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CALCULATING THE PARABOLIC CHERN CHARACTER OF A LOCALLY ABELIAN PARABOLIC BUNDLE

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Abstract. We calculate the parabolic Chern character of a bundle with locally abelian parabolic structure on a smooth strict normal crossings divisor, using the definition in terms of Deligne-Mumford stacks. We obtain explicit formulas for ch_1, ch_2 and ch_3, and verify that these correspond to the formulas given by Borne for ch_1 and Mochizuki for ch_2.

1. Introduction

Let X be a smooth projective variety with a strict normal crossings divisor $D = D_1 + \ldots + D_n \subset X$. The aim of this paper is to give an explicit formula for the parabolic Chern character of a locally abelian parabolic bundle on (X, D) in terms of:
—the Chern character of the underlying usual vector bundle,
—the divisor components D_i in the rational Chow groups of X,
—the Chern characters of the associated-graded pieces of the parabolic filtration along the multiple intersections of the divisor components, and
—the parabolic weights.

After giving a general formula, we compute explicitly the parabolic first, second, third parabolic Chern characters $ch_1^{Par}(E), ch_2^{Par}(E)$ and $ch_3^{Par}(E)$.

The basic idea is to use the formula given in [IS2]. However, their formula did not make clear the contributions of the different elements listed above. In order to adequatly treat this question, we start with a somewhat more general framework of *unweighted parabolic sheaves*. These are like parabolic sheaves except that the real parabolic weights are not specified. Instead, we consider linearly ordered sets Σ_i indexing the parabolic filtrations over the components D_i. Let Σ_i' denote the linearly ordered set of links or adjacent pairs in Σ_i. We also call these "risers" as Σ can be thought of as a set of steps. The parabolic weights are then considered as functions $\alpha_i : \Sigma_i' \to (-1, 0] \subseteq \mathbb{R}$. This division allows us to consider separately some Chern class calculations for the unweighted structures, and then the calculation of the parabolic Chern character using the parabolic weights.

A further difficulty stems from the fact that there are classically two different ways to give a parabolic structure: either as a collection of sheaves included in one

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another; or by fixing a bundle E (typically the zero-weight sheaf) plus a collection of filtrations of $E|_{D_i}$. The formula of [IS2] is expressed in terms of the collection of sheaves, whereas we look for a formula involving the filtrations. Thus, our first task is to investigate the relationship between these two points of view.

An important axiom concerning the parabolic structures considered here, is that they should be *locally abelian*. This means that they should locally be direct sums of parabolic line bundles. It is a condition on the simultaneous intersection of three or more filtrations; up to points where only two divisor components intersect, the condition is automatic. This condition has been considered by a number of authors (Borne [Bo1] [Bo2], Mochizuki [Mo2], Iyer-Simpson [IS1], Steer-Wren [Sr-Wr] and others) and is necessary for applying the formula of [IS2].

An unweighted parabolic sheaf consists then of a collection of sheaves $E_{\sigma_1,\ldots,\sigma_n}$ with $\sigma_i \in \Sigma_i$ on X, whereas an unweighted parabolic structure given by filtrations consists of a bundle E on X together with filtrations $F^i_{\sigma_i} \subset E|_{D_i}$ of the restrictions to the divisor components. In the locally abelian case, these may be related by a long exact sequence (5):

$$0 \longrightarrow E_{\sigma_1,\ldots,\sigma_n} \to E \to \bigoplus_{i=1}^{n} (\xi_i)_\star (L^i_{\sigma_i}) \to \bigoplus_{i<j} (\xi_{i_j})_\star (L^{i_j}_{\sigma_i,\sigma_j}) \to \ldots \to L_{\sigma_1,\ldots,\sigma_n} \longrightarrow 0.$$

Where $L^{i_1,\ldots,i_q}_{\sigma_{i_1},\ldots,\sigma_{i_q}}$ denote the quotient sheaves supported on intersections of the divisors $D_{i_1} \cap \ldots \cap D_{i_q}$.

Using this long exact sequence we get a formula (6) for the Chern characters of $E_{\sigma_1,\ldots,\sigma_n}$ in terms of the Chern character of sheaves supported on intersection of the divisors $D_{i_1} \cap \ldots \cap D_{i_q}$ of the form:

$$ch^{Vb}(E_{\sigma_1,\ldots\sigma_n}) = ch^{Vb}(E) + \sum_{q=1}^{n} (-1)^q \sum_{i_1<i_2<\ldots<i_q} ch\left(\xi_{I,\star}(L^I_{\sigma_{i_1},\ldots,\sigma_{i_q}})\right).$$

The notion of parabolic weight function is then introduced, and the main work of this paper begins: we obtain the Chern characters for the $E_{\alpha_1,\ldots,\alpha_n}$ for any $\alpha_i \in (-1,0]$; these are then put into the formula of [IS2], and the result is computed. This computation requires some combinatorial manipulations with the linearly ordered sets Σ_i notably the associated sets of risers Σ'_i in the ordering. It yields the following formula (3.4) of Theorem 3.4:

$$ch^{Par}(E) = ch^{Vb}(E)e^D +$$

$$e^D . \sum_{q=1}^{n} (-1)^q \sum_{i_1<i_2<\ldots<i_q} \sum_{\lambda_{i_j} \in \Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right) \prod_{j=1}^{q} \left[\frac{e^{D_{i_j}}(1 - e^{-(\alpha_{i_j}(\lambda_{i_j})+1)D_{i_j}})}{e^{D_{i_j}} - 1}\right].$$

In this formula, the associated-graded sheaves corresponding to the multiple filtrations on intersections of divisor components $D_I = D_{i_1} \cap \cdots \cap D_{i_q}$ are denoted by $Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}$. These are sheaves on D_I but are then considered as sheaves on

X by the inclusion $\xi_{I,\star} : D_I \hookrightarrow X$. The Chern character $ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)$ is the Chern character of the coherent sheaf on X. This is not satisfactory, since we want a formula involving the Chern characters of the $Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}$ on D_I. Therefore in §3.1 we use the Grothendieck-Riemann-Roch theorem to interchange ch and $\xi_{I,\star}$, leading to the introduction of Todd classes of the normal bundles of the D_I. Another difficulty is the factor of e^D multiplying the term $ch^{Vb}(E)$; we would like to consider the parabolic Chern class as a perturbation of the Chern class of the usual vector bundle $ch^{Vb}(E)$. Using the same formula for the case of trivial parabolic weights, which must give back $ch^{Vb}(E)$ as an answer, allows us to rewrite the difference between $ch^{Vb}(E)$ and $ch^{Vb}(E)e^D$ in a way compatible with the rest of the formula. After these manipulations the formula becomes (3.14) of Theorem 3.14:

$$ch^{Par}(E) = ch^{Vb}(E) -$$

$$e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<\ldots<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} \prod_{j=1}^{q}\left(\frac{1-e^{-D_{i_j}}}{D_{i_j}}\right).\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right) +$$

$$e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<\ldots<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} \prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

Finally, we would like to compute explicitly the terms $ch_1^{Par}(E)$, $ch_2^{Par}(E)$ and $ch_3^{Par}(E)$. For these, we expand the different terms

$$e^D, \quad \prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right], \quad \prod_{j=1}^{q}\left(\frac{1-e^{-D_{i_j}}}{D_{i_j}}\right)$$

in low-degree monomials of D_{i_j}, and then expand the whole formula dividing the terms up according to codimension. Denoting by $\mathcal{S} := \{1,\ldots,n\}$ the set of indices for divisor components, we get the following formulae:

- $ch_0^{Par}(E) := rank(E).[X]$
- $ch_1^{Par}(E) := ch_1^{Vb}(E) \; - \sum_{i_1\in\mathcal{S}} \sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]$
- $ch_2^{Par}(E) := \; ch_2^{Vb}(E) \; - \sum_{i_1\in\mathcal{S}} \sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\in\Sigma'_{i_1}\\ \lambda_{i_2}\in\Sigma'_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

For $ch_3^{Par}(E)$, see Section 5.

The formula for $ch_1^{Par}(E)$ is well-known (Seshadri *et al*) and, in terms of the definition of Chern classes using Deligne-Mumford stacks, it was shown by Borne in [Bo1]. The formula for $ch_2^{Par}(E)$ was given by Mochizuki in [Mo2], and also stated as a definition by Panov [Pa]. In both cases these coincide with our result (see the discussion on page 27). As far as we know, no similar formula for $ch_3^{Par}(E)$ has appeared in the literature.

Mochizuki defines the Chern classes using the curvature of an adapted metric and obtains his formula as a result of a difficult curvature calculation. It should be noted that our formula concerns the classes defined via Deligne-Mumford stacks in the rational Chow groups of X whereas Mochizuki's definition involving curvature can only define a class in cohomology. The identity of the two formulas shows that the curvature definition and the stack definition give the same result up to degree 2. Of course they must give the same result in general: to prove this for the higher Chern classes this is an interesting question for further study.

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2. Unweighted parabolic structures

2.1. **Index sets.** Let X be a smooth projective variety over an algebraically closed field of characteristic zero and let D be a strict normal crossings divisor on X. Write $D = D_1 + ... + D_n$ where D_i are the irreducible smooth components, meeting transversally. We sometimes denote by $\mathcal{S} := \{1, \ldots, n\}$ the set of indices for components of the divisor D.

Definition 2.1. *For $i = 1, ..., n$, let Σ_i be finite linear ordered sets with notations $\eta_i \leq ... \leq \sigma \leq \sigma' \leq \sigma'' \leq ... \leq \tau_i$ where η_i is the smallest element of Σ_i and τ_i the gratest element of Σ_i.*

Let Σ'_i be the connection between the σ's i.e

$$\Sigma'_i = \{(\sigma, \sigma'),\ s.t\ \sigma < \sigma' \text{ and there exist no } \sigma'' \text{ with } \sigma < \sigma'' < \sigma'\}.$$

Consider the tread functions $m_+ : \Sigma'_i \to \Sigma_i$ *and* $m_- : \Sigma'_i \to \Sigma_i$ *if* $\lambda = (\sigma, \sigma') \in \Sigma'_i$ *then* $\sigma = m_-(\lambda), \sigma' = m_+(\lambda)$. *In the other direction, consider the* riser functions $C_+ : \Sigma_i - \{\tau_i\} \to \Sigma'_i$ *and* $C_- : \Sigma_i - \{\eta_i\} \to \Sigma'_i$ *such that* $C_+(\sigma) = (\sigma, \sigma')$ *where* $\sigma' > \sigma$ *the next element and* $C_-(\sigma) = (\sigma'', \sigma)$ *where* $\sigma'' < \sigma$ *the next smaller element.*

One can think of the elements of Σ_i as the steps or "treads" of a staircase, with η_i and τ_i the lower and upper landings; then Σ_i' is the set of risers between stairs. The tread function sends a riser to the upper and lower treads, while the riser functions send a tread to the upper and lower risers. The upper riser of τ_i and the lower riser of η_i are undefined.

2.2. **Two approaches.**

Definition 2.2. *An* unweighted parabolic sheaf *$E_\cdot$ on (X, D) is a collection of sheaves E_σ indexed by multi-indices $\sigma = (\sigma_1, ..., \sigma_n)$ with $\sigma_i \in \Sigma_i$, together with inclusions of sheaves*

$$E_\sigma \hookrightarrow E_{\sigma'}$$

whenever $\sigma_i \leq \sigma_i'$ (a condition which we write as $\sigma \leq \sigma'$ in what follows), subject to the following hypothesis:

(1) $$E_{\sigma_1,..., \sigma_{i-1}, \eta_i, \sigma_{i+1},...,\sigma_n} = E_{\sigma_1,..., \sigma_{i-1}, \tau_i, \sigma_{i+1},..., \sigma_n}(-D_i).$$

Construction 2.3.

We have inclusions of sheaves

$$E_{\sigma_1,...,\sigma_n} \hookrightarrow E_{\tau_1,...,\tau_{i-1},\sigma_i,\tau_{i+1},...,\tau_n} \hookrightarrow E_{\tau_1,...,\tau_n}.$$

Consider the exact sequence

$$0 \longrightarrow E_{\tau_1,...,\tau_{i-1},\sigma_i,\tau_{i+1},...,\tau_n} \longrightarrow E_{\tau_1,...,\tau_n} \longrightarrow E_{\tau_1,...,\tau_n}/E_{\tau_1,...,\sigma_i,...,\tau_n} \longrightarrow 0,$$

and put $F^i_{\sigma_i} = \dfrac{E_{\tau_1,...,\sigma_i,...,\tau_n}}{E_{\tau_1,...,\eta_i,...,\tau_n}} \subset \dfrac{E_{\tau_1,...,\tau_n}}{E_{\tau_1,...,\eta_i,...,\tau_n}} = E\mid_{D_i}$ then we get the exact sequence

(2)

$$0 \longrightarrow E_{\sigma_1,...,\sigma_n} \longrightarrow E_{\tau_1,...,\tau_n} \longrightarrow \bigoplus_i E_{\tau_1,...,\tau_n}/E_{\tau_1,...,\sigma_i,...,\tau_n} = \bigoplus_i \frac{E_{\tau_1,...,\tau_n}/E_{\tau_1,...,\eta_i,...,\tau_n}}{E_{\tau_1,...,\sigma_i,...,\tau_n}/E_{\tau_1,...,\eta_i,...,\tau_n}}$$

which can be written as

$$0 \longrightarrow E_{\sigma_1,...,\sigma_n} \longrightarrow E_{\tau_1,...,\tau_n} \longrightarrow \bigoplus_i E\mid_{D_i}/F^i_{\sigma_i}.$$

Here, to $E_\cdot$ we associate the usual vector bundle $E := E_{\tau_1,...,\tau_n}$.

Definition 2.4. *Let E be a locally free sheaf on X suppose we have a filtration denoted by $F^i = \{F^i_\sigma \subseteq E\mid_{D_i}, \sigma \in \Sigma_i\}$ of $E\mid_{D_i}$ where $F^i_{\eta_i} = 0$ and $F^i_{\tau_i} = E\mid_{D_i}$ with the remaining terms being saturated subsheaves*

$$0 = F^i_{\eta_i} \subseteq F^i_\sigma \subseteq ... \subseteq F^i_{\tau_i} = E\mid_{D_i}$$

for each $i = 1, ..., n$. We call this a parabolic structure given by filtrations.

The construction 2.3 allows us to pass from an unweighted parabolic sheaf, to a parabolic structure given by filtrations. Suppose we are given an unweighted parabolic sheaf E_σ when all the component sheaves $E_{\sigma_1,\ldots,\sigma_n}$ are vector bundles. Set $E = E_{\tau_1,\ldots,\tau_n}$ and

$$E\mid_{D_i} = E_{\tau_1,\ldots,\tau_n}/E_{\tau_1,\ldots,\tau_{i-1},\eta_i,\ldots,\tau_n}.$$

The image of $E_{\tau_1,\ldots,\tau_{i-1},\sigma_i,\ldots,\tau_n}$ in $E\mid_{D_i}$ is a subsheaf $F^i_{\sigma_i}$, and we assume that it is a saturated subbundle. This gives a parabolic structure given by filtrations.

We can also go in the opposite direction.

Construction 2.5.

Suppose $(E, \{F^i_\sigma\})$ is a parabolic structure given by filtrations. Consider the kernel sheaves

$$0 \longrightarrow E_{\tau_1,\ldots,\sigma_i,\ldots,\tau_n} \longrightarrow E_{\tau_1,\ldots,\tau_n} \longrightarrow E\mid_{D_i}/F^i_{\sigma_i},$$

define a collection of sheaves

$$E_{\sigma_1,\ldots,\sigma_n} = \bigcap_i (E_{\tau_1,\ldots,\sigma_i,\ldots,\tau_n}) \subset E_{\tau_1,\ldots,\tau_n}$$

with has the property that

$$E_{\sigma_1,\ldots,\sigma_{i-1},\eta_i,\sigma_{i+1},\ldots,\sigma_n} = E_{\sigma_1,\ldots,\sigma_{i-1},\tau_i,\sigma_{i+1},\ldots,\sigma_n}(-D_i).$$

Thus we get an unweighted parabolic sheaf.

2.3. **Locally abelian condition.** An unweighted parabolic line bundle is an unweighted parabolic sheaf F such that all the F_σ are line bundles. An important class of examples is obtained as follows: if σ' is a multiindex consisting of $\sigma'_i \in \Sigma'_i$ then we can define an unweighted parabolic line bundle denoted

$$F := \mathcal{O}_X(\sigma') \tag{3}$$

by setting

$$F_{\sigma_1,\ldots,\sigma_n} := \mathcal{O}_X(\sum_{i=1}^n \gamma_i D_i)$$

where each γ_i is equal to -1 or 0; with $\gamma_i = -1$ when $\sigma_i < \sigma'_i$ and $\gamma_i = 0$ when $\sigma_i > \sigma'_i$. Note here that the relations $<, >$ are defined between treads σ_i and risers σ'_i.

On the other hand, if E is a locally -free sheaf on X then it may be considered as an unweighted parabolic sheaf (we say with trivial parabolic structure) by setting E_σ to be $E(\sum_{i=1}^n \gamma_i D_i)$ for $\gamma_i = 0$ if $\sigma_i = \tau_i$ and $\gamma_i = -1$ otherwise.

Definition 2.6. *Suppose E is a vector bundle on X and σ' is a multiindex consisting of $\sigma'_i \in \Sigma'_i$, we can define the unweighted parabolic bundle as follows:*

$$E(\sigma') := E \otimes \mathcal{O}_X(\sigma').$$

Lemma 2.7. *Any unweighted parabolic line bundle has the form $L(\sigma)$ for some σ' and L a line bundle on X.*

Definition 2.8. *An unweighted parabolic sheaf $E_\cdot$, or unweighted parabolic structure given by filtrations $(E, F_\cdot)$, is called* ***locally abelian parabolic bundle*** *if in a Zariski neighbourhood of any point $x \in X$ there is an isomorphism between F and a direct sum of unweighted parabolic line bundles.*

Lemma 2.9. *Suppose $E_{\sigma_1,...,\sigma_n}$ define a locally abelian parabolic bundle on X with respect to $(D_1, ..., D_n)$. Let $E = E_{\tau_1,...,\tau_n}$, which is a sheaf on X. Then E_σ comes from the construction 2.5 as above using unique filtrations $F^i_{\sigma_i}$ of $E\mid_{D_i}$ and we have the follwing properties:*

1) the $E_{\sigma_1,...,\sigma_n}$ are locally free;

2) for each q and collection of indices $(i_1, ..., i_q)$ at each point in the q-fold intersection $p \in D_{i_1} \cap ... \cap D_{i_q}$ the filtrations $F^{i_1}, ..., F^{i_q}$ of E_p admit a common splitting, hence the associated-graded

$$Gr_{j_1}^{F^{i_1}} ... Gr_{j_q}^{F^{i_q}}(E_p)$$

is independent of the order in which it is taken;

3) the functions

$$P \mapsto \text{rk } Gr_{j_1}^{F^{i_1}} ... Gr_{j_q}^{F^{i_q}}(E_p)$$

are locally constant functions of P on the multiple intersections $D_{i_1} \cap ... \cap D_{i_q}$.

Borne [Bo1] shows:

Theorem 2.10. *Suppose given a parabolic structure which is a collection of sheaves $E_{\sigma_1,...,\sigma_n}$ obtained from filtrations on bundle E as above. If the sheaves satisfy condition 1), or if the filtrations satisfy 2) and 3) of the previous lemma, then the parabolic structure is a locally abelian parabolic bundle on (X, D).*

Now we have two directions:

1- If we have a subsheaf structure

$$E(-D) = E_{\eta_1,...,\eta_n} \subseteq E_{\sigma_1,...,\sigma_n} \subseteq E_{\tau_1,...,\tau_n} = E$$

we can define the filtration structures

$$F^i_{\sigma_i} \subseteq E\mid_{D_i}$$

as in Construction 2.3 by using the exact sequence (2). Then, to calculate the Chern character of $E_{\sigma_1,...,\sigma_n}$ in terms of filtration structure, we must be find an extension for the left exact sequence (2) to a long exact sequence.

2- Vice versa.

Suppose we have a locally abelian parabolic structure $\{F^i\}$ given by filtrations, on a vector bundle E with filtrations

$$0 = F^i_{\eta_i} \subseteq F^i_{\sigma} \subseteq ... \subseteq F^i_{\tau_i} = E\mid_{D_i}.$$

Then for $\eta_i \leq \sigma_i \leq \tau_i$ define the quotient sheaves supported on D_i

$$L^i_{\sigma_i} := \frac{E\mid_{D_i}}{F^i_{\sigma_i}}$$

and the parabolic structure E_σ is given by

$$E_{\sigma_1,...,\sigma_n} = Ker(E \longrightarrow \bigoplus_{i=1}^{n} L^i_{\sigma_i}). \tag{4}$$

More generally define a family of multi-index quotient sheaves by

$$L^i_{\sigma_i} := \frac{E\mid_{D_i}}{F^i_{\sigma_i}} \ on\ D_i$$

$$L^i_{\sigma_i,\sigma_j} := \frac{E\mid_{D_i\cap D_j}}{F^i_{\sigma_i}|_{D_{i_j}} + F^j_{\sigma_j}|_{D_{i_j}}} \ on\ D_{i_j} = D_i \cap D_j$$

$$.$$

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$$L_{\sigma_1,...,\sigma_n} := \frac{E\mid_{D_1\cap...\cap D_n}}{F^1_{\sigma_1} + ... + F^n_{\sigma_n}} \ on\ D_1 \cap D_2 \cap ... \cap D_n.$$

In these notations we have $\eta_i \leq \sigma_i \leq \tau_i$, for $i = 1, ..., n$.

If we consider quotient sheaves as corresponding to linear subspaces of the dual projective bundle associated to E, then the multiple quotients above are multiple intersections of the $L^i_{\sigma_i}$. The formula (4) extends to a Koszul-style resolution of the component sheaves of the parabolic structure.

Lemma 2.11. *Suppose that the filtrations give a locally abelian parabolic structure, in particular they satisfy the conditions of Lemma 2.9. Then for any $\eta_i \leq \sigma_i \leq \tau_i$ the following sequence is well defined and exact over X:*
(5)

$$0 \longrightarrow E_{\sigma_1,\ldots,\sigma_n} \to E \to \bigoplus_{i=1}^{n} (\xi_i)_\star(L^i_{\sigma_i}) \to \bigoplus_{i<j} (\xi_{i,j})_\star(L^{i,j}_{\sigma_i,\sigma_j}) \to \ldots \to L_{\sigma_1 \to \ldots,\sigma_n} \longrightarrow 0.$$

Where E is a sheaf over X, $\bigoplus_i (L^i_{\sigma_i})$ are a sheaves over D_i, $\bigoplus_{i<j} (L^{i,j}_{\sigma_i,\sigma_j})$ are sheaves over $D_{i,j} = D_i \cap D_j$, etc.; ξ_i denotes the closed immersion $D_i \hookrightarrow X$, and $(\xi_i)_\star : coh(D_i) \longrightarrow coh(X)$ denotes the associated Gysin map. The general term is a sum over $I = (i_1, ..., i_q)$, where L^I are sheaves over $D_I = D_{i_1} \cap ... \cap D_{i_q}$ pushed forward by the Gysin map $(\xi_I)_\star : coh(D_I) \longrightarrow coh(X)$.

Proof. The proof in [IS2] is modified to cover the unweighted case. The maps in the exact sequence are obtained from the quotient structures of the terms with alternating signs like in the Cech complex. We just have to prove exactness. This a local question. By the locally abelian condition, we may assume that E with its filtrations is a direct sum of rank one pieces. The formation of the sequence, and its exactness, are compatible with direct sums. Therefore we may assume that E has rank one, and in fact $E \cong \mathcal{O}_X$.

In the case $\mathcal{O}_X(\sigma')$ as in (3), the vector bundle E is the trivial bundle and the filtration steps are either 0 or all of $\mathcal{O}_{D_i}$. In particular, there is $\eta_i \leq \sigma'_i \leq \tau_i$ such that $F^i_j = \mathcal{O}_{D_i}$ for $j \geq \sigma'_i$ and $F^i_j = 0$ for $j < \sigma'_i$. Then

$$L^{i_1,\ldots,i_q}_{\sigma_{i_1},\ldots,\sigma_{i_q}} = \mathcal{O}_{D_{i_1},\ldots,D_{i_q}}$$

if $\sigma_{i_j} < \sigma'_{i_j}$ for all $j = 1, ..., q$ and the quotient is zero otherwise. The sequence is defined for each multiindex $\sigma_1, ..., \sigma_n$. Up to reordering the coordinates which doesn't affect the proof, we may assume that there is $p \in [0, n]$ such that $\sigma_i < \sigma'_i$ for $i \leq p$ but $\sigma_i \geqslant \sigma'_i$ for $i > p$. In this case, the quotient is nonzero only when $i_1, ..., i_q \leqslant p$. Furthermore,

$$E_{\sigma_1,\ldots,\sigma_n} = \mathcal{O}(-D_1 - ... - D_p).$$

In local coordinates, the divisors $D_1, ..., D_p$ are coordinate divisors. Everything is constant in the other coordinate directions which we may ignore. The complex in question becomes

$$\mathcal{O}(-D_1 - ... - D_P) \to \mathcal{O} \to \oplus_{1 \leq i \leq p} \mathcal{O}_{D_i} \to \oplus_{1 \leq i \leq j \leq p} \mathcal{O}_{D_i \cap D_j} \to ... \to \mathcal{O}_{D_1 \cap .. \cap D_p}.$$

Etale locally, this is exactly the same as the exterior tensor product of p copies of the resolution of $\mathcal{O}_{\mathbb{A}^1}(-D)$ on the affine line $\mathbb{A}^1$ with divisor D corresponding to the origin,

$$\mathcal{O}_{\mathbb{A}^1}(-D) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_D \to 0.$$

In particular, the exterior tensor product complex is exact except at the beginning where it resolves $\mathcal{O}(-D_1 - ... - D_p)$ as required. □

Using the resolution of the above lemma we can compute the Chern character of $E_{\sigma_1,...\sigma_n}$ in terms of the Chern character of sheaves supported on intersection of the divisors $D_{i_1}\cap, ..., D_{i_q}$. This gives us

$$ch^{Vb}(E_{\sigma_1,...,\sigma_n}) = ch^{Vb}(E) + \sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} ch\left(\xi_{I,\star}(L^I_{\sigma_{i_1},...,\sigma_{i_q}})\right). \tag{6}$$

Definition 2.12. *Let $I = (i_1, ..., i_q)$ for $1 < i_1 < ... < i_q \leq n$ and analyse the quotient $L^{i_1,...,i_q}_{\sigma_{i_1},...,\sigma_{i_q}}$ along the multiple intersection $D_{i_1,...,i_q}$. There, the sheaf $E|_{D_{i_1,...,i_q}}$ has q filtrations $F^{i_j}_{\sigma_{i_j}}|_{D_{i_1,...,D_{i_q}}}$ indexed by $\sigma_{i_j} \in \Sigma_{i_j}$ leading to a multiple associated-graded defined as follows. put*

$$F^{i_1,...,i_q}_{\sigma_{i_1},...,\sigma_{i_q}} := F^{i_1}_{\sigma_{i_1}} \cap ... \cap F^{i_q}_{\sigma_{i_q}} \subset E|_{D_{i_1,...,i_q}}.$$

Where $\sigma_{i_j} \in \Sigma_{i_j}$, $F^{i_1,...,i_q}_{\eta_{i_1},...,\eta_{i_q}} = 0$ and $F^{i_1,...,i_q}_{\tau_{i_1},...,\tau_{i_q}} = E|_{D_{i_1,...,i_q}}$, we have $F^{i_1,...,i_q}_{\sigma_{i_1},...,\sigma_{i_q}} \subset F^{i_1,...,i_q}_{\sigma'_{i_1},...,\sigma'_{i_q}}$ if $\sigma_{i1} \leqslant \sigma'_{i_1}$ and $\sigma_{i_q} \leqslant \sigma'_{i_q}$. Then for a multiindex of risers $\lambda_{i_j} \in \Sigma'_{i_j}$, define

$$Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}} := \frac{F^{i_1,...,i_q}_{m_+(\lambda_{i_1}),...,m_+(\lambda_{i_q})}}{\sum_{j=1}^{q} F^{i_1,...,i_q}_{m_+(\lambda_{i_1}),...,m_+(\lambda_{i_k})-1,...,m_+(\lambda_{i_q})}}$$

where the indices in the denominator are almost all $m_+(\lambda_{i_j})$ but one $m_+(\lambda_{i_k}) - 1$. If the parabolic structure is locally abelian then the filtrations admit a common spliting and we have

$$Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}} = Gr^{F^{i_1}}_{\lambda_{i_1}} Gr^{F^{i_2}}_{\lambda_{i_2}} ... Gr^{F^{i_q}}_{\lambda_{i_q}}(E|_{D_{i_1,...,i_q}}).$$

Lemma 2.13. *Let U be a bundle over Y, and $F^p_1, F^p_2, ..., F^p_q$ are the filtrations such that ∃ common local bases. Then in group of Grothendieck we have*

$$Gr_{F_i} Gr_{F_j} \text{ commute, and } U \approx Gr_{F_1} Gr_{F_2} ... Gr_{F_q}(U).$$

Proof. This may be proven by an inductive argument. □

Theorem 2.14. *Suppose given a locally abelian parabolic bundle. Locally over D_I in the Zariski topology ∃ a finite set $\beta(\lambda_{i_1}, ..., \lambda_{i_q})$ such that we have a base over $E|_{D_i}$ of the form*

$$\{e_{\lambda_{i_1},...,\lambda_{i_q}; b}\}_{\substack{\lambda_{i_j}\in\Sigma'_{i_j} \\ b\in\beta(\lambda_{i_1},...,\lambda_{i_q})}}$$

and for the filtrations $F^{i_1,\ldots,i_q}_{\sigma_{i_1},\ldots,\sigma_{i_q}}$ *admit a base of the form*

$$\{e_{\lambda_{i_1},\ldots,\lambda_{i_q}};b\}_{\lambda_{i_j}<\sigma_{i_j}\ \ iff\ \ m_+(\lambda_{i_j})\leq\sigma_{i_j}}$$

and

$$\{\overline{e}_{\lambda_{i_1},\ldots,\lambda_{i_q}};b\}_{\lambda_{i_j}<\Sigma'_{i_j}}$$

form a base of $Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}$.

Proof. By the locally abelian condition, locally we may assume that the parabolic bundle is a direct sum of parabolic line bundles. For these, the bases have either zero or one elements and we can verify which are nonempty in terms of conditions on the σ_i. □

Corollary 2.15. *In the Grothendieck group of sheaves on* $D_{i_1}\cap\ldots\cap D_{i_q}$*, we have an equivalence*

$$F^{i_1,\ldots,i_q}_{\sigma_{i_1},\ldots,\sigma_{i_q}}\approx\sum_{\substack{\lambda_{i_1}<\sigma_{i_1}\\ \cdots\\ \lambda_{i_q}<\sigma_{i_q}}} Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}} \qquad (a)$$

and

$$\xi_{I,\star}\left(L^{i_1,\ldots,i_q}_{\sigma_{i_1},\ldots,\sigma_{i_q}}\right)\approx\xi_{I,\star}\left(\sum_{\substack{\sigma_{i_1}<\lambda_{i_1}\\ \cdots\\ \sigma_{i_q}<\lambda_{i_q}}} Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}\right) \qquad (b)$$

Now apply the part (b) of the above corollary in equation (6) we get a formula for the Chern character of $E_{\sigma_1,\ldots,\sigma_n}$ in terms of the associated graded as follows :

$$(7)\quad ch^{Vb}(E_{\sigma_1,\ldots,\sigma_n})=ch^{Vb}(E)+\sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\ \sum_{\substack{\sigma_{i_1}<\lambda_{i_1}\\ \cdots\\ \sigma_{i_q}<\lambda_{i_q}}} ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

3. Weighted parabolic structures

The next step is to introduce the notion of *weight function*, providing a real number $\alpha(i)(\lambda_i)$ for each $\lambda_i\in\Sigma'_i$. The weights naturally go with the "risers" of the linearly ordered sets, which is why we introduced the sets Σ'_i above.

We prolong Σ_i by adding its $\mathbb{Z}$-translates. Define

$$\Phi_i=\mathbb{Z}.\Sigma_i:=\mathbb{Z}\times\Sigma_i/\sim$$

s.t $(k,\tau_i)\sim(k+1,\eta_i)$, and

$$\Phi'_i:=\mathbb{Z}\times\Sigma'_i.$$

Prolong the functions

$$C_-,C_+:\mathbb{Z}.\Sigma_i\longrightarrow\mathbb{Z}\times\Sigma'_i$$

by setting $C_+(k,\tau)=(k+1,C_+(\eta))$, and $C_-(k,\eta)=(k-1,C_-(\tau))$.

For any unweighted parabolic sheaf we prolong the notation of $E_{\sigma_1,...,\sigma_n}$ to sheaves $E_{\varphi_1,...,\varphi_n}$ defined for all $\varphi_i = (k_i, \sigma_i) \in \mathbb{Z}.\Sigma_i$ as follows: define

$$E_{\varphi_1,...,\varphi_n} := E_{\sigma_1,...,\sigma_n}(\Sigma k_i D_i).$$

This is well defined modulo the equivalence relation defining Φ_i, because of the condition (1). This gives the property

$$E_{\varphi_1+l_1,...,\varphi_n+l_n} = E_{\varphi_1,...,\varphi_n}(l_1 D_1 + ... + l_n D_n)$$

where $l_i \in \mathbb{Z}$.

Definition 3.1. *A* weight function *is a collection of functions*

$$\alpha(i) : \Sigma_i' \to (-1, 0] \subseteq \mathbb{R}$$

which are increasing, i.e. $\alpha(i)(\lambda') \leq \alpha(i)(\lambda)$ *when* $\lambda' \leq \lambda$.

To transform from unweighted parabolic structure $\longrightarrow$ weighted parabolic structure we must extend the funchtion $\alpha(i)$ to all of $\mathbb{Z}.\Sigma_i'$ by :

$$\alpha(i) : \Sigma_i' \longrightarrow (-1, 0] \longrightarrow \alpha(i) : \Phi_i' \longrightarrow \mathbb{R}$$
$$\text{s.t } \alpha(i)(k, \sigma) = k + \alpha(i)(\sigma).$$

Now define intervals by :

$$Int\,(\alpha(i), \sigma_i) = (\alpha(i)C_-(\sigma_i), \alpha(i)C_+(\sigma_i)]\,,$$

and

$$Int\,(\alpha(i), \varphi_i) = (\alpha(i)C_-(\varphi_i), \alpha(i)C_+(\varphi_i)]\,.$$

We can now define the weighted or usual parabolic sheaf, associated to an unweighted parabolic sheaf and a weight function. Consider the sheaves $E_{\beta_1,...,\beta_n}$ for every $\beta_i \in \mathbb{R}^n$, given $\beta_1, ..., \beta_n$ $\forall\ i$ $\exists!$ $\varphi_i \in \Phi_i'$ such that $\beta_i \in Int(\alpha(i), \varphi_i)$, then we define

$$E_{\beta_1,...,\beta_n} := E_{\varphi_1,...,\varphi_n}$$

This defines a parabolic sheaf in the usual sense [Ma-Yo] [Mo2] [Bo2] [IS2].

Theorem 3.2. *Suppose* E *is a weighted parabolic bundle on* X *with respect to* $D_1, ..., D_n$. *Then we have the following formula for the Chern character of* E *:*

$$ch^{Par}(E) = \frac{\int_{\beta_1=0}^{1} ... \int_{\beta_n=0}^{1} e^{-\Sigma\beta_i D_i} ch^{Vb}(E_{\beta_1,...,\beta_n})}{\int_{\beta_1=0}^{1} ... \int_{\beta_n=0}^{1} e^{-\Sigma\beta_i D_i}}. \tag{8}$$

Proof. See [IS2] (15), p. 35. □

In this formula note the exponentials of real combinations of divisors are interpreted as formal polynomials. The power series for the exponential terminates because the product structure of $CH^{>0}(X)$ is nilpotent.

If the weights are real, then we need the integrals as in the formula, and the result is in $CH(X) \otimes_{\mathbb{Z}} \mathbb{R}$. If the weights are rational, then the integrals may be replaced by sums as in [IS2, Theorem 5.8]. In this case the answer lies in $CH(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. In what follows, if we were to replace the integrals by corresponding sums the answer would come out the same (a factor in the numerator depending on the denominator of the rational weights which are used, will cancel out with the same factor in the numerator). In order to simplify notation we keep to the integral formula.

Let $\varphi_i = (\sigma_i + 1)$ then

$$ch^{Par}(E) = \sum_{\varphi_1 \dots \varphi_n \in \Phi_1 \times \dots \times \Phi_n} \frac{\int_{\beta_1 \in Int(\alpha(1),\varphi_1) \cap (0,1]} \dots \int_{\beta_n \in Int(\alpha(n),\varphi_n) \cap (0,1]} e^{-\Sigma \beta_i D_i} ch^{Vb}(E_{\varphi_1,\dots,\varphi_n})}{\int_0^1 \dots \int_0^1 e^{-\Sigma \beta_i D_i}}$$

Remark 1. *$Int(\alpha(i), \varphi_i) \cap (0,1] = \emptyset$ if $\varphi_i \not\subseteq im(\{1\} \times \Sigma_i) \to \Phi_i$ i.e just if $\varphi_i = \sigma_i + 1$ for $\sigma_i \in \Sigma_i$.*

Definition 3.3. *Let $\varphi_i = \sigma_i + 1$, for $\sigma_i \in \Sigma_i$. Define domains by*

$$Dom(\alpha(i), \sigma_i) := Int(\alpha(i), \sigma_i + 1) \cap (0,1],$$

then

$$ch^{Par}(E) = \sum_{\sigma_1 \dots \sigma_n \in \Sigma_1 \times \dots \times \Sigma_n} \frac{\int_{Dom(\alpha(1),\sigma_1)} \dots \int_{Dom(\alpha(n),\sigma_n)} e^{-\Sigma \beta_i D_i} ch^{Vb}(E_{\varphi_1,\dots,\varphi_n})}{\int_0^1 \dots \int_0^1 e^{-\Sigma \beta_i D_i}}.$$

We have

$$E_{\varphi_1,..,\varphi_n} = E_{\sigma_1,\dots,\sigma_n}(D_1 + \dots + D_n)$$

then

$$ch^{Vb}(E_{\varphi_1,..,\varphi_n}) = ch^{Vb}(E_{\sigma_1,\dots,\sigma_n}) e^{D_1 + \dots + D_n}$$

therefore

$$ch^{Par}(E) = \left(\frac{1}{\int_0^1 \dots \int_0^1 e^{-\Sigma \beta_i D_i}} \right) \times$$

$$\sum_{\sigma_1 \dots \sigma_n \in \Sigma_1 \times \dots \times \Sigma_n} ch^{Vb}(E_{\sigma_1,\dots,\sigma_n}). \left(\int_{\beta_1 = \alpha_-(\sigma_1)+1}^{\alpha_+(\sigma_1)+1} . \int_{\beta_2 = \alpha_-(\sigma_2)+1}^{\alpha_+(\sigma_2)+1} \dots \int_{\beta_n = \alpha_-(\sigma_n)+1}^{\alpha_+(\sigma_n)+1} e^{-(\Sigma \beta_i D_i) + \Sigma D_i} d\beta \right)$$

for $i = 1, \dots, n$, where

$$\alpha_+(\sigma_i) = \begin{cases} \alpha(i)(C_+(\sigma_i)) & \text{or,} \\ 0 & \text{if } \sigma_i = \tau_i \end{cases}$$

and

$$\alpha_-(\sigma_i) = \begin{cases} \alpha(i)(C_-(\sigma_i)) & \text{or,} \\ -1 & \text{if } \sigma_i = \eta_i \end{cases}$$

so $Dom(\alpha(i), \sigma_i) = (\alpha_-(\sigma_i) + 1, \alpha_+(\sigma_i) + 1]$.

Then

$$ch^{Par}(E) = \left(\frac{1}{\int_0^1 ... \int_0^1 e^{-\Sigma \beta_i D_i}} \right) \times$$

$$\sum_{\sigma_1 ... \sigma_n \in \Sigma_1 \times ... \times \Sigma_n} ch^{Vb}(E_{\sigma_1,...,\sigma_n}). \left(\int_{\beta_1 = \alpha_-(\sigma_1)+1}^{\alpha_+(\sigma_1)+1} . \int_{\beta_2 = \alpha_-(\sigma_2)+1}^{\alpha_+(\sigma_2)+1} ... \int_{\beta_n = \alpha_-(\sigma_n)+1}^{\alpha_+(\sigma_n)+1} e^{-\Sigma(\beta_i - 1)D_i} d\beta \right).$$

Take $\gamma_i = \beta_i - 1$ for $i = 1, ..., n$ then

$$ch^{Par}(E) = \left(\frac{1}{\int_0^1 ... \int_0^1 e^{-\Sigma \beta_i D_i}} \right) \times$$

$$\sum_{\sigma_1 ... \sigma_n \in \Sigma_1 \times .. \times \Sigma_n} ch^{Vb}(E_{\sigma_1,...,\sigma_n}) \left(\int_{\gamma_1 = \alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)} e^{-\gamma_1 D_1} d\gamma_1 . \int_{\gamma_2 = \alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)} e^{-\gamma_2 D_2} d\gamma_2 \; .. \int_{\gamma_n = \alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)} e^{-\gamma_n D_n} d\gamma_n \right)$$

and we have

$$ch^{Vb}(E_{\sigma_1,...,\sigma_n}) = ch^{Vb}(E) + \sum_{q=1}^{n} (-1)^q \sum_{i_1 < i_2 < ... < i_q} \sum_{\substack{\sigma_{i_1} < \lambda_{i_1} \\ \sigma_{i_q} < \lambda_{i_q}}} ch \left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}}) \right).$$

Then

$$ch^{Par}(E) = \left(\frac{1}{\int_0^1 ... \int_0^1 e^{-\Sigma \beta_i D_i}} \right) \times$$

$$\sum_{\sigma_1\ldots\sigma_n\in\Sigma_1\times\ldots\times\Sigma_n}\left(ch^{Vb}(E)+\sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\sum_{\substack{\sigma_{i_1}<\lambda_{i_1}\\ \sigma_{i_q}<\lambda_{i_q}}}ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)\right)\times$$

$$\left(\int_{\gamma_1=\alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)}e^{-\gamma_1 D_1}d\gamma_1\int_{\gamma_2=\alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)}e^{-\gamma_2 D_2}d\gamma_2\ \ldots\ \int_{\gamma_n=\alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)}e^{-\gamma_n D_n}d\gamma_n\right)$$

$$=\left(\frac{ch^{Vb}(E)}{\int_0^1\ldots\int_0^1 e^{-\Sigma\beta_i D_i}}\right)\times$$

$$\left(\sum_{\sigma_1\in\Sigma_1}\int_{\gamma_1=\alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)}e^{-\gamma_1 D_1}d\gamma_1\right)\cdot\left(\sum_{\sigma_2\in\Sigma_2}\int_{\gamma_2=\alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)}e^{-\gamma_2 D_2}d\gamma_2\right)\cdots\left(\sum_{\sigma_n\in\Sigma_n}\int_{\gamma_n=\alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)}e^{-\gamma_n D_n}d\gamma_n\right)$$

$$+\ \left(\frac{1}{\int_0^1\ldots\int_0^1 e^{-\Sigma\beta_i D_i}}\right)\cdot\left[\sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\sum_{\lambda_{i_1}\ldots\lambda_{i_q}}ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)\times\right.$$

$$\left(\sum_{\substack{\sigma_1\in\Sigma_1\\ \sigma_1<\lambda_1\ if\ 1\in I}}\int_{\gamma_1=\alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)}e^{-\gamma_1 D_1}d\gamma_1\right)\cdot\left(\sum_{\substack{\sigma_2\in\Sigma_2\\ \sigma_2<\lambda_2\ if\ 2\in I}}\int_{\gamma_2=\alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)}e^{-\gamma_2 D_2}d\gamma_2\right)\cdots$$

$$\left.\cdot\left(\sum_{\substack{\sigma_n\in\Sigma_n\\ \sigma_n<\lambda_n\ if\ n\in I}}\int_{\gamma_n=\alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)}e^{-\gamma_n D_n}d\gamma_n\right)\right]$$

which can be written as

$$ch^{Par}(E)=A+B$$

where

$$A=\left(\frac{ch^{Vb}(E)}{\int_0^1\ldots\int_0^1 e^{-\Sigma\beta_i D_i}}\right)\times$$

$$\left(\sum_{\sigma_1\in\Sigma_1}\int_{\gamma_1=\alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)}e^{-\gamma_1 D_1}d\gamma_1\right)\cdot\left(\sum_{\sigma_2\in\Sigma_2}\int_{\gamma_2=\alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)}e^{-\gamma_2 D_2}d\gamma_2\right)\ldots\left(\sum_{\sigma_n\in\Sigma_n}\int_{\gamma_n=\alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)}e^{-\gamma_n D_n}d\gamma_n\right)$$

One can note that $\int_a^b e^{-\rho D}d\rho = \frac{e^{-aD}-e^{-bD}}{D} = e^{-aD}\frac{(1-e^{(a-b)D})}{D} = e^{-aD}\frac{(a-b)}{td((a-b)D)}$ where td is the Todd class.

We have $\sum_{\sigma_i\in\Sigma_i}\int_{\gamma_i=\alpha_-(\sigma_i)}^{\alpha_+(\sigma_i)} e^{-\gamma_i D_i}d\gamma_i = \int_{-1}^{0} e^{-\gamma_i D_i}d\gamma_i$ for $i = 1, 2, ..., n$ and put $\beta = \gamma$ as integration variable

then

$$A = \left(\frac{ch^{Vb}(E)}{\int_0^1 ... \int_0^1 e^{-\Sigma\gamma_i D_i}}\right).\left(\int_{-1}^{0} e^{-\gamma_1 D_1}d\gamma_1\right).\left(\int_{-1}^{0} e^{-\gamma_2 D_2}d\gamma_2\right)...\left(\int_{-1}^{0} e^{-\gamma_n D_n}d\gamma_n\right)$$

$$= \left[\left(\frac{\int_{-1}^{0} e^{-\gamma_1 D_1}d\gamma_1}{\int_0^1 e^{-\gamma_1 D_1}d\gamma_1}\right).\left(\frac{\int_{-1}^{0} e^{-\gamma_2 D_2}d\gamma_2}{\int_0^1 e^{-\gamma_2 D_2}d\gamma_2}\right)...\left(\frac{\int_{-1}^{0} e^{-\gamma_n D_n}d\gamma_n}{\int_0^1 e^{-\gamma_n D_n}d\gamma_n}\right)\right].ch^{Vb}(E)$$

where

$$A_i = \frac{\int_{-1}^{0} e^{-\gamma_i D_i}d\gamma_i}{\int_0^1 e^{-\gamma_i D_i}d\gamma_i} = \frac{e^{-D_i}-1}{1-e^{-D_i}} = \frac{e^{D_i}(1-e^{-D_i})}{1-e^{-D_i}} = e^{D_i}$$

$$for \quad i = 1, 2, ..., n$$

therefore

$$A = ch^{Vb}(E).e^{D_1}.e^{D_2}...e^{D_n} = ch^{Vb}(E)e^{D}$$

where $D = D_1 + D_2 + ... + D_n$

and

$$B = \left(\frac{1}{\int_0^1 ... \int_0^1 e^{-\Sigma\beta_i D_i}}\right).\left[\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_q}}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right)\times\right.$$

$$\left(\sum_{\substack{\sigma_1\in\Sigma_1\\ \sigma_1<\lambda_1 \ if \ 1\in I}}\int_{\gamma_1=\alpha_-(\sigma_1)}^{\alpha_+(\sigma_1)} e^{-\gamma_1 D_1}d\gamma_1\right).\left(\sum_{\substack{\sigma_2\in\Sigma_2\\ \sigma_2<\lambda_2 \ if \ 2\in I}}\int_{\gamma_2=\alpha_-(\sigma_2)}^{\alpha_+(\sigma_2)} e^{-\gamma_2 D_2}d\gamma_2\right)$$

$$\left....\left(\sum_{\substack{\sigma_n\in\Sigma_n\\ \sigma_n<\lambda_n \ if \ n\in I}}\int_{\gamma_n=\alpha_-(\sigma_n)}^{\alpha_+(\sigma_n)} e^{-\gamma_n D_n}d\gamma_n\right)\right]$$

The sums of integrals can be expressed as single integrals; if $i \notin I$, set formally $\alpha_i(\lambda_i) := 0$ in the following expression:

$$B = \left[\frac{(\int_{-1}^{0} e^{-\gamma_1 D_1} d\gamma_1)(\int_{-1}^{0} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{-1}^{0} e^{-\gamma_n D_n} d\gamma_n)}{(\int_{0}^{1} e^{-\gamma_1 D_1} d\gamma_1)(\int_{0}^{1} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{0}^{1} e^{-\gamma_n D_n} d\gamma_n)}\right] \times$$

$$\left(\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) \times\right.$$

$$\left[\frac{(\int_{-1}^{0} e^{-\gamma_1 D_1} d\gamma_1)(\int_{-1}^{0} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{-1}^{0} e^{-\gamma_n D_n} d\gamma_n)}{(\int_{-1}^{0} e^{-\gamma_1 D_1} d\gamma_1)(\int_{-1}^{0} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{-1}^{0} e^{-\gamma_n D_n} d\gamma_n)}\right] \times$$

$$\left.\left[\frac{(\int_{-1}^{\alpha_1(\lambda_1)} e^{-\gamma_1 D_1} d\gamma_1)(\int_{-1}^{\alpha_2(\lambda_2)} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{-1}^{\alpha_n(\lambda_n)} e^{-\gamma_n D_n} d\gamma_n)}{(\int_{-1}^{0} e^{-\gamma_1 D_1} d\gamma_1)(\int_{-1}^{0} e^{-\gamma_2 D_2} d\gamma_2)...(\int_{-1}^{0} e^{-\gamma_n D_n} d\gamma_n)}\right]\right)$$

$$= e^D . \sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) \prod_{j=1}^{q}\left[\frac{\int_{-1}^{\alpha_{i_j}(\lambda_{i_j})} e^{-\rho D_{i_j}} d\rho}{\int_{-1}^{0} e^{-\rho D_{i_j}} d\rho}\right]$$

$$= e^D . \sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) \prod_{j=1}^{q}\left[\frac{e^{D_{i_j}} - e^{-(\alpha_{i_j}(\lambda_{i_j}))D_{i_j}}}{e^{D_{i_j}} - 1}\right]$$

$$= e^D . \sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) \prod_{j=1}^{q}\left[\frac{e^{D_{i_j}}(1 - e^{-(\alpha_{i_j}(\lambda_{i_j})+1)D_{i_j}})}{e^{D_{i_j}} - 1}\right]$$

therefore we have proven the following

Theorem 3.4.

$$ch^{Par}(E) = ch^{Vb}(E)e^D +$$

(9)

$$e^D . \sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) \prod_{j=1}^{q}\left[\frac{e^{D_{i_j}}(1 - e^{-(\alpha_{i_j}(\lambda_{i_j})+1)D_{i_j}})}{e^{D_{i_j}} - 1}\right].$$

3.1. **Riemann-Roch theorem.** The next step is to use Riemann-Roch theory to interchange *ch* and $\xi_{I,\star}$. Let $K^\circ D_I$ denote the Grothendieck group of vector bundles(locally free sheaves) on D_I. Each vector bundle E determines an element, denoted $[E]$, in $K^\circ D_I$. $K^\circ D_I$ is the free abelian group on the set of isomorphism classes of vector bundles, modulo the relations

$$[E] \;=\; [E'] + [E'']$$

whenever E' is a subbundle of a vector bundle E with quotient bundle $E'' = E/E'$. The tensor product makes $K^\circ D_I$ a ring: $[E].[F] = [E \otimes F]$.

Definition 3.5. *For any morphism $\xi_I : X \to D_I$ there is an induced homomorphism*

$$\xi_I^\star : K^\circ D_I \to K^\circ X,$$

taking $[E]$ to $[\xi_I^\star E]$, where $\xi_I^\star E$ is the pull-back bundle.

The Grothendieck group of coherent sheaves on D_I, denoted by $K_\circ D_I$, is defined to be the free abelian group on the isomorphism class $[\mathcal{F}]$ of coherent sheaves on D_I, modulo the relaions

$$[\mathcal{F}] \;=\; [\mathcal{F}'] + [\mathcal{F}'']$$

for each exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on D_I. Tensor product makes $K_\circ D_I$ a $K^\circ D_I$-module:

$$K^\circ D_I \otimes K_\circ D_I \to K_\circ D_I,$$

$[E].[\mathcal{F}] = [E \otimes_{\mathcal{O}_{\mathcal{D}_\mathcal{I}}} \mathcal{F}]$.

Definition 3.6. *For any proper morphism $\xi_I : D_I \to X$, there is a homomorphism*

$$\xi_{I,\star} : K_\circ D_I \to K_\circ X$$

which takes $[\mathcal{F}]$ to $\sum_{i \geqslant 0} (-1)^i [R^i \xi_{I,\star} \mathcal{F}]$.where $R^i \xi_{I,\star} \mathcal{F}$ is Grothendieck higher direct image sheaf, the sheaf associated to the presheaf

$$U \to H^i(\xi_I^{-1}(U), \mathcal{F})$$

on X.

It is a basic fact the $R^i \xi_{I,\star} \mathcal{F}$ are coherent when $\mathcal{F}$ is coherent and ξ_I is proper. The fact that this push-forward $\xi_{I,\star}$ is well-defined on $K_\circ D_I$ results from the long excat cohomology sequence for the $R^i \xi_{I,\star}$.

Proposition 3.7. *The push-forward and pull-back are related by the usual projection formula:*

$$\xi_{I,\star}(\xi_I^\star a.b) \;=\; a.(\xi_{I,\star}b)$$

for $\xi_I : D_I \to X$ proper, $a \in K^\circ X$, $b \in K_\circ D_I$.

Theorem 3.8. *On any D_I there is a canonical duality homomorphism*

$$K^\circ D_I \to K_\circ D_I$$

which takes a vector bundle to its sheaf of sections. When D_I is non-singular, this duality map is an isomorphism.

Definition 3.9. *Consider D_I which are smooth over a given ground field $\mathbb{C}$. For such D_I we identify $K^\circ D_I$ and $K_\circ D_I$, and write simply $K(D_I)$. There is a homomorphism, called the Chern character*

$$ch : K(D_I) \to A(D_I)_{\mathbb{Q}}$$

determined by the following properties:
i) ch is a homomorphism of rings;
ii) if $\xi_I : X \to D_I$, $ch \circ \xi_I^\star = \xi_I^\star \circ ch$;
iii) if l is a line bundle on D_I, $ch[D_I] = exp(c_1(l)) = \sum_{i \geqslant 0}(1/i!)c_1(l)^i$

Theorem 3.10. *Let $\xi_I : D_I \hookrightarrow X$ be a smooth projective morphism of non-singular quasi-projective varieties. Then for any $Gr_\lambda^I \in K(D_I)$ we have*

$$\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}).td(T_{D_I})\right) = td(T_X).ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)$$

in $A(X) \otimes \mathbb{Q}$. where $Td(X) = td(T_X) \in A(X)_{\mathbb{Q}}$ is the relative tangent sheaf of ξ_I.

See [Fu, pp 286-287].

Theorem 3.11. *If D_I is a non-singular variety set*

$$Td(D_I) = td(T_{D_I}) \in A(D_I)_{\mathbb{Q}}$$

then

if $\xi_I : D_I \hookrightarrow X$, is a closed imbedding of codimension q, and D_I is the intersection of q Cartier divisors $D_{i_1}, ..., D_{i_q}$ on X, then

$$Td(D_I) = \xi_I^\star\left[Td(X).\prod_{j=1}^{q}\left(\frac{1-e^{-Di_j}}{D_{i_j}}\right)\right].$$

See [Fu, p. 293].

So

$$\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}).td(T_{D_I})\right) = \xi_{I,\star}\left[ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}}).\xi_I^\star\left(Td(X).\prod_{j=1}^{q}\left(\frac{1-e^{-Di_j}}{D_{i_j}}\right)\right)\right]$$

$$= \; Td(X).\prod_{j=1}^{q}\left[\frac{1-e^{-Di_j}}{D_{i_j}}\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

Corollary 3.12.

$$ch\left(\xi_{I_\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right) = \prod_{j=1}^{q}\left[\frac{1-e^{-Di_j}}{D_{i_j}}\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)$$

$$= \prod_{j=1}^{q}\left[\left(\frac{e^{-D_{i_j}}(e^{D_{i_j}}-1)}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

Theorem 3.13. *Apply the above corollary in the equation* (3.4) *of Theorem 3.4 we get*

$$ch^{Par}(E) = ch^{Vb}(E)e^D +$$

(10)

$$e^D.\sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\;\sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

Now return to the equation (7) but apply it to the bottom value of $\sigma_i = \eta_i$. Recall that $E_{\eta_1,\ldots,\eta_n} = E(-D)$; but on the other hand $\eta_{i_j} < \lambda_{i_j}$ for any $\lambda_{i_j} \in \Sigma'_{i_j}$. Thus we get

$$ch^{Vb}(E(-D)) = ch^{Vb}(E) + \sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\;\sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

$\Longrightarrow$

$$(11)\;\; ch^{Vb}(E) = ch^{Vb}\left(E(-D)\right) - \sum_{q=1}^{n}(-1)^q\sum_{i_1<i_2<\ldots<i_q}\;\sum_{\lambda_{i_j}\in\Sigma'_{i_j}} ch\left(\xi_{I,\star}(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

Put equation (11) in (3.13) and use Corollary 3.12, to get

$$ch^{Par}(E) = ch^{Vb}\left(E(-D)\right)e^{D} -$$

$$e^{D}.\sum_{q=1}^{n}(-1)^{q}\sum_{i_1<i_2<\ldots<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\prod_{j=1}^{q}\left(\frac{1-e^{-D_{i_j}}}{D_{i_j}}\right).\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)+$$

$$e^{D}.\sum_{q=1}^{n}(-1)^{q}\sum_{i_1<i_2<\ldots<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)$$

We have $ch^{Vb}\left(E(-D)\right)e^{D} = ch^{Vb}(E)$, therefore:

Theorem 3.14.

$$ch^{Par}(E) = ch^{Vb}(E) -$$

$$e^{D}.\sum_{q=1}^{n}(-1)^{q}\sum_{i_1<i_2<\ldots<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\prod_{j=1}^{q}\left(\frac{1-e^{-D_{i_j}}}{D_{i_j}}\right).\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right)+$$

(12)

$$e^{D}.\sum_{q=1}^{n}(-1)^{q}\sum_{i_1<i_2<\ldots<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,\ldots,i_q}_{\lambda_{i_1},\ldots,\lambda_{i_q}})\right).$$

4. Computation of parabolic Chern characters of a locally abelian parabolic bundle E in codimension one and two $ch_1^{Par}(E),\ ch_2^{Par}(E)$

Proposition 4.1. *For an n-dimensional, non singular variety Y, set*

$$A^pY = A_{n-p}Y,$$

where p denotes the codimension, and $n-p$ the dimension. With this indexing by codimension, the product $x\otimes y\to x.y$, reads

$$A^pY\otimes A^qY\to A^{p+q}Y,$$

i.e, the degrees add. Let $1\in A^0Y$ denote the class corresponding to $[Y]$ in A_nY, and set $A^*Y=\bigoplus A^pY$.

Return to the equation (3.14)

$$ch^{Par}(E) = ch^{Vb}(E) -$$
$$e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} \prod_{j=1}^{q}\left(\frac{1-e^{-D_{i_j}}}{D_{i_j}}\right).\xi_{I,\star}\left(ch(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right) +$$
$$e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q} \sum_{\lambda_{i_j}\in\Sigma'_{i_j}} \prod_{j=1}^{q}\left[\left(\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}}\right)\right].\xi_{I,\star}\left(ch(Gr^{i_1,...,i_q}_{\lambda_{i_1},...,\lambda_{i_q}})\right)$$

take

$$\frac{1-e^{-\left(\alpha_{i_j}(\lambda_{i_j})+1\right)D_{i_j}}}{D_{i_j}} = \frac{1-\left(1-(\alpha_{i_j}(\lambda_{i_j})+1)D_{i_j}+\frac{(\alpha_{i_j}(\lambda_{i_j})+1)^2}{2}D_{i_j}^2-\frac{(\alpha_{i_j}(\lambda_{i_j})+1)^3}{6}D_{i_j}^3+...\right)}{D_{i_j}}$$

$$= \left(\alpha_{i_j}(\lambda_{i_j})+1\right) - \frac{\left(\alpha_{i_j}(\lambda_{i_j})+1\right)^2}{2}D_{i_j} + \frac{\left(\alpha_{i_j}(\lambda_{i_j})+1\right)^3}{6}D_{i_j}^2 - ...$$

Let Gr^I_λ be a vector bundles over D_I for $I=(i_1,...,i_q)$ and $\lambda=(\lambda_{i_1},...,\lambda_{i_q})$ with rank r, we have

$$\begin{aligned}\xi_{I,\star}\left(ch(Gr^I_\lambda)\right) &= \xi_{I,\star}\left(ch_0^{D_I}(Gr^I_\lambda)+ch_1^{D_I}(Gr^I_\lambda)\right.\\ &\quad \left.+ch_2^{D_I}(Gr^I_\lambda)+ch_3^{D_I}(Gr^I_\lambda)+...\right)/X\\ &= \xi_{I,\star}\left(ch_0^{D_I}(Gr^I_\lambda)\right)/X+\xi_{I,\star}\left(ch_1^{D_I}(Gr^I_\lambda)\right)/X\\ &\quad +\xi_{I,\star}\left(ch_2^{D_I}(Gr^I_\lambda)\right)/X+\xi_{I,\star}\left(ch_3^{D_I}(Gr^I_\lambda)\right)/X+....\end{aligned}$$

so

$$ch_0^{D_I}(Gr^I_\lambda) = (rank(Gr^I_\lambda))/D_I = \sum_{p\in Irr(D_I)} rank_p(Gr^I_\lambda).[D_p]/D_I$$
$$\text{in } A^0(D_I) = A_{dimD_I}(D_I) = \bigoplus A^0(D_p) = \bigoplus \mathbb{Q}.[D_p]$$

where $D_I = \bigcup_{p\in Irr(D_I)} D_p$ and $Irr(D_I)$ denotes the set of irreducible components of $D_{i_1}\cap D_{i_2}\cap...\cap D_{i_q}$ then

$$\xi_{I,\star}\left(\sum_{p\in Irr(D_I)} rank_p(Gr^I_\lambda).[D_p]/D_I\right) = \left(\sum_{p\in Irr(D_I)} rank_p(Gr^I_\lambda).[D_p]/X\right) \in A^q(X)$$

which is of codimension q,

$ch_1^{D_I}(Gr_\lambda^I) = c_1^{D_I}(Gr_\lambda^I) \in A^1(D_I) = A_{dimD_I-1}(D_I)$ then

$\left(\xi_{I,\star}\left(c_1^{D_I}(Gr_\lambda^I)\right)/X\right) \in A^{1+q}(X)$ which is of codimension $q+1$,

$ch_2^{D_I}(Gr_\lambda^I) = \frac{1}{2}\left[(c_1^{D_I})^2(Gr_\lambda^I) - 2c_2^{D_I}(Gr_\lambda^I)\right] \in A^2(D_I) = A_{dimD_I-2}(D_I)$ then

$\xi_{I,\star}\left(\frac{1}{2}[(c_1^{D_I})^2(Gr_\lambda^I) - 2c_2^{D_I}(Gr_\lambda^I)]\right)/X \in A^{2+q}$ which is of codimension $q+2$,

$ch_3^{D_I}(Gr_\lambda^I) = \frac{1}{6}\left[(c_1^{D_I})^3(Gr_\lambda^I) - 3c_1^{D_I}(Gr_\lambda^I)c_2^{D_I}(Gr_\lambda^I)] + 3c_3^{D_I}(Gr_\lambda^I)\right] \in A^3(D_I)$ then

$\xi_{I,\star}\left(\frac{1}{6}\left[(c_1^{D_I})^3(Gr_\lambda^I) - 3(c_1^{D_I}(Gr_\lambda^I)c_2^{D_I}(Gr_\lambda^I) + 3c_3^{D_I}(Gr_\lambda^I)\right]\right)/X \in A^{3+q}$

which is of codimension $q+3$. Therefore

$$ch^{Par}(E) = ch^{Vb}(E) + e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}$$

$$\prod_{j=1}^{q}\left[\left(\alpha_{i_j}(\lambda_{i_j})+1\right) - \frac{\left(\alpha_{i_j}(\lambda_{i_j})+1\right)^2}{2}D_{i_j} + \frac{\left(\alpha_{i_j}(\lambda_{i_j})+1\right)^3}{6}D_{i_j}^2 - ...\right].$$

$$\Big(\sum_{p\in Irr(D_I)} rank_p(Gr_\lambda^I).[D_p]/X + \xi_{I,\star}(c_1^{D_I}(Gr_\lambda^I))/X + \xi_{I,\star}(\frac{1}{2}[(c_1^{D_I})^2(Gr_\lambda^I) - 2c_2^{D_I}(Gr_\lambda^I)])/X$$

$$+\xi_{I,\star}(\frac{1}{6}[(c_1^{D_I})^3(Gr_\lambda^I) - 3(c_1^{D_I}(Gr_\lambda^I)c_2^{D_I}(Gr_\lambda^I) + 3c_3^{D_I}(Gr_\lambda^I)]))$$

$$-e^D.\sum_{q=1}^{n}(-1)^q \sum_{i_1<i_2<...<i_q}\ \sum_{\lambda_{i_j}\in\Sigma'_{i_j}}\ \prod_{j=1}^{q}\left(1 - \frac{D_{i_j}}{2} + \frac{D_{i_j}^2}{6} - ...\right)\cdot\Bigg[\sum_{p\in Irr(D_I)} r_p(Gr_\lambda^I).[D_p]/X +$$

$$\xi_{I,\star}(c_1^{D_I}(Gr_\lambda^I))/X + \xi_{I,\star}(\frac{1}{2}[(c_1^{D_I})^2(Gr_\lambda^I) - 2c_2^{D_I}(Gr_\lambda^I)])/X +$$

$$\xi_{I,\star}(\frac{1}{6}[(c_1^{D_I})^3(Gr_\lambda^I) - 3(c_1^{D_I}(Gr_\lambda^I)c_2^{D_I}(Gr_\lambda^I) + 3c_3^{D_I}(Gr_\lambda^I)])\Bigg] \tag{13}$$

we have

$ch^{Vb}(E) = rank(E).[X] + ch_1^{Vb}(E) + ch_2^{Vb}(E) + ch_3^{Vb}(E) + ...$

where $rank(E).[X] \in A^0,\ ch_1^{Vb}(E) \in A^1,\ ch_2^{Vb}(E) \in A^2,\ ch_3^{Vb}(E) \in A^3,$

$and\ e^D = 1 + D + \frac{D^2}{2} + \frac{D^3}{6} + ...$ where $1 \in A^0,\ D \in A^1,\ \frac{D^2}{2} \in A^2,\ \frac{D^3}{6} \in A^3$

for $k = 0, 1, ..., n,\ ch_k^{Par}(E) = ch_0^{Par}(E) + ch_1^{Par}(E) + ch_2^{Par}(E) + ch_3^{Par}(E) + ...$

then

$ch_0^{Par}(E) + ch_1^{Par}(E) + ch_2^{Par}(E) = rank(E).[X]/A^0 + ch_1^{Vb}(E)/A^1$

$+ ch_2^{Vb}(E)/A^2 - \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} (\alpha_{i_1}(\lambda_{i_1}) + 1).rank(Gr_{\lambda_{i_1}}^{i_1}).[D_{i_1}]/A^1$

$- \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} (\alpha_{i_1}(\lambda_{i_1}) + 1).(\xi_{i_1})_\star \left(c_1^{D_{i_1}}(Gr_{\lambda_{i_1}}^{i_1}) \right)/A^2$

$- \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} (\alpha_{i_1}(\lambda_{i_1}) + 1).rank(Gr_{\lambda_{i_1}}^{i_1}).[D_{i_1}].[D]/A^2$

$+ \sum_{i_1 < i_2} \sum_{\substack{\lambda_{i_1} \\ \lambda_{i_2}}} \sum_{p \in Irr(D_{i_1} \cap D_{i_2})} (\alpha_{i_1}(\lambda_{i_1}) + 1)(\alpha_{i_2}(\lambda_{i_2}) + 1).rank_p(Gr_{\lambda_{i_1}, \lambda_{i_2}}^{i_1, i_2}).[D_p]/A^2$

$+ \frac{1}{2} \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} (\alpha_{i_1}(\lambda_{i_1}) + 1)^2.rank(Gr_{\lambda_{i_1}}^{i_1}).[D_{i_1}]^2/A^2$

$+ \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} rank(Gr_{\lambda_{i_1}}^{i_1}).[D_{i_1}]/A^1$

$+ \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} (\xi_{i_1})_\star \left(c_1^{D_{i_1}}(Gr_{\lambda_{i_1}}^{i_1}) \right)/A^2$

$$+\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}].[D]/A^2$$

$$-\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]/A^2$$

$$-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2/A^2$$

Lemma 4.2. *In the Grothendieck group, for all $i_1, i_2 \in \mathcal{S}$, for all $\lambda_{i_1} \in \Sigma'_{i_1}$, for all $p \in Irr(D_{i_1} \cap D_{i_2})$, we have:*

$$rank(Gr^{i_1}_{\lambda_{i_1}}) = \sum_{\lambda_{i_2}\in\Sigma'_{i_2}} rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).$$

Now we have $D = \sum_i D_i$ and $[D_{i_1}].[D_{i_2}] = \sum_{p\in Irr(D_{i_1}\cap D_{i_2})} [D_p]$ then

$$[D].[D_{i_1}] = \sum_{i_1\in\mathcal{S}}[D_{i_1}]^2 + \sum_{i_1\neq i_2}[D_{i_1}].[D_{i_2}] = \sum_{i_1\in\mathcal{S}}[D_{i_1}]^2 + \sum_{i_1\neq i_2}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}[D_p].$$

Then

$$ch_0^{Par}(E) + ch_1^{Par}(E) + ch_2^{Par}(E) = rank(E).[X]/A^0 + ch_1^{Vb}(E)/A^1$$

$$+ ch_2^{Vb}(E)/A^2 - \sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]/A^1$$

$$-\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)/A^2$$

$$-\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2/A^2$$

$$-\sum_{i_1\in\mathcal{S}}\sum_{i_1\neq i_2}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\sum_{\lambda_{i_2}\in\Sigma'_{i_2}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]/A^2$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).rank_p(Gr^{i_1i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]/A^2$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]/A^2$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]/A^2$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2/A^2$$

$$+\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2/A^2$$

4.1. **The characteristic numbers for parabolic bundle in codimension one and two.** -For any parabolic bundle E in codimension one, and two, the parabolic first, second Chern characters $ch_1^{Par}(E), and\ ch_2^{Par}(E)$, are obtained as follows:

- $ch_0^{Par}(E) := rank(E).[X]$
- $ch_1^{Par}(E) := ch_1^{Vb}(E) \ - \sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\Sigma'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]$
- $ch_2^{Par}(E) := \ ch_2^{Vb}(E) \ - \sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

In order to compare with Mochizuki's formula, note that

$$\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p] =$$

$$\frac{1}{2}\sum_{i_1\neq i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p]$$

therefore our formula may be written:

$$\bullet\ ch_2^{Par}(E) := ch_2^{Vb}(E) - \sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$+\frac{1}{2}\sum_{i_1\neq i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

This coincides exactly with the formula given by Mochizuki in [Mo2, §3.1.5, p. 30]. It is also the same as the definition given by Panov [Pa]. Note that in Panov's general definition the sum for the last term is written as $\sum_{i,j}$ without the factor of $1/2$ but later he uses it as a sum over $i<j$, so our formula and Mochizuki's also coincide with Panov's formula in the way he uses it.

Mochizuki's formula was for the Chern character in cohomology, which he defined as the integral of the Chern form of the curvature of an adapted metric. Our calculation verifies that this gives the same answer as the method using Deligne-Mumford stacks of [Bi] [Bo1] [IS1] for rational weights. Our formula is valid for the Chern character in the rational or real Chow ring.

Here we explain some of the notation:

- $ch_1^{Vb}(E), ch_2^{Vb}(E)$ denotes the first, second, Chern character of vector bundles E.

- $Irr(D_I)$ denotes the set of the irreducible components of $D_I := D_{i_1}\cap D_{i_2}\cap ...\cap D_{i_q}$.

▪ ξ_I denotes the closed immersion $D_I \longrightarrow X$, and $\xi_{I,\star} : A^k(D_I) \longrightarrow A^{k+q}(X)$ denotes the associated Gysin map.

▪ Let p be an element of $Irr(D_i \cap D_j)$. Then $rank_p(Gr^I_\lambda)$ denotes the rank of Gr^I_λ as an $\mathcal{O}_p$-module.

▪ $[D_{i_j}] \in A^1(X) \otimes \mathbb{Q}$, and $[D_p] \in A^2(X) \otimes \mathbb{Q}$ denote the cycle classes given by D_{i_j} and D_p respectively.

5. Parabolic Chern character of a locally abelian parabolic bundle E in codimension 3, $Ch_3^{Par}(E)$

By the same method of computation as above, we get the following formula, which has not been considered elsewhere in the literature.

$$ch_3^{Par}(E) = ch_3^{Vb}(E) - \frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}].[D]^2$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1)^2.(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D_{i_1}]$$

$$-\frac{1}{6}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1)^3.rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^3$$

$$-\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D]$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1)^2.rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2.[D]$$

$$-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\alpha_{i_1}(\lambda_{i_1})+1).(\xi_{i_1})_\star\left((c_1^{D_{i_1}})^2(Gr^{i_1}_{\lambda_{i_1}})-2c_2^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$-\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}(\alpha_{i_1}(\lambda_{i_1})+1).(\alpha_{i_2}(\lambda_{i_2})+1)^2.rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_{i_2}].[D_p]$$

$$-\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}(\alpha_{i_1}(\lambda_{i_1})+1)^2.(\alpha_{i_2}(\lambda_{i_2})+1).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_{i_1}].[D_p]$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}(\alpha_{i_1}(\lambda_{i_1})+1).(\alpha_{i_2}(\lambda_{i_2})+1).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D].[D_p]$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}(\alpha_{i_1}(\lambda_{i_1})+1).(\alpha_{i_2}(\lambda_{i_2})+1).(\xi_{i_1,i_2})_\star\left(c_1^{D_{i_1}\cap D_{i_2}}(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}})\right)$$

$$-\sum_{i_1<i_2<i_3}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_3}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2}\cap D_{i_3})}(\alpha_{i_1}(\lambda_{i_1})+1).(\alpha_{i_2}(\lambda_{i_2})+1).(\alpha_{i_3}(\lambda_{i_3})+1).rank_p(Gr^{i_1,i_2,i_3}_{\lambda_{i_1},\lambda_{i_2},\lambda_{i_3}}).[D_p]$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}].[D]^2-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D_{i_1}]$$

$$+\frac{1}{6}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^3+\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D]$$

$$-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2.[D]+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}(\xi_{i_1})_\star\left((c_1^{D_{i_1}})^2(Gr^{i_1}_{\lambda_{i_1}})-2c_2^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$-\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D].[D_p]-\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}(\xi_{i_1,i_2})_\star\left(c_1^{D_{i_1}\cap D_{i_2}}(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}})\right)$$

$$+\sum_{i_1<i_2<i_3}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_3}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2}\cap D_{i_3})}rank_p(Gr^{i_1,i_2,i_3}_{\lambda_{i_1},\lambda_{i_2},\lambda_{i_3}}).[D_p]$$

$$+\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_{i_2}].[D_p]$$

$$+\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})} rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_1}}).[D_{i_1}].[D_p]$$

5.1. **The characteristic number for parabolic bundle in codimension three.** -For any parabolic bundle E in codimension 3, the parabolic third Chern character $ch_3^{Par}(E)$, is obtained as follows:

$$ch_3^{Par}(E) = ch_3^{Vb}(E) - \frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}].[D]^2$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\left[\alpha^2_{i_1}(\lambda_{i_1})+2\alpha_{i_1}(\lambda_{i_1})\right].(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D_{i_1}]$$

$$-\frac{1}{6}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}[\alpha^3_{i_1}(\lambda_{i_1})+3\alpha^2_{i_1}(\lambda_{i_1})+3\alpha_{i_1}(\lambda_{i_1})].rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^3$$

$$-\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).[D]$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}[\alpha^2_{i_1}(\lambda_{i_1})+2\alpha_{i_1}(\lambda_{i_1})].rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2.[D]$$

$$-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left((c_1^{D_{i_1}})^2(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$+\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star\left(c_2^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$-\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}[\alpha^2_{i_2}(\lambda_{i_2}).\alpha_{i_1}(\lambda_{i_1})+2\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+\alpha_{i_1}(\lambda_{i_1})+\alpha^2_{i_2}(\lambda_{i_2})+$$
$$2\alpha_{i_2}(\lambda_{i_2})].rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_{i_2}].[D_p]$$

$$-\frac{1}{2}\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}[\alpha^2_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+2\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+\alpha_{i_2}(\lambda_{i_2})+\alpha^2_{i_1}(\lambda_{i_1})+$$
$$2\alpha_{i_1}(\lambda_{i_1})].rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_{i_1}].[D_p]$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}[\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+\alpha_{i_1}(\lambda_{i_1})+\alpha_{i_2}(\lambda_{i_2})].rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D].[D_p]$$

$$+\sum_{i_1<i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}[\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+\alpha_{i_1}(\lambda_{i_1})+\alpha_{i_2}(\lambda_{i_2})].(\xi_{i_1,i_2})_\star\left(c_1^{D_{i_1}\cap D_{i_2}}(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}})\right)$$

$$-\sum_{i_1<i_2<i_3}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_3}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2}\cap D_{i_3})}[\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).\alpha_{i_3}(\lambda_{i_3})+\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2})+$$

$$\alpha_{i_2}(\lambda_{i_2}).\alpha_{i_3}(\lambda_{i_3})+\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_3}(\lambda_{i_3})+\alpha_{i_1}(\lambda_{i_1})+\alpha_{i_2}(\lambda_{i_2})+\alpha_{i_3}(\lambda_i)].rank_p(Gr^{i_1,i_2,i_3}_{\lambda_{i_1},\lambda_{i_2},\lambda_{i_3}}).[D_p].$$

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