

# Solving BSDE with adaptive control variate.

## A note on the rate of convergence of the operator $\mathcal{P}^k$

Emmanuel Gobet\*      Céline Labart<sup>†</sup>

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This note is a complement of the paper "Solving BSDE with adaptive control variate" [1]. It deals with the convergence of the approximating operator  $\mathcal{P}$ , based on a non parametric regression technique called local averaging, and defined in Definition 1.1.

Although the computations are quite standard (see [3], [2]), the specificities of the paper are the following

- the support of the variables is unbounded;
- the error has to be measured using specific  $L_2$ -norms;
- errors on the gradient are provided.

## 1 Definitions

Let us first introduce some notations

- Let  $C_b^{k,l}$  be the set of continuously differentiable functions  $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$  with continuous and uniformly bounded derivatives w.r.t.  $t$  (resp. w.r.t.  $x$ ) up to order  $k$  (resp. up to order  $l$ ).
- $C_p^k$  denotes the set of  $C^{k-1}$  functions whose  $k$ -th derivative is piecewise continuous.
- Constants  $c_{i,j}(\cdot)$  and  $C(d)$ . For any function  $\phi$  in  $C_b^{i,j}$ ,  $c_{i,j}(\phi)$  denotes  $\sum_{k,l=0}^{i,j} |\partial_t^k \partial_x^l \phi|_\infty$ . For  $i = j = 0$ , we set  $c_0(\phi) := c_{0,0}(\phi)$ .  $C(d)$  denotes a constant depending only on  $d$ .
- Functions  $K(T)$ .  $K(\cdot)$  denotes a generic function non decreasing in  $T$  which may depend on  $d, \mu, \beta$ , on the coefficients  $b$  and  $\sigma$  (through  $\sigma_0, \sigma_1, c_{1,3}(\sigma), c_{0,1}(\partial_t \sigma), c_{1,3}(b)$ ) and on other constants appearing in [1, Appendix A]. The parameter  $\beta$  is defined in [1, Section 2.1],  $\mu$  is defined in [1, Section 3.2],  $\sigma_0$  and  $\sigma_1$  are defined in [1, Hypothesis 1].
- Functions  $K_0(T)$ .  $K_0(T)$  are analogous to  $K(T)$  except that they may also depend on the operator  $\mathcal{P}$  (through  $c_1(K_t)$  and  $c_2(K_x)$ , defined in Section [1, Section 7]).

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\*Laboratoire Jean Kuntzmann, Université de Grenoble and CNRS, BP 53, 38041 Grenoble cedex 9, FRANCE(emmanuel.gobet@imag.fr).

<sup>†</sup>LPMA, Université Pierre et Marie Curie, Paris, FRANCE (celine.labart@upmc.fr).

**Definition 1.1.** We approximate a function  $v(t, x)$  by

$$\mathcal{P}^k v(t, x) = \frac{r_n^k(t, x)}{f_n^k(t, x)} g(2^{d+1} T \lambda(B) f_n^k(t, x)), \quad (1.1)$$

where

- $r_n^k(t, x) = \frac{1}{nh_t h_x^d} \sum_{i=1}^n K_t\left(\frac{t-T_i^k}{h_t}\right) K_x\left(\frac{x-X_i^k}{h_x}\right) v(T_i^k, X_i^k)$ ;
- $f_n^k(t, x) = \frac{1}{nh_t h_x^d} \sum_{i=1}^n K_t\left(\frac{t-T_i^k}{h_t}\right) K_x\left(\frac{x-X_i^k}{h_x}\right)$ ;
- the points  $(T_i^k, X_i^k)_{1 \leq i \leq n}$  are uniformly distributed on  $[0, T] \times B$  where  $B := B_\infty(0, a) = [-a, a]^d$ , and  $\mathcal{A}_k$  denotes the set of points  $(T_i^k, X_i^k)_{1 \leq i \leq n}$ ;
- $\lambda(B) = (2a)^d$ ;
- and  $g$  is such that

$$g(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y > 1, \\ -y^4 + 2y^2 & \text{if } y \in [0, 1]; \end{cases} \quad (1.2)$$

- The kernel function  $K_t$  is defined on the compact support  $[-1, 1]$ , bounded, even, non-negative,  $C_p^2$  and  $\int_{\mathbb{R}} K_t(u) du = 1$ ;
- The kernel function  $K_x$  is defined on the compact support  $[-1, 1]^d$ , bounded, and such that  $\forall y = (y_1, \dots, y_d) \in \mathbb{R}^d$ ,  $K_x(y) = \prod_{j=1}^d K_x^j(y_j)$ , where for  $j = 1, \dots, d$   $K_x^j : \mathbb{R} \rightarrow \mathbb{R}$  is an even non-negative  $C_p^2$  function and  $\int_{\mathbb{R}} K_x^j(u) du = 1$ ;
- $\delta_n$  denotes  $\frac{1}{nh_t h_x^d}$ , and  $T \lambda(B) \delta_n \ll 1$ ;
- $h_x \ll a$  and  $h_t \ll \frac{T}{2}$ . Since we study the convergence when  $h_t$  and  $h_x$  tend to 0, we assume in the following that  $h_t \leq 1$  and  $h_x \leq 1$ .

**Remark 1.2.** We give some useful bounds for  $g$  and its first derivative. The function  $G : x \mapsto \frac{g(x)}{x}$  is bounded by 2,  $g'$  is bounded by 2,  $x \mapsto \frac{g'(x)}{x}$  is bounded by 4 and  $x \mapsto \frac{g(x)}{x^2}$  is bounded by 2. Then,  $G'$  is bounded by 6.

**Remark 1.3.** This choice for the operator  $\mathcal{P}^k$  is not harmless.  $\mathcal{P}^k$  should be continuous and differentiable. That's why we multiply  $\frac{r_n^k}{f_n^k}$  by a regularising function  $g$  at the point  $2^{d+1} T \lambda(B) f_n^k$ . Since the function  $f_n^k(t, x)$  converges to  $\frac{1}{T \lambda(B)}$  when  $n$  goes to infinity for  $t \in ]0, T[$  and  $|x_i| < a$ ,  $i = 1, \dots, d$ ,  $g(2^{d+1} T \lambda(B) f_n^k(t, x))$  converges to 1 when  $n$  goes to  $\infty$ . Hence, if  $f_n^k \sim \frac{1}{T \lambda(B)}$ ,  $\mathcal{P}^k v(t, x) = \frac{r_n^k(t, x)}{f_n^k(t, x)}$ , which is a standard estimator. The function  $g$  has an impact on  $\mathcal{P}^k$  only when  $f_n^k$  is strictly positive and small (compared to  $\frac{1}{T \lambda(B)}$ ).

We also introduce the space  $H_{\beta, X}^{m, \mu}$ :

**Definition 1.4** (Space  $H_{\beta, X}^{m, \mu}$ ). Let  $X$  denote the  $\mathbb{R}^d$ -valued process solution of

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad (1.3)$$

where  $W$  is a  $q$ -dimensional standard Brownian motion,  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$ . For any  $m \leq 2, \beta > 0, \mu > 0$ , let  $H_{\beta, X}^{m, \mu}$  define the space of functions  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\|v\|_{H_{\beta, X}^{m, \mu}}^2 = \int_0^T e^{\beta s} \int_{\mathbb{R}^d} e^{-\mu|x|} \sum_{k \leq m} \mathbb{E} |\partial_x^k v(s, X_s^x)|^2 dx ds < \infty.$$

**Definition 1.5** (Function  $\nu_\mu^t$ ). For any  $s, t \in [0, T]$  and any  $x, y \in \mathbb{R}^d$  such that  $t < s$  we define  $\nu_\mu^t(s, y) := \int_{\mathbb{R}^d} e^{-\mu|x|} p(t, x; s, y) dx$ , where  $\mu$  is a positive constant and  $p$  is the transition density function of the process  $X$  defined by (1.3).

**Remark 1.6.** Using the definition of  $\nu$ , we also get  $\|v\|_{H_{\beta, X}^\mu}^2 = \int_0^T e^{\beta s} \int_{\mathbb{R}^d} dy \nu_\mu^0(s, y) |v(s, y)|^2$ .

**Hypothesis 1.** We assume that the coefficients  $\sigma$  and  $b$  are Lipschitz and bounded measurable functions on  $[0, T] \times \mathbb{R}^d$ . We also assume that  $\sigma$  satisfies the ellipticity condition.

## 2 Main results

We aim at proving the following Propositions, which correspond to [1, Theorem 7.4].

**Proposition 2.1.** *Assume Hypothesis 1. We also assume that  $v$  is a  $C^{1,2}([0, T] \times \mathbb{R}^d)$  function and  $v$  and  $\partial_x v$  are bounded by  $c_{0,1}(v)$  and  $v$  satisfies  $\forall t, t' \in [0, T], \forall x \in \mathbb{R}^d, |\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v) \sqrt{|t' - t|}$ , where  $c_{1/2}(v)$  is a positive constant. Then,*

$$\begin{aligned} \mathbb{E} \|\mathcal{P}^k v - v\|_{H_{\beta, X}^\mu}^2 + \mathbb{E} \|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta, X}^\mu}^2 &\leq \epsilon_1(\mathcal{P}) (\mathbb{E} \|v\|_{H_{\beta, X}^{2, \mu}}^2 + \mathbb{E} \|\partial_t v\|_{H_{\beta, X}^\mu}^2) \\ &\quad + e_2(\mathcal{P}) (c_{1/2}^2(v) + c_{0,1}^2(v)), \end{aligned}$$

where  $\epsilon_1(\mathcal{P}) = K_0(T) (h_t^2 + h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2})$ ,  $e_2(\mathcal{P}) = K_0(T) (h_t + e^{-\mu a} \frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}} + \frac{T\lambda(B)\delta_n}{h_x^2})$ . Moreover, if  $v$  is a  $C_b^{1,2}$  function, we get  $\mathbb{E} \|\mathcal{P}^k v - v\|_{H_{\beta, X}^\mu}^2 + \mathbb{E} \|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta, X}^\mu}^2 \leq (\epsilon_1(\mathcal{P}) + e_2(\mathcal{P})) (c_{1/2}^2(v) + c_{1,2}^2(v))$ .

The proof of Proposition 2.1 is done in Sections 4 and 5. Section 4 (resp. Section 5) deals with the bound for  $\mathbb{E} \|\mathcal{P}^k v - v\|_{H_{\beta, X}^\mu}^2$  (resp.  $\mathbb{E} \|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta, X}^\mu}^2$ ).

**Proposition 2.2.** *Under Hypothesis 1, for any random function  $v$  from  $[0, T] \times \mathbb{R}^d$  to  $\mathbb{R}$  independent of  $\mathcal{A}_k$ , one has*

$$\mathbb{E} \|\mathcal{P}^k v\|_{H_{\beta, X}^\mu}^2 + \mathbb{E} \|\partial_x(\mathcal{P}^k v)\|_{H_{\beta, X}^\mu}^2 \leq c_4(\mathcal{P}) \mathbb{E} \|v\|_{H_{\beta, X}^\mu}^2, \text{ where } c_4(\mathcal{P}) = \frac{K_0(T)}{h_x^2}.$$

If  $\mathbb{E}(v(t, x)) = 0$ , one has  $\mathbb{E} \|\mathcal{P}^k v\|_{H_{\beta, X}^\mu}^2 + \mathbb{E} \|\partial_x(\mathcal{P}^k v)\|_{H_{\beta, X}^\mu}^2 \leq \epsilon_4(\mathcal{P}) \mathbb{E} \|v\|_{H_{\beta, X}^\mu}^2$ , where  $\epsilon_4(\mathcal{P}) = K_0(T) \frac{T\lambda(B)\delta_n}{h_x^2}$ .

The proof of Proposition 2.1 is done in Section 6.

**Remark 2.3.** For the sake of clearness, we omit the superscript  $k$  in the definition of  $r_n^k$  and  $f_n^k$ . From now on,  $r_n$  (resp.  $f_n$ ) denotes  $r_n^k$  (resp.  $f_n^k$ ).

### 3 Properties on $f_n$ , $r_n$ and other useful results

In this Section, we only recall some technical results on  $f_n$  and  $r_n$  proved in [4].

**Lemma 3.1** (Lemma 12.13, [4]). *For all  $(s, y) \in [0, T] \times \mathbb{R}^d$ ,*

$$\mathbb{E}[f_n(s, y)] = \frac{1}{T\lambda(B)} \int_{-1\sqrt{\frac{s}{h_t}}}^{1\wedge\frac{T-s}{h_t}} K_t(r) dr \prod_{j=1}^d \int_{-1\sqrt{\frac{-a-y_j}{h_x}}}^{1\wedge\frac{a-y_j}{h_x}} K_x^j(x_j) dx_j, \quad (3.1)$$

and  $\mathbb{E}[f_n(s, y)] \leq \frac{1}{T\lambda(B)}$ . Moreover, for  $(s, y) \in [0, T] \times B$ ,  $\mathbb{E}[f_n(s, y)] \geq \frac{1}{T\lambda(B)2^{d+1}}$ , and for  $(s, y) \in [h_t, T - h_t] \times B_\infty(0, a - h_x)$ ,  $\mathbb{E}[f_n(s, y)] = \frac{1}{T\lambda(B)} = \bar{f}(s, y)$ .

**Proposition 3.2** (Proposition 12.20, [4]). *Assume  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a bounded function. Then, for all  $(s, y) \in [0, T] \times \mathbb{R}^d$ ,*

$$\begin{aligned} (\mathbb{E}[r_n(s, y)])^2 &\leq \frac{c_0^2(v)}{(T\lambda(B))^2} \mathbf{1}_{\{y \in B_\infty(0, a+h_x)\}}, \\ \text{Var}(r_n(s, y)) &\leq 2^{d+1} \frac{c_0^2(K_t)c_0^2(K_x)c_0^2(v)\delta_n}{T\lambda(B)} \mathbf{1}_{\{y \in B_\infty(0, a+h_x)\}}. \end{aligned}$$

Using  $T\lambda(B)\delta_n \ll 1$  yields  $\mathbb{E}(r_n^2(s, y)) \leq 2^{d+2} \frac{c_0^2(K_t)c_0^2(K_x)c_0^2(v)}{(T\lambda(B))^2} \mathbf{1}_{\{y \in B_\infty(0, a+h_x)\}}$ .

Assume  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  is a continuous function. Then,

$$\begin{aligned} \mathbb{E}(r_n(s, y))^2 &\leq \frac{2^{d+1}}{h_t h_x^d (T\lambda(B))^2} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz K_x^2\left(\frac{y-z}{h_x}\right) v^2(r, z), \\ \text{Var}(r_n(s, y)) &\leq \frac{1}{nh_t^2 h_x^{2d} T\lambda(B)} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz K_x^2\left(\frac{y-z}{h_x}\right) v^2(r, z). \end{aligned}$$

Using  $T\lambda(B)\delta_n \ll 1$  yields

$$\mathbb{E}(r_n^2(s, y)) \leq \frac{2^{d+2}}{h_t h_x^d (T\lambda(B))^2} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz K_x^2\left(\frac{y-z}{h_x}\right) v^2(r, z).$$

**Lemma 3.3** (Lemma 12.17, [4]). *For all  $(s, y) \in [0, T] \times \mathbb{R}^d$ , for all  $i \in \{1, \dots, d\}$ , the following assertion holds*

$$\mathbb{E}(\partial_{x_i} f_n(s, y))^2 \leq 2^{2d+3} \frac{c_0^2(K_t)c_{0,1}^2(K_x)}{h_x^2 (T\lambda(B))^2} \mathbf{1}_{\{y \in B_\infty(0, a+h_x)\}}.$$

**Proposition 3.4** (Proposition 12.19, [4]). *Under Hypothesis 1, one has*

$$\int_0^T ds e^{\beta s} \int_B dy \nu^0(s, y) |\mathbb{E}(\partial_{x_i} f_n(s, y))|^2 \leq \frac{K_0(T)}{h_x (T\lambda(B))^2} e^{-\mu a} a^{d-1}.$$

**Lemma 3.5** (Lemma 12.10, [4]). *Assume Hypothesis 1 and let  $f$  be a function from  $[0, T] \times \mathbb{R}^d$  into  $\mathbb{R}^+$ ,  $g_t$  a positive bounded function with compact support in  $[-1, 1]$  and  $g_x$  a positive bounded function with compact support in  $[-1, 1]^d$ . Then,*

$$\begin{aligned} \int_0^T ds e^{\beta s} \int_{\mathbb{R}^d} dy \nu^0(s, y) \int_0^T dr g_t\left(\frac{s-r}{h_t}\right) \int_{\mathbb{R}^d} dz g_x\left(\frac{y-z}{h_x}\right) f(r, z) \leq \\ K(T) c_0(g_t) c_0(g_x) h_t h_x^d \int_0^T dr e^{\beta r} \int_{\mathbb{R}^d} dz \nu^0(r, z) f(r, z). \end{aligned}$$

## 4 Proof of Proposition 2.1: term $\mathbb{E}\|\mathcal{P}^k v - v\|_{H_{\beta,X}^\mu}^2$

The study of  $\mathbb{E}\|\mathcal{P}^k v - v\|_{H_{\beta,X}^\mu}^2$  will be done in two steps. To do so, we add and subtract

$$C_n(s, y) := \frac{r_n(s, y)}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} \quad (4.1)$$

to the term  $v - \mathcal{P}^k v$ .  $C_n$  approximates well  $v$  inside the domain  $[0, T] \times B$ . We get  $\mathbb{E}\|\mathcal{P}^k v - v\|_{H_{\beta,X}^\mu}^2 \leq 2\mathbb{E}\|\mathcal{P}^k v - C_n\|_{H_{\beta,X}^\mu}^2 + 2\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2$ . The two following sections are devoted to the study of  $\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2$  and  $\mathbb{E}\|\mathcal{P}^k v - C_n\|_{H_{\beta,X}^\mu}^2$ .

### 4.1 Study of $\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2$

Using the definition of  $C_n$ , we get

$$C_n(s, y) - v(s, y) = \frac{r_n(s, y) - v(s, y)\mathbb{E}[f_n(s, y)]}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} - v(s, y) \mathbf{1}_{\{s \notin [0, T] \cup y \notin B\}}. \quad (4.2)$$

Then, we split  $\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2$  in two terms, by using the bias-variance decomposition:  $\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2 = \|\mathbb{E}(C_n - v)\|_{H_{\beta,X}^\mu}^2 + \|\text{Std}(C_n - v)\|_{H_{\beta,X}^\mu}^2$ , where  $\text{Std}(Y(s, y)) = \sqrt{\text{Var}(Y(s, y))}$ .

#### 4.1.1 Study of $\|\mathbb{E}(C_n - v)\|_{H_{\beta,X}^\mu}^2$

We have

$$\mathbb{E}(C_n - v)(s, y) = \frac{\mathbb{E}[r_n(s, y)] - v(s, y)\mathbb{E}[f_n(s, y)]}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} - v(s, y) \mathbf{1}_{\{s \notin [0, T] \cup y \notin B\}}.$$

Since  $\mathbb{E}[r_n(s, y)] = \frac{1}{T\lambda(B)h_t h_x^d} \int_B dz \int_0^T dr v(r, z) K_x\left(\frac{y-z}{h_x}\right) K_t\left(\frac{s-r}{h_t}\right)$ , we obtain

$$\mathbb{E}[r_n(s, y)] - v(s, y)\mathbb{E}[f_n(s, y)] = \frac{1}{T\lambda(B)h_t h_x^d} \int_B dz \int_0^T dr K_x\left(\frac{y-z}{h_x}\right) K_t\left(\frac{s-r}{h_t}\right) (v(r, z) - v(s, y)).$$

We use the second property of Lemma 3.1 and the equality  $v(r, z) - v(s, y) = v(r, z) - v(s, z) + v(s, z) - v(s, y)$  to bound  $|\mathbb{E}(C_n - v)(s, y)|$  by  $|A_1(s, y)| + |A_2(s, y)| + |A_3(s, y)|$ , where

$$\begin{aligned} A_1(s, y) &:= \frac{2^{d+1}}{h_t h_x^d} \int_B dz \int_0^T dr K_x\left(\frac{y-z}{h_x}\right) K_t\left(\frac{s-r}{h_t}\right) (v(r, z) - v(s, z)), \\ A_2(s, y) &:= \frac{2^{d+1}}{h_t h_x^d} \int_B dz \int_0^T dr K_x\left(\frac{y-z}{h_x}\right) K_t\left(\frac{s-r}{h_t}\right) (v(s, z) - v(s, y)), \\ A_3(s, y) &:= v(s, y) \mathbf{1}_{\{s \notin [0, T] \cup y \notin B\}}. \end{aligned}$$

We analyze each term in the following three Lemmas.

**Lemma 4.1.** *Let us assume Hypothesis 1 and  $v$  is a function  $C^1$  in time. Then,*

$$\|A_1(s, y)\|_{H_{\beta,X}^\mu}^2 \leq K_0(T) h_t^2 \|\partial_t v\|_{H_{\beta,X}^\mu}^2,$$

*If  $\partial_t v$  is bounded, we get  $\|A_1(s, y)\|_{H_{\beta,X}^\mu}^2 \leq K_0(T) c_{1,0}^2(v) h_t^2$ .*

This Lemma ensues from [4, Lemma 12.36].

**Lemma 4.2.** *Let us assume Hypothesis 1 and  $v$  is a function  $C^1$  in space. There exists a function  $K_0(T)$  such that*

$$\|A_2(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)h_x^2\|\partial_x v\|_{H_{\beta, X}^\mu}^2.$$

*In particular, if  $\partial_x v$  is bounded, we get  $\|A_2(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)c_{0,1}^2(v)h_x^2$ .*

*Proof.* The proof of this Lemma is the same as the one of Lemma 4.1, except that we split the difference  $v(s, z) - v(s, y)$  as a sum of  $d$  terms:  $v(s, z) - v(s, y) = \sum_{i=1}^d v(s, \bar{z}_i) - v(s, \bar{z}_{i-1})$ , where  $\bar{z}_i = (z_1, z_2, \dots, z_i, y_{i+1}, \dots, y_d), \forall i \in \{1, \dots, d\}$ , and  $\bar{z}_0 = y$ . For all  $i \in \{1, \dots, d\}$ , we get  $v(s, \bar{z}_i) - v(s, \bar{z}_{i-1}) = \int_{y_i}^{z_i} dl \partial_{x_i} v(s, \bar{z}_i^l)$ , where  $\bar{z}_i^l = (z_1, \dots, z_{i-1}, l, y_{i+1}, \dots, y_d)$ . ■

**Lemma 4.3.** *Assume Hypothesis 1 and  $v$  is bounded. Then,  $\|A_3(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K(T)c_0^2(v)e^{-\frac{\mu a}{\sqrt{a}}}$ .*

*Proof.* Since  $v$  is bounded, we get  $\|A_3(s, y)\|_{H_{\beta, X}^\mu}^2 \leq c_0^2(v) \int_0^T dr e^{\beta r} \int_{B^c} dy \nu_\mu^0(r, y)$ . To conclude, we use  $\nu_\mu^0(r, y) \leq 2^d K e^{c_2 r} e^{-\mu|y|}$  (see the proof of [1, Proposition 3.8]). ■

Combining Lemmas 4.1, 4.2, 4.3 yields to the following Proposition.

**Proposition 4.4.** *Let us assume Hypothesis 1 and  $v$  is a bounded  $C^{1,1}$  function. Then,*

$$\|\mathbb{E}(C_n - v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)(h_t^2\|\partial_t v\|_{H_{\beta, X}^\mu}^2 + h_x^2\|\partial_x v\|_{H_{\beta, X}^\mu}^2) + c_0^2(v)K(T)e^{-\frac{\mu a}{\sqrt{a}}}.$$

*Moreover, if  $\partial_t v$  and  $\partial_x v$  are bounded, we get  $\|\mathbb{E}(C_n - v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)c_{1,1}^2(v)(h_t^2 + h_x^2 + e^{-\frac{\mu a}{\sqrt{a}}})$ .*

#### 4.1.2 Study of $\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2$

Let us study  $\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2 = \int_0^T ds e^{\beta s} \int_{\mathbb{R}^d} dy \nu_\mu^0(s, y) \text{Var}(C_n - v)(s, y)$ .

**Proposition 4.5.** *Let us assume Hypothesis 1. Then,*

$$\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)T\lambda(B)\delta_n\|v\|_{H_{\beta, X}^\mu}^2.$$

*If  $v$  is bounded, we get  $\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)c_0^2(v)T\lambda(B)\delta_n$ .*

*Proof.* Using (4.2) leads to  $\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2 = \int_0^T ds e^{\beta s} \int_B dy \nu_\mu^0(s, y) \text{Var}(r_n(s, y)) \frac{1}{\mathbb{E}[f_n(s, y)]^2}$ . We use Lemma 3.1 to get

$$\|\text{Std}(C_n - v)\|_{H_{\beta, X}^\mu}^2 \leq 2^{2d+2}(T\lambda(B))^2 \int_0^T ds e^{\beta s} \int_B dy \nu_\mu^0(s, y) \text{Var}(r_n(s, y)).$$

The end of the proof is similar to the one of [4, Proposition 12.34]. ■

### 4.1.3 Conclusion

We combine Propositions 4.4 and 4.5 to get the following result

**Proposition 4.6.** *Let us assume Hypothesis 1 and  $v$  is a bounded  $C^{1,1}$  function. Then,*

$$\mathbb{E}\|C_n - v\|_{H_{\beta,X}^\mu}^2 \leq K_0(T)(T\lambda(B)\delta_n\|v\|_{H_{\beta,X}^\mu}^2 + h_t^2\|\partial_t v\|_{H_{\beta,X}^\mu}^2 + h_x^2\|\partial_x v\|_{H_{\beta,X}^\mu}^2) + c_0^2(v)K(T)e^{-\frac{\mu a}{\sqrt{d}}}.$$

Moreover, if  $\partial_t v$  and  $\partial_x v$  are bounded, we get  $\|\mathbb{E}(C_n - v)\|_{H_{\beta,X}^\mu}^2 \leq K_0(T)c_{1,1}^2(v)(T\lambda(B)\delta_n + h_t^2 + h_x^2 + e^{-\frac{\mu a}{\sqrt{d}}})$ .

### 4.2 Study of $\mathbb{E}\|\mathcal{P}^k v - C_n\|_{H_{\beta,X}^\mu}^2$

By using the definition of  $\mathcal{P}^k v(s, y)$  and  $C_n(s, y)$ , we write

$$\mathcal{P}^k v(s, y) - C_n(s, y) = r_n(s, y) \left[ \frac{1}{f_n(s, y)} g(2^{d+1}T\lambda(B)f_n(s, y)) - \frac{1}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} \right].$$

If  $y \notin B_\infty(0, a + h_x)$ ,  $\mathcal{P}^k v(s, y) - C_n(s, y) = 0$ . If  $y \in B_\infty(0, a + h_x) \setminus B$ ,  $\mathcal{P}^k v(s, y) - C_n(s, y) = \frac{r_n(s, y)}{f_n(s, y)} g(2^{d+1}T\lambda(B)f_n(s, y))$ . Since  $g$  is bounded by 1 and  $|r_n(s, y)| \leq f_n(s, y) \sup_{(s, y) \in [0, T] \times B_\infty(0, a + h_x) \setminus B} |v(s, y)|$ , we get  $|\mathcal{P}^k v(s, y) - C_n(s, y)| \leq c_0(v)$ . If  $y \in B$ ,  $\mathcal{P}^k v(s, y) - C_n(s, y) = \frac{r_n(s, y)}{f_n(s, y)} [g(2^{d+1}T\lambda(B)f_n(s, y)) - \frac{f_n(s, y)}{\mathbb{E}[f_n(s, y)]}]$ . Let us give two upper bounds for  $\mathcal{P}^k v(s, y) - C_n(s, y)$  when  $y \in B$ .

**Lemma 4.7.** *For  $y \in B$ , the two following assertions hold*

$$|\mathcal{P}^k v(s, y) - C_n(s, y)| \leq 2^{d+3}T\lambda(B) \frac{r_n(s, y)}{f_n(s, y)} |f_n(s, y) - \mathbb{E}[f_n(s, y)]|,$$

$$|\mathcal{P}^k v(s, y) - C_n(s, y)| \leq 2^{d+3}(T\lambda(B))^2 |r_n(s, y)| |f_n(s, y) - \mathbb{E}[f_n(s, y)]|.$$

*Proof.* Let  $\tilde{g}(x) := g(2^{d+1}T\lambda(B)x) - \frac{x}{\mathbb{E}[f_n(s, y)]}$ . Then, we use the second property of Lemma 3.1 to get  $\tilde{g}(\mathbb{E}[f_n(s, y)]) = 0$ , and  $\mathcal{P}^k v(s, y) - C_n(s, y) = \frac{r_n(s, y)}{f_n(s, y)} (\tilde{g}(f_n(s, y)) - \tilde{g}(\mathbb{E}[f_n(s, y)]))$ . Moreover, Remark 1.2 leads to  $|\tilde{g}(f_n(s, y)) - \tilde{g}(\mathbb{E}[f_n(s, y)])| \leq 2^{d+3}T\lambda(B) |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . The first result follows. To get the second one, we introduce  $\bar{g}(x) := \frac{g(2^{d+1}T\lambda(B)x)}{x}$ . We have  $\bar{g}(\mathbb{E}[f_n(s, y)]) = \frac{1}{\mathbb{E}[f_n(s, y)]}$  and  $|\bar{g}(f_n(s, y)) - \bar{g}(\mathbb{E}[f_n(s, y)])| \leq 2^{2d+3}(T\lambda(B))^2 |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ .  $\blacksquare$

**Proposition 4.8.** *Assume Hypothesis 1 and  $v$  is bounded. Then,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$ , one has*

$$\begin{aligned} \mathbb{E}\|\mathcal{P}^k v - C_n\|_{H_{\beta,X}^\mu}^2 &\leq K_0(T)\epsilon^2(T\lambda(B))^2\|v\|_{H_{\beta,X}^\mu}^2 \\ &\quad + K_0(T)c_0^2(v)(T\lambda(B))^2\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right). \end{aligned}$$

*Proof.* Using Lemma 4.7, we split  $\mathcal{P}^k v(s, y) - C_n(s, y)$  in two terms, depending on the value of  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  w.r.t. a constant  $\epsilon$ . When  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \leq \epsilon$ , we use the second inequality of Lemma 4.7, otherwise we use the first one. Since  $r_n(s, y) \leq c_0(v)f_n(s, y)$ , we use [4, Proposition 12.16] to get  $\mathbb{E}|\mathcal{P}^k v(s, y) - C_n(s, y)|^2 \leq K(T)\epsilon^2(T\lambda(B))^4\mathbb{E}[r_n^2(s, y)] + K(T)c_0^2(v)\left(\epsilon^2 + \frac{\delta_n}{T\lambda(B)}\right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right)$ . We apply Proposition 3.2 and Lemma 3.5 to conclude.  $\blacksquare$

### 4.3 Conclusion

To conclude, we combine Propositions 4.6 and 4.8 (with  $\epsilon^2 = \frac{\delta_n}{T\lambda(B)}$ ). We obtain

**Proposition 4.9.** *We assume Hypothesis 1 and  $v$  is a bounded  $C^{1,1}$  function. Then,*

$$\begin{aligned} \mathbb{E}\|\mathcal{P}^k v - v\|_{H_{\beta,X}^\mu}^2 &\leq K_0(T)(T\lambda(B)\delta_n\|v\|_{H_{\beta,X}^\mu}^2 \\ &\quad + h_t^2\|\partial_t v\|_{H_{\beta,X}^\mu}^2 + h_x^2\|\partial_x v\|_{H_{\beta,X}^\mu}^2) + c_0^2(v)K(T)(e^{-\frac{\mu a}{\sqrt{d}}} + T\lambda(B)\delta_n). \end{aligned}$$

Moreover, if  $\partial_t v$  and  $\partial_x v$  are bounded, we get  $\|\mathbb{E}(\mathcal{P}^k v - v)\|_{H_{\beta,X}^\mu}^2 \leq K_0(T)c_{1,1}^2(v)(T\lambda(B)\delta_n + h_t^2 + h_x^2 + e^{-\frac{\mu a}{\sqrt{d}}})$ .

## 5 Proof of Proposition 2.1: term $\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta,X}^\mu}^2$

We study  $\mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta,X}^\mu}^2$  componentwise, then we deal with  $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2$ , for  $1 \leq i \leq d$ . The study of this term will be done in two steps. To do so, we add and subtract the term  $\partial_{x_i} C_n(s, y)$  (see (4.1) for the definition of  $C_n$ ) to  $\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} v(s, y)$ .

$$\partial_{x_i} C_n(s, y) = \left( \frac{\partial_{x_i} r_n(s, y)}{\mathbb{E}[f_n(s, y)]} - r_n(s, y) \frac{\mathbb{E}[\partial_{x_i} f_n(s, y)]}{(\mathbb{E}[f_n(s, y)])^2} \right) \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} \quad (5.1)$$

We get  $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2 \leq 2\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|_{H_{\beta,X}^\mu}^2 + 2\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2$ . The two following sections are devoted to the study of  $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2$  and  $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|_{H_{\beta,X}^\mu}^2$ .

### 5.1 Study of $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2$

Using the definition of  $\partial_{x_i} C_n(s, y)$ , we get

$$\begin{aligned} \partial_{x_i} C_n(s, y) - \partial_{x_i} v(s, y) &= -\partial_{x_i} v(s, y) \mathbf{1}_{\{s \notin [0, T] \cup y \notin B\}} \\ &\quad + \left( \frac{\partial_{x_i} r_n(s, y) - \partial_{x_i} v(s, y) \mathbb{E}[f_n(s, y)]}{\mathbb{E}[f_n(s, y)]} - r_n(s, y) \frac{\mathbb{E}[\partial_{x_i} f_n(s, y)]}{(\mathbb{E}[f_n(s, y)])^2} \right) \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}. \end{aligned} \quad (5.2)$$

Then, we split  $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2$  in two terms, by using the bias-variance decomposition:  $\mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta,X}^\mu}^2 = \|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta,X}^\mu}^2 + \|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta,X}^\mu}^2$ .

#### 5.1.1 Study of $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta,X}^\mu}^2$

By using (5.2), we split  $\mathbb{E}(\partial_{x_i} C_n(s, y) - \partial_{x_i} v(s, y))$  in three terms :

$$\begin{aligned} B_1(s, y) &= \frac{\mathbb{E}[\partial_{x_i} r_n(s, y)] - \partial_{x_i} v(s, y) \mathbb{E}[f_n(s, y)]}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}, \\ B_2(s, y) &= -\mathbb{E}[r_n(s, y)] \frac{\mathbb{E}[\partial_{x_i} f_n(s, y)]}{(\mathbb{E}[f_n(s, y)])^2} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}, \\ B_3(s, y) &= -\partial_{x_i} v(s, y) \mathbf{1}_{\{s \notin [0, T] \cup y \notin B\}} \end{aligned}$$



such that  $\mathbb{E}(\partial_{x_i} C_n(s, y) - \partial_{x_i} v(s, y)) = B_1(s, y) + B_2(s, y) + B_3(s, y)$ .

We analyze each term in the three following Lemmas.

**Lemma 5.1.** *Let us assume Hypothesis 1. We also assume  $v$  is a bounded  $C^{0,2}$  function which satisfies  $\forall t, t' \in [0, T], \forall x \in \mathbb{R}^d, |\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v) \sqrt{|t' - t|}$ . Then,*

$$\|B_1(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T) h_x^2 \|\partial_x^2 v\|_{H_{\beta, X}^\mu}^2 + K_0(T) (c_{1/2}^2(v) h_t + c_0^2(v) e^{-\mu a} \frac{a^{d-1}}{h_x}).$$

If  $\partial_x^2 v$  is bounded, we get  $\|B_1(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T) (c_{0,2}^2(v) h_x^2 + c_{1/2}^2(v) h_t + c_0^2(v) e^{-\mu a} \frac{a^{d-1}}{h_x})$ .

*Proof.* Let us recall  $B_1(s, y) = \frac{\mathbb{E}[\partial_{x_i} r_n(s, y) - \partial_{x_i} v(s, y) \mathbb{E}[f_n(s, y)]]}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}$ . We have  $\mathbb{E}(\partial_{x_i} r_n(s, y)) = \frac{1}{h_t h_x^{d+1}} \frac{1}{T \lambda(B)} \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) \int_B dz \partial_{x_i} K_x \left( \frac{y-z}{h_x} \right) v(r, z)$ . We integrate by parts  $\frac{1}{h_x} \int_{-a}^a dz_i (K_x^i)' \left( \frac{y_i - z_i}{h_x} \right) v(r, z)$  and we get  $\frac{1}{h_x} \int_{-a}^a dz_i (K_x^i)' \left( \frac{y_i - z_i}{h_x} \right) v(r, z) = -K_x^i \left( \frac{y_i - a}{h_x} \right) v(r, z_{-a}^i) + K_x^i \left( \frac{y_i + a}{h_x} \right) v(r, z_{-a}^i) + \int_{-a}^a dz_i \partial_{x_i} v(r, z) K_x^i \left( \frac{y_i - z_i}{h_x} \right)$ , where  $z_{-a}^i$  denotes the vector  $(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_d)$ . Then,  $\mathbb{E}[\partial_{x_i} r_n(s, y) - \partial_{x_i} v(s, y) \mathbb{E}[f_n(s, y)]] = \frac{1}{T \lambda(B) h_t h_x^d} \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) \int_B dz K_x \left( \frac{y-z}{h_x} \right) [\partial_{x_i} v(r, z) - \partial_{x_i} v(s, y)] + \frac{1}{T \lambda(B) h_t h_x^d} \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) \int_{[-a, a]^{d-1}} dz^i \prod_{j=1, j \neq i}^d K_x^j \left( \frac{y_j - z_j}{h_x} \right) [-K_x^i \left( \frac{y_i - a}{h_x} \right) v(r, z_{-a}^i) + K_x^i \left( \frac{y_i + a}{h_x} \right) v(r, z_{-a}^i)]$ . Combining this with the bound  $\mathbb{E}[f_n(s, y)] \geq \frac{1}{2^{d+1} T \lambda(B)}$  (see Lemma 3.1) leads to the following upper bound for  $B_1(s, y)$  :  $|B_1(s, y)| \leq B_{11}(s, y) + B_{12}(s, y) + B_{13}(s, y)$ , where

$$\begin{aligned} B_{11}(s, y) &= \frac{2^{d+1}}{h_t h_x^d} \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) \int_B dz K_x \left( \frac{y-z}{h_x} \right) [\partial_{x_i} v(r, z) - \partial_{x_i} v(s, z)] \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}, \\ B_{12}(s, y) &= \frac{2^{d+1}}{h_t h_x^d} \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) \int_B dz K_x \left( \frac{y-z}{h_x} \right) [\partial_{x_i} v(s, z) - \partial_{x_i} v(s, y)] \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}, \\ B_{13}(s, y) &= \frac{2^{d+1}}{h_t h_x} c_0(v) c_0(K_x) \int_0^T dr K_t \left( \frac{s-r}{h_t} \right) (\mathbf{1}_{\{|y+a| \leq h_x\}} + \mathbf{1}_{\{|y-a| \leq h_x\}}) \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}. \end{aligned}$$

By using [4, Lemmas 12.53 and 12.54], we get  $\|B_{11}(s, y)\|_{H_{\beta, X}^\mu}^2 \leq c_{1/2}^2(v) K(T) h_t$ ,  $\|B_{12}(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T) h_x^2 \|\partial_x^2 v\|_{H_{\beta, X}^\mu}^2$ . It remains to bound  $\|B_{13}(s, y)\|_{H_{\beta, X}^\mu}^2$ . We obtain  $\|B_{13}(s, y)\|_{H_{\beta, X}^\mu}^2 \leq \frac{K_0(T) c_0^2(v)}{h_x^2} \int_0^T ds e^{\beta s} \int_{B \setminus B(0, a-h_x)} dy \nu_\mu^0(s, y)$ . Since  $\int_{B \setminus B(0, a-h_x)} dy \nu_\mu^0(s, y) \leq K(T) e^{-\mu(a-h_x)} a^{d-1} h_x$  (see [4, Equation (12.5), page 132]), we get  $\|B_{13}(s, y)\|_{H_{\beta, X}^\mu}^2 \leq \frac{K_0(T) c_0^2(v)}{h_x} e^{-\mu a} a^{d-1}$ . ■

**Lemma 5.2.** *Assume Hypothesis 1 and  $v$  is bounded. Then,  $\|B_2(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T) c_0^2(v) e^{-\mu a} \frac{a^{d-1}}{h_x}$ .*

*Proof.* Since  $v$  is bounded, we get  $\mathbb{E}[r_n(s, y)] \leq c_0(v) \mathbb{E}[f_n(s, y)]$ . Hence,  $|B_2(s, y)| \leq c_0(v) \frac{\mathbb{E}[\partial_{x_i} f_n(s, y)]}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}$ . To conclude, we apply Lemma 3.1 and Proposition 3.4. ■

**Lemma 5.3.** *Assume Hypothesis 1 and  $\partial_x v$  is bounded. Then,  $\|B_3(s, y)\|_{H_{\beta, X}^\mu}^2 \leq K(T)c_{0,1}^2(v)e^{-\frac{\mu a}{\sqrt{d}}}$ .*

*Proof.* We refer to the proof of Lemma 4.3. ■

We combine Lemmas 5.1, 5.2 and 5.3 to get the following Proposition.

**Proposition 5.4.** *Assume Hypothesis 1 and  $v$  is a  $C_b^{0,2} - C_b^{0,1}$  function which satisfies  $\forall t, t' \in [0, T], \forall x \in \mathbb{R}^d, |\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v)\sqrt{|t' - t|}$ . Then,*

$$\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)h_x^2 \|\partial_x^2 v\|_{H_{\beta, X}^\mu}^2 + K_0(T)(c_{1/2}^2(v)h_t + c_0^2(v)e^{-\mu a} \frac{a^{d-1}}{h_x} + c_{0,1}^2(v)e^{-\frac{\mu a}{\sqrt{d}}}).$$

Moreover, if  $\partial_x^2 v$  is bounded, we get  $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)(c_{0,2}^2(v) + c_{1/2}^2(v))(h_x^2 + h_t + e^{-\mu a} \frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$ .

### 5.1.2 Study of $\|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2$

Let us study  $\|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 = \int_0^T ds e^{\beta s} \int_{\mathbb{R}^d} dy \nu_\mu^0(s, y) \text{Var}(\partial_{x_i} C_n - \partial_{x_i} v)(s, y)$ .

**Proposition 5.5.** *Assume Hypothesis 1. It holds*

$$\|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T) \frac{T\lambda(B)\delta_n}{h_x^2} \|v\|_{H_{\beta, X}^\mu}^2.$$

If  $v$  is bounded, we get  $\|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)c_0^2(v) \frac{T\lambda(B)\delta_n}{h_x^2}$ .

*Proof.* We have  $\|\text{Std}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 = \int_0^T ds e^{\beta s} \int_B dy \nu_\mu^0(s, y) \text{Var}(\partial_{x_i} C_n(s, y))$ .

Using (5.1) leads to  $\text{Var}(\partial_{x_i} C_n(s, y)) \leq$

$$2 \left( \frac{1}{(\mathbb{E}[f_n(s, y)])^2} \text{Var}(\partial_{x_i} r_n(s, y)) + \text{Var}(r_n(s, y)) \frac{(\mathbb{E}[\partial_{x_i} f_n(s, y)])^2}{(\mathbb{E}[f_n(s, y)]^4)} \right) \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}}.$$

Then, we use [4, Proposition 12.51] and Lemma 3.1 to get  $\int_0^T ds e^{\beta s} \int_B dy \nu_\mu^0(s, y) \frac{\text{Var}(\partial_{x_i} r_n(s, y))}{(\mathbb{E}[f_n(s, y)]^2)} \leq K_0(T) \frac{T\lambda(B)\delta_n}{h_x^2} \|v\|_{H_{\beta, X}^\mu}^2$ . It remains to bound

$\int_0^T ds e^{\beta s} \int_B dy \nu_\mu^0(s, y) \text{Var}(r_n(s, y)) \frac{(\mathbb{E}[\partial_{x_i} f_n(s, y)]^2)}{(\mathbb{E}[f_n(s, y)]^4)}$ . To do it, we use Lemmas 3.1, 3.3 and the proof of Proposition 4.5. ■

### 5.1.3 Conclusion

We combine Propositions 5.4 and 5.5 to get the following result

**Proposition 5.6.** *Assume Hypothesis 1 and  $v$  is a  $C_b^{1,2} - C_b^{0,1}$  function which satisfies  $\forall t, t' \in [0, T], \forall x \in \mathbb{R}^d, |\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v)\sqrt{|t' - t|}$ . Then,*

$$\begin{aligned} \mathbb{E}\|\partial_{x_i} C_n - \partial_{x_i} v\|_{H_{\beta, X}^\mu}^2 &\leq K_0(T) \left( h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} \right) \|v\|_{H_{\beta, X}^{2, \mu}}^2 \\ &\quad + K_0(T) \left( c_{1/2}^2(v)h_t + c_0^2(v)e^{-\mu a} \frac{a^{d-1}}{h_x} + c_{0,1}^2(v)e^{-\frac{\mu a}{\sqrt{d}}} \right). \end{aligned}$$

Moreover, if  $\partial_x^2 v$  is bounded, we get  $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)(c_{0,2}^2(v) + c_{1/2}^2(v))(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} + h_t + e^{-\mu a} \frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$ .

## 5.2 Study of $\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|_{H_{\beta, X}^\mu}^2$

We have  $\partial_{x_i}(\mathcal{P}^k v)(s, y) = \partial_{x_i} r_n(s, y) \frac{g(2^{d+1} T \lambda(B) f_n(s, y))}{f_n(s, y)} + 2^{2d+2} (T \lambda(B))^2 r_n(s, y) G'(2^{d+1} T \lambda(B) f_n(s, y)) \partial_{x_i} f_n(s, y)$ , where  $G$  has been introduced in Remark 1.2. Using the definition of  $\partial_{x_i} C_n$  (see (5.1)) yields

$$\begin{aligned} \partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y) &= \partial_{x_i} r_n(s, y) \left( \frac{g(2^{d+1} T \lambda(B) f_n(s, y))}{f_n(s, y)} - \frac{1}{\mathbb{E}[f_n(s, y)]} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} \right) \\ &+ r_n(s, y) \left( 2^{d+2} (T \lambda(B))^2 \partial_{x_i} f_n(s, y) G'(2^{d+1} T \lambda(B) f_n(s, y)) - \frac{\mathbb{E}[\partial_{x_i} f_n(s, y)]}{(\mathbb{E}[f_n(s, y)])^2} \mathbf{1}_{\{s \in [0, T]\}} \mathbf{1}_{\{y \in B\}} \right) \end{aligned} \quad (5.3)$$

Let us study  $\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y)$  w.r.t. the value of  $y$ . The first Lemma ensues from the Definition of  $f_n$ .

**Lemma 5.7.** *If  $y \notin B_\infty(0, a + h_x)$ ,  $\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y) = 0$ .*

**Lemma 5.8.** *If  $y \in B_\infty(0, a + h_x) \setminus B$  and  $v$  is bounded, for all  $i \in \{1, \dots, d\}$ ,*

$$\mathbb{E}|\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y)|^2 \leq \frac{K_0(T) c_0^2(v)}{h_x^2}.$$

*Proof.* Let us introduce  $\bar{f}_n^i(s, y) = \frac{1}{nh_t h_x^{d+1}} \sum_{i=1}^n K_t \left( \frac{s - T_i}{h_t} \right) |\partial_{x_i} K_x| \left( \frac{y - X_i}{h_x} \right)$ . The indicators in (5.3) are null. Since  $\frac{g(y)}{y}$  is bounded by 2 (see Remark 1.2 for the bounds for  $g$  and its first derivative),  $\frac{|\partial_{x_i} r_n(s, y)|}{f_n(s, y)} g(2^{d+1} T \lambda(B) f_n(s, y)) \leq 2^{d+2} T \lambda(B) c_0(v) \bar{f}_n^i(s, y)$ . It remains to bound the term containing  $G'$ . To do so, we write  $G'(y) = \frac{g'(y)}{y} - \frac{g(y)}{y^2}$ . Since  $|G(y)| \leq 2$  and  $|r_n(s, y)| \leq c_0(v) f_n(s, y)$ ,  $|r_n(s, y)| \frac{g(2^{d+1} T \lambda(B) f_n(s, y))}{f_n^2(s, y)} |\partial_{x_i} f_n(s, y)| \leq 2^{d+2} T \lambda(B) c_0(v) |\partial_{x_i} f_n(s, y)|$ . Since  $g'$  is bounded by 2,  $2^{d+1} T \lambda(B) |r_n(s, y)| \frac{g'(2^{d+1} T \lambda(B) f_n(s, y))}{f_n(s, y)} |\partial_{x_i} f_n(s, y)|$  is bounded by  $2^{d+2} T \lambda(B) c_0(v) |\partial_{x_i} f_n(s, y)|$ . To conclude, we use  $|\partial_{x_i} f_n(s, y)| \leq \bar{f}_n^i(s, y)$  and [4, Lemma 12.8], which states  $\mathbb{E}(\bar{f}_n^i(s, y))^2 \leq \frac{K_0(T)}{h_x^2 (T \lambda(B))^2}$ .  $\blacksquare$

**Lemma 5.9.** *If  $y \in B$ ,  $\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y) = A(s, y) + B(s, y) + C(s, y)$  where*

$$\begin{aligned} A(s, y) &= \partial_{x_i} r_n(s, y) \left( \frac{g(2^{d+1} T \lambda(B) f_n(s, y))}{f_n(s, y)} - \frac{1}{\mathbb{E}[f_n(s, y)]} \right), \\ B(s, y) &= 2^{2d+2} (T \lambda(B))^2 r_n(s, y) \partial_{x_i} f_n(s, y) [G'(2^{d+1} T \lambda(B) f_n(s, y)) - G'(2^{d+1} T \lambda(B) \mathbb{E}[f_n(s, y)])], \\ C(s, y) &= 2^{2d+2} (T \lambda(B))^2 r_n(s, y) G'(2^{d+1} T \lambda(B) \mathbb{E}[f_n(s, y)]) [\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]]. \end{aligned}$$

*Proof.* We add and subtract  $2^{2d+2} (T \lambda(B))^2 r_n(s, y) \partial_{x_i} f_n(s, y) G'(2^{d+1} T \lambda(B) \mathbb{E}[f_n(s, y)])$  in (5.3) and we use  $G'(2^{d+1} T \lambda(B) \mathbb{E}[f_n(s, y)]) = -\frac{1}{2^{2d+2} (T \lambda(B))^2 \mathbb{E}[f_n(s, y)]^2}$  (since  $2^{d+1} T \lambda(B) \mathbb{E}[f_n(s, y)] \geq 1$ ).  $\blacksquare$

### 5.2.1 Bound for $\mathbb{E}[A^2(s, y)]$

**Lemma 5.10.** *If  $v$  is bounded,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$ , we have*

$$\begin{aligned} \mathbb{E}[A^2(s, y)] &\leq K(T)(T\lambda(B))^4 \epsilon^2 \mathbb{E}[\partial_{x_i} r_n(s, y)]^2 \\ &+ K_0(T) c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left[ \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{T\lambda(B)} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

*Proof.* Studying  $A(s, y)$  boils down to study  $\mathcal{P}^k v - C_n$  where  $r_n$  is replaced by  $\partial_{x_i} r_n$ . First, the second inequality of Lemma 4.7 gives us

$$|A(s, y)| \leq 2^{d+3} (T\lambda(B))^2 |\partial_{x_i} r_n(s, y)| |f_n(s, y) - \mathbb{E}[f_n(s, y)]|,$$

and since  $|\partial_{x_i} r_n(s, y)| \leq c_0(v) \bar{f}_n^i(s, y)$  ( $\bar{f}_n^i(s, y)$  has been introduced in the proof of Lemma 5.8), we also have  $|A(s, y)| \leq 2^{d+3} (T\lambda(B))^2 c_0(v) \bar{f}_n^i(s, y) |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . As in the proof of Proposition 4.8, we split  $A(s, y)$  in two terms, depending on the value of  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  w.r.t. a constant  $\epsilon_0$ . When  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \leq \epsilon_0$ , we use the first inequality, otherwise we use the second one. We get

$$\begin{aligned} |A(s, y)| &\leq 2^{d+3} (T\lambda(B))^2 \epsilon_0 |\partial_{x_i} r_n(s, y)| \\ &+ 2^{d+3} (T\lambda(B))^2 c_0(v) \bar{f}_n^i(s, y) |f_n(s, y) - \mathbb{E}[f_n(s, y)]| \mathbf{1}_{\{|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \geq \epsilon_0\}}. \end{aligned}$$

We split again the second right hand term of the above inequality by introducing  $\pm \mathbb{E}[\bar{f}_n^i(s, y)]$ . We get

$$\begin{aligned} |A(s, y)| &\leq 2^{d+3} (T\lambda(B))^2 \epsilon_0 |\partial_{x_i} r_n(s, y)| \\ &+ 2^{d+3} (T\lambda(B))^2 c_0(v) |\bar{f}_n^i(s, y) - \mathbb{E}[\bar{f}_n^i(s, y)]| |f_n(s, y) - \mathbb{E}[f_n(s, y)]| \mathbf{1}_{\{|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \geq \epsilon_0\}} \\ &+ 2^{d+3} (T\lambda(B)) c_0(v) \frac{K_0(T)}{h_x} |f_n(s, y) - \mathbb{E}[f_n(s, y)]| \mathbf{1}_{\{|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \geq \epsilon_0\}}, \end{aligned}$$

where we have used  $\mathbb{E}(\bar{f}_n^i(s, y))^2 \leq \frac{K_0(T)}{h_x^2 (T\lambda(B))^2}$ . Then, we split the second term of the r.h.s. in two terms, depending on the value of  $|\bar{f}_n^i(s, y) - \mathbb{E}[\bar{f}_n^i(s, y)]|$  w.r.t. a constant  $\epsilon_1$ . We obtain

$$\begin{aligned} |A(s, y)|^2 &\leq K(T)(T\lambda(B))^4 \epsilon_0^2 |\partial_{x_i} r_n(s, y)|^2 \\ &+ K(T)(T\lambda(B))^4 c_0^2(v) \bar{E}^2(s, y) E^2(s, y) \mathbf{1}_{\{E(s, y) \geq \epsilon_0\}} \mathbf{1}_{\{\bar{E}(s, y) \geq \epsilon_1\}} \\ &+ K_0(T)(T\lambda(B))^2 c_0^2(v) \left( \frac{1}{h_x^2} + (T\lambda(B))^2 \epsilon_1^2 \right) E^2(s, y) \mathbf{1}_{\{E(s, y) \geq \epsilon_0\}}. \end{aligned}$$

where  $E(s, y) := |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  and  $\bar{E}(s, y) := |\bar{f}_n^i(s, y) - \mathbb{E}[\bar{f}_n^i(s, y)]|$ . To conclude, it remains to apply [4, Propositions 12.16 and 12.18] (since  $\bar{f}_n^i(s, y)$  is almost  $\partial_{x_i} f_n$ ), Cauchy Schwarz inequality, to choose  $\epsilon_1 = \frac{\epsilon_0}{h_x}$  and to use  $\epsilon_0^2 \leq (T\lambda(B))^{-2}$ . We get

$$\begin{aligned} \mathbb{E}[A^2(s, y)] &\leq K(T)(T\lambda(B))^4 \epsilon_0^2 \mathbb{E}[\partial_{x_i} r_n(s, y)]^2 \\ &+ K_0(T) c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left( \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) \\ &+ K_0(T) c_0^2(v) \frac{(T\lambda(B))^4}{h_x^2} \left[ \left( \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)} \right)^2 \exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))^3} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

Since  $\epsilon_0^2 \leq (T\lambda(B))^{-2}$  and  $T\lambda(B)\delta_n \ll 1$ , we obtain  $(T\lambda(B))^2(\epsilon_0^2 + \frac{\delta_n}{T\lambda(B)})^2 \leq \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)}$ , and the result follows.  $\blacksquare$

### 5.2.2 Bound for $\mathbb{E}[B^2(s, y)]$

**Lemma 5.11.** *If  $v$  is bounded,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$ , we have*

$$\begin{aligned} \mathbb{E}[B^2(s, y)] &\leq C(d)(T\lambda(B))^4 \frac{\epsilon^2}{h_x^2} \mathbb{E}[r_n(s, y)]^2 \\ &+ K_0(T)c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left[ \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

*Proof.* First, we split  $B(s, y)$  in two terms  $B_1 + B_2$  by introducing  $\pm \mathbb{E}[\partial_{x_i} f_n(s, y)]$ . We get

$$\begin{aligned} B_1(s, y) &= 2^{2d+2} (T\lambda(B))^2 r_n(s, y) (\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]) \Delta G, \\ B_2(s, y) &= 2^{2d+2} (T\lambda(B))^2 r_n(s, y) \mathbb{E}[\partial_{x_i} f_n(s, y)] \Delta G, \end{aligned}$$

where  $\Delta G := G'(2^{d+1}T\lambda(B)f_n(s, y)) - G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s, y)])$ .

**Bound for  $B_2$ .** First, we use Lemma 3.3 to bound  $\mathbb{E}[\partial_{x_i} f_n(s, y)]$ :  $\mathbb{E}[\partial_{x_i} f_n(s, y)] \leq \frac{K_0(T)}{T\lambda(B)h_x}$ . Then, we give two bounds for  $B_2$ . The first one uses that  $G'$  is a Lipschitz function on  $[0, \infty[$ . We get  $G'(2^{d+1}T\lambda(B)f_n(s, y)) - G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s, y)]) \leq C(d)T\lambda(B)|f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . Thus,  $|B_2(s, y)| \leq C(d) \frac{(T\lambda(B))^2}{h_x} r_n(s, y) |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . The second bound relies on the inequality  $r_n(s, y) \leq c_0(v)f_n(s, y)$  and the fact that the function  $\tilde{g}(x) := xG'(2^{d+1}T\lambda(B)x) - xG'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s, y)])$  satisfies  $\tilde{g}(\mathbb{E}[f_n(s, y)]) = 0$  and is a Lipschitz function. We get  $|B_2(s, y)| \leq C(d)c_0(v) \frac{T\lambda(B)}{h_x} |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . Once again, we split  $B_2(s, y)$  in two terms, depending on the value of  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  w.r.t. a constant  $\epsilon$ . When  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]| \leq \epsilon$ , we use the first inequality, otherwise we use the second one. By using [4, Proposition 12.16], we get

$$\mathbb{E}[B_2(s, y)]^2 \leq \epsilon^2 \frac{(T\lambda(B))^4}{h_x^2} \mathbb{E}[r_n(s, y)]^2 + K_0(T)c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right).$$

**Bound for  $B_1$ .** As for  $B_2$ , we give two bounds for  $B_1$ :  $|B_1(s, y)| \leq C(d)(T\lambda(B))^3 r_n(s, y) |\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]| |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  and  $|B_1(s, y)| \leq C(d)c_0(v)(T\lambda(B))^2 |\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]| |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$ . Then, we split  $B_1$  in four terms, depending on the value of  $|f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  w.r.t. a constant  $\epsilon_0$  and on the value of  $|\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]|$  w.r.t. a constant  $\epsilon_1$ . We introduce  $E'(s, y) := |\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]|$ . Then, we get

$$\begin{aligned} |B_1(s, y)| &\leq |B_1(s, y)| \mathbf{1}_{\{E' \leq \epsilon_1\}} \mathbf{1}_{\{E \leq \epsilon_0\}} + |B_1(s, y)| \mathbf{1}_{\{E' \leq \epsilon_1\}} \mathbf{1}_{\{E > \epsilon_0\}} \\ &+ |B_1(s, y)| \mathbf{1}_{\{E' > \epsilon_1\}} \mathbf{1}_{\{E \leq \epsilon_0\}} + |B_1(s, y)| \mathbf{1}_{\{E' > \epsilon_1\}} \mathbf{1}_{\{E > \epsilon_0\}}. \end{aligned}$$

We bound the first term (resp. the three other terms) by using the first (resp. second) bound for  $B_1$ . Applying Cauchy-Schwarz inequality, [4, Propositions 12.16 and 12.18] and choosing

$\epsilon_1 = \frac{\epsilon_0}{h_x}$  yield

$$\begin{aligned} \mathbb{E}|B_1(s, y)|^2 &\leq K(T)(T\lambda(B))^6 \frac{\epsilon_0^4}{h_x^2} \mathbb{E}[r_n(s, y)]^2 \\ &+ K(T)c_0^2(v) \frac{(T\lambda(B))^4 \epsilon_0^2}{h_x^2} \left( \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) \\ &+ K(T)c_0^2(v) \frac{(T\lambda(B))^4}{h_x^2} \left[ \left( \epsilon_0^2 + \frac{\delta_n}{T\lambda(B)} \right)^2 \exp\left(-\frac{c\epsilon_0^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))^3} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

For  $\epsilon_0^2 \leq (T\lambda(B))^{-2}$ , we get  $(T\lambda(B))^6 \frac{\epsilon_0^4}{h_x^2} \leq (T\lambda(B))^4 \frac{\epsilon_0^2}{h_x^2} \leq \frac{(T\lambda(B))^2}{h_x^2}$ . Hence, the first two terms of the bound for  $\mathbb{E}|B_1(s, y)|^2$  are smaller than the terms bounding  $\mathbb{E}|B_2(s, y)|^2$ . We end the proof as in Lemma 5.10.  $\blacksquare$

### 5.2.3 Bound for $\mathbb{E}[C^2(s, y)]$

**Lemma 5.12.** *If  $v$  is bounded,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$ , we have*

$$\begin{aligned} \mathbb{E}[C^2(s, y)] &\leq K(T)(T\lambda(B))^4 \frac{\epsilon^2}{h_x^2} \mathbb{E}[r_n(s, y)]^2 \\ &+ K_0(T)c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left[ \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

*Proof.* We recall  $C(s, y) = 2^{2d+2}(T\lambda(B))^2 r_n(s, y) G'(2^{d+1}T\lambda(B)\mathbb{E}[f_n(s, y)])[\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]]$ . We use that  $G'$  is bounded and we split  $C(s, y)$  in two terms depending on the value of  $E' = |\partial_{x_i} f_n(s, y) - \mathbb{E}[\partial_{x_i} f_n(s, y)]|$  w.r.t. a constant  $\epsilon_1$ . We get

$$|C(s, y)|^2 \leq C(d)(T\lambda(B))^4 \epsilon_1^2 |r_n(s, y)|^2 + C(d)(T\lambda(B))^4 c_0^2(v) |f_n(s, y)|^2 (E')^2 \mathbf{1}_{\{E' > \epsilon_1\}},$$

where we have used  $r_n(s, y) \leq c_0(v)f_n(s, y)$ . Then, we split the second term of the r.h.s. of the above inequality by introducing  $\pm \mathbb{E}[f_n(s, y)]$ . Since  $\mathbb{E}[f_n(s, y)] \leq \frac{1}{T\lambda(B)}$ , we obtain

$$\begin{aligned} |C(s, y)|^2 &\leq K(T)(T\lambda(B))^4 \epsilon_1^2 |r_n(s, y)|^2 + K(T)(T\lambda(B))^2 c_0^2(v) (E')^2 \mathbf{1}_{\{E' > \epsilon_1\}} \\ &+ K(T)(T\lambda(B))^4 c_0^2(v) (E')^2 \mathbf{1}_{\{E' > \epsilon_1\}} |f_n(s, y) - \mathbb{E}[f_n(s, y)]|^2. \end{aligned}$$

Finally, we split the last term of the above inequality in two terms depending on the value of  $E = |f_n(s, y) - \mathbb{E}[f_n(s, y)]|$  w.r.t. a constant  $\epsilon_0$ . We get

$$\begin{aligned} |C(s, y)|^2 &\leq K(T)(T\lambda(B))^4 \epsilon_1^2 |r_n(s, y)|^2 + C(d)(T\lambda(B))^4 \left( \frac{1}{(T\lambda(B))^2} + \epsilon_0^2 \right) c_0^2(v) (E')^2 \mathbf{1}_{\{E' > \epsilon_1\}} \\ &+ K(T)(T\lambda(B))^4 c_0^2(v) (E')^2 \mathbf{1}_{\{E' > \epsilon_1\}} E^2 \mathbf{1}_{\{E > \epsilon_0\}}. \end{aligned}$$

Combining Cauchy Schwarz inequality, [4, Propositions 12.16 and 12.18], choosing  $\epsilon_1 = \frac{\epsilon_0}{h_x}$  and using  $\epsilon_0 \leq (T\lambda(B))^{-1}$  and  $T\lambda(B)\delta_n \ll 1$  lead to the result.  $\blacksquare$

### 5.2.4 Conclusion

Combining Lemmas 5.10, 5.11 and 5.12 leads to the following Proposition.

**Proposition 5.13.** *If  $v$  is bounded,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$  and  $y \in B$ , we have*

$$\begin{aligned} \mathbb{E}[|\partial_{x_i}(\mathcal{P}^k v)(s, y) - \partial_{x_i} C_n(s, y)|^2] &\leq C(d)(T\lambda(B))^4 \epsilon^2 \left( \mathbb{E}[\partial_{x_i} r_n(s, y)]^2 + \frac{1}{h_x^2} \mathbb{E}[r_n(s, y)]^2 \right) \\ &+ K_0(T) c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left[ \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{(T\lambda(B))} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

Combining Lemmas 5.7, 5.8, Proposition 5.13 and following the same proof as [4, Theorem 12.50] yields

**Proposition 5.14.** *Assume Hypothesis 1 and  $v$  is bounded. Then,  $\forall \epsilon \geq 0$  such that  $\epsilon^2 \leq (T\lambda(B))^{-2}$ , we have*

$$\begin{aligned} \mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v) - \partial_{x_i} C_n\|_{H_{\beta, X}^\mu}^2 &\leq K_0(T)(T\lambda(B))^2 \epsilon^2 (\|\partial_x v\|_{H_{\beta, X}^\mu}^2 + \frac{1}{h_x^2} \|v\|_{H_{\beta, X}^\mu}^2) + c_0^2(v) \frac{K_0(T)}{h_x} e^{-\mu a} a^{d-1} \\ &+ K_0(T) c_0^2(v) \frac{(T\lambda(B))^2}{h_x^2} \left[ \left( \epsilon^2 + \frac{\delta_n}{T\lambda(B)} \right) \exp\left(-\frac{c\epsilon^2 T\lambda(B)}{\delta_n}\right) + \frac{\delta_n}{T\lambda(B)} \exp\left(-\frac{c}{T\lambda(B)\delta_n}\right) \right]. \end{aligned}$$

### 5.3 Conclusion

We combine Propositions 5.14 and 5.6 with  $\epsilon^2 = \frac{\delta_n}{T\lambda(B)}$  to get the following result

**Proposition 5.15.** *Assume Hypothesis 1 and  $v$  is a  $C^{1,2} - C_b^{0,1}$  function satisfying  $\forall t, t' \in [0, T], \forall x \in \mathbb{R}^d, |\partial_x v(t, x) - \partial_x v(t', x)| \leq c_{1/2}(v) \sqrt{|t' - t|}$ . Then,*

$$\begin{aligned} \mathbb{E}\|\partial_x(\mathcal{P}^k v) - \partial_x v\|_{H_{\beta, X}^\mu}^2 &\leq K_0(T) \left( h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} \right) \|v\|_{H_{\beta, X}^{2, \mu}}^2 \\ &+ K_0(T) \left( c_{1/2}^2(v) h_t + c_0^2(v) e^{-\mu a} \frac{a^{d-1}}{h_x} + c_{0,1}^2(v) e^{-\frac{\mu a}{\sqrt{d}}} + c_0^2(v) \frac{T\lambda(B)\delta_n}{h_x^2} \right). \end{aligned}$$

Moreover, if  $\partial_x^2 v$  is bounded, we get  $\|\mathbb{E}(\partial_{x_i} C_n - \partial_{x_i} v)\|_{H_{\beta, X}^\mu}^2 \leq K_0(T)(c_{0,2}^2(v) + c_{1/2}^2(v))(h_x^2 + \frac{T\lambda(B)\delta_n}{h_x^2} + h_t + e^{-\mu a} \frac{a^{d-1}}{h_x} + e^{-\frac{\mu a}{\sqrt{d}}})$ .

## 6 Proof of Proposition 2.2

### 6.1 Bound for $\mathbb{E}\|\mathcal{P}^k v\|_{H_{\beta, X}^\mu}^2$

From the definition of  $\mathcal{P}^k$  and since  $\frac{g(x)}{x}$  is bounded by 2, we deduce  $|\mathcal{P}^k v(s, y)|^2 \leq 2^{2d+3}(T\lambda(B))^2 |r_n|^2(s, y)$ . Then, Proposition 3.2 gives

$$\begin{aligned} \mathbb{E}(|\mathcal{P}^k v(s, y)|^2) &\leq 2^{2d+3}(T\lambda(B))^2 \mathbb{E}(|r_n|^2(s, y)), \\ &\leq \frac{K_0(T)}{h_t h_x^d} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz K_x^2\left(\frac{y-z}{h_x}\right) \mathbb{E}(v^2(r, z)). \end{aligned}$$

Using the definition of  $\|\cdot\|_{H_{\beta,X}^\mu}$  and Lemma 3.5 yields  $\mathbb{E}\|\mathcal{P}^k v\|_{H_{\beta,X}^\mu}^2 \leq K_0(T)\mathbb{E}\|v\|_{H_{\beta,X}^\mu}^2$ . If  $v$  is unbiased,  $\mathbb{E}(|r_n|^2(s,y)) = \text{Var}(r_n(s,y))$ . Proposition 3.2 gives

$$\begin{aligned} \mathbb{E}(|\mathcal{P}^k v(s,y)|^2) &\leq 2^{2d+3}(T\lambda(B))^2 \text{Var}(r_n(s,y)), \\ &\leq \frac{K_0(T)T\lambda(B)}{nh_t^2 h_x^{2d}} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz K_x^2\left(\frac{y-z}{h_x}\right) \mathbb{E}(v^2(r,z)), \end{aligned}$$

and we get  $\mathbb{E}\|\mathcal{P}^k v\|_{H_{\beta,X}^\mu}^2 \leq K_0(T)T\lambda(B)\delta_n \mathbb{E}\|v\|_{H_{\beta,X}^\mu}^2$ .

## 6.2 Bound for $\mathbb{E}\|\partial_x(\mathcal{P}^k v)(t,x)\|_{H_{\beta,X}^\mu}^2$

We have  $\partial_{x_i} \mathcal{P}^k v(s,y) = 2^{d+1}T\lambda(B)\partial_{x_i} r_n(s,y)G(2^{d+1}T\lambda(B)f_n(s,y)) + (2^{d+1}T\lambda(B))^2 r_n(s,y)\partial_{x_i} f_n(s,y)G'(2^{d+1}T\lambda(B)f_n(s,y))$  where  $G$  has been introduced in Remark 1.2. Using the bounds for  $G$  and  $G'$ , we obtain  $|\partial_{x_i} \mathcal{P}^k v(s,y)| \leq 2^{d+2}T\lambda(B)|\partial_{x_i} r_n(s,y)| + 6 * 2^{2d+2}(T\lambda(B))^2 |r_n(s,y)| |\partial_{x_i} f_n(s,y)|$ , and

$$\mathbb{E}\|\partial_{x_i}(\mathcal{P}^k v)\|_{H_{\beta,X}^\mu}^2 \leq C(d)(T\lambda(B))^2 (\mathbb{E}\|\partial_{x_i} r_n\|_{H_{\beta,X}^\mu}^2 + (T\lambda(B))^2 \mathbb{E}\|(r_n \partial_{x_i} f_n)\|_{H_{\beta,X}^\mu}^2).$$

### 6.2.1 Bound for $(T\lambda(B))^2 \mathbb{E}\|\partial_{x_i} r_n\|_{H_{\beta,X}^\mu}^2$

We write  $\mathbb{E}(|\partial_{x_i} r_n(s,y)|^2) = (\mathbb{E}(\partial_{x_i} r_n(s,y)))^2 + \text{Var}(\partial_{x_i} r_n(s,y))$ . As in Proposition 3.2, we get  $(\mathbb{E}(\partial_{x_i} r_n(s,y)))^2 \leq \frac{1}{h_t h_x^{d+2}} \frac{1}{(T\lambda(B))^2} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz (\partial_{x_i} K_x)^2\left(\frac{y-z}{h_x}\right) \mathbb{E}(v^2(r,z))$ ,  $\text{Var}(\partial_{x_i} r_n(s,y)) \leq \frac{1}{nh_t h_x^{d+2} T\lambda(B)} \int_0^T dr K_t^2\left(\frac{s-r}{h_t}\right) \int_B dz (\partial_{x_i} K_x)^2\left(\frac{y-z}{h_x}\right) \mathbb{E}(v^2(r,z))$ . Since we assume  $T\lambda(B)\delta_n \ll 1$ , Lemma 3.5 yields  $(T\lambda(B))^2 \mathbb{E}\|\partial_{x_i} r_n\|_{H_{\beta,X}^\mu}^2 \leq \frac{K_0(T)}{h_x^2} \mathbb{E}\|v\|_{H_{\beta,X}^\mu}^2$ . If  $v$  is unbiased, we get  $(T\lambda(B))^2 \mathbb{E}\|\partial_{x_i} r_n\|_{H_{\beta,X}^\mu}^2 \leq K_0(T) \frac{T\lambda(B)\delta_n}{h_x^2} \mathbb{E}\|v\|_{H_{\beta,X}^\mu}^2$ .

### 6.2.2 Bound for $(T\lambda(B))^4 \mathbb{E}\|r_n \partial_{x_i} f_n\|_{H_{\beta,X}^\mu}^2$

First, we develop the product  $r_n^2(s,y)(\partial_{x_i} f_n)^2(s,y)$  by using the following formulae  $(\partial_{x_i} f_n(s,y))^2 = \frac{1}{n^2 h_t^2 h_x^{2d+2}} \left( \sum_{j=1}^n K_t^2(j)(\partial_{x_i} K_x)^2(j) + \sum_{i,j=1, i \neq j}^n K_t(i)K_t(j)(\partial_{x_i} K_x)(i)(\partial_{x_i} K_x)(j) \right) := A + B$ ,  $r_n^2(s,y) = \frac{1}{n^2 h_t^2 h_x^{2d}} \left( \sum_{k=1}^n K_t^2(k)K_x^2(k)v^2(k) + \sum_{k,l=1, k \neq l}^n K_t(k)K_t(l)K_x(k)K_x(l)v(k)v(l) \right) := C + D$ , where  $K_t(j) := K_t\left(\frac{s-T_j}{h_t}\right)$ ,  $K_x(j) := K_x\left(\frac{y-X_j}{h_x}\right)$ ,  $(\partial_{x_i} K_x)(j) := (\partial_{x_i} K_x)\left(\frac{y-X_j}{h_x}\right)$  and  $v(k) := v(T_k, X_k)$ . Developing  $A \times C$  leads to

$$\begin{aligned} \mathbb{E}(A \times C) &= \frac{1}{n^4 h_t^4 h_x^{4d+2}} (n \mathbb{E}(K_t^4(1)(\partial_{x_i} K_x)^2(1)K_x^2(1)v^2(1)) \\ &\quad + n(n-1) \mathbb{E}(K_t^2(1)(\partial_{x_i} K_x)^2(1)) \mathbb{E}(K_t^2(1)K_x^2(1)v^2(1))). \end{aligned}$$

Since  $\mathbb{E}(K_t^2(1)(\partial_{x_i} K_x)^2(1))$  is bounded by  $K_0(T) \frac{h_t h_x^d}{T\lambda(B)}$ , we get  $\mathbb{E}(A \times C) \leq \frac{\delta_n^3}{h_t h_x^{d+2}} \mathbb{E}(K_t^4(1)(\partial_{x_i} K_x)^2(1)K_x^2(1)v^2(1)) + \frac{\delta_n^2}{T\lambda(B)h_t h_x^{d+2}} \mathbb{E}(K_t^2(1)K_x^2(1)v^2(1))$ . We write terms



of type  $\mathbb{E}(g_t(1)g_x(1)v^2(1))$  as  $\frac{1}{T\lambda(B)} \int_0^T dr g_t \left( \frac{s-r}{h_t} \right) \int_B dz g_x \left( \frac{y-z}{h_x} \right) \mathbb{E}(v^2(r, z))$ , where  $g_t$  (resp  $g_x$ ) represents a function depending on  $K_t$  (resp. on  $K_x$  and  $\partial_x K_x$ ). Finally, by using the same procedure for the other terms, we obtain  $\mathbb{E}((\partial_{x_i} f_n(s, y))^2 (r_n^2(s, y))) \leq \frac{K_0(T)}{(T\lambda(B))^4 h_t h_x^{d+2}} \int_0^T dr g_t \left( \frac{s-r}{h_t} \right) \int_B dz g_x \left( \frac{y-z}{h_x} \right) \mathbb{E}(v^2(r, z))$ . Applying Lemma 3.5 yields  $(T\lambda(B))^4 \mathbb{E} \|r_n \partial_{x_i} f_n\|_{H_{\beta, X}^\mu}^2 \leq \frac{K_0(T)}{h_x^2} \mathbb{E} \|v\|_{H_{\beta, X}^\mu}^2$ .

If  $v$  is unbiased, terms like  $(\mathbb{E}(K_t(1)K_x(1)v(1)))^2$  are null and this leads to  $(T\lambda(B))^4 \mathbb{E} \|r_n \partial_{x_i} f_n\|_{H_{\beta, X}^\mu}^2 \leq \frac{K_0(T)T\lambda(B)\delta_n}{h_x^2} \mathbb{E} \|v\|_{H_{\beta, X}^\mu}^2$ .

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