give conditions on b, σ, η, f and g to obtain equality (1.5). In the following theorem, we show that the value function associated to our optimal control problem verifies the Hamilton-Jacobi-Bellman equation.

We introduce the following set :

$$C_p^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R}) = \left\{ V \in C^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R}) \text{ such that } V, \frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial x^2} \text{ and } \frac{\partial V}{\partial y} \right. \\
\left. \text{satisfy a polynomial growth condition} \right\}.$$

Theorem 3.1 *Hamilton-Jacobi-Bellman equation.*

Let $V \in C_p^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R})$ such that it verifies (1.4), (1.5) and Assumptions (A.4) and (A.5). Then $V(s, x, y)$ solves the following Hamilton-Jacobi-Bellman equation :

$$\begin{aligned}
&\sup_{(u,v) \in U} \left\{ \frac{\partial V}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + \frac{\partial V}{\partial x}(s, x, y) b(s, x, y, z, u, v) \right. \\
&+ \int_{\mathbb{R}} \left\{ V(s, x + \eta(s, x, y, z, u, v, k), y) - V(s, x, y) - \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \right\} \lambda(dk) \\
&\left. + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(s, x, y) + f(s, x, y, u, v) \right\} = 0
\end{aligned} \tag{3.1}$$

with terminal condition

$$V(T, x, y) = g(x, y).$$

Proof. Fix $(s, x, y) \in [0, T) \times \mathbb{R}^2$, $\varepsilon_1 > 0$ and $(u, v) \in U$ (u and v are control values). Let X be given by (1.1) with $u(s) = u(s, x, y, z) \equiv u$ and $v(s) = v(s, x, y, z) \equiv v$, and fix $\xi \in C([-\delta, 0]; \mathbb{R})$ and $\nu \in C([0, T]; \mathbb{R})$ such that $x = \xi(0) = X(s)$ and $y = \int_{-\delta}^0 e^{\rho\tau} \nu(\tau) \xi(\tau) d\tau = Y(s)$. Then the dynamic programming principle implies

$$V(s, x, y) \geq \mathbb{E}^{s, \xi} \left[\int_s^{s+\varepsilon_1} f(t, X(t), Y(t), u(t), v(t)) dt + V(s + \varepsilon_1, X(s + \varepsilon_1), Y(s + \varepsilon_1)) \right].$$

Dividing by ε_1 and rearranging the terms we have,

$$\frac{\mathbb{E}^{s, \xi} [V(s + \varepsilon_1, X(s + \varepsilon_1), Y(s + \varepsilon_1))] - V(s, x, y)}{\varepsilon_1} + \mathbb{E}^{s, \xi} \left[\frac{1}{\varepsilon_1} \int_s^{s+\varepsilon_1} f(t, X(t), Y(t), u(t), v(t)) dt \right] \leq 0.$$

As $V \in C_p^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R})$, $\int_0^\cdot (V(t, X(t) + \eta(t, X(t), Y(t), Z(t), u(t), v(t), k), Y(t)) - V(t, X(t), Y(t))) \tilde{N}(dt, dk)$ and $\int_0^\cdot \frac{\partial V}{\partial x}(t, X(t), Y(t)) \sigma(t, X(t), Y(t), Z(t), u(t), v(t)) dB(t)$ are martingales (see e.g. [5]). Thus applying Itô's formula to $V(s + \varepsilon_1, X(s + \varepsilon_1), Y(s + \varepsilon_1))$ we get

$$\begin{aligned} \mathbb{E}^{s, \xi} \left[\frac{1}{\varepsilon_1} \int_s^{s+\varepsilon_1} \left(\frac{\partial V}{\partial t}(t, X(t), Y(t)) + \frac{\partial V}{\partial x}(t, X(t), Y(t)) b(t, X(t), Y(t), Z(t), u(t), v(t)) \right. \right. \\ \left. \left. + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, X(t), Y(t)) \sigma^2(t, X(t), Y(t), Z(t), u(t), v(t)) \right. \right. \\ \left. \left. - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(t, X(t), Y(t)) \eta(t, X(t), Y(t), Z(t), u(t), v(t), k) \lambda(dk) \right. \right. \\ \left. \left. + (v(t)X(t) - e^{-\rho\delta} Z(t) - \rho Y(t)) \frac{\partial V}{\partial y}(t, X(t), Y(t)) + f(t, X(t), Y(t), u(t), v(t)) \right. \right. \\ \left. \left. + \int_{\mathbb{R}} \left(V(t, X(t) + \eta(t, X(t), Y(t), Z(t), u(t), v(t), k), Y(t)) \right. \right. \right. \\ \left. \left. \left. - V(t, X(t), Y(t)) \right) \lambda(dk) \right) dt \right] \leq 0. \end{aligned}$$

Letting ε_1 tend to 0, we obtain

$$\begin{aligned}
& \frac{\partial V}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + \frac{\partial V}{\partial x}(s, x, y) b(s, x, y, z, u, v) \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(s, x, y) \\
& + \int_{\mathbb{R}} \left(V(s, x + \eta(s, x, y, z, u, v, k), y) - V(s, x, y) \right) \lambda(dk) + f(s, x, y, u, v) \leq 0.
\end{aligned}$$

This holds for any $(u, v) \in U$, so

$$\begin{aligned}
& \sup_{(u,v) \in U} \left\{ \frac{\partial V}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + \frac{\partial V}{\partial x}(s, x, y) b(s, x, y, z, u, v) \right. \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(s, x, y) \\
& + \int_{\mathbb{R}} \left(V(s, x + \eta(s, x, y, z, u, v, k), y) - V(s, x, y) \right) \lambda(dk) \\
& \left. + f(s, x, y, u, v) \right\} \leq 0. \tag{3.2}
\end{aligned}$$

Conversely, if we take $\varepsilon_2 > 0$, we can find u and v such that

$$\begin{aligned}
V(s, x, y) - \varepsilon_1 \varepsilon_2 \leq \mathbb{E}^{s, \xi} \left[\int_s^{s+\varepsilon_1} f(t, X(t), Y(t), u(t), v(t)) dt \right. \\
\left. + V(s + \varepsilon_1, X(s + \varepsilon_1), Y(s + \varepsilon_1)) \right]. \tag{3.3}
\end{aligned}$$

Applying Itô's formula to $V(s + \varepsilon_1, X(s + \varepsilon_1), Y(s + \varepsilon_1))$ in inequality (3.3), dividing by ε_1 and letting ε_1 tend to 0, we obtain :

$$\begin{aligned}
-\varepsilon_2 & \leq \frac{\partial V}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + \frac{\partial V}{\partial x}(s, x, y) b(s, x, y, z, u, v) \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(s, x, y) \\
& + f(s, x, y, u, v) + \int_{\mathbb{R}} \left(V(s, x + \eta(s, x, y, z, u, v, k), y) - V(s, x, y) \right) \lambda(dk) \\
& \leq \sup_{(u,v) \in U} \left\{ \frac{\partial V}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + f(s, x, y, u, v) \right. \\
& + \frac{\partial V}{\partial x}(s, x, y) b(s, x, y, z, u, v) - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) \\
& \left. + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(s, x, y) + \int_{\mathbb{R}} \left(V(s, x + \eta(s, x, y, z, u, v, k), y) \right. \right.
\end{aligned}$$

$$\left. -V(s, x, y) \right\} \lambda(dk). \quad (3.4)$$

Inequalities (3.2) and (3.4) give us (3.1). □

We now prove a verification theorem which is used in particular to solve some problems with singular control such as the famous Merton problem. For more details, see [11] or [12], Chapter 4. This theorem says that if there exists a solution to the Hamilton-Jacobi-Bellman equation then this function is the value function of the optimal control problem considered and the equality (1.5) holds.

Theorem 3.2 Verification theorem.

Let $W \in C_p^{1,2,1}([0, T] \times \mathbb{R}^2; \mathbb{R}) \cap C([0, T] \times \mathbb{R}^2; \mathbb{R})$.

(i) Let us suppose that

$$\begin{aligned} & \sup_{(u,v) \in U} \left\{ \frac{\partial W}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 W}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + f(s, x, y, u, v) \right. \\ & + \frac{\partial W}{\partial x}(s, x, y) b(s, x, y, z, u, v) - \int_{\mathbb{R}} \frac{\partial W}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) \\ & + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial W}{\partial y}(s, x, y) + \int_{\mathbb{R}} \left(W(s, x + \eta(s, x, y, z, u, v, k), y) \right. \\ & \left. \left. - W(s, x, y) \right) \lambda(dk) \right\} \leq 0, \end{aligned} \quad (3.5)$$

$$W(T, x, y) \geq g(x, y). \quad (3.6)$$

Then $W \geq \tilde{V}$ pointwise on $[0, T] \times \mathbb{R}^2$.

(ii) Let us also suppose that $W(T, \cdot, \cdot) = g(\cdot, \cdot)$ and that for all $(s, x, y) \in [0, T] \times \mathbb{R}^2$, there exist $(u^*, v^*) \in U$ such that :

$$\begin{aligned} & \sup_{(u,v) \in U} \left\{ \frac{\partial W}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 W}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u, v) + f(s, x, y, u, v) \right. \\ & + \frac{\partial W}{\partial x}(s, x, y) b(s, x, y, z, u, v) - \int_{\mathbb{R}} \frac{\partial W}{\partial x}(s, x, y) \eta(s, x, y, z, u, v, k) \lambda(dk) \\ & + (vx - e^{-\rho\delta} z - \rho y) \frac{\partial W}{\partial y}(s, x, y) + \int_{\mathbb{R}} \left(W(s, x + \eta(s, x, y, z, u, v, k), y) \right. \\ & \left. \left. - W(s, x, y) \right) \lambda(dk) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial W}{\partial s}(s, x, y) + \frac{1}{2} \frac{\partial^2 W}{\partial x^2}(s, x, y) \sigma^2(s, x, y, z, u^*, v^*) + f(s, x, y, u^*, v^*) \\
&\quad + \frac{\partial W}{\partial x}(s, x, y) b(s, x, y, z, u^*, v^*) - \int_{\mathbb{R}} \frac{\partial W}{\partial x}(s, x, y) \eta(s, x, y, z, u^*, v^*, k) \lambda(dk) \\
&\quad + (v^* x - e^{-\rho\delta} z - \rho y) \frac{\partial W}{\partial y}(s, x, y) + \int_{\mathbb{R}} \left(W(s, x + \eta(s, x, y, z, u^*, v^*, k), y) \right. \\
&\quad \left. - W(s, x, y) \right) \lambda(dk) = 0
\end{aligned}$$

and that the SDDEJ :

$$\begin{aligned}
dX(t) &= b(t, X(t), Y(t), Z(t), u^*(t), v^*(t)) dt + \sigma(t, X(t), Y(t), Z(t), u^*(t), v^*(t)) dB(t) \\
&\quad + \int_{\mathbb{R}} \eta(t^-, X(t^-), Y(t^-), Z(t^-), u^*(t^-), v^*(t^-), k) \tilde{N}(dt, dk)
\end{aligned}$$

has a unique solution. Then $W = \tilde{V}$ pointwise on $[0, T] \times \mathbb{R}^2$ and u^* and v^* are optimal controls.

Proof. Let $(u, v) \in \mathcal{A}_s$. By Itô's formula and by the same arguments as for Theorem 3.1, we have,

$$\begin{aligned}
\mathbb{E}^{s, \xi}[W(T, X(T), Y(T))] &= W(s, x, y) + \mathbb{E}^{s, \xi} \left[\int_s^T \frac{\partial W}{\partial t}(t, X(t), Y(t), Z(t), u(t), v(t)) \right. \\
&\quad + \frac{1}{2} \frac{\partial^2 W}{\partial x^2}(t, X(t), Y(t)) \sigma^2(t, X(t), Y(t), Z(t), u(t), v(t)) \\
&\quad + \frac{\partial W}{\partial x}(t, X(t), Y(t)) b(t, X(t), Y(t), Z(t), u(t), v(t)) \\
&\quad - \int_{\mathbb{R}} \frac{\partial W}{\partial x}(t, X(t), Y(t)) \eta(t, X(t), Y(t), Z(t), u(t), v(t), k) \lambda(dk) dt \\
&\quad + \int_s^T \left((v(t)X(t) - e^{-\rho\delta} Z(t) - \rho(t)) \frac{\partial W}{\partial y}(t, X(t), Y(t)) \right) dt \\
&\quad + \int_s^T \int_{\mathbb{R}} \left(W(t, X(t) + \eta(t, X(t), Y(t), Z(t), u(t), v(t), k)) Y(t) \right. \\
&\quad \left. - W(t, X(t), Y(t)) \right) \lambda(dk) dt \Big].
\end{aligned}$$

From (3.5),

$$\mathbb{E}^{s, \xi}[W(T, X(T), Y(T))] \leq W(s, x, y) - \mathbb{E}^{s, \xi} \left[\int_s^T f(t, X(t), Y(t), u(t), v(t)) dt \right].$$

Hence, using (3.6),

$$W(s, x, y) \geq \mathbb{E}^{s, \xi} \left[\int_s^T f(t, X(t), Y(t), u(t), v(t)) dt + g(X(T), Y(T)) \right]$$

$$\geq J(s, \xi; u, v).$$

Since $(u, v) \in \mathcal{A}_s$ were arbitrary, this proves (i). For (ii), we apply the above arguments to u^* and v^* and the inequality becomes equality. \square

We just proved that if we find a solution for the Hamilton-Jacobi-Bellman equation then it is the value function of our optimal control problem. As we search for a solution which is only depending on s, x and y , one might expect that some conditions are needed on b, σ, η, f and g . We investigate the conditions for the following equation :

$$\begin{aligned} dX(t) = & [\bar{\mu}(t, X(t), Y(t), Z(t)) + \alpha(t, X(t), Y(t), u(t), v(t))]dt \\ & + \bar{\sigma}(t, X(t), Y(t), v(t))dB(t) + \int_{\mathbb{R}} \bar{\eta}(t, X(t), Y(t), v(t), k) \tilde{N}(dt, dk) \end{aligned}$$

and show :

Theorem 3.3 *If the HJB equation has a solution V which satisfies Condition (1.5) then*

$$\bar{\mu}(s, x, y, z) = \beta(s, x, y) + \theta(x, y)z$$

and the following conditions have to be verified :

$$\frac{\partial \hat{\beta}}{\partial y}(s, x, y) - e^{\rho\delta} \theta(x, y) \frac{\partial \hat{\beta}}{\partial x}(s, x, y) = 0 \quad (3.7)$$

with $\hat{\beta}(s, x, y) = \beta(s, x, y) - \rho y \theta(s, x, y)$,

$$\frac{\partial F}{\partial y} \left(s, x, y, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2} \right) - e^{\rho\delta} \theta(x, y) \frac{\partial F}{\partial x} \left(s, x, y, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2} \right) \quad (3.8)$$

with $F(s, x, y, V_1, V_2, V_3) = \sup_{(u,v) \in U} \left\{ \frac{1}{2} V_3(s, x, y) \sigma(s, x, y, v)^2 + V_2(s, x, y) \alpha(s, x, y, u, v) \right.$
 $+ \int_{\mathbb{R}} (V_1(s, x + \eta(s, x, y, v, k), y) - V_1(s, x, y) - V_2(s, x, y) \eta(s, x, y, v, k)) \lambda(dk)$
 $\left. + v x e^{\rho\delta} V_2(s, x, y) \theta(x, y) + f(s, x, y, u, v) \right\}$

and

$$\frac{\partial g}{\partial y}(x, y) - e^{\rho\delta} \theta(x, y) \frac{\partial g}{\partial x}(x, y) = 0. \quad (3.9)$$

Proof. We know that if V only depends on s, x and y , then V satisfies the HJB equation

$$\begin{aligned}
& \frac{\partial V}{\partial s}(s, x, y) + \frac{\partial V}{\partial x}(s, x, y)\bar{\mu}(s, x, y, z) - (e^{-\rho\delta}z + \rho y)\frac{\partial V}{\partial y}(s, x, y) \\
& + \sup_{(u,v) \in U} \left\{ \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y)\bar{\sigma}(s, x, y, v)^2 + \frac{\partial V}{\partial x}(s, x, y)\alpha(s, x, y, u, v) \right. \\
& + \int_{\mathbb{R}} \left(V(s, x + \bar{\eta}(s, x, y, v, k), y) - V(s, x, y) - \frac{\partial V}{\partial x}(s, x, y)\bar{\eta}(s, x, y, v, k) \right) \lambda(dk) \\
& \left. + vx \frac{\partial V}{\partial y}(s, x, y) + f(s, x, y, u, v) \right\} = 0
\end{aligned} \tag{3.10}$$

with terminal condition

$$V(T, x, y) = g(x, y).$$

We wish to obtain necessary conditions on $\bar{\mu}, \bar{\sigma}$ and $\bar{\eta}$ that ensure that Equation (3.10) has a solution independent of z . Differentiating with respect to z , we obtain :

$$\frac{\partial V}{\partial y}(s, x, y) = e^{\rho\delta} \frac{\partial V}{\partial x}(s, x, y) \frac{\partial \bar{\mu}}{\partial z}(s, x, y, z).$$

Replacing $\frac{\partial V}{\partial y}(s, x, y)$ in Equation (3.10), we have :

$$\begin{aligned}
& \frac{\partial V}{\partial s}(s, x, y) + \frac{\partial V}{\partial x}(s, x, y)\bar{\mu}(s, x, y, z) - (e^{-\rho\delta}z + \rho y)e^{\rho\delta} \frac{\partial V}{\partial x}(s, x, y) \frac{\partial \bar{\mu}}{\partial z}(s, x, y, z) \\
& + \sup_{(u,v) \in U} \left\{ \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y)\bar{\sigma}(s, x, y, v)^2 + \frac{\partial V}{\partial x}(s, x, y)\alpha(s, x, y, u, v) \right. \\
& + \int_{\mathbb{R}} \left(V(s, x + \bar{\eta}(s, x, y, v, k), y) - V(s, x, y) - \frac{\partial V}{\partial x}(s, x, y)\bar{\eta}(s, x, y, v, k) \right) \lambda(dk) \\
& \left. + vxe^{\rho\delta} \frac{\partial V}{\partial x}(s, x, y) \frac{\partial \bar{\mu}}{\partial z}(s, x, y, z) + f(s, x, y, u, v) \right\} = 0
\end{aligned}$$

Let us take $\bar{\mu}(s, x, y, z)$ as

$$\bar{\mu}(s, x, y, z) = \beta(s, x, y) + \theta(x, y)z.$$

Then Equation (3.10) takes the form :

$$\begin{aligned} & \frac{\partial V}{\partial s}(s, x, y) + \frac{\partial V}{\partial x}(s, x, y)[\beta(s, x, y) - \rho y \theta(s, x, y)] + \sup_{(u, v) \in U} \left\{ \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, x, y) \bar{\sigma}(s, x, y, v)^2 \right. \\ & + \frac{\partial V}{\partial x}(s, x, y) \alpha(s, x, y, u, v) + \int_{\mathbb{R}} \left(V(s, x + \bar{\eta}(s, x, y, v, k), y) - V(s, x, y) \right. \\ & \left. \left. - \frac{\partial V}{\partial x}(s, x, y) \eta(s, x, y, v, k) \right) \lambda(dk) + v x e^{\rho \delta} \frac{\partial V}{\partial x}(s, x, y) \theta(x, y) + f(s, x, y, u, v) \right\} = 0 \end{aligned}$$

and does not contain any z . The last step is to ensure the equality :

$$\frac{\partial V}{\partial y}(s, x, y) - e^{\rho \delta} \theta(x, y) \frac{\partial V}{\partial x}(s, x, y) = 0. \quad (3.11)$$

If we introduce a new variable \tilde{y} such that

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y} - e^{\rho \delta} \theta(x, y) \frac{\partial}{\partial x},$$

then Equation (3.11) states that $\frac{\partial V}{\partial \tilde{y}}(s, x, y) = 0$ and V have to be independent of \tilde{y} . Consequently, Conditions (3.7) - (3.9) must be verified. \square

4 Necessary and sufficient maximum principles

The necessary maximum principle shows that if the controls are optimal for Problem 1.1 then they satisfy the maximum principle conditions whereas the sufficient one shows that if the controls satisfy the maximum principle conditions then they are optimal for Problem 1.1. These two theorems give an efficient alternative to the Hamilton-Jacobi-Bellman equation and its verification theorem since the latter involves a complicated integro-differential equation.

To establish these results, we define the Hamiltonian $H : [0, T] \times \mathbb{R}^3 \times U \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathcal{R} \mapsto \mathbb{R}$ as

$$\begin{aligned} & H(t, X(t), Y(t), Z(t), u(t), v(t), p(t), q(t), r(t, \cdot)) \\ & = f(t, X(t), Y(t), u(t), v(t)) + b(t, X(t), Y(t), Z(t), u(t), v(t)) p_1(t) \\ & \quad + (v(t) X(t) - e^{-\rho \delta} Z(t) - \rho Y(t)) p_2(t) + \sigma(t, X(t), Y(t), Z(t), u(t), v(t)) q_1(t) \\ & \quad + \int_{\mathbb{R}} \eta(t, X(t), Y(t), Z(t), u(t), v(t), k) r_1(t, k) \lambda(dk) \end{aligned} \quad (4.1)$$

with $p = (p_1, p_2, p_3)$, $q = (q_1, q_2)$, $r = (r_1, r_2)$ and \mathcal{R} the set of functions $r_1 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $r_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that the integral in (4.1) converges.

Theorem 4.1 Necessary maximum principle.

Assume that the HJB equation has a solution $V \in C^{2,3,2}([0, T] \times \mathbb{R}^2; \mathbb{R})$. Let $u^*(\cdot)$ and $v^*(\cdot)$ be optimal controls for Problem 1.1 and $X^*(\cdot), Y^*(\cdot), Z^*(\cdot)$ the associated solutions of system (1.1). Then there are p_i , $i = 1, 2, 3$ and q_j, r_j , $j = 1, 2$ such that

$$\begin{aligned} dp_1^*(t) &= -\frac{\partial H}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot))dt + q_1^*(t)dB(t) \\ &\quad + \int_{\mathbb{R}} r_1^*(t^-, k)\tilde{N}(dt, dk) \\ dp_2^*(t) &= -\frac{\partial H}{\partial y}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot))dt + q_2^*(t)dB(t) \\ &\quad + \int_{\mathbb{R}} r_2^*(t^-, k)\tilde{N}(dt, dk) \\ dp_3^*(t) &= -\frac{\partial H}{\partial z}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r(t, \cdot))dt \end{aligned}$$

$$p_1^*(T) = \frac{\partial g}{\partial x}(X(T), Y(T))$$

$$p_2^*(T) = \frac{\partial g}{\partial y}(X(T), Y(T))$$

$$p_3^*(T) = 0.$$

Moreover, $u^*(\cdot)$ and $v^*(\cdot)$ satisfy

$$\begin{aligned} \frac{\partial H}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) &= 0 \\ \frac{\partial H}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) &= 0 \end{aligned}$$

and

$$p_3^*(t) = 0, \text{ for all } t \in [0, T].$$

Proof. Let $u^*(\cdot)$ and $v^*(\cdot)$ be optimal controls for Problem 1.1. Then the Hamilton-Jacobi-Bellman equation holds :

$$\begin{aligned} &\frac{\partial V}{\partial t}(t, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, x, y) \sigma^2(t, x, y, z, u^*, v^*) + \frac{\partial V}{\partial x}(t, x, y) b(t, x, y, z, u^*, v^*) \\ &\quad - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(t, x, y) \eta(t, x, y, z, u^*, v^*, k) \lambda(dk) + (v^* x - e^{-\rho \delta} z - \rho y) \frac{\partial V}{\partial y}(t, x, y) \\ &\quad + f(t, x, y, u^*, v^*) + \int_{\mathbb{R}} \left(V(t, x + \eta(t, x, y, z, u^*, v^*, k), y) - V(t, x, y) \right) \lambda(dk) = 0. \end{aligned}$$

Differentiating this equation with respect to x and evaluating the result at $x = X^*(t)$, $y = Y^*(t)$ and $z = Z^*(t)$, we obtain :

$$\begin{aligned}
& \frac{\partial^2 V}{\partial x \partial t}(t, X^*(t), Y^*(t)) + \frac{1}{2} \frac{\partial^3 V}{\partial x^3}(t, X^*(t), Y^*(t)) \sigma^2(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \frac{\partial \sigma}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial b}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial \eta}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \lambda(dk) \\
& + \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) b(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& - \int_{\mathbb{R}} \left(\frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \right) \lambda(dk) \\
& + v^*(t) \frac{\partial V}{\partial y}(t, X^*(t), Y^*(t)) + (v^*(t) X^*(t) - e^{-\rho \delta} Z^*(t) - \rho Y^*(t)) \frac{\partial^2 V}{\partial x \partial y}(t, X^*(t), Y^*(t)) \\
& + \int_{\mathbb{R}} \left(\frac{\partial V}{\partial x}(t, X^*(t) + \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k), Y^*(t)) \right. \\
& \cdot \left. \left(1 + \frac{\partial \eta}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \right) - \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \right) \lambda(dk) \\
& + \frac{\partial f}{\partial x}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) = 0.
\end{aligned}$$

Let us now denote

$$G(t) = \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t))$$

and apply Itô's formula on it.

$$\begin{aligned}
dG(t) &= \left(\frac{\partial^2 V}{\partial t \partial x}(t, X^*(t), Y^*(t)) + \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) b(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \right. \\
& + \frac{1}{2} \frac{\partial^3 V}{\partial x^3}(t, X^*(t), Y^*(t)) \sigma^2(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& - \int_{\mathbb{R}} \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \lambda(dk) \\
& + (v^*(t) X^*(t) - e^{-\rho \delta} Z^*(t) - \rho Y^*(t)) \frac{\partial^2 V}{\partial x \partial y}(t, X^*(t), Y^*(t)) \\
& \left. + \int_{\mathbb{R}} \left(\frac{\partial V}{\partial x}(t, X^*(t) + \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k), Y^*(t)) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \lambda(dk) \Big) dt \\
& + \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \sigma(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) dB(t) \\
& + \int_{\mathbb{R}} \left(\frac{\partial V}{\partial x}(t^-, X^*(t^-) + \eta(t^-, X^*(t^-), Y^*(t^-), Z^*(t^-), u^*(t^-), v^*(t^-), k), Y^*(t^-)) \right. \\
& \left. - \frac{\partial V}{\partial x}(t^-, X^*(t^-), Y^*(t^-)) \right) \tilde{N}(dt, dk)
\end{aligned}$$

The next step is to use the differentiated form of the HJB equation. By the definition of the Hamiltonian, we have :

$$\begin{aligned}
\frac{\partial H}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) &= \frac{\partial f}{\partial x}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial b}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) p_1^*(t) + \frac{\partial \sigma}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) q_1^*(t) \\
& + v^*(t) p_2^*(t) + \int_{\mathbb{R}} \frac{\partial \eta}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) r_1^*(t, k) \lambda(dk),
\end{aligned}$$

and if we write :

$$p_1^*(t) = \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)), \quad (4.2)$$

$$q_1^*(t) = \sigma(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)), \quad (4.3)$$

$$\begin{aligned}
r_1^*(t, \cdot) &= \frac{\partial V}{\partial x}(t, X^*(t) + \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k), Y^*(t)) \\
& - \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)),
\end{aligned} \quad (4.4)$$

$$p_2^*(t) = \frac{\partial V}{\partial y}(t, X^*(t), Y^*(t)), \quad (4.5)$$

we obtain :

$$\begin{aligned}
dp_1^*(t) &= - \frac{\partial H}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) dt + q_1^*(t) dB(t) \\
& + \int_{\mathbb{R}} r_1^*(t^-, k) \tilde{N}(dt, dk).
\end{aligned}$$

Then applying the same method on $I(t) = \frac{\partial V}{\partial y}(t, X^*(t), Y^*(t))$ and on $J(t) = \frac{\partial V}{\partial z}(t, X^*(t), Y^*(t)) = 0$, we derive the expressions of $dp_2^*(t)$ and $dp_3^*(t)$. In particular, this permits to prove that $p_3^*(t) = 0$.

The first order conditions for the Hamilton-Jacobi-Bellman equation are :

$$\begin{aligned}
& \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \frac{\partial \sigma}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \sigma(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial b}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial \eta}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \lambda(dk) \\
& + \frac{\partial f}{\partial u}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) + \int_{\mathbb{R}} \frac{\partial \eta}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \\
& \cdot \frac{\partial V}{\partial x}(t, X^*(t) + \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k), Y^*(t)) \lambda(dk) = 0
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2 V}{\partial x^2}(t, X^*(t), Y^*(t)) \frac{\partial \sigma}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \sigma(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial b}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) \\
& - \int_{\mathbb{R}} \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) \frac{\partial \eta}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \lambda(dk) \\
& + \frac{\partial f}{\partial v}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) + \int_{\mathbb{R}} \frac{\partial \eta}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \\
& \cdot \frac{\partial V}{\partial x}(t, X^*(t) + \eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k), Y^*(t)) \lambda(dk) \\
& + X^*(t) \frac{\partial V}{\partial x}(t, X^*(t), Y^*(t)) = 0.
\end{aligned}$$

Using Equations (4.2), (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{\partial \eta}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) r_1^*(t, k) \lambda(dk) + \frac{\partial f}{\partial u}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial \sigma}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) q_1^*(t) + \frac{\partial b}{\partial u}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) p_1^*(t) \\
& = 0
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{\partial \eta}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) r_1^*(t, k) \lambda(dk) + \frac{\partial f}{\partial v}(t, X^*(t), Y^*(t), u^*(t), v^*(t)) \\
& + \frac{\partial \sigma}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) q_1^*(t) + \frac{\partial b}{\partial v}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t)) p_1^*(t) \\
& + X^*(t) p_1^*(t) = 0.
\end{aligned}$$

These conditions are exactly the ones we searched for and the proof is complete. \square

For the second part of this section, we define the adjoint processes as :

$$\begin{aligned}
dp_1(t) &= -\frac{\partial H}{\partial x}(t, X(t), Y(t), Z(t), u(t), v(t), p(t), q(t), r(t, \cdot))dt + q_1(t)dB(t) \\
&\quad + \int_{\mathbb{R}} r_1(t^-, k)\tilde{N}(dt, dk) \\
dp_2(t) &= -\frac{\partial H}{\partial y}(t, X(t), Y(t), Z(t), u(t), v(t), p(t), q(t), r(t, \cdot))dt + q_2(t)dB(t) \\
&\quad + \int_{\mathbb{R}} r_2(t^-, k)\tilde{N}(dt, dk) \\
dp_3(t) &= -\frac{\partial H}{\partial z}(t, X(t), Y(t), Z(t), u(t), v(t), p(t), q(t), r(t, \cdot))dt \\
p_1(T) &= \frac{\partial g}{\partial x}(X(T), Y(T)) \\
p_2(T) &= \frac{\partial g}{\partial y}(X(T), Y(T)) \\
p_3(T) &= 0.
\end{aligned}$$

Theorem 4.2 Sufficient maximum principle.

Let $X^*(\cdot), Y^*(\cdot), Z^*(\cdot)$ and $p(\cdot), q(\cdot), r(\cdot, \cdot)$ be the solutions of the system (1.1) and the adjoint equations respectively. Suppose we have

$$\begin{aligned}
&H(t, \cdot, \cdot, \cdot, \cdot, \cdot, p(t), q(t), r(t, \cdot)(t)) \text{ and } g(\cdot, \cdot) \text{ are concave, for all } t \in [0, T], \\
&H(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p(t), q(t), r(t, \cdot)) \\
&= \sup_{(u,v) \in U} H(t, X^*(t), Y^*(t), Z^*(t), u, v, p(t), q(t), r(t, \cdot)), \text{ for all } t \in [0, T], \\
&p_3(t) = 0, \text{ for all } t \in [0, T].
\end{aligned}$$

Then $u^*(\cdot)$ and $v^*(\cdot)$ are optimal controls for the initial problem.

Proof. Let $X(t), Y(t), Z(t)$ be the solution of system (1.1). The goal of the proof is to show that for all $(u, v) \in \mathcal{A}_0$,

$$\begin{aligned}
&J(0, \xi; u^*, v^*) - J(0, \xi; u, v) \\
&= \mathbb{E}^{0, \xi} \left[\int_0^T (f(t, X^*(t), Y^*(t), u^*(t), v^*(t)) - f(t, X(t), Y(t), u(t), v(t)))dt \right. \\
&\quad \left. + g(X^*(T), Y^*(T)) - g(X(T), Y(T)) \right] \geq 0.
\end{aligned}$$

By concavity of g , we have :

$$\begin{aligned} & \mathbb{E}^{0,\xi}[g(X^*(T), Y^*(T)) - g(X(T), Y(T))] \\ & \geq \mathbb{E}^{0,\xi} \left[(X^*(T) - X(T)) \frac{\partial g}{\partial x}(X^*(T), Y^*(T)) + (Y^*(T) - Y(T)) \frac{\partial g}{\partial y}(X^*(T), Y^*(T)) \right] \\ & \geq \mathbb{E}^{0,\xi}[(X^*(T) - X(T))p_1^*(T)] + \mathbb{E}^{0,\xi}[(Y^*(T) - Y(T))p_2^*(T)]. \end{aligned}$$

Using integration by parts formula for jump processes (which is derived from the Itô formula), we get

$$\begin{aligned} & \mathbb{E}^{0,\xi}[(X^*(T) - X(T))p_1^*(T)] + \mathbb{E}^{0,\xi}[(Y^*(T) - Y(T))p_2^*(T)] \\ & = \mathbb{E}^{0,\xi} \left[\int_0^T (X^*(t^-) - X(t^-)) dp_1^*(t) + \int_0^T (Y^*(t^-) - Y(t^-)) dp_2^*(t) \right. \\ & \quad + \int_0^T p_1^*(t) d(X^*(t) - X(t)) + \int_0^T p_2^*(t) d(Y^*(t) - Y(t)) \\ & \quad + \int_0^T \left(\sigma(t, X^*(t), Y^*(t), u^*(t), v^*(t)) - \sigma(t, X(t), Y(t), u(t), v(t)) \right) q_1^*(t) dt \\ & \quad + \int_0^T \left(\int_{\mathbb{R}} \left(\eta(t^-, X^*(t^-), Y^*(t^-), u^*(t^-), v^*(t^-), k) \right. \right. \\ & \quad \left. \left. - \eta(t^-, X(t^-), Y(t^-), u(t^-), v(t^-), k) \right) r_1^*(t, k) \lambda(dk) \right) dt \left. \right] \\ & = \mathbb{E}^{0,\xi} \left[\int_0^T (X^*(t) - X(t)) \left(-\frac{\partial H}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) \right) dt \right. \\ & \quad + \int_0^T (Y^*(t) - Y(t)) \left(-\frac{\partial H}{\partial y}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) \right) dt \\ & \quad + \int_0^T p_1^*(t) (b(t, X^*(t), Y^*(t), u^*(t), v^*(t)) - b(t, X(t), Y(t), u(t), v(t))) dt \\ & \quad + \int_0^T (\sigma(t, X^*(t), Y^*(t), u^*(t), v^*(t)) - \sigma(t, X(t), Y(t), u(t), v(t))) q_1^*(t) dt \\ & \quad + \int_0^T p_2^*(t) ((v^*(t)X^*(t) - e^{-\rho\delta}Z^*(t) - \rho Y^*(t)) - (v(t)X(t) - e^{-\rho\delta}Z(t) - \rho Y(t))) dt \\ & \quad + \int_0^T \int_{\mathbb{R}} (\eta(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), k) \\ & \quad \left. - \eta(t, X(t), Y(t), Z(t), u(t), v(t), k)) r_1^*(t, k) \lambda(dk) dt \right]. \end{aligned}$$

Using the definition of H and compiling the last results, we obtain the following inequality :

$$\begin{aligned}
& J(0, \xi; w^*) - J(0, \xi; w) \\
& \geq \mathbb{E}^{0, \xi} \left[\int_0^T \left(H(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) \right. \right. \\
& \quad \left. \left. - H(t, X(t), Y(t), Z(t), u(t), v(t), p^*(t), q^*(t), r^*(t, \cdot)) \right) dt \right. \\
& \quad \left. + \int_0^T (X^*(t) - X(t)) \left(-\frac{\partial H}{\partial x}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) \right) dt \right. \\
& \quad \left. + \int_0^T (Y^*(t) - Y(t)) \left(-\frac{\partial H}{\partial y}(t, X^*(t), Y^*(t), Z^*(t), u^*(t), v^*(t), p^*(t), q^*(t), r^*(t, \cdot)) \right) dt \right].
\end{aligned}$$

We conclude the proof using the concavity of H and the maximality of u^* and v^* .

The arguments used are the same as in the deterministic case. See for example [18].

□

5 Applications

In this section, we use our results to solve two optimal control problems in finance. The first one is to choose a portfolio such that the variance of the terminal wealth of an agent is minimal under an expectation constraint. The second one is a standard consumption and portfolio problem. Our goal is to maximize the expected utility of an agent during his life under his budget constraint. In these two examples, we take $\rho = 0$ in order to apply Theorem 3.3.

We consider a financial market with two assets : A non-risky one with price $(S_0(t))_{t \geq 0}$ that follows the equation :

$$dS_0(t) = b(t)S_0(t)dt, \quad S_0(0) = 1.$$

and a risky asset with price process $(S_1(t))_{t \geq 0}$ governed by a stochastic delay equation of the form

$$\begin{cases} dS_1(t) = \left(\mu(t)S_1(t) + \alpha(t) \int_{-\delta}^0 S_1(t+s)ds + \beta(t)S_1(t-\delta) \right) dt \\ \quad + \sigma(t)S_1(t)dB(t) + \int_{\mathbb{R}} \eta(t^-, k)\tilde{N}(dt, dk); \quad t \geq 0 \\ S_1(s) = \kappa(s) \quad s \in [-\delta, 0] \end{cases}$$

with $\eta(t, z) > -1$ for almost all t, z .

Consider an agent who is free to invest in both the above assets, and whose wealth process is defined as

$$X(t) = n_0(t)S_0(t) + n_1(t)S_1(t),$$

where $n_0(t)$ and $n_1(t)$ respectively the number of shares held in the riskless and the risky assets.

Let us define $\pi(t)$ as the proportion of wealth invested in the risky asset at time t and denote by $\tilde{\pi}(s, t)$ the quantity :

$$\tilde{\pi}(s, t) = \frac{n_1(t)}{n_1(t+s)}\pi(t+s), \forall s \in \mathbb{R}.$$

We obtain by this notation $\tilde{\pi}(0, t) = \pi(t)$ and by convention we take $\tilde{\pi}(s, t) = 0$ if $n_1(t+s) = 0$. We also recall that

$$Y(t) = \int_{-\delta}^0 \tilde{\pi}(s, t)X(t+s)ds \quad \text{and} \quad Z(t) = \tilde{\pi}(-\delta, t)X(t-\delta).$$

Example 1 : Mean-variance portfolio selection

The objective is to find an admissible portfolio which minimizes the variance of the terminal wealth of the agent under the condition that $\mathbb{E}[X(T) + \theta Y(T)] = A$, where $A \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$. We refer to [15] for the solution without delay.

By the Lagrange multiplier method, we have to solve the following problem without any constraint :

$$\min_{\pi} \mathbb{E}[(X(T) + \theta Y(T) - A)^2] - \lambda(\mathbb{E}[X(T) + \theta Y(T)] - A)$$

where λ is the Lagrange multiplier. By computation, we have :

$$\begin{aligned} & \mathbb{E}[(X(T) + \theta Y(T) - A)^2] - \lambda(\mathbb{E}[X(T) + \theta Y(T)] - A) \\ &= \mathbb{E} \left[(X(T) + \theta Y(T))^2 - 2 \left(A + \frac{\lambda}{2} \right) (X(T) + \theta Y(T)) + A^2 + \lambda A \right] \\ &= \mathbb{E} \left[\left(X(T) + \theta Y(T) - \left(A + \frac{\lambda}{2} \right) \right)^2 \right] - \frac{\lambda^2}{4} \\ &= \mathbb{E}[(X(T) + \theta Y(T) - a)^2] - \frac{\lambda^2}{4}. \end{aligned}$$

Thus, the initial problem is equivalent the following one :

Problem 5.1 Find an admissible portfolio π such that

$$\mathbb{E} \left[-\frac{1}{2}(X(T) + \theta Y(T) - a)^2 \right]$$

is maximal under the wealth constraint

$$\begin{cases} dX(t) = [(\mu(t) - b(t))\pi(t)X(t) + b(t)X(t) + \alpha(t)Y(t) + \beta Z(t)]dt + \sigma(t)\pi(t)X(t)dB(t) \\ \quad + \pi(t^-)X(t^-) \int_{\mathbb{R}} \eta(t^-, k) \tilde{N}(dt, dk) \\ X(s) = \xi(s), \quad \pi(s) = \nu(s), \quad s \in [-\delta, 0]. \end{cases}$$

Proposition 5.2 The optimal portfolio $\pi^*(t)$ for Problem 5.1 is given by

$$\pi^*(t) = \frac{\mu(t) - b(t) + \beta}{X^*(t)(\sigma(t)^2 + \int_{\mathbb{R}} \eta(t, k)^2 \lambda(dk))} \left(a \exp \left(\int_t^T b(s) ds \right) - X^*(t) - \beta Y^*(t) \right).$$

Proof. To solve this problem, we apply our maximum principle Theorems 4.1 and 4.2. The hamiltonian is defined by :

$$\begin{aligned} H(t, X(t), Y(t), Z(t), v(t), p(t), q(t), r(t, \cdot)) &= [(\mu(t) - b(t))\pi(t)X(t) + b(t)X(t) + \alpha(t)Y(t) \\ &\quad + \beta Z(t)]p_1(t) + \sigma(t)\pi(t)X(t)q_1(t) + (\pi(t)X(t) - e^{-\rho\delta}Z(t) - \rho Y(t))p_2(t) \\ &\quad + \int_{\mathbb{R}} \pi(t)X(t)\eta(t, k)r_1(t, k)\lambda(dk). \end{aligned}$$

The associated adjoint equations are :

$$\begin{aligned} dp_1(t) &= - \left[((\mu(t) - b(t))\pi(t) + b(t))p_1(t) + \sigma(t)\pi(t)q_1(t) + \pi(t)p_2(t) \right. \\ &\quad \left. + \int_{\mathbb{R}} \pi(t)\eta(t, k)r_1(t, k)\lambda(dk) \right] dt + q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, k)\tilde{N}(dt, dk), \\ dp_2(t) &= -[\alpha(t)p_1(t) - \rho p_2(t)]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, k)\tilde{N}(dt, dk), \\ dp_3(t) &= -[\beta(t)p_1(t) - p_2(t)e^{-\rho\delta}]dt, \\ p_1(T) &= -(X(T) + \theta Y(T) - a), \\ p_2(T) &= -\theta(X(T) + \theta Y(T) - a), \\ p_3(T) &= 0. \end{aligned} \tag{5.1}$$

We assume that $\theta = \beta e^{\rho\delta}$ and $p_1(t) = \Phi(t)(X(t) + \beta e^{\rho\delta}Y(t)) + \Psi(t)$. Then the differentiated form of $p_1(t)$ is :

$$\begin{aligned} dp_1(t) = & \left(\Phi'(t)X(t) + \Phi(t)((\mu(t) - b(t))\pi(t)X(t) + b(t)X(t) + \alpha(t)Y(t) + \beta Z(t)) \right. \\ & \left. + \Phi'(t)\beta e^{\rho\delta}Y(t) + \Phi(t)\beta(\pi(t)X(t) - e^{-\rho\delta}Z(t) - \rho Y(t)) + \Psi'(t) \right) dt \\ & + \Phi(t)\sigma(t)\pi(t)X(t)dB(t) + \Phi(t) \int_{\mathbb{R}} \pi(t^-)X(t^-)\eta(t, k)\tilde{N}(dt, dk). \end{aligned}$$

Identifying with equation (5.1), we obtain

$$q_1(t) = \Phi(t)\sigma(t)\pi(t)X(t),$$

$$r_1(t, k) = \Phi(t)\pi(t)X(t)\eta(t, k),$$

and

$$\begin{aligned} \pi(t) \left[(\mu(t) - b(t))(\Phi(t)(X(t) + \beta e^{\rho\delta}Y(t)) + \Psi(t)) + e^{\rho\delta}\beta(\phi(t)(X(t) + \beta e^{\rho\delta}Y(t)) \right. \\ \left. + \Psi(t)) + \Phi(t)(\mu(t) - b(t))X(t) + \Phi(t)\beta e^{\rho\delta}X(t) \right] + \pi(t)^2 \left[\sigma(t)^2\Phi(t)X(t) \right. \\ \left. + \phi(t)X(t) \int_{\mathbb{R}} \eta(t, k)^2\lambda(dk) \right] + b(t)(\phi(t)(X(t) + \beta e^{\rho\delta}Y(t)) + \Psi(t)) + \Phi'(t)X(t) \\ + \Phi(t)(b(t)X(t) + \alpha(t)Y(t) + \beta Z(t)) + \Phi'(t)\beta e^{\rho\delta}Y(t) - \Phi(t)\beta e^{\rho\delta}(e^{-\rho\delta}Z(t) + \rho Y(t)) \\ + \Psi'(t) = 0. \end{aligned} \quad (5.2)$$

Let $\pi^*(t)$ be an optimal control, by Theorem 4.1, it maximizes the Hamiltonian and we obtain :

$$(\mu(t) - b(t))p_1(t) + \sigma(t)q_1(t) + p_2(t) + \int_{\mathbb{R}} \eta(t, k)r_1(t, k)\lambda(dk) = 0.$$

Replacing $p_1(t), q_1(t)$ and $r_1(t, \cdot)$ by their values we have :

$$\begin{aligned} (\mu(t) - b(t)) \left[\Phi(t)(X^*(t) + \beta e^{\rho\delta}Y^*(t)) + \Psi(t) \right] + \sigma(t)^2\Phi(t)\pi^*(t) + \beta e^{\rho\delta} \left[\Phi(t)(X^*(t) \right. \\ \left. + \beta e^{\rho\delta}Y^*(t)) + \Psi(t) \right] + \pi^*(t)\Phi(t) \int_{\mathbb{R}} \eta(t, k)^2\lambda(dk) = 0. \end{aligned}$$

So the value of $\pi^*(t)$ is given by :

$$\pi^*(t) = - \frac{(\mu(t) - b(t) + \beta e^{\rho\delta})(\Phi(t)(X^*(t) + \beta e^{\rho\delta}Y^*(t)) + \Psi(t))}{\Lambda(t)\Phi(t)X(t)}, \quad (5.3)$$

where $\Lambda(t) = \sigma(t)^2 + \int_{\mathbb{R}} \eta(t, k)^2 \lambda(dk)$.

Replacing the value of π^* in (5.2) and identifying the terms in X^* , Y^* and Z^* we obtain the two equations :

$$\begin{aligned}\Phi'(t) &= \left(\frac{(\mu(t) - b(t) + \beta e^{\rho\delta})^2}{\Lambda(t)} - 2b(t) \right) \Phi(t), \quad \Phi(T) = -1, \\ \Psi'(t) &= \left(\frac{(\mu(t) - b(t) + \beta e^{\rho\delta})^2}{\Lambda(t)} - b(t) \right) \Psi(t), \quad \Psi(T) = a,\end{aligned}$$

but also the condition :

$$\alpha(t) = \beta e^{\rho\delta} b(t).$$

The solutions of the two equations are :

$$\begin{aligned}\Phi(t) &= - \exp \left(\int_t^T \left\{ \frac{(\mu(s) - b(s) + \beta e^{\rho\delta})^2}{\Lambda(s)} - 2b(s) \right\} ds \right), \\ \Psi(t) &= a \exp \left(\int_t^T \left\{ \frac{(\mu(s) - b(s) + \beta e^{\rho\delta})^2}{\Lambda(s)} - b(s) \right\} ds \right)\end{aligned}$$

and $\pi^*(t)$ is given by (5.3). □

Example 2 : Optimal consumption and portfolio problem

The objective here is to find an admissible portfolio and an admissible consumption process which maximize the expected utility of consumption and the terminal wealth of an agent. In this example, the parameters are time-independent and there is no jump part. The problem to solve is :

Problem 5.3 *Find an admissible portfolio π and an admissible consumption rate c such that*

$$\mathbb{E} \left[\int_0^T e^{-\rho t} \frac{c(t)^\gamma}{\gamma} dt + \frac{1}{\gamma} (\theta_1 X(T) + \theta_2 Y(T))^\gamma \right]$$

is maximal under the wealth constraint

$$\begin{cases} dX(t) &= [(\mu - b)\pi(t)X(t) + bX(t) + \alpha Y(t) + \beta Z(t) - c(t)]dt + \sigma\pi(t)X(t)dB(t) \\ X(s) &= \xi(s), \quad \pi(s) = \nu(s), \quad s \in [-\delta, 0] \end{cases}$$

for $\gamma < 1$, θ_1 and $\theta_2 \in \mathbb{R}$.

To solve this problem, we use the Hamilton-Jacobi-Bellman equation and the associated verification theorem.

By Theorem 3.1, the HJB equation gives :

$$\sup_{(c,\pi) \in (C,U)} \left\{ \frac{\partial V}{\partial t}(t, x, y) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, x, y) \sigma^2 \pi^2 x^2 + \frac{\partial V}{\partial x}(t, x, y) ((\mu - b)\pi x + bx + \alpha y + \beta z - c) \right. \\ \left. + (\pi x - e^{-\rho\delta} z - \rho y) \frac{\partial V}{\partial y}(t, x, y) + e^{-\rho\delta} \frac{c^\gamma}{\gamma} \right\} = 0,$$

First order conditions hold :

$$c^*(t) = \left[e^{st} \frac{\partial V}{\partial x}(t, x, y) \right]^{\frac{1}{\gamma-1}}, \\ \pi^*(t) = - \frac{\frac{\partial V}{\partial x}(t, x, y)(\mu - b) + \frac{\partial V}{\partial y}(t, x, y)}{\sigma^2 x \frac{\partial^2 V}{\partial x^2}(t, x, y)}.$$

Then HJB equation becomes :

$$\frac{\partial V}{\partial t}(t, x, y) + (bx + \alpha y + \beta z) \frac{\partial V}{\partial x}(t, x, y) - (e^{-\rho\delta} z + \rho y) \frac{\partial V}{\partial y}(t, x, y) \\ + \frac{1-\gamma}{\gamma} e^{\frac{1}{\gamma-1}st} \frac{\partial V}{\partial x}^{\frac{\gamma}{\gamma-1}}(t, x, y) - \frac{\left(\frac{\partial V}{\partial x}(t, x, y)(\mu - b) + \frac{\partial V}{\partial y}(t, x, y) \right)^2}{2\sigma^2 x \frac{\partial^2 V}{\partial x^2}(t, x, y)} = 0, \quad (5.4)$$

with $V(T, x, y) = \frac{1}{\gamma} (\theta_1 X(T) + \theta_2 Y(T))^\gamma$.

Let us now assume that $\beta = \frac{\alpha}{\rho + b} e^{-\rho\delta}$, $\theta_1 = b$, $\theta_2 = \frac{b\alpha}{\rho + b}$ and $V(t, x, y) = h(t) \left(bx + \frac{b\alpha}{\rho + b} y \right)^\gamma$.

Using these hypothesis, equation (3.1) is transformed into this ODE :

$$\dot{h}(t) + a(t) h(t)^{\frac{\gamma}{\gamma-1}} + Ah(t) = 0, \quad h(T) = \frac{1}{\gamma}$$

with

$$a(t) = \frac{1-\gamma}{\gamma} e^{\frac{1}{\gamma-1}st} (\gamma b)^{\frac{\gamma}{\gamma-1}}$$

and

$$A = \gamma b + \frac{\gamma(\mu - b + \frac{\alpha}{\rho+b})^2}{2\sigma^2(1-\gamma)}.$$

The solution of this equation is :

$$h(t) = \frac{- \int_t^T a(u) e^{-\frac{Au}{1-\gamma}} du + (1-\gamma)(\gamma e^{-AT})^{-\frac{1}{1-\gamma}}}{(1-\gamma)e^{At} \left(\frac{1-\gamma}{\gamma} \left(-\frac{1}{\gamma} \int_t^T a(u) e^{-\frac{Au}{1-\gamma}} du + \frac{1-\gamma}{\gamma} (\gamma e^{-AT})^{-\frac{1}{1-\gamma}} \right) \right)^\gamma}.$$

As $V(t, x, y) = h(t) \left(bx + \frac{b\alpha}{\rho + b} y \right)^\gamma$ solves the HJB equation, by Theorem 3.2 and by Theorem 3.3 it is the value function of the problem we consider. Finally, optimal consumption and portfolio are :

$$c^*(t) = [e^{st} \gamma b h(t)]^{\frac{1}{\gamma-1}} \left(bX^*(t) + \frac{b\alpha}{\rho + b} Y^*(t) \right), \quad \forall t \in [0, T]$$

and

$$\pi^*(t) = \frac{\mu - b + \frac{\alpha}{\rho + b}}{\sigma^2 b X^*(t) (1 - \gamma)} \left(bX^*(t) + \frac{b\alpha}{\rho + b} Y^*(t) \right), \quad \forall t \in [0, T].$$

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