



HAL
open science

Strange duality for Verlinde spaces of exceptional groups at level one

Arzu Boysal, Christian Pauly

► **To cite this version:**

Arzu Boysal, Christian Pauly. Strange duality for Verlinde spaces of exceptional groups at level one. 2009. hal-00373573v1

HAL Id: hal-00373573

<https://hal.science/hal-00373573v1>

Preprint submitted on 6 Apr 2009 (v1), last revised 14 Sep 2009 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STRANGE DUALITY FOR VERLINDE SPACES OF EXCEPTIONAL GROUPS AT LEVEL ONE

ARZU BOYSAL AND CHRISTIAN PAULY

ABSTRACT. The moduli stack $\mathcal{M}_X(E_8)$ of principal E_8 -bundles over a smooth projective curve X carries a natural divisor Δ . We study the pull-back of the divisor Δ to the moduli stack $\mathcal{M}_X(P)$, where P is a semi-simple and simply connected group such that its Lie algebra $\mathrm{Lie}(P)$ is a maximal conformal subalgebra of $\mathrm{Lie}(E_8)$. We show that the divisor Δ induces “Strange Duality”-type isomorphisms between the Verlinde spaces at level one of the following pairs of groups $(\mathrm{SL}(5), \mathrm{SL}(5))$, $(\mathrm{Spin}(8), \mathrm{Spin}(8))$, $(\mathrm{SL}(3), E_6)$ and $(\mathrm{SL}(2), E_7)$.

1. INTRODUCTION

Let X be a smooth complex projective curve of genus g and let G be a simple and simply connected complex Lie group. We denote by $\mathcal{M}_X(G)$ the moduli stack parametrizing principal G -bundles over the curve X and by \mathcal{L}_G the ample line bundle over $\mathcal{M}_X(G)$ generating its Picard group. The starting point of our investigation is the observation (see e.g. [So], [F1], [F2]) that

$$\dim H^0(\mathcal{M}_X(E_8), \mathcal{L}_{E_8}) = 1.$$

for any genus g . In other words, the moduli stack $\mathcal{M}_X(E_8)$ carries a natural divisor Δ . Unfortunately a geometric interpretation of this divisor is not known.

In this paper we study the pull-back of this mysterious divisor Δ under the morphisms $\mathcal{M}_X(P) \rightarrow \mathcal{M}_X(E_8)$ induced by the group homomorphisms $\phi : P \rightarrow E_8$, where we assume that P is simply connected and that the differential $d\phi : \mathfrak{p} = \mathrm{Lie}(P) \rightarrow \mathfrak{e}_8 = \mathrm{Lie}(E_8)$ is a *conformal* embedding of Lie algebras. Note that we only assume \mathfrak{p} to be semi-simple. The maximal conformal subalgebras of \mathfrak{e}_8 with Dynkin (multi-)index one have been classified by [BB] and [SW] and the full list is

$$(1) \quad \mathfrak{so}(16), \mathfrak{sl}(9), \mathfrak{sl}(5) \oplus \mathfrak{sl}(5), \mathfrak{sl}(3) \oplus \mathfrak{e}_6, \mathfrak{sl}(2) \oplus \mathfrak{e}_7, \mathfrak{g}_2 \oplus \mathfrak{f}_4.$$

In Table (2) we list the corresponding simply connected Lie groups P and the finite kernel N of their natural maps to E_8 (see e.g. [CG] Lemma 3.3).

$$(2) \quad \begin{array}{|c|c|c|c|c|c|c|} \hline P & \mathrm{Spin}(16) & \mathrm{SL}(9) & \mathrm{SL}(5) \times \mathrm{SL}(5) & \mathrm{SL}(3) \times E_6 & \mathrm{SL}(2) \times E_7 & G_2 \times F_4 \\ \hline N & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} & \mathbb{Z}/5\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 1 \\ \hline \end{array}$$

Note that N is a subgroup of the center of P . We introduce the finite abelian group $\mathcal{M}_X(N)$ of principal N -bundles over X , which acts on $\mathcal{M}_X(P)$ by twisting P -bundles with N -bundles. Since N is the kernel of ϕ , the group $\mathcal{M}_X(N)$ acts on the fibers of the induced stack morphism $\tilde{\phi} : \mathcal{M}_X(P) \rightarrow \mathcal{M}_X(E_8)$ and, since $\tilde{\phi}^* \mathcal{L}_{E_8} = \mathcal{L}_P$, we obtain a canonical linear action of the group $\mathcal{M}_X(N)$ on the space of global sections $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$. If P has two simple factors,

2000 *Mathematics Subject Classification*. Primary 14D20, 14H60, 17B67.

the line bundle \mathcal{L}_P denotes the tensor product of the ample generators on each factor (see section 2.1).

With this notation our main result can be stated as follows.

Theorem 1. Let P be a Lie group of Table (2) and let

$$\phi_P : H^0(\mathcal{M}_X(E_8), \mathcal{L}_{E_8}) \longrightarrow H^0(\mathcal{M}_X(P), \mathcal{L}_P)$$

be the linear map on global sections induced by the homomorphism $\phi : P \rightarrow E_8$. Then

- (i) The linear map ϕ_P is nonzero.
- (ii) In all cases except $P = G_2 \times F_4$, its one-dimensional image $\text{im } \phi_P = \langle \sigma \rangle$ coincides with the $\mathcal{M}_X(N)$ -invariant subspace in $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$.

Unfortunately, even in the cases $\text{SL}(9)$ and $\text{Spin}(16)$, we are not able to give a geometric description of the zero-divisor of the $\mathcal{M}_X(N)$ -invariant section σ in the moduli stacks $\mathcal{M}_X(\text{SL}(9))$ and $\mathcal{M}_X(\text{Spin}(16))$ — see section 7.1 for further discussion.

A word about the proof of Theorem 1. We make use of the identification of the space of generalized G -theta functions $H^0(\mathcal{M}_X(G), \mathcal{L}_G)$ with the space of conformal blocks $\mathcal{V}_0^\dagger(X, \mathfrak{g})$ associated to the curve X with one marked point labelled with the zero weight. Here $\mathfrak{g} = \text{Lie}(G)$. Under this identification the linear map ϕ_P becomes the map induced by the natural inclusion of the basic highest weight modules $\mathcal{H}_0(\mathfrak{p}) \hookrightarrow \mathcal{H}_0(\mathfrak{e}_8)$ of the affine Lie algebras $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{e}}_8$. The proof has essentially two steps. First, we use a result by P. Belkale ([B] Proposition 5.8) saying that the linear map ϕ_P has constant rank when the curve X varies in a family of smooth curves. Here the fact, that the embedding $\mathfrak{p} \subset \mathfrak{e}_8$ is conformal, is crucial, since it ensures that ϕ_P is projectively flat with respect to the WZW connections on both sheaves of vacua over any family of smooth curves. Secondly, we study in section 4 the behaviour of the factorization rules of the spaces of conformal blocks associated to \mathfrak{g} under any conformal embedding $\mathfrak{p} \subset \mathfrak{g}$. This will follow once one has decomposed the “sewing-procedure” tensor $\tilde{\gamma}_\lambda \in \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]]$ under the decomposition of the $\hat{\mathfrak{g}}$ -modules $\mathcal{H}_\lambda(\mathfrak{g})$ and $\mathcal{H}_{\lambda^\dagger}(\mathfrak{g})$ into irreducible $\hat{\mathfrak{p}}$ -modules. Finally, combining these two steps allows us to show by induction on the genus of X that ϕ_P is non-zero.

In the cases when P is not simple and N not trivial, an argument using the representation theory of Heisenberg groups allows us to show the following result, which can be seen as an instance of Strange Duality for exceptional groups at level one.

Theorem 2. Let (A, B) be one of the three pairs $(\text{SL}(5), \text{SL}(5))$, $(\text{SL}(3), E_6)$, $(\text{SL}(2), E_7)$. Then the unique $\mathcal{M}_X(N)$ -invariant section

$$\sigma \in H^0(\mathcal{M}_X(A \times B), \mathcal{L}_{A \times B}) = H^0(\mathcal{M}_X(A), \mathcal{L}_A) \otimes H^0(\mathcal{M}_X(B), \mathcal{L}_B)$$

introduced in Theorem 1 for $P = A \times B$ induces an isomorphism

$$\sigma : H^0(\mathcal{M}_X(A), \mathcal{L}_A)^* \longrightarrow H^0(\mathcal{M}_X(B), \mathcal{L}_B).$$

A similar isomorphism is obtained for the pair $(\text{Spin}(8), \text{Spin}(8))$ — see section 7.2.1. We would like to emphasize that Theorem 2 is independent of Theorem 1. The striking point is that all “Strange Duality” isomorphisms of Theorem 2 are obtained by restricting the E_8 -theta divisor Δ to special subgroups P .

The last few years have seen important progress on “Strange Duality” or “rank-level” duality for Verlinde spaces. For a survey we refer e.g. to the papers [MO], [Po] or [Pa].

We would like to thank Laurent Manivel and Nicolas Ressayre for helpful comments.

Acknowledgements. The first author wishes to thank the Consejo Superior de Investigaciones Científicas (CSIC, Spain) and the Université de Montpellier II (France) for financial support of research visits. The second author was partially supported by the Ministerio de Educación y Ciencia (Spain) through the grant SAB2006-0022.

2. NOTATION AND PRELIMINARIES

2.1. Moduli stacks and line bundles.

2.1.1. *Dynkin index.* Let \mathfrak{p} and \mathfrak{g} be two simple Lie algebras and let $\varphi : \mathfrak{p} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism. There exists [D] a unique integer $d = d_\varphi$, called the Dynkin index of the homomorphism φ , satisfying

$$(\varphi(x), \varphi(y))_{\mathfrak{g}} = d_\varphi(x, y)_{\mathfrak{p}}, \quad \text{for all } x, y \in \mathfrak{p},$$

where $(\ , \)_*$ denotes the invariant bilinear form on \mathfrak{p} and \mathfrak{g} normalized such that $(\theta, \theta) = 2$ for the maximal roots θ of \mathfrak{p} and \mathfrak{g} . If \mathfrak{p} is semi-simple with two components $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, then the Dynkin multi-index of $\varphi = \varphi_1 \oplus \varphi_2 : \mathfrak{p}_1 \oplus \mathfrak{p}_2 \rightarrow \mathfrak{g}$ is given by $d_\varphi = (d_{\varphi_1}, d_{\varphi_2})$, where d_{φ_i} is defined using $\varphi_i : \mathfrak{p}_i \rightarrow \mathfrak{g}$.

2.1.2. *Line bundles over the moduli stack $\mathcal{M}_X(P)$.* If P is a simple and simply connected complex Lie group we refer to [LS] and [So] for the description of the ample generator \mathcal{L}_P of the Picard group of the moduli stack $\mathcal{M}_X(P)$. If $P = P_1 \times P_2$ with P_i simple and simply connected, we put $\mathcal{L}_P = \mathcal{L}_{P_1} \boxtimes \mathcal{L}_{P_2}$ and we note $\mathcal{L}_P^d = \mathcal{L}_{P_1}^{d_1} \boxtimes \mathcal{L}_{P_2}^{d_2}$ for a multi-index $d = (d_1, d_2)$. The following lemma follows easily from [LS] and [KNR].

Lemma 2.1. *Let $\phi : P \rightarrow G$ be a homomorphism between simply-connected complex Lie groups with G simple and P semi-simple. Let $\tilde{\phi} : \mathcal{M}_X(P) \rightarrow \mathcal{M}_X(G)$ be the induced stack morphism. Then we have the equality*

$$\tilde{\phi}^* \mathcal{L}_G = \mathcal{L}_P^{d_\varphi},$$

where d_φ is the Dynkin-(multi) index of the differential $\varphi = d\phi : \mathfrak{p} \rightarrow \mathfrak{g}$.

2.2. Spaces of conformal blocks.

2.2.1. *The case \mathfrak{g} simple.* Let \mathfrak{g} be a simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We denote by $(\ , \)$ the normalized Cartan-Killing form and we will use the same notation for the restricted form on \mathfrak{h} and for the induced form on \mathfrak{h}^* . We consider (see [K] Chapter 7) the non-twisted affine Lie algebra associated to \mathfrak{g}

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with Lie bracket

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) \text{Res}_{z=0}(gdf) \cdot c, \quad [\mathfrak{g}, c] = 0, \quad [d, c] = 0, \quad [d, x(n)] = nx(n),$$

for $x, y \in \mathfrak{g}$, $f, g \in \mathbb{C}((z))$ and $n \in \mathbb{Z}$. Here we put $x(n) = x \otimes z^n$. We identify \mathfrak{g} with the subalgebra $\mathfrak{g} \otimes 1$ of $\hat{\mathfrak{g}}$. The subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is the Cartan subalgebra of the affine Lie algebra $\hat{\mathfrak{g}}$. We extend $\lambda \in \mathfrak{h}^*$ to a linear form on $\hat{\mathfrak{h}}$ by putting $\langle \lambda, \mathbb{C}c \oplus \mathbb{C}d \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the standard pairing. We define the elements Λ_0 and δ in the dual $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ by $\langle \delta, d \rangle = \langle \Lambda_0, c \rangle = 1$ and $\langle \delta, \mathfrak{h} \oplus \mathbb{C}c \rangle = \langle \Lambda_0, \mathfrak{h} \oplus \mathbb{C}d \rangle = 0$. We extend the form (\cdot, \cdot) to $\hat{\mathfrak{h}}^*$ by putting

$$(\mathfrak{h}^*, \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta) = 0, \quad (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0, \quad (\delta, \Lambda_0) = 1.$$

The Weyl group of \mathfrak{g} is denoted by $W(\mathfrak{g})$, or simply W if no confusion arises. Call $w_0^{\mathfrak{g}} \in W(\mathfrak{g})$ its longest element. Later we will need the following fact

Proposition 2.2. *Let $\mathfrak{p} \subset \mathfrak{g}$ be an embedding of semi-simple Lie algebras and choose Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{g}}$. Then there exists an element $\tilde{w} \in W(\mathfrak{g})$ which preserves the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{g}}$ and such that the restriction $\tilde{w}|_{\mathfrak{h}_{\mathfrak{p}}}$ coincides with the longest element $w_0^{\mathfrak{p}}$.*

Proof. This can be deduced from a more general fact — see e.g. Theorem 2.1.4 [BS]. Moreover if \mathfrak{p} and \mathfrak{g} have the same rank (all our cases except $\mathfrak{g}_2 \oplus \mathfrak{f}_4$), i.e. $\mathfrak{h}_{\mathfrak{p}} = \mathfrak{h}_{\mathfrak{g}}$, we have a canonical inclusion $W(\mathfrak{p}) \subset W(\mathfrak{g})$. \square

Next, we need to recall some representation theory of the affine Lie algebra $\hat{\mathfrak{g}}$ from [K] Chapter 12. Given a positive integer k , called level, we consider the finite set

$$P_k(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) \mid (\lambda, \theta) \leq k\} \subset \mathfrak{h}^*,$$

where $P_+(\mathfrak{g})$ is the set of dominant integral weights of \mathfrak{g} . We denote by $P(\mathfrak{g})$ the weight lattice of \mathfrak{g} . There is an involution of the set $P_k(\mathfrak{g})$ given by $\lambda \mapsto \lambda^\dagger$, where $-\lambda^\dagger$ is the lowest weight of the irreducible right \mathfrak{g} -module V_λ^* , the dual of V_λ . In other words, $\lambda^\dagger = -w_0^{\mathfrak{g}}(\lambda)$.

We introduce the set

$$\widehat{P}_k(\mathfrak{g}) := \{\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \mid \lambda \in P_+(\mathfrak{g}), \zeta \in \mathbb{C}\} \subset \hat{\mathfrak{h}}^*$$

Note that there is a projection map

$$\widehat{P}_k(\mathfrak{g}) \rightarrow P(\mathfrak{g}), \quad \hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \mapsto \lambda.$$

We will view $P_k(\mathfrak{g})$ as a subset of $\widehat{P}_k(\mathfrak{g})$ under the mapping $\lambda \mapsto \hat{\lambda} = \lambda + k\Lambda_0$. We extend the involution $\lambda \mapsto \lambda^\dagger$ to $\widehat{P}_k(\mathfrak{g})$ by mapping

$$\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \quad \mapsto \quad \hat{\lambda}^\dagger = -w_0^{\mathfrak{g}}(\lambda) + k\Lambda_0 + \zeta\delta.$$

Given $\lambda \in P_k(\mathfrak{g})$ we denote by $\mathcal{H}_\lambda(\mathfrak{g})$, or simply \mathcal{H}_λ if no confusion arises, the integrable $\hat{\mathfrak{g}}$ -module with highest weight λ and of level k , i.e. c acts on \mathcal{H}_λ as $k \cdot \text{Id}$. Note that $d \in \hat{\mathfrak{h}}$ acts trivially on the highest weight vector $v_\lambda \in \mathcal{H}_\lambda$. More generally, for any $\hat{\lambda} = \lambda + k\Lambda_0 + \zeta\delta \in \widehat{P}_k(\mathfrak{g})$ with $\lambda \in P_k(\mathfrak{g})$ we denote by $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ the integrable $\hat{\mathfrak{g}}$ -module with highest weight vector v_λ such that $d \in \hat{\mathfrak{h}}$ acts as $d.v_\lambda = \zeta v_\lambda$. Note that \mathcal{H}_λ and $\mathcal{H}_{\hat{\lambda}}$ become isomorphic as modules over the subalgebra $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c$. The $\hat{\mathfrak{g}}$ -module $\mathcal{H}_0(\mathfrak{g})$ with zero weight and level 1 is called the *basic* $\hat{\mathfrak{g}}$ -module.

Let s be a positive integer. For $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_s) \in P_k(\mathfrak{g})^s$ we define

$$\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \otimes \dots \otimes \mathcal{H}_{\lambda_s}.$$

Given s points $\vec{p} = (p_1, \dots, p_s)$ with $p_i \in X$ we consider the open subset $U = X \setminus \{p_1, \dots, p_s\}$ and we denote the space of covacua associated to the data $(X, \vec{p}, \vec{\lambda})$ by

$$\mathcal{V}_{\vec{\lambda}}(X, \mathfrak{g}) := \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(U) \cdot \mathcal{H}_{\vec{\lambda}}.$$

We refer e.g. to [U] for the precise definition of the action of the Lie algebra $\mathfrak{g}(U)$ on $\mathcal{H}_{\vec{\lambda}}$ and for basic properties of the vector space $\mathcal{V}_{\vec{\lambda}}(X, \mathfrak{g})$. The space of vacua or *space of conformal blocks* is by definition the dual of $\mathcal{V}_{\vec{\lambda}}(X, \mathfrak{g})$ and is denoted by

$$\mathcal{V}_{\vec{\lambda}}^\dagger(X, \mathfrak{g}) \hookrightarrow \mathcal{H}_{\vec{\lambda}}^\dagger.$$

We denote by $\langle | \rangle$ the natural pairing between $\mathcal{H}_{\vec{\lambda}}$ and its dual $\mathcal{H}_{\vec{\lambda}}^\dagger$.

The construction of the space of conformal blocks can be carried out for a family $\mathcal{X} \rightarrow S$ of pointed nodal curves and provides a sheaf over the base scheme S , called the sheaf of vacua and denoted by $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})$. A fundamental property is that the sheaf of vacua is locally free and that it commutes with any base change (see e.g. [U] Theorem 4.4.2).

2.2.2. *The case \mathfrak{g} semi-simple.* We now adapt the previous constructions to semi-simple Lie algebras \mathfrak{g} . For our purposes it is enough to deal with the case when \mathfrak{g} is the direct sum of two simple Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. By [K] section 12.9 we define the affine Lie algebra associated to \mathfrak{g} by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d,$$

with Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d$. Similar to the case of simple algebras, one defines a Lie bracket and a non-degenerate bilinear form on $\hat{\mathfrak{h}}$ and on its dual $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0^{(1)} \oplus \mathbb{C}\Lambda_0^{(2)} \oplus \mathbb{C}\delta$.

Given a multi-index $k = (k_1, k_2)$, with k_i positive integers, we introduce the sets

$$\begin{aligned} P_k(\mathfrak{g}) &= P_{k_1}(\mathfrak{g}_1) \times P_{k_2}(\mathfrak{g}_2), \\ \widehat{P_k(\mathfrak{g})} &= \{\hat{\lambda} = \lambda + k_1\Lambda_0^{(1)} + k_2\Lambda_0^{(2)} + \zeta\delta \mid \lambda \in P_+(\mathfrak{g}), \zeta \in \mathbb{C}\} \subset \hat{\mathfrak{h}}^*, \end{aligned}$$

and we associate to a weight $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in P_k(\mathfrak{g})$ the integrable $\hat{\mathfrak{g}}$ -module

$$\mathcal{H}_\lambda(\mathfrak{g}) = \mathcal{H}_{\lambda^{(1)}}(\mathfrak{g}_1) \otimes \mathcal{H}_{\lambda^{(2)}}(\mathfrak{g}_2).$$

Similarly, we introduce for any $\hat{\lambda} \in \widehat{P_k(\mathfrak{g})}$ the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$.

With these definitions it is easy to deduce the following decomposition of the spaces of conformal blocks

$$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{g}) = \mathcal{V}_{\vec{\lambda}^{(1)}}^\dagger(\mathfrak{g}_1) \otimes \mathcal{V}_{\vec{\lambda}^{(2)}}^\dagger(\mathfrak{g}_2) \quad \text{with } \vec{\lambda} = (\vec{\lambda}^{(1)}, \vec{\lambda}^{(2)}) \in P_k(\mathfrak{g})^s.$$

2.3. Generalized theta functions and the Verlinde formula. For the convenience of the reader we recall (see [LS], [F3], [KNR]) that there is an isomorphism between the space $H^0(\mathcal{M}_X(G), \mathcal{L}_G^k)$ of generalized G -theta functions of level k and the space of conformal blocks $\mathcal{V}_0^\dagger(X, \mathfrak{g})$ associated to the curve X with one marked point labelled with the zero weight at level k , i.e.,

$$(3) \quad H^0(\mathcal{M}_X(G), \mathcal{L}_G^k) \xrightarrow{\sim} \mathcal{V}_0^\dagger(X, \mathfrak{g}).$$

The dimension of this space is given by the Verlinde formula (see [F3], [T]) and its values for the groups G at level one can be computed straightforwardly and are given in the following table

$$(4) \quad \begin{array}{c|cccccc} G & \text{SL}(n) & \text{Spin}(2n) & E_6 & E_7 & E_8 & G_2 \text{ and } F_4 \\ \hline \dim H^0(\mathcal{M}_X(G), \mathcal{L}_G) & n^g & 4^g & 3^g & 2^g & 1 & \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1} \end{array}$$

3. CONFORMAL PAIRS: PROPERTIES OF THEIR REPRESENTATIONS

3.1. The Virasoro algebra and its representation on $\mathcal{H}_\lambda(\mathfrak{g})$. The Virasoro algebra

$$\text{Vir} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}d_n \oplus \mathbb{C}\tilde{c},$$

is defined by the relations $[d_i, d_j] = (i-j)d_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j}\tilde{c}$ and $[d_i, \tilde{c}] = 0$.

If \mathfrak{g} is simple, we define for any level k and any $\hat{\lambda} \in \widehat{P}_k(\mathfrak{g})$, the central charge $c(\mathfrak{g}, k)$ and the trace anomaly $\Delta_{\hat{\lambda}}(\mathfrak{g})$ as

$$c(\mathfrak{g}, k) = \frac{k \dim \mathfrak{g}}{\check{h}(\mathfrak{g}) + k}, \quad \text{and} \quad \Delta_{\hat{\lambda}}(\mathfrak{g}) = \frac{(\hat{\lambda}, \hat{\lambda} + 2\hat{\rho})}{2(\check{h}(\mathfrak{g}) + k)}.$$

Here $\check{h}(\mathfrak{g})$ is the dual Coxeter number of \mathfrak{g} and $\hat{\rho} = \rho + \check{h}(\mathfrak{g})\Lambda_0$, with ρ the half-sum of the positive roots of \mathfrak{g} . We choose dual bases $\{u_i\}$ and $\{u^i\}$ of the simple algebra \mathfrak{g} and introduce for any $n \in \mathbb{Z}$ the Sugawara operator (see [KW] section 3.2)

$$(5) \quad L_n^{\mathfrak{g}} = \frac{1}{2(k + \check{h}(\mathfrak{g}))} \sum_{j \in \mathbb{Z}} \sum_i : u_i(-j)u^i(j+n) :,$$

where the notation $: \cdot :$ stands for the normal ordering. Then $L_n^{\mathfrak{g}}$ acts linearly on $\mathcal{H}_\lambda(\mathfrak{g})$ and, by putting $d_n = L_n^{\mathfrak{g}}$ and $\tilde{c} = c(\mathfrak{g}, k)\text{Id}$, we obtain a representation of Vir on $\mathcal{H}_\lambda(\mathfrak{g})$.

If \mathfrak{g} is semi-simple with $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we define for $k = (k_1, k_2)$, see [KW] formula (1.4.7))

$$c(\mathfrak{g}, k) = c(\mathfrak{g}_1, k_1) + c(\mathfrak{g}_2, k_2), \quad \text{and} \quad \Delta_{\hat{\lambda}}(\mathfrak{g}) = \Delta_{\hat{\lambda}^{(1)}}(\mathfrak{g}_1) + \Delta_{\hat{\lambda}^{(2)}}(\mathfrak{g}_2),$$

and we put $L_n^{\mathfrak{g}} = L_n^{\mathfrak{g}_1} + L_n^{\mathfrak{g}_2}$. As in the simple case, we obtain a representation of Vir on $\mathcal{H}_\lambda(\mathfrak{g})$.

For later use we recall the following relation ([KW] formula (1.4.5))

$$(6) \quad \Delta_{\hat{\lambda}+n\delta}(\mathfrak{g}) = \Delta_{\hat{\lambda}}(\mathfrak{g}) + n.$$

The endomorphism $L_0^{\mathfrak{g}}$ of the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ can be diagonalized

$$\mathcal{H}_{\hat{\lambda}} = \bigoplus_{m=0}^{\infty} \mathcal{H}_{\hat{\lambda}}(m) \quad \text{with} \quad \mathcal{H}_{\hat{\lambda}}(m) := \{u \in \mathcal{H}_{\hat{\lambda}} \mid L_0^{\mathfrak{g}}(u) = (\Delta_{\hat{\lambda}}(\mathfrak{g}) + m)u\}.$$

We recall that the endomorphism $L_0^{\mathfrak{g}}$ of $\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})$ defined by (5) can also be written as ([KW] (3.2.6) or [K] Corollary 12.8 (b))

$$(7) \quad L_0^{\mathfrak{g}} = \Delta_{\hat{\lambda}}(\mathfrak{g})\text{Id} - d.$$

Note that $\mathcal{H}_{\hat{\lambda}}(0)$ equals the irreducible \mathfrak{g} -module V_{λ} with highest weight λ .

3.2. Definition of conformal pair.

Definition 3.1. *Let \mathfrak{p} be a semisimple subalgebra of a simple Lie algebra \mathfrak{g} , and let ℓ denote the Dynkin (multi-)index of the inclusion homomorphism $\mathfrak{p} \subset \mathfrak{g}$. We say that \mathfrak{p} is a conformal subalgebra of \mathfrak{g} at level k if $c(\mathfrak{p}, k\ell) = c(\mathfrak{g}, k)$.*

The equality $c(\mathfrak{p}, k\ell) = c(\mathfrak{g}, k)$ in the above definition in fact holds only if $k = 1$. Classification of conformal pairs are given in [BB] and [SW]. We recall (see [KW]) that, since \mathfrak{p} is semisimple, $\mathfrak{p} \subset \mathfrak{g}$ is a conformal subalgebra is equivalent to the statement that any irreducible $\hat{\mathfrak{g}}$ -module $\mathcal{H}_{\lambda}(\mathfrak{g})$ of level 1 decomposes into a finite sum of irreducible $\hat{\mathfrak{p}}$ -modules of level ℓ .

A fundamental property of a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$ is the following.

Proposition 3.2 ([KW] Proposition 3.2 (c)). *If $\mathfrak{p} \subset \mathfrak{g}$ is a conformal embedding, then*

$$L_n^{\mathfrak{g}} = L_n^{\mathfrak{p}} \in \text{End}(\mathcal{H}_{\hat{\lambda}}(\mathfrak{g})) \quad \text{for all } n \in \mathbb{Z}.$$

In the above statement, we warn that (though implicit in the notation), the operator $L_n^{\mathfrak{g}}$ is acting at level $k = 1$ and $L_n^{\mathfrak{p}}$ at level ℓ . In all the cases we will consider the integer ℓ is equal to 1, see Table 11.

3.3. The pairing on $\mathcal{H}_{\lambda} \times \mathcal{H}_{\lambda^{\dagger}}$.

First we recall the following lemma

Lemma 3.3 ([U] Lemma 4.4.3). *There exists a bilinear pairing*

$$(\cdot|\cdot) : \mathcal{H}_{\lambda} \times \mathcal{H}_{\lambda^{\dagger}} \longrightarrow \mathbb{C},$$

unique up to a multiplicative constant such that

$$(8) \quad (X(n)u|v) + (u|X(-n)v) = 0,$$

for any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $u \in \mathcal{H}_{\lambda}$ and $v \in \mathcal{H}_{\lambda^{\dagger}}$. Moreover the restriction of this pairing to $\mathcal{H}_{\lambda}(m) \times \mathcal{H}_{\lambda^{\dagger}}(m')$ is zero if $m \neq m'$, and non-degenerate if $m = m'$.

Consider the restriction of the pairing $(\cdot|\cdot)$ to $\mathcal{H}_{\lambda}(0) \times \mathcal{H}_{\lambda^{\dagger}}(0) = V_{\lambda} \times V_{\lambda^{\dagger}}$. By definition $V_{\lambda^{\dagger}} = V_{\lambda}^*$, so that $V_{\lambda} \otimes V_{\lambda^{\dagger}} = \text{End}(V_{\lambda})$. The pairing on $V_{\lambda} \times V_{\lambda}^*$ is given by a multiple of the natural evaluation map. More generally, the pairing $(\cdot|\cdot)$ induces for any integer m an isomorphism $\phi_m : \mathcal{H}_{\lambda^{\dagger}}(m) \xrightarrow{\sim} \mathcal{H}_{\lambda}(m)^*$ and therefore a distinguished element, which we denote by $\gamma_{\lambda}(m)$ and which corresponds to the identity element in $\text{End}(\mathcal{H}_{\lambda}(m))$ under the isomorphism

$$(9) \quad \text{id} \otimes \phi_m : \mathcal{H}_{\lambda}(m) \otimes \mathcal{H}_{\lambda^{\dagger}}(m) \xrightarrow{\sim} \text{End}(\mathcal{H}_{\lambda}(m)) \quad \gamma_{\lambda}(m) \mapsto \text{Id}_{\mathcal{H}_{\lambda}(m)}.$$

Note that the family $\{\gamma_{\lambda}(m)\}_{m \in \mathbb{Z}_+}$ is uniquely defined up to a multiplicative constant.

Consider a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$ and an integrable $\hat{\mathfrak{g}}$ -module $\mathcal{H}_\lambda(\mathfrak{g})$ of level one. Then we have a decomposition as $\hat{\mathfrak{p}}$ -module

$$\mathcal{H}_\lambda(\mathfrak{g}) = \bigoplus_{\hat{\mu} \in B(\lambda)} M(\hat{\mu}, \lambda) \otimes \mathcal{H}_{\hat{\mu}}(\mathfrak{p}),$$

where $B(\lambda)$ is a finite subset of $\widehat{P_\ell(\mathfrak{p})}$ and the $M(\hat{\mu}, \lambda)$ are finite-dimensional vector spaces. The integer $\dim M(\hat{\mu}, \lambda)$ is the multiplicity of the representation $\mathcal{H}_{\hat{\mu}}(\mathfrak{p})$ in $\mathcal{H}_\lambda(\mathfrak{g})$. Note that the weights $\hat{\mu} \in B(\lambda)$ do not necessarily lie in $P_\ell(\mathfrak{p})$. We can write $\hat{\mu} = \mu + \sum_i \ell_i \Lambda_0^{(i)} - n_\mu \delta$.

Using Proposition 3.2 we deduce an equality between the trace anomalies

$$(10) \quad \Delta_\lambda(\mathfrak{g}) = \Delta_{\hat{\mu}}(\mathfrak{p}) \quad \text{for any } \hat{\mu} \in B(\lambda).$$

Moreover by (6) we also have $\Delta_{\hat{\mu}}(\mathfrak{p}) = \Delta_\mu(\mathfrak{p}) - n_\mu$. Since $-n_\mu$ is the d -eigenvalue of the highest weight vector $v_\mu \in \mathcal{H}_{\hat{\mu}}(\mathfrak{p}) \subset \mathcal{H}_\lambda(\mathfrak{g})$ and since all d -eigenvalues of $\mathcal{H}_\lambda(\mathfrak{g})$ are negative, we conclude that $n_\mu \in \mathbb{Z}_+$.

Remark 3.4. We immediately deduce from the above that $n_\mu = 0$ if and only if the \mathfrak{p} -module $V_\mu = \mathcal{H}_\mu(\mathfrak{p})(0)$ appears in the decomposition into irreducible \mathfrak{p} -modules of the \mathfrak{g} -module $V_\lambda = \mathcal{H}_\lambda(\mathfrak{g})(0)$.

Thus we conclude that given $\mu \in P_\ell(\mathfrak{p})$ there exists at most one $\hat{\mu} \in B(\lambda)$ — since n_μ is given by the difference of the trace anomalies. So we will write

$$\text{mult}_\lambda(\mu, \mathfrak{p}) := \dim M(\hat{\mu}, \lambda)$$

for the multiplicity of occurrence of $\mathcal{H}_\mu(\mathfrak{p})$ in $\mathcal{H}_\lambda(\mathfrak{g})$.

Proposition 3.5. *We have the equality*

$$B(\lambda^\dagger) = B(\lambda)^\dagger := \{\hat{\mu} \in \widehat{P_\ell(\mathfrak{p})} \mid \hat{\mu}^\dagger \in B(\lambda)\}.$$

Moreover $\text{mult}_\lambda(\mu, \mathfrak{p}) = \text{mult}_{\lambda^\dagger}(\mu^\dagger, \mathfrak{p})$.

Proof. For $\hat{\nu} \in P_k(\hat{\mathfrak{g}})$ and $\lambda \in P_k(\mathfrak{g})$ we will denote by $V_{\hat{\nu}}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})$ the weight space of the $\hat{\mathfrak{g}}$ -module $\mathcal{H}_\lambda(\mathfrak{g})$ associated to the weight $\hat{\nu}$. It follows from relation (7) that $V_{\hat{\nu}}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})(m)$ if and only if $\hat{\nu}$ is of the form $\hat{\nu} = \nu + k\Lambda_0 - m\delta$. By Lemma 3.3 we know that $\mathcal{H}_\lambda(\mathfrak{g})(m)$ and $\mathcal{H}_{\lambda^\dagger}(\mathfrak{g})(m)$ are dual spaces, and it follows from relation (8) that the weight spaces

$$V_{\nu+k\Lambda_0-m\delta}^\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})(m) \quad \text{and} \quad V_{-\nu+k\Lambda_0-m\delta}^{\lambda^\dagger} \subset \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})(m)$$

are dual to each other. Hence their dimensions coincide

$$\text{mult}_{\mathcal{H}_\lambda}(\nu + k\Lambda_0 - m\delta) = \text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\nu + k\Lambda_0 - m\delta).$$

Consider $\hat{\mu} \in B(\lambda)$. Then by (10) we have $\Delta_\lambda(\mathfrak{g}) = \Delta_{\hat{\mu}}(\mathfrak{p})$ and since the bilinear form (\cdot, \cdot) is invariant under the finite Weyl group and $w_0(\rho) = -\rho$ we also have $\Delta_{\lambda^\dagger}(\mathfrak{g}) = \Delta_{\hat{\mu}^\dagger}(\mathfrak{p})$. By [K] Proposition 12.11 there exists a weight $\hat{\nu}$ of $\mathcal{H}_\lambda(\mathfrak{g})$ such that $\hat{\nu}|_{\hat{\mathfrak{p}}} = \hat{\mu}$. By Proposition 2.2 there exists an element $\tilde{w} \in W(\mathfrak{g})$ which restricts to $w_0^\mathfrak{p} \in W(\mathfrak{p})$ and by [K] Proposition 12.5 (c)

$$\text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\nu + k\Lambda_0 - m\delta) = \text{mult}_{\mathcal{H}_{\lambda^\dagger}}(-\tilde{w}(\nu) + k\Lambda_0 - m\delta).$$

But $-\tilde{w}(\nu) + k\Lambda_0 - m\delta|_{\hat{\mathfrak{p}}} = \hat{\mu}^\dagger$, so again by [K] Proposition 12.11 we obtain that $\mu^\dagger \in B(\lambda^\dagger)$. The same reasoning combined with [K] formula (12.11.1) shows that $\text{mult}_\lambda(\mu, \mathfrak{p}) = \text{mult}_{\lambda^\dagger}(\mu^\dagger, \mathfrak{p})$. \square

3.4. Conformal pairs and the sewing procedure. Let q be a formal variable. Given $\lambda \in P_k(\mathfrak{g})$ we define the element (cf. section 3.3)

$$\tilde{\gamma}_\lambda := \sum_{m=0}^{\infty} \gamma_\lambda(m) q^m \in \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]].$$

Note that $\tilde{\gamma}_\lambda$ is well-defined up to a multiplicative constant.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a conformal subalgebra. We suppose for simplicity that $\text{mult}_\lambda(\mu, \mathfrak{p}) = 1$ for all $\hat{\mu} \in B(\lambda)$ — this will be the case of all our examples (see table(11) below).

Proposition 3.6. *We have a decomposition in $\mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})[[q]]$*

$$\tilde{\gamma}_\lambda = \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \tilde{\gamma}_\mu,$$

where the positive integer n_μ equals $\Delta_\mu(\mathfrak{p}) - \Delta_\lambda(\mathfrak{g})$.

Proof. We consider the decomposition into d -eigenspaces

$$\mathcal{H}_\lambda(\mathfrak{g})(m) = \bigoplus_{\hat{\mu} \in B(\lambda)} \mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m).$$

Note that $\mathcal{H}_{\hat{\mu}}(\mathfrak{p})(m) = \mathcal{H}_\mu(\mathfrak{p})(m - n_\mu)$ for all $m \in \mathbb{Z}_+$ and that $\mathcal{H}_\mu(\mathfrak{p})(l) = \{0\}$ for $l < 0$. The identity transformation of $\mathcal{H}_\lambda(\mathfrak{g})(m)$ obviously decomposes as

$$\text{Id}_{\mathcal{H}_\lambda(\mathfrak{g})(m)} = \sum_{\hat{\mu} \in B(\lambda)} \text{Id}_{\mathcal{H}_\mu(\mathfrak{p})(m - n_\mu)} \quad \text{for any } m \in \mathbb{Z}_+.$$

It is easy to check that the pairing $(\cdot | \cdot)$ on $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$ restricts to the corresponding pairing on $\mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger}$ for any $\hat{\mu} \in B(\lambda)$, so that

$$\gamma_\lambda(m) = \sum_{\hat{\mu} \in B(\lambda)} \gamma_\mu(m - n_\mu) \quad \text{for any } m \in \mathbb{Z}_+.$$

Here we put $\gamma_\mu(l) = 0$ if $l < 0$. Multiplying with q^m and summing over \mathbb{Z}_+ gives the relation

$$\begin{aligned} \tilde{\gamma}_\lambda &= \sum_{m=0}^{\infty} \gamma_\lambda(m) q^m = \sum_{\hat{\mu} \in B(\lambda)} \sum_{m \geq n_\mu} \gamma_\mu(m - n_\mu) q^{m - n_\mu} q^{n_\mu} \\ &= \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \tilde{\gamma}_\mu. \end{aligned}$$

□

The following table is extracted from [KS] page 2235 and gives for any conformal subalgebra \mathfrak{p} of Table (1) the decomposition of the basic representation $\mathcal{H}_0(\mathfrak{e}_8)$ as $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ -module, in particular its Dynkin (multi-)index ℓ , its subset $B(0)$ and the action of the involution $\mu \mapsto \mu^\dagger$. We use Bourbaki's notation for the fundamental weights of a simple Lie algebra.

(11)

\mathfrak{p}	ℓ	$B(0)$	$\varpi \mapsto \varpi^\dagger$
$\mathfrak{so}(16)$	1	$\{0, \varpi_7\}$	$\varpi_7^\dagger = \varpi_7$
$\mathfrak{sl}(9)$	1	$\{0, \varpi_3, \varpi_6\}$	$\varpi_3^\dagger = \varpi_6$
$\mathfrak{sl}(5) \oplus \mathfrak{sl}(5)$	(1,1)	$\{(0, 0), (\varpi_1, \varpi_2), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_4, \varpi_3)\}$	$(\varpi_i, \varpi_j)^\dagger = (\varpi_{5-i}, \varpi_{5-j})$
$\mathfrak{sl}(3) \oplus \mathfrak{e}_6$	(1,1)	$\{(0, 0), (\varpi_1, \varpi_1), (\varpi_2, \varpi_6)\}$	$(\varpi_1, \varpi_1)^\dagger = (\varpi_2, \varpi_6)$
$\mathfrak{sl}(2) \oplus \mathfrak{e}_7$	(1,1)	$\{(0, 0), (\varpi_1, \varpi_7)\}$	$(\varpi_1, \varpi_7)^\dagger = (\varpi_1, \varpi_7)$
$\mathfrak{g}_2 \oplus \mathfrak{f}_4$	(1,1)	$\{(0, 0), (\varpi_2, \varpi_4)\}$	$(\varpi_2, \varpi_4)^\dagger = (\varpi_2, \varpi_4)$

We recall that if $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{b}$, then we have the decomposition as $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}]$ -module

$$\mathcal{H}_0(\mathfrak{e}_8) = \bigoplus_{(\lambda_1, \lambda_2) \in B(0)} \mathcal{H}_{\lambda_1}(\mathfrak{a}) \otimes \mathcal{H}_{\lambda_2}(\mathfrak{b}).$$

4. CONFORMAL PAIRS: FACTORIZATION RULES

In this section we describe how the factorization rules behave under conformal embeddings. First we recall the factorization rules. Let X_0 be a nodal (not necessarily irreducible) curve with one node x_0 . We call \tilde{X} the normalization of X_0 with $\pi : \tilde{X} \rightarrow X_0$ and $\pi^{-1}(x_0) = \{a, b\}$.

Proposition 4.1 (Factorization rules, [U] Theorem 4.4.9). *Let \mathfrak{g} be a semi-simple Lie algebra and X_0 a nodal curve with s marked points with labels $\vec{\lambda} \in P_k(\mathfrak{g})^s$. There is an isomorphism*

$$\bigoplus_{\lambda \in P_k(\mathfrak{g})} \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \xrightarrow{\oplus \iota_\lambda} \mathcal{V}_{\vec{\lambda}}^\dagger(X_0, \mathfrak{g}).$$

We will denote by \mathcal{O} the ring of formal power series $\mathbb{C}[[q]]$ and by $K = \mathbb{C}((q))$ its field of fraction. We consider a family of curves \mathcal{X} over $\text{Spec } \mathcal{O}$ such that its special fiber \mathcal{X}_0 is a nodal curve X_0 over \mathbb{C} and its generic fiber \mathcal{X}_K a smooth curve over the field K . Consider the sheaf of conformal blocks $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})$ for the family \mathcal{X} , which is an \mathcal{O} -module of finite type. Moreover since the formation of the sheaf of conformal blocks commutes with base change, the fibre $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})_0$ over the closed point $0 \in \text{Spec } \mathcal{O}$ of $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})$ coincides with $\mathcal{V}_{\vec{\lambda}}^\dagger(X_0, \mathfrak{g})$. We thus obtain a restriction map

$$r_0 : \mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\lambda}}^\dagger(X_0, \mathfrak{g}).$$

On the other hand, there exists for every $\lambda \in P_k(\mathfrak{g})$ a \mathbb{C} -linear map — the so-called sewing procedure, see [U] formula (4.4.3) and Lemma 4.4.5

$$\mathfrak{s}_\lambda : \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g}), \quad \psi_\lambda \mapsto \widetilde{\psi}_\lambda := \mathfrak{s}_\lambda(\psi_\lambda).$$

The linear maps ι_λ and \mathfrak{s}_λ are defined as follows: for $\psi_\lambda \in \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g})$

$$\langle \iota_\lambda(\psi_\lambda) | u \rangle := \langle \psi_\lambda | u \otimes \gamma_\lambda(0) \rangle \in \mathbb{C} \quad \text{and} \quad \langle \widetilde{\psi}_\lambda | \tilde{u} \rangle := \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\lambda \rangle \in \mathcal{O}$$

for any vectors $u \in \mathcal{H}_{\vec{\lambda}}$ and $\tilde{u} \in \mathcal{H}_{\vec{\lambda}}[[q]]$. We recall (see [U] Lemma 4.4.6) that $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g})$ identifies with the subset of linear forms in $\mathcal{H}_{\vec{\lambda}}^\dagger[[q]]$ satisfying the formal gauge condition. It is clear from these definitions that the map \mathfrak{s}_λ is an extension of ι_λ , i.e., $\iota_\lambda = r_0 \circ \mathfrak{s}_\lambda$.

We consider now a conformal embedding $\mathfrak{p} \subset \mathfrak{g}$. We assume that all level one representations $\mathcal{H}_\lambda(\mathfrak{g})$ decompose with multiplicities one, i.e. $\text{mult}_\lambda(\mu, \mathfrak{p}) = 1$ for all $\hat{\mu} \in B(\lambda)$. For $\vec{\lambda} =$

$(\lambda_1, \dots, \lambda_s)$ define $B(\vec{\lambda}) = B(\lambda_1) \times \dots \times B(\lambda_s)$. Consider $\vec{\mu} \in B(\vec{\lambda})$ and the corresponding inclusion

$$\mathcal{H}_{\vec{\mu}}(\mathfrak{p}) \hookrightarrow \mathcal{H}_{\vec{\lambda}}(\mathfrak{g}).$$

The gauge condition is preserved under restriction of linear forms to $\mathcal{H}_{\vec{\mu}}(\mathfrak{p})$ — see [NT] formula (2.9), so that we obtain an \mathcal{O} -linear map

$$(12) \quad \alpha : \mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{X}, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\mu}}^\dagger(\mathcal{X}, \mathfrak{p}).$$

The restriction of α to $0 \in \text{Spec } \mathcal{O}$ gives a \mathbb{C} -linear map

$$\alpha_0 : \mathcal{V}_{\vec{\lambda}}^\dagger(X_0, \mathfrak{g}) \longrightarrow \mathcal{V}_{\vec{\mu}}^\dagger(X_0, \mathfrak{p}).$$

The main result of this section is the following description of the map α_0

Proposition 4.2. *Using the isomorphisms given by the factorization rules, the linear map α_0 decomposes as*

$$\alpha_0 = (\alpha_0^{\lambda, \mu}) : \bigoplus_{\lambda \in P_1(\mathfrak{g})} \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \longrightarrow \bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{\vec{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p}),$$

where the linear map $\alpha_0^{\lambda, \mu} : \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g}) \rightarrow \mathcal{V}_{\vec{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p})$

- is identically zero, if $\Delta_\mu(\mathfrak{p}) \neq \Delta_\lambda(\mathfrak{g})$.
- is induced by the natural inclusion $\mathcal{H}_{\vec{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\vec{\mu}}(\mathfrak{p}) \otimes \mathcal{H}_{\vec{\mu}^\dagger}(\mathfrak{p}) \hookrightarrow \mathcal{H}_{\vec{\lambda}}(\mathfrak{g}) \otimes \mathcal{H}_\lambda(\mathfrak{g}) \otimes \mathcal{H}_{\lambda^\dagger}(\mathfrak{g})$, if $\Delta_\mu(\mathfrak{p}) = \Delta_\lambda(\mathfrak{g})$.

Proof. We fix $\lambda \in P_1(\mathfrak{g})$ and consider an element $\psi_\lambda \in \mathcal{V}_{\vec{\lambda}, \lambda, \lambda^\dagger}^\dagger(\tilde{X}, \mathfrak{g})$. In order to compute the decomposition of $\alpha_0(\iota_\lambda(\psi_\lambda))$ in the direct sum $\bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{\vec{\mu}, \mu, \mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p})$, we will first decompose the extension $\alpha(\widetilde{\psi_\lambda})$ and then restrict. In fact, we have $\alpha_0(\iota_\lambda(\psi_\lambda)) = r_0(\alpha(\widetilde{\psi_\lambda}))$. On the other hand, using Proposition 3.6 and the definition of $\widetilde{\psi_\lambda}$, we easily obtain that for any $\tilde{u} \in \mathcal{H}_{\vec{\mu}}[[q]] \hookrightarrow \mathcal{H}_{\vec{\lambda}}[[q]]$

$$\begin{aligned} \langle \alpha(\widetilde{\psi_\lambda}) | \tilde{u} \rangle &= \langle \widetilde{\psi_\lambda} | \tilde{u} \rangle = \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\lambda \rangle \\ &= \sum_{\hat{\mu} \in B(\lambda)} q^{n_\mu} \langle \psi_\lambda | \tilde{u} \otimes \tilde{\gamma}_\mu \rangle \end{aligned}$$

Now we restrict this expression to the special fiber, i.e., we put $q = 0$, which leads to

$$\alpha_0(\iota_\lambda(\psi_\lambda)) = \sum_{\substack{\hat{\mu} \in B(\lambda) \\ n_\mu = 0}} \iota_\mu(\alpha_0^{\lambda, \mu}(\psi_\lambda)),$$

where $\alpha_0^{\lambda, \mu}$ is the linear map described in the proposition. \square

5. PROOF OF THEOREM 1

5.1. Proof of part (i). First of all, we will restate Theorem 1 in terms of spaces of conformal blocks. By [B] Proposition 5.2 the linear map ϕ_P identifies under the ‘‘Verlinde’’ isomorphism (3) with the linear map

$$\phi_{X, \mathfrak{p}} : \mathcal{V}_0^\dagger(X, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(X, \mathfrak{p}),$$

induced by the inclusion of the basic $\hat{\mathfrak{p}}$ -module $\mathcal{H}_0(\mathfrak{p})$ into the basic $\hat{\mathfrak{e}}_8$ -module $\mathcal{H}_0(\mathfrak{e}_8)$. We recall that we choose a point $p \in X$ and we denote $U = X \setminus \{p\}$. It is therefore equivalent to show that $\phi_{X, \mathfrak{p}}$ is non-zero for any smooth curve X .

We then observe that the rank of the linear map $\phi_{X,\mathfrak{p}}$ is constant when the smooth curve X varies by [B] Proposition 5.8 and Lemma A.1. It is therefore sufficient to show that there exists a smooth curve X for which the map $\phi_{X,\mathfrak{p}}$ is non-zero. We will prove that by induction on the genus g of X .

The case $g = 0$ is easily seen as follows. Over the projective line we have $\mathfrak{g}(U) \cdot \mathcal{H}_0(\mathfrak{g}) = \bigoplus_{m>0} \mathcal{H}_0(\mathfrak{g})(m)$ for any semi-simple Lie algebra \mathfrak{g} . Hence the one-dimensional space of covacua $\mathcal{V}_0(\mathbb{P}^1, \mathfrak{g})$ is generated by the image under the projection

$$\mathcal{H}_0(\mathfrak{g}) \longrightarrow \mathcal{V}_0(\mathbb{P}^1, \mathfrak{g}) = \mathcal{H}_0(\mathfrak{g})/\mathfrak{g}(U) \cdot \mathcal{H}_0(\mathfrak{g})$$

of the highest weight vector $v_0(\mathfrak{g}) \in \mathcal{H}_0(\mathfrak{g})(0) = V_0 = \mathbb{C}$. Since the trivial \mathfrak{e}_8 -module $V_0 = \mathbb{C}$ restricts to the trivial \mathfrak{p} -module, the highest weight vector $v_0(\mathfrak{p}) \in \mathcal{H}_0(\mathfrak{p})(0) = V_0$ coincides with the highest weight vector $v_0(\mathfrak{g})$ under the inclusion $\mathcal{H}_0(\mathfrak{p}) \hookrightarrow \mathcal{H}_0(\mathfrak{g})$, which implies that $\phi_{\mathbb{P}^1,\mathfrak{p}}$ is non-zero.

Next, we consider as in section 4 a family \mathcal{X} of genus g curves parametrized by $\text{Spec } \mathcal{O}$ such that $\mathcal{X}_0 = X_0$ is a nodal curve and \mathcal{X}_K a smooth curve defined over $K = \mathbb{C}((q))$. We also consider (12) the \mathcal{O} -linear map α associated to the conformal embedding $\mathfrak{p} \subset \mathfrak{e}_8$ and the trivial weights $\lambda = 0$ and $\mu = 0$

$$\alpha : \mathcal{V}_0^\dagger(\mathcal{X}, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\mathcal{X}, \mathfrak{p}).$$

By Proposition 4.2 and by the induction hypothesis, the restriction α_0 of the map α to the special fiber is non-zero: in fact, the genus of the normalization \tilde{X} equals $g - 1$ and α_0 decomposes as follows

$$\alpha_0 = (\alpha_0^{0,\mu}) : \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{e}_8) \xleftarrow{\sim} \mathcal{V}_{0,0,0}^\dagger(\tilde{X}, \mathfrak{e}_8) \longrightarrow \bigoplus_{\mu \in P_\ell(\mathfrak{p})} \mathcal{V}_{0,\mu,\mu^\dagger}^\dagger(\tilde{X}, \mathfrak{p}),$$

where the first isomorphism is the so-called *propagation of vacua* isomorphism (see e.g. [U] Theorem 3.3.1). We note that $P_1(\mathfrak{e}_8) = \{0\}$. By Remark 3.4 and Proposition 4.2 we have $\alpha_0^{0,\mu} = 0$ if $\mu \neq 0$ and, again due to the propagation of vacua isomorphism $\mathcal{V}_{0,0,0}^\dagger(\tilde{X}, \mathfrak{p}) \xrightarrow{\sim} \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{p})$, the map $\alpha_0^{0,0}$ is identified with the map $\phi_{\tilde{X},\mathfrak{p}} : \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\tilde{X}, \mathfrak{p})$, which is non-zero by the induction hypothesis. Hence α_0 is also non-zero.

By semi-continuity we conclude that the restriction $\alpha_K = \phi_{\mathcal{X}_K,\mathfrak{p}} : \mathcal{V}_0^\dagger(\mathcal{X}_K, \mathfrak{e}_8) \longrightarrow \mathcal{V}_0^\dagger(\mathcal{X}_K, \mathfrak{p})$ of α to the generic fiber is non-zero. Hence, again by [B] Proposition 5.8 and Lemma A.1, the K -linear map $\phi_{X_K,\mathfrak{p}}$ is non-zero for any genus g curve X_K defined over the field $K = \mathbb{C}((q))$. Given a genus g curve X defined over \mathbb{C} , the result then follows from the equality $\phi_{X,\mathfrak{p}} \otimes_{\mathbb{C}} K = \phi_{X \otimes_{\mathbb{C}} K,\mathfrak{p}}$.

5.2. Irreducible representations of Heisenberg groups. Before pursuing the proof of Theorem 1, we need to recall some known facts on Heisenberg groups and their irreducible representations. We consider a semi-simple simply connected group P with center Z of the following table

(13)

P	Spin(4n)	SL(n)	E_6	E_7	SL(n) \times SL(n)	SL(3) \times E_6	SL(2) \times E_7
Z	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

The finite abelian group $\mathcal{M}_X(Z)$ of principal Z -bundles acts on $\mathcal{M}_X(P)$ by twisting P -bundles with Z -bundles. Note that $|\mathcal{M}_X(Z)| = |Z|^{2g}$. We denote by t_ζ the automorphism of

$\mathcal{M}_X(P)$ induced by the twist with $\zeta \in \mathcal{M}_X(Z)$. We introduce the Mumford group associated to the line bundle \mathcal{L}_P

$$\mathcal{G}(\mathcal{L}_P) := \{(\zeta, \psi) \mid \zeta \in \mathcal{M}_X(Z) \text{ and } \psi : t_\zeta^* \mathcal{L}_P \xrightarrow{\sim} \mathcal{L}_P\}.$$

The Mumford group is a central extension of the group $\mathcal{M}_X(Z)$ by \mathbb{C}^* and acts via $s \mapsto \psi(t_\zeta^*(s))$ on the space of global sections $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$. Note that the center \mathbb{C}^* of $\mathcal{G}(\mathcal{L}_P)$ acts by scalar multiplication.

We will need the following result.

Lemma 5.1 ([F1] Lemma 16). *The Mumford group $\mathcal{G}(\mathcal{L}_P)$ is isomorphic as an extension to a finite Heisenberg group $\mathcal{G}(\delta)$. The type δ depends only on the center Z and on the genus g of X .*

We refer e.g. to [M] page 294 for the definition of the Heisenberg group $\mathcal{G}(\delta)$.

Corollary 5.2. *For all groups P of the Table (13) the space of global sections $H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ is an irreducible representation of $\mathcal{G}(\mathcal{L}_P)$.*

Proof. This is an immediate consequence of the fact there exists a unique irreducible representation of $\mathcal{G}(\delta)$ on which \mathbb{C}^* acts as scalar multiplication (see e.g. [M] Proposition 3), which has dimension $|Z|^g$, and from the numerical identity $\dim H^0(\mathcal{M}_X(P), \mathcal{L}_P) = |Z|^g$ provided by the Verlinde formula. \square

5.3. Proof of part (ii). Part (ii) follows immediately from the fact that the subgroup $\mathcal{M}_X(N)$ is a maximal isotropic subgroup of $\mathcal{M}_X(Z)$. Here isotropic means with respect to the standard symplectic form on $\mathcal{M}_X(Z)$ induced by the commutators in the Mumford group (see e.g. [M] page 293). The canonical lift $\mathcal{M}_X(N) \hookrightarrow \mathcal{G}(\mathcal{L}_P)$ is given by the isomorphism $\tilde{\phi}^* \mathcal{L}_{E_8} = \mathcal{L}_P$ and the fact that $\mathcal{M}_X(N)$ acts on the fibers of $\tilde{\phi}$. Hence by [M] Proposition 3 and by Corollary 5.2 the subspace of $\mathcal{M}_X(N)$ -invariant sections is one-dimensional.

6. PROOF OF THEOREM 2

The main observation is that, in all three cases, the finite group $N \subset P = A \times B$ projects isomorphically to the center of A and B . Hence $N = Z(A) = Z(B)$. Moreover the $\mathcal{M}_X(N)$ -invariance of the section $\sigma \in H^0(\mathcal{M}_X(P), \mathcal{L}_P)$ translates into the equivariance of the linear map $\sigma : H^0(\mathcal{M}_X(A), \mathcal{L}_A)^* \longrightarrow H^0(\mathcal{M}_X(B), \mathcal{L}_B)$ under the group $\mathcal{G}(\mathcal{L}_A) = \mathcal{G}(\mathcal{L}_B)$. Since both spaces are irreducible representations of $\mathcal{G}(\mathcal{L}_A) = \mathcal{G}(\mathcal{L}_B)$ by Corollary 5.2, the non-zero map σ is an isomorphism by Schur's lemma.

7. FURTHER REMARKS

7.1. Invariant sections.

7.1.1. $G = \mathrm{SL}(9)$. In this case the restriction of the E_8 -theta divisor is the unique (up to a scalar) $\mathrm{Jac}(X)[3]$ -invariant section in $H^0(\mathcal{M}_X(\mathrm{SL}(9)), \mathcal{L}_{\mathrm{SL}(9)})$. Here the group $\mathrm{Jac}(X)[3]$ of 3-torsion line bundles over X acts by tensor product on the moduli stack of rank-9 vector bundles with trivial determinant. Note that we take here the linear action of $\mathrm{Jac}(X)[3]$ on $H^0(\mathcal{M}_X(\mathrm{SL}(9)), \mathcal{L}_{\mathrm{SL}(9)})$ induced by the isomorphism $\tilde{\phi}^* \mathcal{L}_{E_8} = \mathcal{L}_{\mathrm{SL}(9)}$. This leads to the natural question (see also [F2] section 6): does there exist a geometrical description of the zero-divisor of this invariant section?

7.1.2. $G = \text{Spin}(16)$. The $\mathcal{M}_X(N)$ -invariant section $\sigma \in H^0(\mathcal{M}_X(\text{Spin}(16)), \mathcal{L}_{\text{Spin}(16)})$ can be described as follows. The center Z of $\text{Spin}(16)$ equals $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\pm 1, \pm\gamma\}$, where $\{\pm 1\}$ is the kernel K of the homomorphism $\text{Spin}(16) \rightarrow \text{SO}(16)$ and $\pm\gamma$ covers the element $-\text{Id} \in \text{SO}(16)$. Note that $N = \{1, \gamma\}$. By [PR] Proposition 8.2 the space $H^0(\mathcal{M}_X(\text{Spin}(16)), \mathcal{L}_{\text{Spin}(16)})$ admits a basis $\{s_\kappa\}$ indexed by the 4^g theta-characteristics κ of the curve X and such that the set-theoretical support of the zero-divisor of s_κ equals

$$D_\kappa = \{E \in \mathcal{M}_X(\text{Spin}(16)) \mid \dim H^0(X, E(\mathbb{C}^{16}) \otimes \kappa) > 0\}.$$

Here $E(\mathbb{C}^{16})$ denotes the orthogonal rank-16 vector bundle associated to E . The group $\mathcal{M}_X(N)$ identifies with the group $\text{Jac}(X)[2]$ of 2-torsion line bundles over X and under the canonical lift $\text{Jac}[2] \hookrightarrow \mathcal{G}(\mathcal{L}_{\text{Spin}(16)})$ we have that $\alpha \cdot s_\kappa = s_{\alpha\kappa}$ for some suitably normalized basis $\{s_\kappa\}$. It is then clear that $\sigma = \sum_\kappa s_\kappa$, where κ runs over all theta-characteristics. As in the previous case, a geometrical description of its zero-divisor is still missing.

7.2. Other conformal subalgebras of \mathfrak{e}_8 .

7.2.1. $(\text{Spin}(8), \text{Spin}(8))$. We observe that $\mathfrak{p} = \mathfrak{so}(8) \oplus \mathfrak{so}(8)$ is a *non-maximal* conformal subalgebra of \mathfrak{e}_8 . In fact, the inclusion $\mathfrak{p} \subset \mathfrak{e}_8$ factorizes through $\mathfrak{so}(16)$. We therefore obtain a group homomorphism $\phi : P = \text{Spin}(8) \times \text{Spin}(8) \rightarrow \text{Spin}(16) \rightarrow E_8$ with kernel $N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We note that N sits diagonally in $Z(\text{Spin}(8)) \times Z(\text{Spin}(8))$. Theorem 1 and Theorem 2 also hold in this case.

7.2.2. (G_2, F_4) . As indicated in Table (4) the spaces $H^0(\mathcal{M}_X(G_2), \mathcal{L}_{G_2})$ and $H^0(\mathcal{M}_X(F_4), \mathcal{L}_{F_4})$ have the same dimension. Theorem 1 says that the E_8 -theta divisor Δ induces a non-zero linear map $\sigma : H^0(\mathcal{M}_X(G_2), \mathcal{L}_{G_2})^* \rightarrow H^0(\mathcal{M}_X(F_4), \mathcal{L}_{F_4})$. Since G_2 and F_4 have no center, the argument used in the proof of Theorem 2 breaks down in this case.

REFERENCES

- [BB] A. Bais, P. Bouwknegt: A classification of subgroup truncations of the bosonic string, Nuclear Physics B279 (1987), 561-570
- [B] P. Belkale: Strange duality and the Hitchin/WZW connection, arXiv:0705.0717
- [BS] A. Berenstein, R. Sjamaar: Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion, J. Amer. Math. Soc. 13, No. 2 (2000), 433-466
- [CG] A. Cohen, R. Griess: On finite simple subgroups of the complex Lie group of type E_8 , The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proceedings of Symposia in Pure Mathematics, Volume 47, Part 2, AMS (1987)
- [D] E.B. Dynkin: Semisimple subalgebras of semisimple Lie algebras, Am. Math. Soc. Transl. (Ser. II) 6 (1957), 111-244
- [F1] G. Faltings: Theta functions on moduli spaces of G -bundles, J. Alg. Geom. 18 (2009), 309-369
- [F2] G. Faltings: Thetafunktionen auf Modulräumen von Vektorbündeln, Jahresbericht der DMV, 110. Band (2008), Heft 1, 3-17
- [F3] G. Faltings: A proof for the Verlinde formula, J. Alg. Geom. 3 (1994), 347-374
- [K] V. Kac: Infinite dimensional Lie algebras, Third Edition, Cambridge University Press (1990)
- [KS] V. Kac, M. Sanielevici: Decomposition of representations of exceptional affine algebras with respect to conformal subalgebras, Physical Review D, Vol. 37, No. 8 (1988), 2231-2237
- [KW] V. Kac, M. Wakimoto: Modular and conformal invariance constraints in representation theory of affine algebras, Advances in Math. 70 (1988), 156-234
- [KNR] S. Kumar, M.S. Narasimhan, A. Ramanathan: Infinite Grassmannians and moduli spaces of G -bundles, Math. Ann. 300 (1994), 41-75
- [LS] Y. Laszlo, C. Sorger: The line bundles on the moduli of parabolic G -bundles over curves and their sections, Ann. Scient. Ec. Norm. Sup. 30 (1997), 499-525

- [MO] A. Marian, D. Oprea: A tour of theta dualities on moduli spaces of sheaves, Curves and abelian varieties, Contemporary Mathematics, 465, AMS, Rhode Island (2008), 175-202
- [M] D. Mumford: On the Equations Defining Abelian Varieties I, Invent. Math. 1 (1966), 287-354
- [NT] T. Nakanishi, A. Tsuchiya: Level-Rank Duality of WZW Models in Conformal Field Theory, Comm. Math. Phys. 144 (1992), 351-372
- [PR] C. Pauly, S. Ramanan: A duality for Spin Verlinde spaces and Prym Theta functions, J. London Math. Soc. 63 (2001), 513-532
- [Pa] C. Pauly: La dualité étrange, Séminaire Bourbaki No. 994, June 2008
- [Po] M. Popa: Generalized theta linear series on moduli spaces of vector bundles on curves, arXiv:0712.3192
- [SW] A. Schellekens, N. Warner: Conformal subalgebras of Kac-Moody algebras, Physical Review D, Vol. 34, No. 10 (1986), 3092-3096
- [So] C. Sorger: On moduli of G -bundles on a curve for exceptional G , Ann. Scient. Éc. Norm. Sup. 32 (1999), 127-133
- [T] C. Teleman: Lie algebra cohomology and the fusion rules, Commun. Math. Phys. 173 (1995), 265-311
- [U] K. Ueno: Introduction to conformal field theory with gauge symmetries, in Geometry and Physics, Lecture Notes in Pure and Applied Mathematics 184, Marcel Dekker (1996), 603-745

MATEMATİK BÖLÜMÜ, BOĞAZIÇI ÜNİVERSİTESİ, 34342, İSTANBUL, TURKEY
E-mail address: arzu.boysal@boun.edu.tr

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONTPELLIER II - CASE COURRIER 051, PLACE
 EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE
E-mail address: pauly@math.univ-montp2.fr