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Contents

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Introduction

A complex analytic space S is *holomorphically convex* if there is a proper holomorphic morphism $\pi : S \to T$ with $\pi_*\mathcal{O}_S = \mathcal{O}_T$ such that T is a Stein space. T is then called the *Cartan-Remmert reduction* of S.

The so-called Shafarevich conjecture of holomorphic convexity predicts that the universal covering space $\widetilde{X^{\text{univ}}}$ of a complex compact projective manifold X should be holomorphically convex.

This is trivial if the fundamental group is finite. The Shafarevich conjecture is a corollary of the Riemann uniformization theorem in dimension 1. The study of the Shafarevich conjecture for smooth projective surfaces was initiated in the mid 80s by Gurjar and Shastri [GurSha85] and Napier [Nap90].

In the mid nineties, new ideas introduced by J. Kóllar and independently by F. Campana have revolutionized the subject. The outcome was what is still the best result available with no assumption on the fundamental group, namely the construction of the Shafarevich map (aka Γ-reduction) [Cam94, Kol93, Kol95].

At the same time, Corlette and Simpson [Cor88, Cor93, Sim88, Sim92, Sim94] were developing Nonabelian Hodge theory. A bit later a p- adic version of Nonabelian Hodge theory in degree 1 was developed by Gromov and Schoen [GroSch92].

The idea that Nonabelian Hodge theory can be used to prove Shafarevich conjecture was introduced in 1994 by the second author. He proved Shafarevich conjecture for nilpotent fundamental groups [Kat97]. At about the same time the second and the fourth author proved the Shafarevich conjecture for smooth projective surfaces with fundamental group admitting faithful Zariski dense representation, in a reductive complex algebraic group [KatRam98].

The first author then found a way to extend this Nonabelian Hodge theoretic argument to higher dimension and showed that the Shafarevich conjecture holds for any smooth projective variety with fundamental group having a faithful representation, Zariski dense in a reductive complex algebraic group - see [Eys04].

Several influential contributions to these and closely related topics were also made by Lasell and the fourth author [LasRam96], Mok [Mok92] and Zuo [Zuo94].

The present article studies the conjecture in the case when $\pi_1(X,x)$ has a finite dimensional complex linear representation with infinite monodromy group. It combines and develops further some known techniques in Non abelian Hodge theory. In particular we prove the conjecture for projective manifolds X whose fundamental group admits a faithful representation in $GL_n(\mathbb{C})$.

The general strategy has two main steps. First we use the given faithful linear representation to construct certain complex variations of mixed Hodge structures ($\mathbb{C}$-VMHS). Then we utilize the associated period mappings to construct a Shafarevich morphism. This is quite similar to the way period maps for complex variations of pure Hodge structures ($\mathbb{C}$-VHS) were used in [Eys04]. Once the Shafarevich morphism is constructed, holomorphic convexity is much simpler to obtain here. The crucial point in the construction of the Shafarevich morphism is a rather subtle rationality lemma which turned out to rely on Mixed Hodge Theory.

The paper is organized as follows. Section 2 introduces Absolute Constructible Sets and recalls results from [Eys04]. Section 3 introduces a $\mathbb{C}$-VMHS constructed in [EysSim09] which serves as a main ingredient of the proof. Section 4 contains the proof of an important strictness statement. Section 5 contains a rationality lemma and the reduction to finite number of local systems. Section 6 contains the construction of the Shafarevich morphism and the proof of the main theorem.

Given present-day technology, it seems difficult to go significantly further in the direction of proving the Shafarevich conjecture. Perhaps, the generalization to the Kähler case or understanding sufficient conditions for holomorphic convexity of the universal covering space of a singular projective variety might produce interesting developments. Several interesting observations have been made in cases of nonresidually finite fundamental groups. Bogomolov and the second author suggest [BoKa98] that the Shafarevich conjecture might fail in the case of nonresidually finite fundamental groups. From another point of view, papers by Bogomolov and de Oliveira [BoDe0O5, BoDe0O6] suggest that big part of universal coverings of smooth projective varieties might still be holomorphically convex.

Notations

In what follows, X will denote a connected projective algebraic complex manifold, $x \in X$ a point, $\overline{\mathbb{Q}} \subset \ell \subset \mathbb{C}$ a field of definition for X , and Z a connected projective algebraic variety.

Statement of the Main Theorem

Theorem 1 *Let G be a reductive algebraic group defined over $\mathbb{Q}$. Let $M = M_B(X,G)$ be the character scheme of $\pi_1(X,x)$ with values in G.*

(a) *Let $\widetilde{H_M^\infty} \subset \pi_1(X,x)$ be the intersection of the kernels of all representations $\pi_1(X,x) \to G(A)$, where A is an arbitrary $\mathbb{C}$-algebra of finite type. Then,*

the associated Galois covering space of X:

$$\widetilde{X_M^\infty} = \widetilde{X^{univ}}/\widetilde{H_M^\infty}$$

is holomorphically convex.

(b) *There exists a natural non-increasing family*

$$\widetilde{H_M^1} \subseteq \widetilde{H_M^2} \subseteq \cdots \subseteq \widetilde{H_M^k} \subseteq \cdots \subseteq \widetilde{H_M^\infty} \trianglelefteq \pi_1(X,x)$$

of normal subgroups in $\pi_1(X,x)$. For a given k the group $\widetilde{H_M^k}$ corresponds to representations $\pi_1(X,x) \to G(A)$, with A an Artin local algebra, and such that the Zariski closure of their monodromy group has k-step unipotent radical. For every $\widetilde{H_M^k}$ the associated cover

$$\widetilde{X_M^k} = \widetilde{X^{univ}}/\widetilde{H_M^k}$$

is holomorphically convex.

Remarks. *Let $\pi : \widetilde{X_M^k} \to \widetilde{S_M^k}(X)$ be the Cartan-Remmert reduction of $\widetilde{X_M^k}$. The quotient group $\pi_1(X,x)_M^k := \pi_1(X,x)/\widetilde{H_M^k}$ acts properly discontinuously on $\widetilde{X_M^k}$ and π is equivariant. We then define the Shafarevich variety as the normal compact complex space $Sh_M^k(X) = \widetilde{S_M^k}(X)/\pi_1(X,x)_M^k$. The resulting Shafarevich morphism $sh_M^k : X \to Sh_M^k(X)$ is then independent on $k \in \mathbb{N}^* \cup \infty$.*

For every subgroup $H \subset \pi_1(X,x)$ such that $\widetilde{H_M^\infty} \subset H \subset \widetilde{H_M^1}$ the covering space $\widetilde{X^{univ}}/H$ is holomorphically convex as well.

See section 3.1 for the precise definition of $\widetilde{H_M^k}$. If $G = GL_1$, this theorem is a restatement of [Kat97]. Actually, the theorem is likely to hold when we replace $M_B(X,G)$ by an arbitrary absolutely closed set M defined over $\mathbb{Q}$ [Sim93].

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We have to apologize for the excessive delay between our first announcement talks on this subject and the availability of a text in preprint form.

1 Absolute Constructible Sets

1.1 Basic facts

Let G be an algebraic reductive group defined over $\overline{\mathbb{Q}}$. The *representation scheme* of $\pi_1(X,x)$ is an affine $\overline{\mathbb{Q}}$-algebraic scheme described by its functor of points:

$$R(\pi_1(X,x),G)(\mathrm{Spec}(A)) := \mathrm{Hom}(\pi_1(X,x),G(A))$$

for any $\overline{\mathbb{Q}}$ algebra A. The *character scheme* of $\pi_1(X,x)$ with values in G is the affine scheme

$$M_B(X,G) = R(\pi_1(X,x),G)//G.$$

Let $\bar{k}$ be an algebraically closed field of characteristic zero. Then $M_B(X,G)(\bar{k})$ is the set of $G(\bar{k})$-conjugacy classes of *reductive* representations of $\pi_1(X,x)$ with values in $G(\bar{k})$, see [LuMa85].

Character schemes of fundamental groups of complex projective manifolds are rather special. In [Sim94], two additional quasi-projective schemes over ℓ are constructed: $M_{DR}(X,G)$ and $M_{Dol}(X,G)$. The $\mathbb{C}$-points of $M_{DR}(X,G)$ are in bijection with the equivalence classes of flat G-connections with reductive monodromy, and the $\mathbb{C}$-points of $M_{Dol}(X,G)$ are in bijection with the isomorphism classes of polystable G-Higgs G-bundles with vanishing first and second Chern class. Whereas the notion of a polystable Higgs bundle depends on the choice of a polarization on X the moduli space $M_{Dol}(X,G)$ does not, i.e. - all moduli spaces one constructs for the different polarizations are naturally isomorphic, [Sim94]. This is analogous to the classical statement that the usual Hodge decomposition on the de Rham cohomology is purely complex analytic, i.e. independent of a choice of a Kähler metric[1]. $M_{Dol}(X,G)$ is acted upon algebraically by the multiplicative group $\mathbb{C}^*$. There is furthermore a complex analytic biholomorphic map

$$RH : M_B(X,G)(\mathbb{C}) \to M_{DR}(X,G)(\mathbb{C})$$

and a real analytic homeomorphism

$$KH : M_B(X,G)(\mathbb{C}) \to M_{Dol}(X,G)(\mathbb{C}).$$

RH and KH are also independant of the choice of a Kähler metric. When $l = \overline{\mathbb{Q}}$, one defines an *absolute constructible subset* of $M_B(X,G)(\mathbb{C})$ to be a subset M such that:

- M is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_B(X,G)$,
- $RH(M)$ is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{DR}(X,G)$,
- $KH(M)$ is a $\mathbb{C}^*$-invariant set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{Dol}(X,G)$.

There is a rich theory describing the structure of absolutely constructible subsets in $M_B(X,G)$. Here we briefly summarize only those properties of absolutely constructible sets that we will need later. Full proofs and details can be found in [Sim93].

- The full moduli space $M_B(X,G)$ of representations of $\pi_1(X,x)$ in G defined in [Sim94] is absolutely constructible and quasi compact (acqc).

[1]The harmonic representative of a cohomology class depends in general on the Kähler metric. A helpful remark in the present context is that the harmonic representative of a degree 1 cohomology class actually does not depend on the Kähler metric.

- The closure (in the classical topology) of an acqc subset is also acqc.
- Whenever ρ is an isolated point in $M_B(X,G)$, $\{\rho\}$ is acqc.
- Absolute constructibility is invariant under standard geometric constructions. For instance, for any morphism $f : Y \to X$ of smooth connected projective varieties, the property of a subset being absolutely constructible is preserved when taking images and preimages via $f^* : M_B(X,G) \to M_B(Y,G)$. Similarly, for any homomorphism $\mu : G \to G'$ of reductive groups, taking images and preimages under $\mu_* : M_B(X,G) \to M_B(X,G')$ preserves absolute constructibility.
- Given a dominant morphism $f : Y \to X$ and $i \in \mathbb{N}$ the set $M_f^i(X, GL_n)$ of local systems V on Y such that $R^i f_* V$ is a local system is ac. Also, taking images and inverse images under $R^i f_* : M_f^i(X, GL_n) \to M_B(Y, GL_{n'})$ preserves acqc sets.
- The complex points of a closed acqc set M are stable under the $\mathbb{C}^*$ action defined by [Sim88] in terms of Higgs bundles. By [Sim88] the fixed point set $M^{\mathrm{VHS}} := M^{\mathbb{C}^*}$ consists of representations underlying polarizable complex Variations of Hodge structure ($\mathbb{C}$-VHS, for short). Furthermore M is then the smallest closed acqc set in $M_B(X,G)$ containing M^{VHS}.

1.2 Reductive Shafarevich conjecture

After complete results were obtained for surfaces in [KatRam98], the Shafarevich conjecture on holomorphic convexity for reductive linear coverings of arbitrary projective algebraic manifolds over $\mathbb{C}$ was settled affirmatively in [Eys04].

Theorem 1.1 *Let $M \subset M_B(X,G)$ be an absolute constructible set of conjugacy classes of linear reductive representations of $\pi_1(X,x)$ in some reductive algebraic group G over $\overline{\mathbb{Q}}$.*

Define a normal subgroup $H_M \subset \pi_1(X,x)$ by:

$$H_M = \bigcap_{\rho \in M(\overline{\mathbb{Q}})} \ker(\rho).$$

The Galois covering space $\widetilde{X_M} = \widetilde{X^{univ}}/H_M$ is holomorphically convex.

Without a loss of generality we may assume in this theorem that M is a closed absolutely constructible set since $\widetilde{X_M} = \widetilde{X_{\overline{M}}}$.

Let Γ_M be the quotient group defined by

$$\Gamma_M = \pi_1(X,x) / \bigcap_{\rho \in M(\overline{\mathbb{Q}})} \ker(\rho).$$

Γ_M is the Galois group of $\widetilde{X_M}$ over X. and acts in a proper discontinuous fashion on the Cartan-Remmert reduction $\widetilde{S_M}(X)$ of $\widetilde{X_M}$, which is a normal complex space. The quotient space

$$Sh_M(X) = \widetilde{S_M}(X)/\Gamma_M$$

is then a normal projective variety and the quotient morphism $sh_M : X \to Sh_M(X)$ is called the Shafarevich morphism attached to M. This morphism is a fibration, i.e.: is surjective with connected fibers.

Its fibers Z are connected, have the property that $\pi_1(Z) \to \Gamma_M$ has finite image and are maximal with respect to these properties.

Corollary 1.2 *If $\pi_1(X, x)$ is almost reductive (i.e. has a Zariski dense representation with finite kernel in a reductive algebraic group over $\mathbb{C}$) then the Shafarevich conjecture holds for X.*

2 $\mathbb{C}$-VMHS attached to an absolute closed set

We will first review some of the results in [Hai98] and [EysSim09] that enable one to construct various $\mathbb{C}$-VMHS on X out of M.

The results in [Hai98] are important, general and abstract since they deal with general compactifiable Kähler spaces. The results in [EysSim09] deal with the less general situation of a compact Kähler manifold but are more explicit and give some useful properties we have not been able to deduce from [Hai98]. More to the point, [EysSim09] will be sufficient for proving the Shafarevich conjecture in the case when $\pi_1(X, x)$ has a faithful complex linear representation. On the other hand, the results in [Hai98, Sections 1-12] are needed for the optimal form of our results.

2.1 $\mathbb{C}$-VMHS, definition, basic properties

The notion of polarized $\mathbb{C}$-VHS was introduced in [Sim88] as a straightforward variant of [Gri73]. We will use another equivalent definition:

Definition 2.1 *A $\mathbb{C}$-VHS (polarized complex variation of Hodge structures) on X of weight $w \in \mathbb{Z}$ is a 5-tuple $(X, \mathbb{V}, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, S)$ where:*

1. *$\mathbb{V}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces,*
2. *S a non degenerate flat sesquilinear pairing on $\mathbb{V}$,*
3. *$\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$ by locally free coherent analytic sheaves such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$,*
4. *$\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{\bar{X}}$ by locally free coherent antianalytic sheaves such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1} \otimes \Omega^1_{\bar{X}}$,*

5. *for every point $x \in X$ the fiber at x $(\mathbb{V}_x, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-MHS polarized by S_x.*

The conjugate $\mathbb{C}$-VHS is the $\mathbb{C}$-VHS obtained on $\overline{\mathbb{V}}$ setting $\mathcal{F}^\bullet_{\overline{\mathbb{V}}} = \overline{\mathcal{G}}^\bullet$, etc. The local system $\mathbb{V} \oplus \overline{\mathbb{V}}$ carries a real polarized Variation of Hodge Structures.

Recall that a real reductive algebraic group E is said to be of Hodge type if there is a morphism $h : U(1) \to Aut(E)$ such that $h(-1)$ is a Cartan involution of E, see [Sim92, p.46]. By definition, h is a Hodge structure on E. Connected groups of Hodge type are precisely those admitting a compact Cartan subgroup. A Hodge representation of E is a finite dimensional complex representation $\alpha : E(\mathbb{R}) \to GL(\mathbb{V}_\mathbb{C})$ such that h fixes $\ker(\alpha)$.In this case, $\mathbb{V}_\mathbb{C}$ inherits a pure polarized Hodge structures of weight zero. The adjoint representation of a Hodge group is Hodge. Thus the Lie algebra $\mathfrak{E}$ of E has a natural real Hodge structure of weight 0 compatible with the Lie bracket. The Lie algebra action $\mathfrak{E}_\mathbb{C} \otimes \mathbb{V}_\mathbb{C} \to \mathbb{V}_\mathbb{C}$ respects the Hodge structures.

The real Zariski closure E_ρ of the monodromy group of a representation $\rho : \pi_1(X, x) \to G(\mathbb{C})$ underlying a $\mathbb{C}$-VHS is a group of Hodge type. We have $E_\rho \subset R_{\mathbb{C}|\mathbb{R}} G_\mathbb{C}$, where $R_{\mathbb{C}|\mathbb{R}}$ is the Weil restriction of scalars functor. Every Hodge representation α of E gives rise to

$$\alpha \circ \rho : \pi_1(X, x) \to GL(\mathbb{V}_\alpha)$$

a representation that underlies a $\mathbb{C}$-VHS [Sim92, Lemma 5.5].

The notion of $\mathbb{C}$-VMHS (or graded-polarized variation of $\mathbb{C}$-mixed Hodge structures) used in [EysSim09] is a straightforward generalization of that given in [StZ85, Usu83]:

Definition 2.2 *A $\mathbb{C}$-VMHS on X is a 6-tuple $(X, \mathbb{V}, \mathbb{W}_\bullet, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, (S_k)_{k\in\mathbb{Z}})$ where:*

1. *$\mathbb{V}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces,*
2. *$\mathbb{W}_\bullet = (\mathbb{W}_k)_{k\in\mathbb{Z}}$ is a decreasing filtration of $\mathbb{V}$ by local subsystems,*
3. *$\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_\mathbb{C} \mathcal{O}_X$ by locally free coherent analytic sheaves such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$,*
4. *$\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V} \otimes_\mathbb{C} \mathcal{O}_{\bar{X}}$ by locally free coherent antianalytic sheaves such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1} \otimes \Omega^1_{\bar{X}}$,*
5. *$\forall x \in X$ the stalk $(\mathbb{V}_x, \mathbb{W}_{\bullet,x}, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-MHS,*
6. *S_k is flat sesquilinear non degenerate pairing on $Gr^\mathbb{W}_k\mathbb{V}$,*
7. *$(X, Gr^\mathbb{W}_k\mathbb{V}, Gr^{\mathbb{W}\otimes_\mathbb{C}\mathcal{O}_X}_k \mathcal{F}^\bullet, Gr^{\mathbb{W}\otimes_\mathbb{C}\mathcal{O}_{\bar{X}}}_k \overline{\mathcal{G}}^\bullet, S_k)$ is a $\mathbb{C}$-VHS.*

We use the following terminology in the sequel:

Definition 2.3 *A homomorphism of groups $\rho : \Gamma \to \Gamma'$ will be called trivial if $\rho(\Gamma) = \{e\}$. A VMHS will be called trivial (or constant) if its monodromy representation is trivial.*

2.2 Mixed Hodge theory for the relative completion

In [Hai98, Theorem 13.10], certain $\mathbb{R}$-VMHS are attached to a $\mathbb{R}$-VHS on compact Kähler manifold. In this section we review the results of [Hai98] relevant to our discussion and complement them with some explicit examples. We will omit the proofs of the statements that are not essential to our present goals but we will describe in greater detail the examples we need.

2.2.1 Hain's theorems

Let us first review [Hai98, Sections 1-12]. Another reference where this material (and much more) has been nicely rewritten in a more general form is [Pri07, Section 6]. Let E^ρ be a real reductive group of Hodge type. Let $\rho : \pi_1(X,x) \to E^\rho(\mathbb{R})$ be a Zariski dense representation underlying a VHS, and let

$$1 \to \mathcal{U}_x^\rho \to \mathcal{G}_x^\rho \to E^\rho \to 1, \quad a : \pi_1(X,x) \to \mathcal{G}_x^\rho(\mathbb{R})$$

be its relative completion [Hai98].

$\mathcal{G}_x^\rho$ is a proalgebraic group over $\mathbb{R}$ and $\mathcal{U}_x^\rho$ is its prounipotent radical.

Let $k \geq 1$ be an integer, $\mathcal{U}_{x,k}^\rho$ be the k-th term of the lower central series of $\mathcal{U}_x^\rho$ and $\mathcal{G}_{x,k}^\rho$ be $\mathcal{G}_x^\rho/\mathcal{U}_{k_x}^\rho$.

The commutative Hopf algebra $\mathbb{R}[\mathcal{G}_x^\rho]$ of the regular functions on $\mathcal{G}_x^\rho$ carries a compatible $\mathbb{R}$-MHS with nonnegative weights. The increasing weight filtration is described by the formula:

$$W_k\mathbb{R}[\mathcal{G}_x^\rho] = \mathbb{R}[\mathcal{G}_{x,k}^\rho]$$

where $\mathbb{R}[\mathcal{G}_{x,k}^\rho]$ is identified with its image in $\mathbb{R}[\mathcal{G}_x^\rho]$. Although these MHS are not necessarily finite dimensional, they are always filtered direct limits of finite dimensional ones.

Let $\mathbb{M}_x = (M_x, W_\bullet, F^\bullet)$ be a finite dimensional complex mixed Hodge structure and consider $\alpha : \mathcal{G}_x^\rho(\mathbb{R}) \to GL(M_x)$ a representation of $\mathcal{G}_x^\rho$. We will say α is a Mixed Hodge representation iff α is the representation arising from the real points of a rational representation of $\mathcal{G}_x^\rho$ in M_x and the coaction

$$\alpha^* : M_x \to M_x \otimes \mathbb{C}[\mathcal{G}_x^\rho]$$

respects the natural MHS.

The main result of [Hai98, Section 13] can now be stated as follows.

Proposition 2.4 *Let α be a Mixed Hodge representation. The representation $\alpha \circ a : \pi_1(X,x) \to GL(M_x)$ underlies a $\mathbb{C}$-VMHS. Moreover, any $\mathbb{C}$-VMHS $\mathbb{M}$ whose graded constituents $Gr_W^k\mathbb{M}$ are VHS such that their monodromy representations $\pi_1(X,x) \to GL(Gr_W^k\mathbb{M}_x)$ factor through ρ is of this type. A similar statement holds for $\mathbb{R}$-VMHS.*

The recent preprint [Ara09] gives among other things an alternative approach to this material.

2.2.2 Example

In [Hai98] Hain describes the steps $\mathbb{W}_k\mathbb{M}$ of the weight filtration in Proposition 2.4 through iterated integrals. This however is somewhat technical and goes beyond the scope of the present paper. Instead of discussing the general construction, we will spell out the definition of the $\mathbb{C}$-VMHS underlying some very specific $\mathbb{C}$-Mixed Hodge representations of $\mathcal{G}^{\rho}_{x,1}$ which will play a prominent role in our considerations.

Let $(X, \mathbb{V}, \mathcal{F}^{\bullet}, \overline{\mathcal{G}}^{\bullet}, S)$ be a $\mathbb{C}$-VHS that will be assumed with no loss of generality of weight 0. We will write $\mathbb{V}$ for short, since this will not cause any confusion.

Let $\mathcal{E}^{\bullet}(X, \mathbb{V})$ be the de Rham complex of $\mathbb{V}$. This de Rham complex inherits a Hodge filtration from $\mathcal{F}$ and the Hodge filtration on $\mathcal{E}^{\bullet}(X)$ and an anti-Hodge filtration from $\mathcal{G}$ and the anti-Hodge filtration on $\mathcal{E}^{\bullet}(X)$. The resulting two filtrations on its cohomology groups define on $H^p(X, \mathbb{V})$ a $\mathbb{C}$-Hodge structure of weight p. Furthermore, once we fix a Kähler form on X, there is a subspace $\mathcal{H}^p(X, \mathbb{V}) \subset \mathcal{E}^p(X, \mathbb{V})$ consisting of harmonic forms in a suitable sense such that the composite map $[\_] : \mathcal{H}^p(X, \mathbb{V}) \subset \mathcal{Z}^p(X, \mathbb{V}) \to H^p(X, \mathbb{V})$ is an isomorphism. This is standard and can be found in e.g. [Zuc79] for $\mathbb{V}$ a $\mathbb{R}$-VHS. The general $\mathbb{C}$-VHS case follows in exactly the same way.

Remark 2.5 *When $p = 1$, the space of harmonic forms is actually independent of the Kähler metric and of the polarization S. Furthermore if $Y \to X$ is a morphism $f^*\mathcal{H}^1(X, \mathbb{V}) \subset \mathcal{H}^1(Y, \mathbb{V})$. Indeed $\mathcal{H}^1(X, \mathbb{V}) = \ker(D') \cap \ker(D'') \cap \mathcal{E}^1(X, \mathbb{V})$.*

Consider $\alpha \in \mathcal{H}^1(X, \mathbb{V})$ such that $[\alpha]$ is of pure Hodge type (P, Q). Then, for all $y \in X$, $\alpha(p) \in \mathbb{V}_y^{P-1,Q} \otimes \Omega^{1,0} \oplus \mathbb{V}_y^{P,Q-1} \otimes \Omega^{0,1}$.

Let $(\alpha_i)_{i \in I}$ be a $\mathbb{C}$-basis of $\mathcal{H}^1(X, \mathbb{V})$ such that each $[\alpha_i]$ is of pure Hodge type. Let $([\alpha_i]^*)_{i \in I}$ be the dual basis of the dual vector space $H^1(X, \mathbb{V})^*$ and define

$$\Omega \in \mathcal{E}^1(X, \mathbb{V} \otimes H^1(X, \mathbb{V})^*)$$

by the formula:

$$\Omega = \sum_i \alpha_i \otimes [\alpha_i]^*.$$

Note that Ω does not depend on the chosen basis. Now we define a new connection on the vector bundle underlying the local system $\mathbb{M}_0 = \mathbb{C} \oplus \mathbb{V}^* \otimes H^1(X, \mathbb{V})$ on X a new connection by setting:

$$d_{\mathbb{M}} = \begin{pmatrix} d_{\mathbb{C}} & \Omega \\ 0 & d_{\mathbb{V}^*} \otimes Id_{H^1(X,\mathbb{V})} \end{pmatrix}.$$

The duality pairings $H^1(X, \mathbb{V}) \otimes H^1(X, \mathbb{V})^* \to \mathbb{C}$ and $\mathbb{V} \otimes \mathbb{V}^* \to \mathbb{C}$ are tacitly used in this formula. Since $d_{\mathbb{V}}\alpha_i = 0$, the connection follows that $d_{\mathbb{M}}$ is a flat connection and this gives rise to a local system $\mathbb{M}$. Furthermore, the connection $d_{\mathbb{M}}$ respects the 2-step filtration $W^0\mathbb{M} = \mathbb{C}$ $W^1\mathbb{M} = \mathbb{M}$, hence $\mathbb{M}$ is a filtered local system whose graded parts are $Gr^0_W\mathbb{M} = \mathbb{C}$ and $Gr^1_W\mathbb{M} = \mathbb{V}^* \otimes H^1(X, \mathbb{V})$.

We now define a Hodge filtration $\mathcal{F}^\bullet$ of the smooth vector bundle underlying $\mathbb{M}$ by the formula valid for every $p \in X$:

$$\mathcal{F}_p^k = \begin{cases} \mathcal{F}^k_{\mathbb{V}_p^* \otimes H^1(X,\mathbb{V})} \subset \mathbb{V}_p^* \otimes H^1(X,\mathbb{V}) & \text{if } \ k > 0, \\ \mathbb{C}_p \oplus \mathcal{F}^k_{\mathbb{V}_p^* \otimes H^1(X,\mathbb{V})} \subset \mathbb{C}_p \oplus \mathbb{V}_p^* \otimes H^1(X,\mathbb{V}) & \text{if } \ k \leq 0. \end{cases}$$

Similarly, one defines an anti-Hodge filtration on $\mathbb{M}$ which we denote by $\overline{\mathcal{G}}^\bullet$ by the formula valid for every $p \in X$:

$$\overline{\mathcal{G}}_p^k = \begin{cases} \overline{\mathcal{G}}^k_{\mathbb{V}_p^* \otimes H^1(X,\mathbb{V})} \subset \mathbb{V}_p^* \otimes H^1(X,\mathbb{V}) & \text{if } \ k > 0, \\ \mathbb{C}_p \oplus \overline{\mathcal{G}}^k_{\mathbb{V}_p^* \otimes H^1(X,\mathbb{V})} \subset \mathbb{C}_p \oplus \mathbb{V}_p^* \otimes H^1(X,\mathbb{V}) & \text{if } \ k \leq 0. \end{cases}$$

It defines on each stalk $\mathbb{M}_p$ a $\mathbb{C}$-MHS such that $Gr_W^0 \mathbb{M}_p$ is the trivial Hodge structure on $\mathbb{C}$ and $Gr_W^1 \mathbb{M}_p$ is the given Hodge Structure on $\mathbb{V}_p^* \otimes H^1(X,\mathbb{V})$.

Lemma 2.6 *$\mathcal{F}^k$ is a holomorphic subbundle of the holomorphic vector bundle $\mathcal{M}$ underlying $\mathbb{M}$ and satisfies Griffiths transversality.*

Proof: First observe that the $\bar\partial$ operator of $\mathcal{M}$ is given by $d_{\mathbb{M}}^{0,1}$. Consider the original flat connection

$$d = \begin{pmatrix} d_{\mathbb{C}} & 0 \\ 0 & d_{\mathbb{V}^*} \otimes Id_{H^1(X,\mathbb{V})} \end{pmatrix}.$$

Obviously

$$d_{\mathbb{M}}^{0,1} = d^{0,1} + \begin{pmatrix} 0 & \Omega^{0,1} \\ 0 & 0 \end{pmatrix}.$$

Since $d^{0,1}$ preserves $\mathcal{F}^k$, it follows that $\mathcal{F}_k$ is an holomorphic subbundle of $\mathcal{M}$ iff

$$\begin{pmatrix} 0 & \Omega^{0,1} \\ 0 & 0 \end{pmatrix} \mathcal{F}^k \subset \Omega^{0,1} \otimes \mathcal{F}^k,$$

where $\Omega^{0,1} \in \mathcal{E}^{0,1}(X,\mathbb{V}) \otimes H^1(X,\mathbb{V})^*$ is the $(0,1)$-component of Ω. This condition is equivalent to $\Omega^{0,1} \cdot \mathcal{F}^k_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0$ if $k > 0$.

We are thus reduced to checking that for every $\alpha \in \mathcal{H}^1(X,\mathbb{V})$ such that $[\alpha]$ is of pure Hodge type (P,Q) and $[\beta]^\vee \in (H^1(X,\mathbb{V})^*)^{-P,-Q}$

$$\alpha^{0,1} \otimes [\beta]^\vee \cdot \mathcal{F}^k_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0, \text{ if } k > 0.$$

It is enough to check that $[\alpha] \otimes [\beta]^\vee \cdot H^{k,1-k}_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0$, or further decomposing in Hodge type that

$$\alpha^{0,1} \otimes [\beta]^\vee \cdot h^{-P'+k,-Q'-k+1} \otimes [\beta]^{P',Q'} = O$$

where $h^{-P'+k,-Q'-k+1} \in (\mathbb{V}^*)^{-P'+k,-Q'-k+1}$ and $[\beta]^{P',Q'} \in H^1(X,\mathbb{V})^{P',Q'}$. The only non trivial case is when $P' = P, Q' = Q$ and this reduces to showing that $\mathbb{V}^{P,Q-1} \otimes \Omega^{0,1}.(\mathbb{V}^*)^{-P+k,-Q-k+1} = 0$ which is the case since $k > 0$.

Griffiths transversality is the statement that $d^{1,0}_{\mathbb{M}}\mathcal{F}^k \subset \mathcal{F}^{k-1}\otimes\Omega^{1,0}$ and follows from the same argument. □

Antiholomorphicity and Griffiths anti-transversality for $\overline{\mathcal{G}^\bullet}$ can be proved by the same method. Hence we have defined on $\mathbb{M}$ a graded polarizable VMHS with weights $0, 1$, the polarizations being the natural ones. In [HaiZuc87], the case of $\mathbb{V} = \mathbb{C}_X$ is treated. In that case, the VMHS is actually defined over $\mathbb{Z}$.

Definition 2.7 $\mathbb{M} = \mathbb{M}(\mathbb{V}) := (X, \mathbb{M}, \mathbb{W}_\bullet, \mathcal{F}^\bullet_M, \overline{\mathcal{G}}^\bullet_M, (S_k)_{k=0,1})$ *is the* 1*-step* $\mathbb{C}$*-VMHS attached to* $\mathbb{V}$*.*

2.3 Mixed Hodge theory for the deformation functor

In this paragraph, we review the construction of [EysSim09]. The 'new' aspects of this construction actually grew out of the previous example. The older aspects, on the other hand, were part of Goldman-Millson's theory of deformations for representations of Kähler groups [GolMil88].

In this paragraph, we fix $N \in \mathbb{N}$ and assume that $G = GL_N$ and $M = M_B(X, GL_N)$. Let $\rho : \pi_1(X, x) \to GL_N(\mathbb{C})$ be the monodromy represention of a $\mathbb{C}$-VHS. Let $\hat{\mathcal{O}}_\rho$ be the complete local ring of $[\rho] \in R(\pi_1(X, x), GL_N)(\mathbb{C})$. Let

$$\mathsf{obs}_2 = [-;-] : S^2H^1(X, \mathrm{End}(\mathbb{V}_\rho)) \to H^1(X, \mathrm{End}(\mathbb{V}_\rho))$$

be the Goldman-Millson obstruction to deforming ρ. Define $I_2, (I_n)_{n\geq 2}, (\Pi_n)_{n\geq 0}$, as follows:

$$\begin{aligned}
\Pi_0 &= \mathbb{C}\\
\Pi_1 &= H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
I_2 &= \mathrm{Im}({}^t\mathsf{obs}_2) \subset S^2H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
I_n &= I_2S^{n-2}H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
\Pi_n &= S^nH^1(X, \mathrm{End}(\mathbb{V}_\rho))^*/I_n
\end{aligned}$$

Then the complete local $\mathbb{C}$-algebra

$$(\hat{\mathcal{O}}_T, \mathfrak{m}) := (\sum_{n\geq 0}\Pi_n, \sum_{n\geq 1}\Pi_n)$$

is the function algebra of a formal scheme T which is the germ at 0 of the quadratic cone

$$\mathsf{obs}_2^{-1}(0) \subset H^1(X, \mathrm{End}(\mathbb{V}_\rho)).$$

We endow $\hat{\mathcal{O}}_T$ with a split mixed Hodge structure with non positive weights, whose weight filtration is given by the formula $W_k\hat{\mathcal{O}}_T = \mathfrak{m}^{-k}$ for $k \leq 0$, arising from the identifications:

$$\hat{\mathcal{O}}_T = \sum_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1} = \sum_{n\in\mathbb{N}}\Pi_n,$$

Π_n being endowed with its natural $\mathbb{C}$-Hodge structure of weight $-n$. This mixed Hodge structure is infinite dimensional, but can be described as the limit of the resulting finite dimensional MHS on $\hat{\mathcal{O}}_T/\mathfrak{m}^n$.

In [GolMil88], an isomorphism between $Spf(\hat{\mathcal{O}}_\rho)$ and $T \times A$ is constructed, where A is the germ at zero of a finite dimensionnal vector space. In [EysSim09], this constuction is revisited. A slight reinterpretation of Goldman-Millson theory is that one can realize the formal local scheme T as a hull of the deformation functor for ρ. Actually, there are *three* preferred such realizations $\mathcal{GM}^c, \mathcal{GM}', \mathcal{GM}''$ which are given by three canonical representations:

$$\rho_T^{GM} : \pi_1(X,x) \to GL_N(\widehat{\mathcal{O}}_T)$$
$$\rho_T^{GM'} : \pi_1(X,x) \to GL_N(\widehat{\mathcal{O}}_T)$$
$$\rho_T^{GM''} : \pi_1(X,x) \to GL_N(\widehat{\mathcal{O}}_T)$$

These three representations are conjugate up to an isomorphism of T.

We can now summarize the results developped by [EysSim09] in the form we shall need:

Definition 2.8 *Let $\eta_1, \ldots, \eta_b \in E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$ form a basis of the subspace $\mathcal{H}^1(X, \mathrm{End}(\mathbb{V}_\rho))$ of harmonic twisted one forms, each η_i being of pure Hodge type (P_i, Q_i) for the Deligne-Zucker $\mathbb{C}$-Mixed Hodge Complex $E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$. Then $\{\eta_i\}$ is a basis of $H^1(X, End(\mathbb{V}_\rho))$ whose dual basis we denote by $(\{\eta_1\}^*, \ldots, \{\eta_b\}^*)$.*

The $\mathrm{End}(\mathbb{V}_\rho) \otimes \Pi_1$-valued one-form α_1^v is defined by the formula:

$$\alpha_1^v = \sum_{i=1}^{b} \eta_i \otimes \{\eta_i\}^*.$$

Proposition 2.9 *For $k \geq 2$, we can construct a unique D''-exact form $\alpha_k^v \in E^1(X, \mathrm{End}(\mathbb{V}_\rho)) \otimes \Pi_k$ such that the following relation holds:*

$$D'\alpha_k^v + \alpha_{k-1}^v \alpha_1^v + \alpha_{k-2}^v \alpha_2^v + \ldots + \alpha_1^v \alpha_{k-1}^v = 0.$$

Proposition 2.10 *Let $A_v = \sum \alpha_k^v$ acting on the vector bundle underlying the filtered local system $(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T, W_k(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T) = \mathbb{V}_\rho \otimes \mathfrak{m}^{k - wght(\mathbb{V}_\rho)})$, whose connection will be denoted by D.*

Then, $D + A^v$ respects this weight filtration, satisfies Griffiths' transversality for the Hodge filtration $\mathcal{F}^\bullet$ defined by

$$\mathcal{F}^p = \bigoplus_{k=-n}^{0} \mathcal{F}^p(\mathbb{V}_\rho \otimes \Pi_{-k})$$

and we can construct an anti-Hodge filtration so that the resulting structure is a graded polarizable $\mathbb{C}$-VMHS whose monodromy representation is $\rho_T^{GM''}$.

A detailed proof of this proposition is given in [EysSim09]. The essential part is the construction of the anti-Hodge filtration which is similar in spirit but somewhat subtler than construction given in Example 2.2.2.

Definition 2.11 *The $\mathbb{C}$-VMHS obtained by reduction* $\mod \mathfrak{m}^n$ *corresponds to*

$$\rho_{T,n} := (\rho_T^{cGM^v} \mod \mathfrak{m}^n) : \pi_1(X,x) \to GL_N(\widehat{\mathcal{O}}_T/\mathfrak{m}^n).$$

It will be called the n-th deformation of $\mathbb{V}_\rho$ and will be denoted by $\mathbb{D}_n(\mathbb{V}_\rho)$.

By construction, $D + A^v$ is an $\hat{\mathcal{O}}_T$-linear connection. As a consequence of the methods in [EysSim09, pp 18-23], we also have:

Proposition 2.12 *There is a MHS on $\hat{\mathcal{O}}_T$ whose weight filtration is given by the powers of the maximal ideal and such that the natural map $\hat{\mathcal{O}}_T \to \mathrm{End}_{\mathbb{C}}(\mathbb{D}_n(\mathbb{V}_\rho))$ respects the natural MHS.*

This MHS is *not* the split MHS constructed above. This split MHS is just the weight graded counterpart of the true object. These MHS and VMHS are not uniquely defined when the deformation functor of ρ is not prorepresentable. This phenomenon does not occur when the representation is irreducible.

Remark 2.13 *The restriction $G = GL_n$ in the above considerations was introduced only for convenience. It is not essential. In [EysSim09], similar statements are proven for arbitrary reductive groups G.*

3 Subgroups of $\pi_1(X,x)$ attached to M

Let G be a reductive algebraic group defined over $\overline{\mathbb{Q}}$. Suppose as before $M \subset M_B(X,G)$ is an absolute closed subset.

3.1 Definitions

Definition 3.1 *Let M^{VHS} be the subset of $M(\mathbb{C})$ consisting of the conjugacy classes of $\mathbb{C}$-VHS that is $M^{VHS} := KH^{-1}(M_{Dol}(X,G)^{\mathbb{C}^*}(\mathbb{C}))$.*

We choose a set M^* of reductive representations $\rho : \pi_1(X,x) \to G(\mathbb{C})$ mapping onto M^{VHS} under the natural map $R(\pi_1(X,x),G) \to M_B(X,G)$. To be more precise, we define M^* to be the union of the closed G-orbits on $R(\pi_1(X,x),G)$ -or equivalently the set of reductive complex representations- whose equivalence class lie in M^{VHS}. Similarly, we define M' to be the union of the closed G-orbits on $R(\pi_1(X,x),G)$ whose equivalence class lie in M. To each $\rho \in M^*$ we attach E^ρ the real Zariski closure of its monodromy group and the other constructions reviewed in paragraph 2.2.1.

Definition 3.2 *The tannakian categories $\mathcal{T}_M^{VHS}$ and $\mathcal{T}_M$ are defined as follows:*

$\mathcal{T}_M^{VHS}$ is the full Tannakian subcategory of the category of local systems on X generated by the elements of M^{VHS}.

$\mathcal{T}_M$ *is the full Tannakian subcategory of the category of local systems on* X *generated by the elements of* M*.*

Every object in $\mathcal{T}_M^{VHS}$ is isomorphic to an object which is a subquotient of $\alpha_1(\rho_1)\otimes\ldots\otimes\alpha_s(\rho_s)$, where $\rho_1,\ldots,\rho_s$ are elements of M^* and α_i is a complex linear finite dimensionnal representation of $E^{\rho_i}(\mathbb{R})$. Let M^{**} be the set of all such subquotients. The objects of $\mathcal{T}_M^{VHS}$ underly polarizable $\mathbb{C}$-VHS.

Let T_M^{VMHS} be the thick Tannakian subcategory of ($\mathbb{C}$-VMHS) whose graded constituents are objects of $\mathcal{T}_M^{VHS}$. The full subcategory of $\mathcal{T}_M^{VMHS}$ with a weight filtration of length at most $k+1$ will be denoted by $T_M^{VMHS}(k)$.

Example 3.3 *For every* $\rho\in M^{**}$*,* α *as above and* $\sigma=\alpha\circ\rho$*,* $\mathbb{D}_k(\mathbb{V}_\sigma)$ *is an object of* $T_M^{VMHS}(k)$*.*

Definition 3.4 *Given* X*,* G*, and* $M\subset M_B(X,G)$ *as above, and* $k\in\mathbb{N}$ *we define the following natural quotients of* $\pi_1(X,x)$*:*

Γ_M^∞ *is the quotient of* $\pi_1(X,x)$ *by the intersection* H_M^∞ *of the kernels of the objects of* $\mathcal{T}_M^{VMHS}$ *and of the objects of* M*.*

$\widetilde{\Gamma_M^\infty}$ *is the quotient of* $\pi_1(X,x)$ *by the intersection* $\widetilde{H_M^\infty}$ *of the kernels of the monodromy representation of* $\mathbb{D}_n(\mathbb{V}_\sigma)$*,* $\sigma\in M^{**}$*,* $n\in\mathbb{N}$*, and of the objects of* M*.*

Γ_M^k *is the quotient of* $\pi_1(X,x)$ *by the intersection* H_M^k *of the kernels of the objects of* $\mathcal{T}_M^{VMHS}(k)$ *and of the objects of* M*.*

$\widetilde{\Gamma_M^k}$ *is the quotient of* $\pi_1(X,x)$ *by the intersection* $\widetilde{H_M^k}$ *of the kernels of the monodromy representation of* $\mathbb{D}_k(\mathbb{V}_\sigma)$*,* $\sigma\in M^{**}$*, and of the objects of* M*.*

It is likely that the canonical quotient morphism $\Gamma_M^k\to\widetilde{\Gamma_M^k}$ is an isomorphism but we do not have a proof of this fact yet. We will thus have to work with the above slightly clumsy notation.

Note that we have the inclusions:

$$\Gamma_M^\infty=\bigcap_{k\in\mathbb{N}}\Gamma_M^k\subset\Gamma_M^{k+1}\subset\Gamma_M^k\subset\Gamma_M^0=\Gamma_M,$$
$$\widetilde{\Gamma_M^\infty}=\bigcap_{k\in\mathbb{N}}\widetilde{\Gamma_M^k}\subset\widetilde{\Gamma_M^{k+1}}\subset\widetilde{\Gamma_M^1}\subset\widetilde{\Gamma_M^0}=\widetilde{\Gamma_M}.$$

It should be noted that since H_M^k (respectively $\widetilde{H_M^k}$) is normal the various base point changing isomorphisms $\pi_X(X,x')\to\pi_1(X,x)$ respect H_M^k (respectively $\widetilde{H_M^k}$). Hence, dropping the base point dependance in the notation H_M^k (respectively $\widetilde{H_M^k}$) is harmless.

For future reference, we state the following lemma whose proof is tautological.

Lemma 3.5 H_M^k *is the intersection of* Γ_M *and the kernels of* $a_k^\rho:\pi_1(X,x)\to\mathcal{G}_{x,k}^\rho(\mathbb{R})$*.*

3.2 Strictness

Let $z \in Z$ be a base point in the connected projective variety Z.

Proposition 3.6 *For every $f : (Z, z) \to (X, x)$ such that $\pi_1(Z, z) \to \Gamma_M$ is trivial, the following are equivalent:*

1. *For every $\mathbb{V}$ in $\mathcal{T}_M^{VHS}$, $H^1(X, \mathbb{V}) \to H^1(Z, \mathbb{V})$ is trivial,*
2. *$\pi_1(Z, z) \to \Gamma_M^1$ is trivial,*
3. *$\pi_1(Z, z) \to \widetilde{\Gamma_M^1}$ is trivial,*
4. *For every $\mathbb{V}$ in $\mathcal{T}_M^{VHS}$, for every $\widehat{Z}_i \to Z$ a resolution of singularities of an irreducible component, the VMHS $M(\mathbb{V})_{\widehat{Z}_i}$ is trivial.*
5. *For every $\sigma \in M^{**}$ and $k \in \mathbb{N}$, for every $\widehat{Z}_i \to Z$ a resolution of singularities of an irreducible component, the VMHS $\mathbb{D}_k(\mathbb{V}_\sigma)_{\widehat{Z}_i}$ is trivial.*
6. *$\pi_1(Z, z) \to \widetilde{\Gamma_M^\infty}$ is trivial.*
7. *$\pi_1(Z, z) \to \Gamma_M^\infty$ is trivial.*

Proof:

$(1 \iff 2)$. Fix $\rho \in M^{**}$. Denote by E^ρ the real Zariski closure of $\rho(\pi_1(X, x))$. By hypothesis $\rho(\pi_1(Z, z)) = \{e\}$ and thanks to [Hai98], section 11, [2] we have a diagram:

$$\begin{array}{ccccc} \pi_1(Z,z) & \stackrel{a_Z}{\mapsto} & \hat{\pi}_1^{DR}(Z,z) & & \\ \downarrow & & \downarrow & & \\ \pi_1(X,x) & \stackrel{a_X}{\mapsto} & \mathcal{U}_x^\rho & \subset & \mathcal{G}_x^\rho \end{array}$$

where $\pi_1(Z, z) \stackrel{a_Z}{\mapsto} \hat{\pi}_1^{DR}(Z, z) = \mathcal{U}^e(Z, z) = \mathcal{G}^e(Z, z)$ is the Malcev completion of $\pi_1(Z, z)$, i.e.: its relative completion with respect to the trivial representation.

Let $\{V_\alpha\}_\alpha$ be a set of representatives of all isomorphism classes of complex irreducible left E^ρ-modules.

The prounipotent group morphism $f_* : \hat{\pi}_1^{DR}(Z, z) \to \mathcal{U}_x^\rho$ gives rise to a morphism of proalgebraic complex vector groups (=limits of finite dimensional complex vector spaces viewed as algebraic groups):

$$H_1(\hat{\pi}_1^{DR}(Z, z))(\mathbb{C}) \to H_1(\mathcal{U}_x^\rho)(\mathbb{C})$$

where $H_1(\mathcal{U}) = \mathcal{U}/\mathcal{U}'$ is the abelianization. One has identifications (see [Hai98] p 73):

[2] Stricto sensu, in order to apply [Hai98] sect. 11, we need that Z be a smooth connected manifold. We can replace Z by a neighborhood U of it in some embedding in $\mathbf{P}^N(\mathbb{C})$ such that $Z \to U$ is an homotopy equivalence and apply [Hai98] sect. 11 to U.

$$H_1(\hat{\pi}_1^{DR}(Z,z))(\mathbb{C}) = H_1(Z,\mathbb{C}),$$
$$H_1(\mathcal{U}_x^\rho)(\mathbb{C}) = \prod_\alpha H_1(X,\mathbb{V}_\alpha)\otimes V_\alpha^*$$

where $\mathbb{V}_\alpha$ is the local system attached to ρ and V_α. The map is the transpose of the map

$$\bigoplus_\alpha H^1(X,\mathbb{V}_\alpha^*)\otimes V_\alpha \to H^1(X,\mathbb{C})$$

given on each factor by the composition:

$$H^1(X,\mathbb{V}_\alpha^*)\otimes V_\alpha \overset{i_Z^*\otimes \mathrm{id}_{V_\alpha}}{\to} H^1(Z,\mathbb{V}_\alpha^*)\otimes V_\alpha = H^1(Z,\mathbb{C})\otimes V_\alpha^*\otimes V_\alpha \overset{\mathrm{id}\otimes\mathrm{tr}}{\to} H^1(Z,\mathbb{C}).$$

For the middle equality in this formula, we used that $\mathbb{V}_{\alpha|Z}$ is the trivial local system, which follows from the assumption that $\pi_1(Z,z)\to\Gamma_M$ is trivial.

Hence condition 1 is equivalent to $H_1(\hat{\pi}_1^{DR}(Z,z)(\mathbb{C}))\to H_1(\mathcal{U}_x^\rho)(\mathbb{C})$ being zero which in turn is equivalent to condition 2.

($2\Longrightarrow 3$) Condition 3 is obviously implied by condition 2.

($3\Longrightarrow 1$) If 3 holds $\mathbb{D}_1(\mathbb{V}_\sigma)|_Z$ is a trivial local system . But, by construction this local system is a deformation of a trivial local system by a one-step nilpotent matrix of closed one forms written in the following block form:

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

hence, in the same basis, its monodromy on any $\gamma\in\pi_1(Z,z)$ is given by:

$$\begin{pmatrix} 1 & \int_\gamma A \\ 0 & 1 \end{pmatrix}.$$

Hence the triviality of $\mathbb{D}_1(\mathbb{V}_\sigma)|_Z$ implies that $\int_\gamma A = 0$, or that the cohomology class of A is zero. But by construction, the cohomology class of A is zero iff condition 1 holds.

($1\Longrightarrow 4$) The cohomology class of the form α_1 vanishes after restriction to Z and so vanishes after pullback to $\widehat{Z}_i$. We denote by f_i the composition of f with the map $\widehat{Z}_i\to Z$. But $\alpha_1\in\ker(D')\cap\ker(D'')$. Hence $f_i^*\alpha_1\in\ker(D')_{i,1}\cap\ker(D'')_{i,1}$ where $D'_{i,1}, D''_{i,1}$ are the usual D', D'' acting on $E^1(\widehat{Z}_i,\mathrm{End}(f_i^*\mathbb{V}_\rho))$. As mentioned before, Hodge theory implies that:

$$\ker(D')_{i,1}\cap\ker(D'')_{i,1} = \mathcal{H}^1(\widehat{Z}_i,\mathrm{End}(f_i^*\mathbb{V}_\rho)).$$

Hence $f_i^*\alpha_1$ is the harmonic representative of its class. From this it follows that $f_i^*\alpha_i = 0$. This implies that $f_i^*\mathbb{M}$ is the trivial deformation of $f_i^*\mathbb{M}_0$ and condition 4 follows.

($4\Longrightarrow 1$) The method we used to prove ($3\Longrightarrow 1$) works to yield that $H^1(X,\mathbb{V})\to H^1(\hat{Z}_i,\mathbb{V})$ is zero. But this implies by the argument we used to

show $(1 \Longrightarrow 4)$ that $f_i^* \alpha_1 = 0$. This in turn implies that $i_A^* \alpha_1 = 0$ if $f(Z) = \coprod A$ is a smooth stratification. Hence the holonomy of $M(\mathbb{V})_Z$ is trivial. Applying once again the method for $(3 \Longrightarrow 1)$ completes the argument.

$(1 \Longrightarrow 5)$ Continuing the same line of reasoning as in proving $(1 \Longrightarrow 4)$ and using the fact that the (α_k^v) constructed in proposition 2.9 are uniquely determined, it follows that $(f_i^* \alpha_k^v)$ is the family of twisted forms one gets from applying the construction of proposition 2.9 starting with $f_i^* \alpha_1 = 0$. Hence $f_i^* \alpha_k^v = 0$ and $f_i^* A^v = 0$. Condition 5 then follows.

$(1 \Longrightarrow 6)$ Continuing this line of reasoning, the argument made in $(4 \Longrightarrow 1)$ implies that the restriction of $\mathbb{D}_k(\mathbb{V}_\sigma)$ to Z has trivial monodromy, which is equivalent to condition 6.

$(6 \Longrightarrow 2)$ is trivial.

$(6 \Longrightarrow 5)$ comes from the fact that condition 6 implies that the restriction of $\mathbb{D}_k(\mathbb{V}_\sigma)$ to Z has trivial monodromy, condition 5 follows a fortiori.

$(7 \Longrightarrow 3)$ is trivial.

$(1 \Longrightarrow 7)$ The proof is an easy adaptation of the argument of [Kat97], section 2. We nevertheless feel it is necessary to give some details.

The Lie algebras $L(Z,z) = \mathrm{Lie}(\hat{\pi}_1^{DR}(Z,z))$ and $\mathfrak{U}_x^\rho = \mathrm{Lie}(\mathcal{U}_x^\rho)$ are nilpotent and so come equiped with a decreasing filtration given by their lower central series. The map i_Z gives rise to a Lie algebra morphism $(i_Z)_* : L(Z,z) \to \mathfrak{U}_x^\rho$. It is enough to show that $(i_Z)_* = 0$

By relabelling we can convert the lower central series into an increasing filtration $B^\bullet L(Z,z)$ and $B^\bullet \mathfrak{U}_x^\rho$ with indices ≤ -1. For both Lie algebras $Gr_B^{-1}(\_) = H_1(\_)$ and $Gr_B^{-1}(L(Z,z))$ generates the graded Lie algebra $Gr_B^\bullet(L(Z,z))$. Hence condition 3 implies that $Gr_B^\bullet(i_Z)_* : Gr_B^\bullet L(Z,z) \to Gr_B^\bullet \mathfrak{U}_x^\rho$ is zero.

First consider the case where Z is smooth. Then, by [Hai87] [Hai98], both $L(Z,z)$ and $\mathfrak{U}_x^\rho$ carry a functorial Mixed Hodge structure whose weight filtration is $B^\bullet$. Hence, since the map $(i_Z)_*$ respects the Mixed Hodge structures, it is strict for the weight filtration and $Gr_B^\bullet(i_Z)_* = 0 \Rightarrow (i_Z)_* = 0$.

Next we consider the case where $H_1(Z)$ is pure of weight one. We recall, see [Hai87], that $\mathbb{R}[\widehat{\pi}_1^{DR}(Z,z)] = H^0(\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R}))$ where $\overline{\mathsf{B}}$ is the reduced bar construction and $E^\bullet(Z)$ is a multiplicative mixed Hodge complex computing $H^\bullet(Z)$ endowed with a base point at z. $\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R})$ carries an increasing filtration $\mathfrak{B}_\bullet$, the bar filtration. It follows from [Hai87] that $\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R})$ endowed with the bar filtration is a filtered mixed Hodge complex so that the bar filtration on $\mathbb{R}[\widehat{\pi}_1^{DR}(Z,z)]$ is a filtration by MHS. The Eilenberg-Moore spectral sequence which is the spectral sequence associated to the bar filtration is a spectral sequence in the category of MHS and, since $H^1(Z)$ is pure of weight one, $E_1^{s,-s} = H^1(Z)^{\otimes s}$ is pure of weight s. Hence $Gr_{\mathfrak{B}}^k \mathbb{R}[\widehat{\pi}_1^{DR}(Z,z)]$ is pure of weight k. Since the bar filtration is a refinement of the weight filtration, it follows that the bar filtration and the weight filtration coincide. Combining this with the preceding argument, one easily finishes the proof of the case when $H_1(Z)$ is pure of weight one.

Finally note that by passing to a hyperplane section we may assume that Z is a curve which without a loss of generality can be taken to be seminormal

and the argument of [Kat97] p. 340-341 applies verbatim. One concludes using lemma 2.4 p. 342 in [Kat97].

□

Remark 3.7 *If we skip items 2 and 7 in the previous proposition we obtain a strictness statement which can be proved without relying on [Hai98].*

Remark 3.8 *As far as the equivalence of condition 7 with the rest is concerned, we believe that one can adapt the explicit argument made for* $(1 \Longrightarrow 6)$ *using the more sophisticated iterated integrals of [Hai98].*

Except perhaps for condition 7, that depends on X *being projective, the proposition is valid in the compact Kähler case.*

Remark 3.9 *A generalization to the Kähler case of the main result in [Kat97] with an alternative proof has been given in the unpublished thesis [Ler99] (see also [Cla08]) as a byproduct of her exegesis of [Hai87] and [Hai85b]. The core of her argument could be reformulated in such a way that it becomes equivalent to the special case of the present one where* $G = \{e\}$ *is the trivial group.*

3.3 Reduction to using VSHM

Proposition 3.10 *Let* n *be a non negative integer. Let* H_n *be the intersection of the kernels of all linear representations* $\pi_1(X) \to GL_n(A)$*,* A *being an arbitrary* $\mathbb{C}$*-algebra. Let* $M = M(X, GL_n)$*. Then,* $H_n = \widetilde{H^{\infty}_M}$*.*

Proof: The inclusion $H_n \subset \widetilde{H^{\infty}_M}$ is obvious. Now let $\gamma \in \widetilde{H^{\infty}_M}$. Then γ defines a matrix valued regular function F on $R(\pi_1(X,x), GL_n)$ (i.e.: $F \in \mathrm{Mat}_{n\times n}(\mathbb{C}[R(\pi_1(X,x), GL_n)])$) which reduces to the constant function with value I_n on $T_\rho \subset R(\pi_1(X,x), GL_n)$ for every element $\rho \in M^{VHS}$. Goldman-Millson theory implies that the tautological representation $\pi_1(X,x) \to GL_n(\widehat{\mathcal{O}_\rho})$ is conjugate to the pull back by cGM of ρ_T^{cGM}. Hence F induces the trivial matrix valued function when reduced to $Spf(\widehat{\mathcal{O}_\rho})$. Hence F induces the constant matrix valued function with value I_n on some complex analytic neighborhood of M^{VHS}.

Let $\widetilde{\rho}$ be a semisimple complex representation mapping to $M - M^{VHS}$. Then, by [Sim88], $\widetilde{\rho}$ correspond to a polystable Higgs bundle $(\mathcal{E}, \theta)$. For $t \in \mathbb{C}^*$, let $\tilde{\rho}(t)$ corresponds to $(\mathcal{E}, t.\theta)$. By applying the Goldman-Millson construction to each $\rho\tilde{(t)}$, we get a real analytic family of flat connections $(D_t)_{t\in\mathbb{C}^*}$ on the smooth vector bundle underlying $\mathcal{E} \otimes \mathcal{O}_{\tilde{\rho}(t)}$ (see for instance [Pri06, pp. 21]) such that the image F_t of the matrix function F in the complete local ring at $\tilde{\rho}(t)$, satisfies $F_t = \mathrm{hol}(D_t) \in \mathrm{Mat}_{n\times n}(\mathcal{O}_{\tilde{\rho}(t)})$. Since $F_t = I_n$ for small t then $F_1 = I_n$. Hence F maps to I_n in $\mathrm{Mat}_{n\times n}(\mathbb{C}[(\Gamma, GL_n)]_{\tilde{\rho}})$. Hence $F = I_n$ in a complex analytic neighborhood of the set of semisimple representations.

Given a non semi simple representation ρ^{arb} we may find a sequence $(\rho_m)_{m\in\mathbb{N}}$ of conjugate representations converging to a semi simple one we see that $\rho_m(\gamma) = Id_n$ for $m \gg 0$ hence $\rho^{arb}(\gamma) = Id_n$ one concludes that $F = I_n$ or in other

words that γ lies in the kernel of every representation $\pi_1(X) \to GL_n(A)$, for an arbitrary $\mathbb{C}$-algebra A. In particular $\gamma \in H_n$. $\square$

Corollary 3.11 *Assume $\pi_1(X,x)$ has a faithful representation in $GL_n(\mathbb{C})$. Then $\widetilde{H_M^\infty} = \{1\}$.*

4 Rationality lemma

4.1 Some pure Hodge substructures attached to an absolute closed set M and a fiber of sh_M

Let $f : Z \to X$ be a morphism and $M \subset M_B(X,G)$ an absolute closed subset.

For $\mathbb{V}$ be an object of $\mathcal{T}_M$, we denote by $\mathrm{tr} : \mathbb{V} \otimes \mathbb{V}^* \to \mathbb{C}$ the natural contraction. Consider the subspace $P_{\mathbb{V}}(Z/X) \subset H^1(Z,\mathbb{C})$ defined by:

$$P_{\mathbb{V}}(Z/X) := \mathrm{Im}\left[f^*H^1(X,\mathbb{V}) \otimes H^0(Z,\mathbb{V}^*) \xrightarrow{\cup} H^1(Z,\mathbb{V}\otimes\mathbb{V}^*) \xrightarrow{\mathrm{tr}} H^1(Z,\mathbb{C})\right].$$

In this formula, we denoted by $\mathbb{V}$ the local system on Z defined as $f^*\mathbb{V}$. Obviously, no confusion can arise from this slight abuse of notation.

Definition 4.1 *We also define $P_M(Z/X), \overline{P}_M(Z/X) \subset H^1(Z,\mathbb{C})$ as follows:*

$P_M(Z/X) \subset H^1(Z,\mathbb{C})$: the subspace of $H^1(Z,\mathbb{C})$ spanned by the $P_{\mathbb{V}}(Z/X)$, when $\mathbb{V}$ runs over all objects in $\mathcal{T}_M^{VHS}$.

$\overline{P}_M(Z/X) \subset H^1(Z,\mathbb{C})$: the subspace of $H^1(Z,\mathbb{C})$ spanned by the $P_{\mathbb{V}}(Z/X)$, when $\mathbb{V}$ runs over all objects in $\mathcal{T}_M$.

$H^1(Z,\mathbb{C})$ is defined over $\mathbb{Z}$ since it is the complexification of $H^1_{sing}(Z,\mathbb{Z})$. This Betti integral structure is the one we will tacitly use.

Lemma 4.2 *$P_M(Z/X)$ is a pure $\mathbb{C}$-Hodge substructure of weight one of the $\mathbb{C}$-MHS underlying Deligne's MHS on $H^1(Z,\mathbb{C})$.*

Proof: Since each $\mathbb{V}$ is a $\mathbb{C}$-VHS of weight zero, and X is smooth, it follows that $H^1(X,\mathbb{V})$ is a pure $\mathbb{C}$-Hodge structure of weight one. Also by [Del71-75] the mixed Hodge structures on the cohomology of varieties with coefficients in variations of Hodge structures are functorial and hence $P_{\mathbb{V}}(Z/X)$ is a $\mathbb{C}$-Hodge substructure of $H^1(Z,\mathbb{C})$. Finally by strictness [Del71-75] the span $P_M(Z/X)$ of the $P_{\mathbb{V}}(Z/X)$'s will also be pure and of weight one. $\square$

Lemma 4.3 *If G is defined over $\mathbb{Q}$ and that the absolutely closed subset $M \subset M_B(X,G)$ is defined over $\mathbb{Q}$, $\overline{P}_M(Z/X)$ is defined over $\mathbb{Q}$*

Assume now that $f(Z)$ is contained in a fiber of the reductive Shafarevich morphism for M or that equivalently a finite étale cover of Z lifts to a compact analytic subspace of $\widetilde{X_M}$. Then after a finite etale cover we may assume that $f_*\pi_1(Z,z) \subset H_M$, ie that every object ρ in $\mathcal{T}_M$ satisfies $\rho(\pi_1(Z,z)) = \{e\}$. The rationality lemma is the following statement:

Theorem 4.4 *Assume G is defined over $\mathbb{Q}$ and $M = M_B(X, G)$. Assume that $f_*\pi_1(Z, z) \subset H_M$. If $\pi_1(Z) \to \Gamma_M$ is trivial then $P_M(Z/X) = \overline{P}_M(Z/X)$.*

Corollary 4.5 *If G and $M = M_B(X, G)$ are defined over $\mathbb{Q}$, then $P_M(Z/X)$ is also defined over $\mathbb{Q}$.*

The rest of this section will be devoted to the proof of theorem 4.4.

We will also assume $\dim M > 0$ since the result is obvious for an absolute closed subset consisting of isolated points. The proof will be done in several steps which reduce the general statement to special situations.

Remark 4.6 *It seems likely that Theorem 4.4 holds true for arbitrary absolute closed subsets defined over $\mathbb{Q}$. One basically needs to adapt [EysSim09] to this situation.*

4.2 Reduction to the smooth case

First we reduce to the case when Z is smooth. We need the following lemma:

Lemma 4.7 *$\overline{P}_M(Z/X)$ is a pure weight one substructure of Deligne's MHS on $H^1(Z)$.*

Proof: Let $\mathbb{V}$ be an object of $\mathcal{T}_M$ By [Sim97, Theorem 4.1] the space $H^1(X, \mathbb{V})$ carries a pure twistor structure of weight one. Furthermore by [Sim97, Theorem 5.2] the space $H^1(Z, \mathbb{V})$ carries a canonical mixed twistor structure and $f^*H^1(X, \mathbb{V}) \subset H^1(Z, \mathbb{V})$ is a twistor substructure. By functoriality $P_{\mathbb{V}}(Z/X) \subset H^1(Z, \mathbb{C})$ will be a pure weight one twistor substructure and hence the span $\overline{P}_M(Z/X) \sum_{\mathbb{V}} P_{\mathbb{V}}(Z/X) \subset H^1(Z, \mathbb{C})$ is a pure weight one twistor substructure of the mixed Hodge structure $H^1(Z, \mathbb{C})$. However the Dolbeault realization of $\overline{P}_M(Z/X)$ is clearly preserved by $\mathbb{C}^*$ since by assumption $\mathbb{C}^*$ leaves M^{Dol} invariant. Therefore $\overline{P}_M(Z/X)$ is a sub Hodge structure. □

In order to prove Theorem 4.4, since $P_M(Z/X) \subset \overline{P}_M(Z/X)$ is pure of weight one, it is enough to prove that $Gr_1^W P_M(Z/X) = Gr_1^W \overline{P}_M(Z/X)$. Hence, without a loss of generality, we can assume that Z is smooth.

4.3 Reduction to a finite number of local systems

Lemma 4.8 *There is a finite set S of objects of $\mathcal{T}_M^{VHS}$ such that whenever a morphism $Z \to X$ has the property $im[\pi_1(Z) \to \Gamma_M] = 0$ it follows that*

$$P_M(Z/X) = \sum_{\mathbb{V} \in S} P_{\mathbb{V}}(Z/X).$$

Similarly, there is a finite set $\overline{S}$ of objects of $\mathcal{T}_M$, so that $\overline{P}_M(Z/X) = \sum_{\mathbb{V} \in \overline{S}} P_{\mathbb{V}}(Z/X)$. Furthermore the set $\overline{S}$ can be chosen so that for any Higgs bundle (E, θ) corresponding to a $\mathbb{V} \in \overline{S}$ the $\mathbb{C}$-VHS associated to $\lim_{t \to 0}(E, t.\theta)$ belongs to S.

Proof: Consider $(S_\alpha)_\alpha$ a stratification of $Sh_M(X)$ by locally closed smooth algebraic subsets such that $s_\alpha := sh_M|_{(sh_M)^{-1}(S_\alpha)} : (sh_M)^{-1}(S_\alpha) \to S_\alpha$ is a topological fibration. Fix $p_\alpha \in S_\alpha$. Let $Z_\alpha = s_\alpha^{-1}(p_\alpha)$, let $Z_{\alpha,o}$ be a connected component and let $Z'_{\alpha,o} \to Z_{\alpha,o}$ be the topological covering space defined by $Z'_{\alpha,o} = \widetilde{Z^{univ}_{\alpha,o}}/\ker(\pi_1(Z_{\alpha,o}) \to \Gamma_M)$. $Z'_{\alpha,o} \to Z_{\alpha,o}$.

Since $H^1(Z_{\alpha,o}, \mathbb{C})$ is finite dimensional, it follows that a finite set S exists with the required properties for $Z = Z_{\alpha,o}$. Since the cohomology classes coming from X are flat under the Gauss Manin connection, this statement holds true for all fibers of s_α. Since every $f : Z \to X$ with the required properties factors through one of the $Z_{\alpha,o}$'s, the lemma follows. □

4.4 Hodge theoretical argument

From now on, we really need to assume that $M = M_B(X, G)$, and that G is defined over $\mathbb{Q}$.

Let A be a noetherian $\mathbb{C}$-algebra and $\rho_A : \pi_1(X, x) \to GL_N(A)$ be a representation. Let $\mathbb{V}_A$ be the local system of free A-modules attached to ρ_A and $\mathbb{V}_A^\vee = \mathrm{Hom}_A(\mathbb{V}_A, A)$ be the local system associated to ${}^t\rho^{-1}$. We define:

$$P(A) = \mathrm{Im}\left[H^1(X, \mathbb{V}_A) \otimes_A H^0(Z, \mathbb{V}_A^\vee) \to H^1(Z, \mathbb{C}) \otimes_{\mathbb{C}} A\right]$$

$P(A)$ is an A-submodule of the free A-module $H^1(Z, \mathbb{C}) \otimes_{\mathbb{C}} A$.

Let σ in M^{**} be a non-isolated point. In subsection 2.3, we recalled the construction and basic properties of $T_\sigma \subset R(\pi_1(X, x), G)$ a formal local subscheme which gives rise to a hull of the deformation functor of σ. In follows from [GolMi90], that this formal subscheme is actually the formal neigborhood of σ in an analytic germ $T^{an}_\sigma \subset R(\pi_1(X, x), G)$. If we decompose the reduced germ of T^{an}_σ into the union $T^{an,red}_\sigma = \cup_i T^{an,i}$ of its analytic irreductibile components, then we will denote by T^i the formal neighborhood of σ in $T^{an,i}$. The irreducible components of an analytic germ being in one to one correspondance with the irreducible components of the associated formal germ, it follows that $T^{red}_\sigma = \cup_i T^i$ is still the irreducible decomposition of the reduced formal local subscheme underlying T_σ. Note that T^i is an integral formal subscheme of T_σ and so its ideal $\mathfrak{P}^i$ is a minimal prime of $\widehat{\mathcal{O}}_{T_\sigma}$.

Lemma 4.9 *The weight and Hodge filtrations on $\widehat{\mathcal{O}}_{T_\sigma}$ induce on $\mathfrak{P}^i \subset \widehat{\mathcal{O}}_{T_\sigma}$ a sub-MHS structure.*

Proof: First observe that the minimal associated primes of the graded ring $Gr^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$ are graded ideals and also split subMHS of the split MHS on $Gr^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$ since the $Res_{\mathbb{C}|\mathbb{R}}\mathbb{C}^*$-action defining the Hodge decomposition is compatible with the ring structure.

By construction, there is a ring isomorphism $\widehat{\mathcal{O}}_{T_\sigma} \to Gr^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$. This ring isomorphism takes minimal associated primes to minimal associated primes. Hence, $Gr^{\mathfrak{m}^\bullet}\mathfrak{P}^i$ is a sub Hodge Structure of $Gr^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$.

There is no canonical choice for this isomorphism but it can be chosen in such a way that it respects the weight and Hodge filtrations - but not the three

filtrations. This implies that the trace of the Hodge filtration of $Gr^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$ on $Gr^{\mathfrak{m}^\bullet}\mathfrak{P}^i$ is the filtration induced by the trace of the Hodge filtration of $\widehat{\mathcal{O}}_{T_\sigma}$ on $\mathfrak{P}^i$. The anti Hodge filtration satisfies a similar statement.

These two facts imply that $\mathfrak{P}^i \subset \widehat{\mathcal{O}}_{T_\sigma}$ is a sub-MHS structure. □

Hence the complete local algebra $\widehat{O}_{T^i}$ carries a $\mathbb{C}$-MHS and $\rho_{\widehat{O}_{T^i}} : \pi_1(X,x) \to G(\widehat{O}_{T^i})$ is the monodromy of the local system $\mathbb{D}(\mathbb{V}_\sigma) \otimes_{\widehat{O}_{T_\sigma}} \widehat{O}_{T^i}$. Thanks to lemma 2.12 and lemma 4.9, this local system underlies a $\mathbb{C}$-VMHS whose weight filtration corresponds to the powers of the maximal ideal in $\widehat{O}_{T^i}$.

By, construction the tautological representation $\rho_{\mathcal{O}_{T^{an,i}}} : \pi_1(X,x) \to G(\mathcal{O}_{T^{an,i}})$ is a holomorphic family of representations parametrized by a reduced germ of complex space.

If there is a proper closed analytic subset $Z^i \subset T^{an,i}$ such that $\forall p \in T^{an,i} - Z^i$ the representation $\rho_{\mathcal{O}_{T^{an,i}}}(p)$ is a reductive representation, then the inclusion $f_*\pi_1(Z,z) \subset H_M$ implies that the restriction of $\rho_{\mathcal{O}_{T^{an,i}}}(p)$ to $\pi_1(Z,z)$ is trivial for $p \notin Z^i$. Hence the restriction of $\rho_{O_{T^{an,i}}}$ and $\rho_{\widehat{O}_{T^i}}$ to $\pi_1(Z,z)$ are trivial as well.

If not, then for each irreducible component $M' \subset M$ containing σ, take a component R' via σ of the preimage $\pi^{-1}(M') \in R(\pi_1(X,x),G)$ which dominates M'. Let $(R')^{red} \subset R'$ be its maximal reduced subscheme. Consider the semisimplification of the representation attached to the generic point of the subscheme $(R')^{red} \subset R(\pi_1(X,x),G)$. It is conjugate to a Zariski dense representation with values in with values in some $G' \subset G$, where G' is reductive over $\overline{\mathbb{Q}}$. But $\mathrm{Im}(M_B(X,G') \to M_B(X,G))$ is a closed acqc set and so $M' \subset \mathrm{Im}(M_B(X,G') \to M_B(X,G))$. So without a loss of generality we may replace G by G' and also replace T^{an}_σ by an analytic Goldman-Millson slice through σ in $R(\pi_1(X,x),G')$. With this new definition, the restriction of $\rho_{\widehat{O}_{T^i}}$ to $\pi_1(Z,z)$ is trivial too and the corresponding local system on Z is the constant local system $\mathbb{V}_\sigma \otimes_{\mathbb{C}} \widehat{O}_{T^i}$.

In particular, we have a canonical isomorphism of VMHS $\mathbb{V}^\vee_{\widehat{O}^i_T/\mathfrak{m}^k}|_Z \simeq \mathbb{V}^\vee_\sigma|_Z \otimes_{\mathbb{C}} \widehat{O}^i_T/\mathfrak{m}^k$. It now follows that, for all $k \in \mathbb{N}$:

$$P(\widehat{O}_{T^i}/\mathfrak{m}^k) = \mathrm{Im}(H^1(X, \mathbb{V}_{\widehat{O}^i_T/\mathfrak{m}^k}) \otimes_{\mathbb{C}} H^0(Z, \mathbb{V}^\vee_\sigma) \overset{H_{\widehat{O}^i_T/\mathfrak{m}^k}}{\to} H^1(Z,\mathbb{C}) \otimes_{\mathbb{C}} \widehat{O}_{T^i}/\mathfrak{m}^k).$$

$H_k := H_{\widehat{O}_{T^i}/\mathfrak{m}^k}$ preserves the natural Mixed Hodge structures.

Proposition 4.10 $P_k := P(\widehat{O}_{T^i}/\mathfrak{m}^k) \subset P_1 \otimes \widehat{O}_{T^i}/\mathfrak{m}^k \subset P_M(Z/X) \otimes \widehat{O}_{T^i}/\mathfrak{m}^k$.

Proof: If $k = 1$ this is trivial: by construction, $P_1 \subset P_M(Z/X)$. We now argue by induction and assume that the result holds for $k' < k$.

The representation $\rho_k = \rho_{\widehat{O}^i_T/\mathfrak{m}^k}$ underlies a variation of complex mixed Hodge structure $\mathbb{M}_k$ on X. The weight filtration is given by the powers of $\mathfrak{m}$. Since ρ_k is trivial on $\pi_1(Z)$ then its restriction to Z is the trivial VMHS $\mathbb{H} \otimes_{\mathbb{C}} \widehat{O}^i_T/\mathfrak{m}^k$ where $\mathbb{H}$ is some Hodge structure of weight zero (with a possibly

non trivial Hodge vector) and, on $\widehat{O}_{T^i}/\mathfrak{m}^k$, the weight filtration is described by the powers of $\mathfrak{m}$.

$$P_k = \mathrm{Im}\left[H^1(X, \mathbb{M}_k \otimes_{\mathbb{C}} \mathbb{H}) \xrightarrow{H_k} H^1(Z, \mathbb{C}) \otimes_{\mathbb{C}} \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

The weights of $\mathbb{M}_k$ are $0, \ldots, -k+1$. Consider the following diagram of MHS, in which the rows are exact:

$$\begin{array}{ccccc} H^1(X, W_{-k+1}\mathbb{M}_k \otimes_{\mathbb{C}} \mathbb{H}) & \longrightarrow & H^1(X, \mathbb{M}_k \otimes_{\mathbb{C}} \mathbb{H}) & \longrightarrow & H^1(X, \mathbb{M}_{k-1} \otimes_{\mathbb{C}} \mathbb{H}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(Z) \otimes \mathfrak{m}^{k-1}/\mathfrak{m}^k & \longrightarrow & H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k & \longrightarrow & H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^{k-1} \end{array}$$

Remember we assume Z to be smooth. The weights of the MHS in the first row are $2-m$, in the second $2-m, ..., 1$, in the third one $3-m, ..., 1$. Hence the second line is just the canonical exact sequence

$$W_{2-k}\left[H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right] \longrightarrow H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k \longrightarrow Gr^W_{3-k}\left[H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

The main observation is now that, by strictness, we have:

$$W_{2-k}P_k = Im\left[H^1\left(X, W_{-k+1}\left(\mathbb{M}_k \otimes_{\mathbb{C}} \mathbb{H}\right)\right) \longrightarrow H^1(Z) \otimes \widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

From this it follows that $W_{2-k}P_k \in P_1 \otimes \mathfrak{m}^{k-1}/\mathfrak{m}^k$. By induction, $P_{k-1} \subset H^1(Z) \otimes \widehat{O}_{T^i}/\mathfrak{m}^{k-1} \subset P_1 \otimes \widehat{O}_{T^i}/\mathfrak{m}^{k-1}$. But P_k (respectively P_{k-1}) is the image of the map in the third column (resp. the second). It follows that $P_k \subset P_1 \otimes \widehat{O_{T^i}}/\mathfrak{m}^k$.

□

4.5 Proof of theorem 4.4 if $M = M_B(X, G)$

It follows from proposition 4.10 that:

$$P(\mathcal{O}_{T^{an,i}}) \subset P_M(Z/X) \otimes \mathcal{O}_{T^{an,i}}.$$

It follows that for all p in the complex analytic germ $T^{an,i}$ we have:

$$P_{\mathbb{V}_{\rho(p)}}(Z/X) \subset P_M(Z/X).$$

Since there is a complex analytic neighborhood U of σ in M such that every point of U has a (semisimple) representative in $T^{an,i}$, it follows that for every $\mathbb{V} \in U$ we have $P_{\mathbb{V}}(Z/X) \subset P_M(Z/X)$.

Now let $\overline{S}$ be the finite set from Lemma 4.8. Suppose $\mathbb{V} \in \overline{S}$ with an associated Higgs bundle (E, θ), and let $(\mathbb{V}_t)_{t\in\mathbb{C}^*}$ be the local systems corresponding to the Higgs bundle $(E, t\theta)$. For a small enough t we have:

$$P_{\mathbb{V}_t}(Z/X) \subset P_M(Z/X)$$

Fix t small enough and non zero. It follows that $\dim(\sum_{\mathbb{V}\in\bar{S}} P_{\mathbb{V}_t}(Z/X)) \leq \dim P_M(Z/X)$.

Consider:

$$P^{Dol}_{\mathbb{V}_t} = \mathrm{Im}\left[H^1_{Dol}(X, \mathbb{V}_t) \otimes H^0_{Dol}(Z, \mathbb{V}_t^*) \xrightarrow{\mathrm{Id}\otimes\mathrm{tr}} H^1_{Dol}(Z)\right].$$

Using Simpson's Dolbeault isomorphism we have:

$$\dim\left(\sum_{\mathbb{V}\in\bar{S}} P^{Dol}_{\mathbb{V}_t}(Z/X)\right) \leq \dim P_M(Z/X).$$

Recall there is a natural isomorphism $s(t) : H^\bullet_{Dol}(-, \mathbb{V}) \to H^\bullet_{Dol}(-, \mathbb{V}_t)$. Let (E, θ) be a polystable Higgs bundle representing $\mathbb{V}$. Then $H^\bullet_{Dol}(X, \mathbb{V}) := \mathbb{H}^\bullet(X, (E \otimes \Omega^\bullet_X, \theta))$. We can construct an quasi-isomorphism $(E \otimes \Omega^\bullet_X, \theta) \to (E \otimes \Omega^\bullet_X, t.\theta)$ by the formula:

$$\begin{array}{ccccccc}
E & \xrightarrow{\theta} & E\otimes\Omega^1_X & \xrightarrow{\theta\wedge} & E\otimes\Omega^2_X & \xrightarrow{\theta\wedge} & \dots \\
\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t^2} & & \\
E & \xrightarrow{t\theta} & E\otimes\Omega^1_X & \xrightarrow{t\theta\wedge} & E\otimes\Omega^1_X & \xrightarrow{t\theta\wedge} & \dots
\end{array}$$

Since $(E, t\theta)$ is the polystable Higgs bundle representing $\mathbb{V}_t$ this quasi isomorphism defines indeed an isomorphism $s(t) : H^\bullet_{Dol}(-, \mathbb{V}) \to H^\bullet_{Dol}(-, \mathbb{V}_t)$.

In case (E, θ) is kept fixed by $\mathbb{C}^*$ which means that there is an isomorphism $\psi(t) : (E, \theta) \to (E, t\theta)$, $a(t) = \psi(t)^{-1} \circ s(t)$ is an automorphism of $H^1_{Dol}(X, \mathbb{V})$ which comes from an action of $\mathbb{C}^*$. Here (E, θ) is not kept fixed by $\mathbb{C}^*$ but its restriction to Z is. This gives a diagram:

$$\begin{array}{ccc}
H^1_{Dol}(X, \mathbb{V}) & \longrightarrow & H^1_{Dol}(Z, \mathbb{V}) \\
\downarrow{\scriptstyle s(t)} & & \downarrow{\scriptstyle a(t)} \\
H^1_{Dol}(X, \mathbb{V}_t) & \longrightarrow & H^1_{Dol}(Z, \mathbb{V})
\end{array}$$

By functoriality and the definition of $P^{Dol}_{\mathbb{V}_t}(Z/X)$, we get a commutative diagram:

$$\begin{array}{ccc}
P^{Dol}_{\mathbb{V}}(Z/X) & \longrightarrow & H^1_{Dol}(Z) \\
\downarrow{\scriptstyle \tilde{s}(t)} & & \downarrow{\scriptstyle a(t)} \\
P^{Dol}_{\mathbb{V}_t}(Z/X) & \longrightarrow & H^1_{Dol}(Z)
\end{array}$$

and $\tilde{s}(t)$ is an isomorphism.

Hence $\dim(\sum_{\mathbb{V}\in\bar{S}} P_{\mathbb{V}}^{Dol}(Z/X)) \leq \dim P_M(Z/X)$. Since, by Simpson's Dolbeault isomorphism, the l.h.s is $\dim \overline{P}_M(Z/X)$ the theorem is proved.

5 Construction of the Shafarevich morphism

5.1 Preliminary considerations

5.1.1 Pure weight one rational subspaces of $H^1(Z)$

Let Z be a complex projective variety.

The possibly non zero Hodge numbers of Deligne's Mixed Hodge structure [Del71-75] on the first cohomology group $H^1(Z,\mathbb{Z})$ of the connected projective variety Z are $h^{0,0}, h^{0,1}, h^{1,0}$.

In particular, we have an extension of $\mathbb{Q}$-MHS of a pure weight one HS by a pure weight zero HS:

$$0 \to W_0(H^1(Z,\mathbb{Q})) \to H^1(Z,\mathbb{Q}) \to Gr_1^W(H^1(Z,\mathbb{Q})) \to 0. \tag{1}$$

Let $Z^{sn} \to Z$ be the seminormalisation of Z (see [Kol96] Chap. I Definition 7.2.1, p. 84 and the original references therein) $H^1(Z) \to H^1(Z^{sn})$ is an isomorphism of MHS since $Z^{sn}(\mathbb{C}) \to Z(\mathbb{C})$ is an homeomorphism [Kol96] I.(7.2.1.1).

Let $\mathcal{A}$ be a pure weight one $\mathbb{Q}$-HS. There is an abelian variety (well defined up to isogeny) such that $H^1(A,\mathbb{Q}) = \mathcal{A}$.

Lemma 5.1 *Let $\widetilde{\phi} : \mathcal{A} \to H^1(Z,\mathbb{Q})$ be a morphism of MHS. Then there exists a rational number $d \neq 0$ and a morphism $\psi : Z^{sn} \to A$ such that $d\widetilde{\phi} = H^1(\psi)$.*

We may as well assume Z is seminormal. Assume moreover that Z is a curve. Consider more generally $\phi : \mathcal{A} \to Gr_1^W H^1(Z)$ a morphism $\mathbb{Q}$-HS of pure weight one.

Pulling back the extension (1) by the morphism ϕ defines a extension of $\mathbb{Q}$-MHS

$$0 \to W_0(H^1(Z,\mathbb{Q})) \to \mathcal{A}' \to \mathcal{A} \to 0. \tag{2}$$

Proof: Let $\nu : Z^\nu \to Z$ be the normalisation of Z. Thanks to [Del71-75] lemme 10.3.1. the extension (1) is isomorphic to

$$0 \to W_0 \to H^1(Z) \xrightarrow{\nu^*} H^1(Z^\nu) \to 0.$$

Let $\gamma : [0,1] \to Z$ be a loop which is based at a singular point, meets the singular locus of Z at finitely many points and is smooth outside these points. The preimage of γ in Z^ν γ is a finite union $\gamma_1, \ldots, \gamma_n$ of paths possibly lying in several connected components of Z^ν. This defines a linear form $\int_\gamma : \omega \mapsto$

$\sum_i \int_{\gamma_i} \omega$ on $H^0(Z^\nu, \Omega^1)$ and, upon composition with ϕ, a linear form $\phi^* \int_\gamma$ on $\mathcal{A}^{1,0}$.

It follows from [Car87] theorem (1.13) -see also the enlightening example (1.17) - that (2) is split if and only if for every γ as above $\phi^* \int_\gamma$ is a rational multiple of a period of $\mathcal{A}$, i.e. lies in the image of $H_1(A, \mathbb{Q})$.

The datum $\widetilde{\phi}$ gives actually such a splitting and the Abel-Jacobi construction gives a continuous mapping $Z \to A$ with the required property which is holomorphic when pulled back to Z^ν. Since Z is seminormal this continuous mapping actually underlies a morphism.

The general case readily follows from the curve case. Assume first Z is irreducible. Let $\lambda : Z^\nu \to Z$ be the normalisation of Z. Then we can construct a morphism $\psi^\nu : Z^\nu \to A$ and an integer d such that $dH^1(\lambda) \circ \tilde{\phi} = H^1(\psi)$. This morphism is locally constant on the fibers of $\lambda : Z^\nu \to Z$. On the other hand we can always find a connected curve C passing through each connected component of a given positive dimensionnal fiber F of λ. Consider $C^{sn} \to C$ the seminormalization of C. This is a homeomorphism which identifies $H^1(C)$ and $H^1(C^{sn})$ with their respective Mixed Hodge structures. The morphism $\psi^\nu|_C : C^{sn} \to A$ is isogenous to the one predicted by lemma 5.1 applied to C and the resulting $\tilde{\phi}_C : \mathcal{A} \to H^1(C)$. Let C' be the image of C in Z. Since $\tilde{\phi}_C$ factors through $H^1(C')$ it follows that $\psi^\nu|_C$ is constant on the finite fiber of $C^{sn} \to C'$. Hence ψ^ν assume the same value on all connected components of F. Hence it descends to morphism $\psi : Z \to A$ since Z is seminormal.

In general, Z has m irreducible components, there are $m-1$ constants of integration to take care of and a connected curve in Z meeting every connected component of the smooth locus do the bookkeeping. □

5.1.2 Period mappings for $\mathbb{C}$-VMHS

$\mathbb{R}$-MHS have period domains and $\mathbb{R}$-VMHS period mappings generalizing those constructed by Griffiths for $\mathbb{R}$-VHS, [Usu83], see also [Car87] [3].

Recall that X is a complex projective manifold and let $(X, \mathbb{V}, \mathcal{F}^\bullet, S)$ be a $\mathbb{R}$-VHS of weight zero and let M be the real Zariski closure of its monodromy group computed at some basepoint $x \subset X$. Let $U \subset M$ be the isotropy group of the Hodge filtration on $\mathbb{V}_x$. Then the period domain of $\mathbb{V}$ is the complex manifold $D(\mathbb{V}) := M/U$. It is endowed with a certain horizontal distribution which can be described in terms of the Hodge structure on the Lie algebra $\mathfrak{m}$ of M. It is actually the actually the period domain attached to the Hodge semisimple group M^{ad}. Furthermore $D(\mathbb{V})$ is a moduli space of Hodge structures on M, see [GriSch69] for more details.

Let $(X, \mathbb{V}, \mathbb{W}_\bullet, \mathcal{F}^\bullet, (S_k)_{k\in\mathbb{Z}})$ be a $\mathbb{R}$-VMHS. Again we have a period domain $MD(\mathbb{V})$ for this variation and a holomorpic fibration of period domains $\psi : MD(\mathbb{V}) \to \prod_k D(Gr_k^{\mathbb{W}}\mathbb{V})$ which is compatible to the horizontal distributions.

[3] Actually, $\mathbb{C}$-VMHS have also period mappings of their own but since this would not give additional information, we will stick to the usual conventions used in the litterature

The domain $MD(\mathbb{V})$ is a homogenous space of the form H/U', where H is the subgroup of $W_0GL(\mathbb{V}_x)$ mapping to $\prod_k M(S_k)$ under the natural surjection $W_0GL(\mathbb{V}_x) \to GL(Gr_0^{\mathbb{W}}\mathbb{V}_x)$.

Accordingly there is an equivariant holomorphic horizontal period mapping $\phi_{\mathbb{V}} : \widetilde{X^{univ}} \to MD(\mathbb{V})$ with a commutative diagram:

$$\begin{array}{ccc} \widetilde{X^{univ}} & \xrightarrow{\phi_{\mathbb{V}}} & MD(\mathbb{V}) \\ & \searrow & \downarrow \psi \\ & & \prod_k D(Gr_k^{\mathbb{W}}\mathbb{V}). \end{array}$$

Let $\mathbb{M}$ be a $\mathbb{R}$-VMHS of weights -1 , 0 and MD be its period domain. Let D be product of the the period domains corresponding to the graded parts of $\mathbb{M}$. The map $MD \to D$ is then an affine bundle.

The following lemma can be extracted from [Car87, p. 200].

Lemma 5.2 *$MD \to D$ is a holomorphic vector bundle.*

The fiber $V(H_{-1}, H_0)$ of $MD \to D$ at (H_{-1}, H_0) is canonically isomorphic to $\mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}}/F^0$ where $\mathrm{Hom}(H_0, H_{-1})$ is endowed of its natural Hodge structure of weight -1.

Consider $f : Z \to X$ a morphism such that $f^*Gr_i\mathbb{M}$ is a VHS with trivial monodromy. Let $P_{\mathbb{M}} = \sum_i P_{f^*Gr_i\mathbb{M}}(Z/X) \subset H^1(Z)$, then we have the following lemma:

Lemma 5.3 *There is a commutative diagram*

$$\begin{array}{ccc} \widetilde{Z^{univ}} & \to & (P_{\mathbb{M}}^{1,0})^* \\ & \searrow & \downarrow g_{\mathbb{M}} \\ & & V(H_{-1}, H_0) \end{array}$$

where $g_{\mathbb{M}}$ is linear and injective and, when Z is smooth, the horizontal map is given by integration of closed holomorphic forms.

Proof: The proof is straightforward and is left to the reader. The case when $\mathbb{V} = \mathbb{C}$ is standard and the general case follows by the same reasoning. □

5.2 Proof of Theorem 1

5.2.1 Notations

In what follows, $M = M_B(X, G)$ where G is a reductive group defined over $\mathbb{Q}$.

Lemma 5.4 *There exists an object $\mathbb{M}_1$ of $\mathcal{T}_M^{MVHS}(1)$ such that for every $f : Z \to X$ for which $\pi_1(Z) \to \Gamma_M$ is trivial, we have that $P_{\mathbb{M}_1} = P_M(Z/X)$ and that $g_{\mathbb{M}_1}$ is injective*

Proof: Take

$$\mathbb{M}_1 := \sum_{\sigma \in S} \left(\mathbb{D}_1(\mathbb{V}_\sigma) + \overline{\mathbb{D}_1(\mathbb{V}_\sigma)} \right),$$

where $S \subset \mathcal{T}_M^{VHS}$ is the finite set constructed in lemma 4.8, and $\mathbb{D}_1(\mathbb{V}_\sigma)$ is the $\mathbb{C}$-VMHS from Definition 2.11. $\square$

Let $\widetilde{X_M^k}$ be the covering space of X defined as $\widetilde{X^{univ}}/\widetilde{H_M^k}$. This covering is Galois with Galois group Γ_M^k.

Consider the local systems that belong to the finite set S in $\mathcal{T}_M^{VHS}$ from lemma 4.8. Without loss of generality we may assume that they underly real VHS of weight zero. Every ρ in S underlies a Zariski dense representation $\pi_1(X) \to G_\rho$ where G_ρ is a real Lie group of Hodge type. Let $\rho_S : \pi_1(X) \to G_S = \prod_{\rho \in S} G_\rho$ be the direct sum representation.

5.2.2 Construction of the Shafarevich morphism in case $k = 1$

In this paragraph, we assume that $k = 1$. Choose a finite dimensionnal real representation as in lemma 5.4 of $\mathcal{G}_S^1(\mathbb{R})$ such that the associated local system $\mathbb{W}(1)$ underlies a graded polarizable real variation of mixed Hodge structure with the finite weight filtration: $0 = \mathbb{W}_{-2} \subset \mathbb{W}_{-1} \subset \mathbb{W}_0 = \mathbb{W}(1)$.

Associated with $\mathbb{W}(1)$, we have a holomorphic Griffiths' transversal period mapping $q_S^1 : \widetilde{X^{univ}} \to \mathcal{D}_S^1$ where $\mathcal{D}_S^1$ is is the corresponding period domain for MHS. The period domain $\mathcal{D}_S^1$ has a holomorphic fibration $\pi : \mathcal{D}_S^1 \to \mathcal{D}_S$ which makes it an affine fibration over the period domain $\mathcal{D}_S$. The composition $\pi \circ q_S^1$ is the period mapping for the associated graded object of $\mathcal{T}_M^{VHS}$.

The map q_S^1 factors through a holomorphic horizontal map $Q_M^1 \widetilde{X_M^1} \to \mathcal{D}_S^1$.

Consider the holomorphic map $q_S : \widetilde{X_M^1} \stackrel{Q_M^1 \times Sh_M}{\longrightarrow} \mathcal{D}_S^1 \times Sh_M(X)$.

Lemma 5.5 *Every connected component of a fiber of q is compact.*

Proof: Such a component Φ is contained in the lift of some fiber Z of $X \to Sh_M(X)$. Replacing Z by an etale cover, we may assume $\rho(\pi_1(Z)) = \{e\}$ whenever the conjugacy class of the reductive representation ρ is in M.

Hence Φ is a connected component of a fiber of the map q' defined as q_S restricted to $\widetilde{Z_M^1} = \widetilde{Z^{univ}}/\ker(\pi_1(Z) \to \widetilde{\Gamma_M^1})$.

Now $\pi \circ q'$ is the constant map and Φ is a connected component of a fiber of an holomorphic map $\psi : \widetilde{Z_M^1} \to V$ where V is a complex vector space which is a fiber of π.

Apply lemma 5.1 to $X = Z$ and $\mathcal{A} = P_M(Z/X)$. The rationality hypothesis is fulfilled thanks to Theorem 4.4. We find a map to an abelian variety $Z^{sn} \to A$ and using $P_M^{1,0}(Z/X)^* \to A$ the universal covering space of A a proper holomorphic map $\psi' : \widetilde{Z_M^1}^{sn} \to P_M^{1,0}(Z/X)^*$.

Our claim follows from the fact that we have a commutatative diagram:

$$\begin{array}{ccc} \widetilde{Z_M^1}^{sn} & \stackrel{\psi}{\longrightarrow} & P_M^{1,0}(Z/X)^* \\ s \downarrow & & i \downarrow \\ \widetilde{Z_M^1} & \stackrel{\psi}{\longrightarrow} & V \end{array}$$

Where s is the seminormalisation and i an injective linear map.

□

Next, recall the following classical result:

Lemma 5.6 *([Car], vol 2, pp. 797-811) Let X, S be two complex spaces and $f : X \to S$ a morphism. Assume a connected component F of a fiber of f is compact. Then, F has a neighbourhood V such that $g(V)$ is a local analytic subvariety of S and $V \to g(V)$ is proper.*

Assume furthermore any connected component of a fiber of f is compact and X and S are normal. Then, the set $\bar{S}$ of connected components of a fiber of f can be endowed with a structure of normal complex space such that the quotient mapping $e : X \to \bar{S}$ is holomorphic, proper, with connected fibers.

Using this lemma, we construct a surjective proper holomorphic mapping with connected fibers to a normal complex space $r^1_M : \widetilde{X^1_M} \to \widetilde{S^1_M}(X)$ such that its fibers are precisely the connected components of the fibers of q. Since q is Γ^1_M-equivariant it folllows that r^1_M is Γ^1_M-equivariant too. Note that Γ^1_M acts on $\widetilde{S}^1_M(X)$ in a proper discontinuous fashion and hence has at most finite stabilizers.

Lemma 5.7 *The fibers of r^1_M are precisely the maximal connected analytic subvarieties of $\widetilde{X^1_M}$.*

Proof: It is enough to show that whenever Z is a connected compact analytic subvariety of $\widetilde{X^1_M}$, r^1_M is constant. Fix such a Z.

The map $f : Z \to X$ has the property that the group homomorphism $\pi_1(Z) \to \Gamma^1_M$ induced by $\pi_1(f)$ has finite image. Let Z' be a connected étale cover of Z such that $\pi_1(Z') \to \Gamma^1_M$ is trivial. Abusing notation, let $f : Z' \to X$ be the resulting map. Then, for every representation ρ in M, $f^*\rho$ is trivial and for every object $\mathbb{V}$ of $\mathcal{T}^{VHS}_M$, the restriction map $H^1(X, \mathbb{V}) \to H^1(Z', \mathbb{V})$ is zero. This implies, through the proof of lemma 5.5 that q is constant on Z' and thus r^1_M. □

Remark. In fact it can be shown that $\widetilde{S^1_M}X)/\Gamma^1_M$ is a normal algebraic variety. This follows from recent work of G. Pearlstein but is not used in the the main theorem and so we will not discuss it here.

5.2.3 Stein property in the case $k = 1$

Proposition 5.8 *$\widetilde{X^1_M}$ is holomorphically convex and r^1_M is its Cartan-Remmert factorisation.*

Proof: Consider the natural period mapping $\widetilde{S_M}(X) \to \mathcal{D}_S$ and the affine bundle $V_S(X) = \widetilde{S_M}(X) \times_{\mathcal{D}_S} \mathcal{D}^1_S \to \widetilde{S_M}(X)$. The previous consideration imply that $\widetilde{S^1_M}(X) \to V_S(X)$ is proper and finite to one, hence finite.

Being an affine bundle over a Stein space $V_S(X)$ is Stein, hence $\widetilde{S^1_M}(X)$ is Stein. □

5.2.4 General case

Theorem 5.9 *Let $\widetilde{X} = \widetilde{X^{univ}}/\Gamma$ be a Galois covering space of X with $\widetilde{H^{\infty}_M} \subset \Gamma \subset \widetilde{H^1_M}$. Then $\widetilde{X}$ is holomorphically convex.*

Proof: Consider the map $q : \widetilde{X} \to \widetilde{S^1_M}(X)$.

We claim that every connected component Φ of a fiber of q is compact. Indeed Φ has to be a connected lift of a projective variety $Z \subset X$ which is mapped to a point in $S^1_M(X)$. Replacing Z by an étale cover, we may assume $\pi_1(Z) \to \Gamma^1_M$ is trivial hence $Im(\pi_1(Z) \to \pi_1(X)) \subset \Gamma$ by Proposition 3.6. This implies that Φ is compact.

In particular we may construct its Stein factorization $\widetilde{X} \to \widetilde{S}$ and it follows from the previous argument that $p : \widetilde{S} \to \widetilde{S^1_M}(X)$ has the following property:

Lemma 5.10 *Every point $x \in \widetilde{S^1_M}(X)$ has a neighbourhood U such that $p^{-1}(U)$ is the disjoint union of open sets V and $p|_V$ is a quotient map by a finite group G.*

This certainly implies that $\widetilde{S}$ is Stein.

In fact the finite group in question is $\ker(\pi_1((r^1_M)^{-1}(x)) \to \Gamma/\Gamma^1_M)$ and injects into the real points of a prounipotent proalgebraic group. It is thus a trivial group hence $\widetilde{S} \to \widetilde{S^1_M}(X)$ is a topological covering map. □

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