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Christian Duval, Péter A. Horvathy. Non-relativistic conformal symmetries and Newton-Cartan structures. 2009. hal-00373011v1

HAL Id: hal-00373011

<https://hal.science/hal-00373011v1>

Preprint submitted on 3 Apr 2009 (v1), last revised 26 Sep 2009 (v5)

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Non-relativistic conformal symmetries and Newton-Cartan structures

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3 April 2009

Abstract

Non-relativistic conformal infinitesimal transformations are derived directly from the structure of Galilei spacetime. They form, as originally found by Henkel et al., an infinite dimensional Virasoro-like Lie algebra. Its finite-dimensional subalgebras are labeled by the “dynamical exponent” $z = 2/q$, where q is some rational number. Viewed as projective Newton-Cartan symmetries, they yield, for timelike geodesics, the usual Schrödinger Lie algebra, with $z = 2$. For lightlike geodesics, they yield the Conformal Galilean Algebra of Lukierski, Stichel and Zakrzewski, with $z = 1$. The purpose of the present article is to provide a unifying classification of the various conformal infinitesimal symmetries of Newton-Cartan spacetime. Physical systems which realize these symmetries include, e.g., classical systems of massive and massless non-relativistic particles.

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1 Introduction

Non-relativistic conformal symmetries, which are attracting much present interest [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], are of two types.

Firstly, it has been recognized almost forty years ago [11, 13] that the free Schrödinger equation of a massive particle has, beyond the obvious Galilean symmetry, two more “conformal” symmetries. Those are generated by the “Schrödinger” *dilation*

$$D = 2t \frac{\partial}{\partial t} + x^A \frac{\partial}{\partial x^A} \tag{1.1}$$

and *expansion* (or inversion)

$$K = t^2 \frac{\partial}{\partial t} + tx^A \frac{\partial}{\partial x^A} \tag{1.2}$$

spacetime vector fields, where the dummy index A runs from 1 to d , the dimension of space.

Schrödinger dilations and expansions span, with time translations, $E = \partial/\partial t$, a Lie algebra isomorphic to $\mathfrak{so}(2, 1)$. Adding dilations and expansions to the Galilei group yields a two-parameter extension of the latter, dubbed as the [unextended] *Schrödinger group*, $\text{Sch}(d)$.¹

Using the word “conformal” has been contested [1], hinting at its insufficiently clear relation to some conformal structure. This criticism is only half-justified, however. The Schrödinger symmetry has in fact been related to the Newton-Cartan structure of non-relativistic spacetime [18, 19, 20, 21, 22], but this relation has remained rather confidential.

A different point of view was put forward in Ref. [23], where it has been shown that non-relativistic theories can be studied in a “Kaluza-Klein type” framework, whereas the “non-relativistic conformal” transformations appear as those, genuine, conformal transformations of a relativistic spacetime in one higher dimension, which commute with translations in the “vertical” direction. The latter provides us, furthermore, with the central extension required by the mass [24, 25]. See also [12].

Secondly, in [1, 2, 3, 8, 9, 10], however, attention has been directed to another, less-known aspect. It has been shown, in fact, that a specific group contraction, applied to the relativistic conformal group $O(d+1, 2)$, provides, for vanishing mass, $m = 0$, a *second type of conformal extension* of the Galilei group. Since group contraction does not change the number of generators, the new extension, called the *Conformal Galilean Group* has the same dimension as its relativistic counterpart. Its Lie algebra, the *Conformal Galilei Algebra* is spelled as the CGA in the above-mentioned reference. The CGA is spanned by the vector fields²

$$X = \left(\frac{1}{2}\kappa t^2 + \lambda t + \varepsilon \right) \frac{\partial}{\partial t} + \left(\omega_B^A x^B + \lambda x^A + \kappa t x^A - \frac{1}{2}\alpha^A t^2 + \beta^A t + \gamma^A \right) \frac{\partial}{\partial x^A} \quad (1.3)$$

with $\omega \in \mathfrak{so}(3)$, $\alpha, \beta, \gamma \in \mathbb{R}^3$, and $\lambda, \kappa, \varepsilon \in \mathbb{R}$. The new dilations and expansions, associated with λ and κ ; close, with time translations parametrized by ε , into an $\mathfrak{so}(2, 1)$ Lie subalgebra [1, 2, 3, 8], acting *differently* from that of the Schrödinger case: unlike the “Schrödinger” one, (1.1), the CGA dilation in (1.3) dilates space and time at the same rate. The vector α generates “accelerations” [1].

¹ The physical realizations of the Schrödinger group, in spatial dimension $d \geq 3$, admit one more parameter, associated with the *mass*. Adding it yields the *extended Schrödinger group*, which is the “non-relativistic conformal” extension of the one-parameter central extension of the Galilei group, called the *Bargmann group*. In the plane, $d = 2$, the Galilei group also has a second, “exotic”, central extension, widely studied during the last decade [14, 15, 16, 17].

²The central extensions of the CGA have been discussed in Refs. [1, 3].

The Lie algebra (1.3) can be further generalized [2, 3, 8], in terms of infinitesimal “time redefinition” and time-dependent translations,

$$X = h(t)\frac{\partial}{\partial t} + (h'(t)x^A + \beta^A(t))\frac{\partial}{\partial x^A} \quad (1.4)$$

where h , and β are arbitrary functions of time. The new expansions and accelerations are plainly recovered choosing $h(t) = \frac{1}{2}\kappa t^2$ and $\beta(t) = -\frac{1}{2}\alpha t^2$, respectively. Promoting the infinitesimal rotations, $X = \omega_B^A(t)x^B\partial_A$, to be also time-dependent yields an infinite dimensional conformal extension of the CGA.

The purpose of the present paper, a sequel and natural extension of earlier work devoted to Galilean isometries [22], is to trace back all these “conformal” symmetries to the *structure of non-relativistic spacetime*.

Our clue is to define *non-relativistic conformal transformations* in the framework of Newton-Cartan spacetime [26, 27, 29, 30], ideally suited to deal with those symmetries in a purely geometric way.

Both the Schrödinger and Conformal Galilean transformations turn out to be special cases, related to our choice of the relative strength of space and time dilations, characterized by a dynamical exponent [2, 3].

Our paper is organized as follows. After reviewing the Newton-Cartan structures — the structure of non-relativistic spacetime — we define Galilei conformal transformations. They only concern the (singular) “metric”, γ , and the “clock”, represented by a one-form θ . Their symmetries form infinite dimensional Virasoro-like Lie algebras.

Newton-Cartan structures also involve a connection, Γ , which is *not* entirely determined by the previous structures. Preserving the geodesic equations adds, for timelike curves, extra conditions, reducing the symmetry to that of Schrödinger for timelike geodesics. We get, in turn, for lightlike geodesics, novel conformal extensions of the [unextended] Galilei group. For rational dynamical exponent, we get finite-dimensional Lie algebras of the CGA type.

The general theory is illustrated on two examples, namely on Galilean massive and massless non-relativistic particle models.

2 Newton-Cartan structures

Recall that a Newton-Cartan (NC) spacetime structure, $(M, \gamma, \theta, \Gamma)$, consists of a smooth, connected, $(d + 1)$ -dimensional manifold M , a twice-contravariant semi-positive symmetric tensor field $\gamma = \gamma^{ab}\partial_a \otimes \partial_b$ (where $a, b = 0, 1, \dots, d$) whose

kernel is spanned by the time one-form $\theta = \theta_a dx^a$. Also Γ is a Galilei connection, i.e., a symmetric linear connection compatible with γ and θ [26, 18, 22, 29, 30, 33]. Now, in contradistinction to the relativistic framework, such a connection is *not* uniquely determined by the Galilei spacetime structure (M, γ, θ) . Therefore, in order to reduce the ambiguity, one usually introduces NC-connection as Galilei connections subject to the nontrivial symmetry of the curvature: $R_a{}^b{}_c{}^d = R_c{}^d{}_a{}^b$ (where $R_a{}^b{}_c{}^d \equiv \gamma^{bk} R_{akc}{}^d$); the latter may be thought of as part of the covariant Newtonian gravitational field equations [27, 29, 28, 30]. Under mild geometric conditions, the quotient $T = M/\ker(\theta)$ is a well-behaved one-dimensional manifold, either compact or non-compact, interpreted as the time axis endowed with the closed one-form θ , interpreted as the Galilei *clock*. The tensor field γ then defines a Riemannian metric on each of the (spacelike) fibers of the projection $M \rightarrow T$.

The standard example of a NC-structure is given by $M \subset \mathbb{R} \times \mathbb{R}^d$ together with $\gamma = \delta^{AB} \partial_A \otimes \partial_B$ (where $A, B = 1, \dots, d$), and $\theta = dx^0$; the nonzero components of the connection, $\Gamma_{00}^A = \partial_A V$, host the Newtonian scalar potential, V . The above coordinate system (x^0, \dots, x^d) will be called Galilean.

The *flat* NC-structure corresponds to the trivial subcase where $M = \mathbb{R} \times \mathbb{R}^d$, and

$$\gamma^{ab} = \delta_A^a \delta_B^b \delta^{AB}, \quad \theta_a = \delta_a^0, \quad \Gamma_{ab}^c = 0 \quad (2.1)$$

for all $a, b, c = 0, \dots, d$. Such a coordinate system will be called (NC-)inertial.

Since we will be dealing with “conformal” Galilean spacetime transformations that preserve the directions of the Galilei structure, we must bear in mind that the transformation law of the NC-connection, Γ , will have to be specified independently of that of the Galilei “metric” (γ, θ) , which is clearly due to the fact that there are extra degrees of freedom associated with NC-connections. Let us, hence, describe the precise geometric content of NC-connections.

It has been shown [29] that any NC-connection can be decomposed according to³

$$\Gamma_{ab}^c = {}^U\Gamma_{ab}^c + \theta_{(a} F_{b)k} \gamma^{kc} \quad (2.2)$$

where [39]

$${}^U\Gamma_{ab}^c = \gamma^{ck} \left(\partial_{(a} {}^U\gamma_{b)k} - \frac{1}{2} \partial_k {}^U\gamma_{ab} \right) + \partial_{(a} \theta_{b)} U^c \quad (2.3)$$

is the unique NC-connection for which the unit spacetime vector field U (i.e., such that $\theta_a U^a = 1$) is geodesic and curlfree, F being an otherwise arbitrary closed two-form. Here ${}^U\gamma$ is the symmetric, twice-covariant, tensor field uniquely determined

³Round brackets denote symmetrization.

by ${}^U\gamma_{ak}\gamma^{kb} = \delta_a^b - U^b\theta_a$ and ${}^U\gamma_{ak}U^k = 0$. From a mechanical standpoint, the above two-form, F , of M encodes Coriolis-like accelerations relatively to the observer U .

For example, if $M \subset \mathbb{R} \times \mathbb{R}^3$, the constant, future-pointing, vector field $U = \partial_0$ will represent the four-velocity of an observer. Now, F being closed, one has, locally, $F = dA$ for some one-form A , e.g., $A = -V(t, \mathbf{x})dt + \omega(t)_{BC} x^B dx^C$, where $V(t, \mathbf{x})$ is the Newtonian (plus centrifugal) potential, and $\omega(t) \in \mathfrak{so}(3)$ the time-dependent angular velocity of the observer relatively to the Galilei frame associated with the coordinates $t = x^0$, and $\mathbf{x} = (x^1, x^2, x^3)$. Anticipating the introduction of the equations of free fall, we check that the equations of NC-geodesics (4.1) with the choice of time, t , as an affine parameter, yield, with the help of (2.2), the familiar equations

$$\ddot{t} = 0, \quad \ddot{\mathbf{x}} = -\mathbf{grad}(V) + \dot{\omega} \times \mathbf{x} + 2\omega \times \dot{\mathbf{x}} \quad (2.4)$$

governing the motion of a massive particle in a rotating Galilei coordinate system.⁴

We have thus seen that, in view of (2.2), we can usefully parametrize NC-connections, Γ , by the above pairs (U, F) .

Let us mention [29, 30] that for a given, *fixed*, Galilei structure (γ, θ) , the pair (U', F') defines the same NC-connection, Γ , as (U, F) does iff both are *gauge*-related by the so-called *Milne boosts*⁵

$$U' = U + \gamma(\Psi), \quad F' = F + d\Phi \quad (2.5)$$

where $\Psi = \Psi_a dx^a$ is an arbitrary one-form of M , which may be interpreted as a boost, and $\Phi = \Phi_a dx^a$ is such that

$$\Phi_a = \Psi_a - \left(\Psi_b U^b + \frac{1}{2} \gamma^{bc} \Psi_b \Psi_c \right) \theta_a. \quad (2.6)$$

The infinitesimal versions of the preceding gauge transformations read, accordingly,

$$\delta U = \gamma(\psi), \quad \delta F = d\phi \quad (2.7)$$

where ψ is an arbitrary one-form of M (an infinitesimal boost), and

$$\phi = {}^U\gamma(\delta U). \quad (2.8)$$

One readily checks that, indeed, $\delta\Gamma = 0$.

⁴The non-trivial components of the NC-connection (2.2) read, in this case, $\Gamma_{00}^A = \partial_A V - \dot{\omega}_B^A x^B$, and $\Gamma_{B0}^A = -\omega_B^A$, for all $A, B = 1, 2, 3$.

⁵See [21, 22]. This special feature of NC-connections will play a central rôle in the definition of (lightlike) NC-conformal transformations; see Section 4.3 below.

3 Galilei-conformal transformations, Schrödinger-Virasoro Lie algebra

3.1 General definition of Galilei conformal transformations

In close relationship to the Lorentzian framework, we will call *Galilei-conformal* transformation of (M, γ, θ) any diffeomorphism of M that preserves the direction of γ . In view of the fundamental constraint $\gamma^{ab}\theta_b = 0$, it follows that Galilei-conformal transformation automatically preserve the direction of the time one-form θ .

In terms of infinitesimal transformations, a *Galilei-conformal* vector field of (M, γ, θ) is a vector field, X , of M that Lie-transport the direction of γ ; we will thus define $X \in \mathbf{cgal}(M, \gamma, \theta)$ iff

$$L_X \gamma = f \gamma \quad \text{hence} \quad L_X \theta = g \theta \quad (3.1)$$

for some smooth functions f, g of M , depending on X . Then, $\mathbf{cgal}(M, \gamma, \theta)$ becomes a Lie algebra whose bracket is the Lie bracket of vector fields.

The one-form being parallel transported by the NC-connection, one has $d\theta = 0$; this yields $dg \wedge \theta = 0$, implying that g is (the pull-back of) a smooth function on T , i.e., that g is an arbitrary function of $t = x^0$, which locally parametrizes the time axis.

Let us work out the expression of the generators of the *Galilei conformal Lie algebra*, $\mathbf{cgal}(d) = \mathbf{cgal}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta)$ of the flat NC-structure (2.1). Those are the vector fields, $X = X^0 \partial_0 + X^A \partial_A$, solutions of (3.1), namely such that⁶

$$\partial_A X_B + \partial_B X_A = -f \delta_{AB} \quad (3.2)$$

$$\partial_A X^0 = 0 \quad (3.3)$$

$$\partial_0 X^0 = g \quad (3.4)$$

for all $A, B = 1, \dots, d$. We readily find that $X \in \mathbf{cgal}(d)$ iff⁷

$$X = h(t) \frac{\partial}{\partial t} + \left(\omega_B^A(t) x^B + \beta^A(t) + \kappa^A(t) x_B x^B - 2x^A \kappa_B(t) x^B + \chi(t) x^A \right) \frac{\partial}{\partial x^A} \quad (3.5)$$

where $\omega \in \mathfrak{so}(d)$, β , κ , χ , and $h = X^0$ are arbitrary functions of time, t ; those are clearly interpreted as infinitesimal rotations, space translations, expansions (or inversions), dilations, and time reparametrization.

⁶Let us recall the general expressions of the Lie derivatives of γ and θ along the vector field $X = X^a \partial_a$ of M , namely $L_X \gamma^{ab} = X^c \partial_c \gamma^{ab} - 2\partial_c X^{(a} \gamma^{b)c}$, and $L_X \theta_a = \partial_a (\theta_b X^b)$.

⁷We will assume $d > 1$.

We note, *en passant*, that the $\mathbf{cgal}(d)$ -generators (3.5) project as vector fields of the time axis; therefore, there exists a canonical Lie algebra homomorphism: $\mathbf{cgal}(d) \rightarrow \text{Vect}(\mathbb{R})$ given by $X \mapsto h(t)\partial_t$, onto the Lie algebra of vector fields of $T \cong \mathbb{R}$, i.e., the (centerless) *Virasoro Lie algebra*.

3.2 Galilei conformal transformations with dynamical exponent z

One can, at this stage, try and seek non-relativistic avatars of general relativistic infinitesimal conformal transformations. Given a Lorentzian (ore, more generally, a pseudo-Riemannian) manifold (M, g) , the latter Lie algebra is generated by the vector fields, X , of M such that

$$L_X(g^{-1} \otimes g) = 0 \quad (3.6)$$

where g^{-1} denotes the inverse of the metric $g : TM \rightarrow T^*M$.

It has been shown [31] that one can expand a Lorentz metric in terms of the small parameter $1/c^2$, where c stands for the speed of light, as $g = c^2\theta \otimes \theta - U\gamma + \mathcal{O}(c^{-2})$, and $g^{-1} = -\gamma + c^{-2}U \otimes U + \mathcal{O}(c^{-4})$, with the same notation as before. Then, a non-relativistic limit of Equation (3.6) would be $L_X \lim_{c \rightarrow \infty} (c^{-2} g^{-1} \otimes g) = 0$, that is

$$L_X(\gamma \otimes \theta \otimes \theta) = 0. \quad (3.7)$$

This is merely one of the possibilities at hand in our formalism. In fact, having at our disposal a Galilei structure on M , we will introduce, instead of (3.7), a more flexible condition. Indeed, owing to the degeneracy of the Galilei “metric” (γ, θ) , we will deal with the following condition, viz.,

$$L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0 \quad (3.8)$$

for some $m = 1, 2, 3, \dots$, and $n = 0, 1, 2, \dots$, to be further imposed on the vector fields $X \in \mathbf{cgal}(M, \gamma, \theta)$. This is equivalent to Equation (3.1) together with the extra condition

$$f + qg = 0 \quad \text{where} \quad q = \frac{n}{m}. \quad (3.9)$$

Indeed, $L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0$ implies $L_X\gamma = f\gamma$ and $L_X\theta = g\theta$ for some functions f and g of M such that $mf + ng = 0$. Equation (3.7) plainly corresponds to the special case $m = 1, n = 2$. From now on, we will call *dynamical exponent* the quantity

$$z = \frac{2}{q} \quad (3.10)$$

where q is as in (3.9). This quantity will be shown to match the ordinary notion of dynamical exponent; see, e.g., [32, 2].

We will, hence, introduce the Galilean avatars, $\mathbf{cgal}_z(M, \gamma, \theta)$, of the Lie algebra $\mathfrak{so}(d+1, 2)$ of conformal vector fields of a pseudo-Riemannian structure of signature $(d, 1)$ as the Lie algebras spanned by the vector fields X of M satisfying the conditions (3.8) — or (3.9). We will call $\mathbf{cgal}_z(M, \gamma, \theta)$ the *conformal Galilei Lie algebra with dynamical exponent z* (see (3.10)).

The Lie algebra

$$\mathfrak{sv}(M, \gamma, \theta) = \mathbf{cgal}_2(M, \gamma, \theta) \quad (3.11)$$

is the obvious curved case generalization of the *Schrödinger-Virasoro* Lie algebra $\mathfrak{sv}(d) = \mathfrak{sv}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta)$ introduced in [32] (see also [2]) from a different viewpoint in the case of a flat NC-structure. The representations of the Schrödinger-Virasoro group and of its Lie algebra, $\mathfrak{sv}(d)$, as well as the deformations of the latter have been thoroughly studied and investigated in [34].

An easy calculation using (3.2), (3.4), the new constraint (3.9) and (3.10) shows that $X \in \mathbf{cgal}_z(d)$ iff

$$X = z h(t) \frac{\partial}{\partial t} + \left(\omega_B^A(t) x^B + \beta^A(t) + h'(t) x^A \right) \frac{\partial}{\partial x^A}, \quad (3.12)$$

where $\omega \in \mathfrak{so}(d)$, β , and h are arbitrary, smooth, functions of T . Equation (3.12) generalizes (1.4) from $z = 1$ to any z .

The Lie algebra $\mathbf{cgal}_\infty(M, \gamma, \theta)$ corresponding to the case $q = 0$ is interesting (see below, Section 5.2). We have, indeed, $X \in \mathbf{cgal}_\infty(M, \gamma, \theta)$ iff

$$L_X \gamma = 0. \quad (3.13)$$

In the case of a flat NC-structure, $\mathbf{cgal}_\infty(d) = \mathbf{cgal}_\infty(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta)$ is spanned by the vector fields

$$X = h(t) \frac{\partial}{\partial t} + \left(\omega_B^A(t) x^B + \beta^A(t) \right) \frac{\partial}{\partial x^A} \quad (3.14)$$

where $\omega \in \mathfrak{so}(d)$, β , and h are arbitrary, smooth, functions of T .

4 Newton-Cartan conformal transformations

4.1 Transformation law of NC-connections under Galilei-conformal rescalings

As previously emphasized, NC-connections are quite independent geometric objects; they, hence, deserve a special treatment. The idea pervading earlier work [18, 20, 22]

on non-relativistic symmetries is that specifying explicitly the transformation law of the NC-connection is mandatory in a number of cases, e.g., those relevant to geometric mechanics and non-relativistic physical theories.

We will, henceforth, focus attention on the notion of Newtonian geodesic; more particularly, we will insist that the above-mentioned Galilean conformal transformations should, in addition, permute the NC-geodesics.

The geodesics of a NC-structure $(M, \gamma, \theta, \Gamma)$ are plainly geodesics of (M, Γ) , i.e., the solutions of the differential equations

$$\ddot{x}^c + \Gamma_{ab}^c \dot{x}^a \dot{x}^b = \mu \dot{x}^c \quad (4.1)$$

for all $c = 0, \dots, d$, where μ is some smooth function of TM ; here, we have put $\dot{x}^a = dx^a/d\tau$ where τ is an otherwise arbitrary curve-parameter.

Let us remind that Equation (4.1) models *free fall* in NC theory [26, 27, 29], just as it does in general relativity. By putting $\dot{t} = \theta_a \dot{x}^a$, we characterize⁸

$$\textit{timelike} \text{ geodesics by: } \dot{t} \neq 0, \quad (4.2)$$

$$\textit{lightlike} \text{ geodesics by: } \dot{t} = 0. \quad (4.3)$$

We recall that spacetime transformations which permute the geodesics of (M, Γ) , i.e., preserve the form of the geodesic equation (4.1), are *projective* transformations; they form the projective group of the affine structure. Infinitesimal projective transformations generate a Lie algebra which, hence, consists of vector fields, X , of M satisfying

$$L_X \Gamma_{ab}^c = \delta_a^c \varphi_b + \delta_b^c \varphi_a \quad (4.4)$$

for a certain one-form $\varphi = \varphi_a dx^a$ of M depending on X .

The rest of the article will be devoted to the specialization of projective transformations to the specific case of timelike (4.2) and lightlike (4.3) NC-geodesics.

Let us now work out the general form of the variation, $\delta\Gamma$, of a NC-connection, Γ , under infinitesimal conformal rescalings of the Galilei structure (γ, θ) of M , namely

$$\delta\gamma = f\gamma \quad \text{hence} \quad \delta\theta = g\theta \quad (4.5)$$

where f is an arbitrary function of M , and g an arbitrary function of T (compare Equation (3.1)). We will furthermore put, in full generality,

$$\delta U = -gU + \gamma(\psi) \quad (4.6)$$

⁸The condition $\dot{t} = 0$ is clearly a first-integral of Equation (4.1). Lightlike geodesics are, hence, spacelike; the origin of the terminology will be explained later, in Section 5.

in order to comply with the constraint $\theta_a U^a = 1$, where ψ is an arbitrary one-form of M interpreted as an infinitesimal Milne boost (cf. (2.7)).

Starting from (2.2), we get $\delta\Gamma_{ab}^c = (\delta^U\Gamma)_{ab}^c + \delta\theta_{(a}F_{b)k}\gamma^{ck} + \theta_{(a}\delta F_{b)k}\gamma^{ck} + \theta_{(a}F_{b)k}\delta\gamma^{ck}$. Then, using (4.5) and (4.6) applied to the expression (2.3) of the NC-connection ${}^U\Gamma$, we find $\delta({}^U\Gamma)_{ab}^c = -\delta_{(a}^c\partial_{b)}f + U^c\theta_{(a}\partial_{b)}(f+g) + \frac{1}{2}(\gamma^{ck}\partial_k f){}^U\gamma_{ab} - \theta_{(a}d\phi_{b)k}\gamma^{ck}$, where $\phi = {}^U\gamma(\delta U)$ is the one-form (2.8) associated with the ‘‘Milne’’ variation (4.6) of the observer U . Then, with the help of Equation (4.5), and of general result $\delta F = d\phi$, see (2.7), we can finally claim that

$$\delta\Gamma_{ab}^c = -\delta_{(a}^c\partial_{b)}f + U^c\theta_{(a}\partial_{b)}(f+g) + \frac{1}{2}(\gamma^{ck}\partial_k f){}^U\gamma_{ab} + (f+g)\gamma^{ck}\theta_{(a}F_{b)k} \quad (4.7)$$

for all $a, b, c = 0, \dots, d$.

Equation (4.7) is of central importance in our study; it yields the general form of the variations of the NC-connection compatible with the constraints $\nabla_c\gamma^{ab} = 0$, and $\nabla_a\theta_b = 0$, and associated with the conformal Galilei rescalings (4.5).

4.2 The Schrödinger Lie algebra: timelike geodesics

Let us first cope with the case of generic, *timelike*, geodesics of (M, Γ) defined by $\dot{t} \neq 0$, cf. Equation (4.2), and representing the worldlines of *massive non-relativistic particles*. From now on, we choose to enforce preservation of their equations (4.1), in addition to that, (4.5), of the direction of the Galilei structure (γ, θ) .

Clearly, the variations (4.7) of the NC-connection will preserve the equations of these geodesics, i.e., will define infinitesimal projective transformations (4.4) of (M, Γ) , provided $\varphi = -df$, and $f+g=0$ in order to guarantee the vanishing of the fourth term (hence of the second term) in the right hand-side of (4.7). Now $f = -g$ depending on time only, we automatically find that $\gamma(df) = 0$, hence that the third term brings no contribution.

The variation (4.7) of the NC-connection will, hence, be an infinitesimal projective transformation of the NC-structure iff

$$f + g = 0 \quad (4.8)$$

$$\varphi + f'\theta = 0 \quad (4.9)$$

where f' stands for the (time-)derivative of the function f of T . This readily implies that [18, 20]

$$\delta\Gamma_{ab}^c = -f'\delta_{(a}^c\theta_{b)}. \quad (4.10)$$

The next natural step consists in demanding that the variations (4.5) of the Galilei structure, and those (4.10) of the NC-connection are, indeed, generated by

infinitesimal spacetime transformations. We will thus define a new Lie algebra, denoted $\mathfrak{sch}(M, \gamma, \theta, \Gamma)$, as the Lie algebra of those vector fields that are infinitesimal (i) Galilei-conformal transformation of (M, γ, θ) , and (ii) projective transformations of (M, Γ) . We call $\mathfrak{sch}(M, \gamma, \theta, \Gamma)$ the *Schrödinger Lie algebra*, which is therefore spanned by the vector fields, X , of M such that [18, 19, 20]

$$L_X \gamma^{ab} = -g \gamma^{ab}, \quad L_X \theta_a = g \theta_a \quad \& \quad L_X \Gamma_{ab}^c = g' \delta_{(a}^c \theta_{b)} \quad (4.11)$$

for all $a, b, c = 0, 1, \dots, d$.

The Schrödinger Lie algebra (4.11) is, indeed, a *finite-dimensional* Lie subalgebra of the Schrödinger-Virasoro Lie algebra (3.11), i.e.,

$$\mathfrak{sch}(M, \gamma, \theta, \Gamma) \subset \mathfrak{sv}(M, \gamma, \theta). \quad (4.12)$$

Let us now work out the form of the Schrödinger Lie algebra in the flat case. We will thus determine the generators of the Lie algebra $\mathfrak{sch}(d) = \mathfrak{sch}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta, \Gamma)$ in the special case (2.1).

The system (4.11) to solve for the vector fields $X = X^0 \partial_0 + X^A \partial_A$ therefore reads⁹

$$\partial_A X_B + \partial_B X_A = g \delta_{AB} \quad (4.13)$$

$$\partial_A X^0 = 0 \quad (4.14)$$

$$\partial_0 X^0 = g \quad (4.15)$$

$$\partial_0 \partial_0 X^A = 0 \quad (4.16)$$

$$\partial_0 \partial_B X^A = \frac{1}{2} g' \delta_B^A \quad (4.17)$$

$$\partial_A \partial_B X^C = 0 \quad (4.18)$$

for all $A, B, C = 1, \dots, d$. We deduce, from (4.18) that $X^A = M_B^A(t) x^B + C^A(t)$, and, using (4.17), we find $M_B^A(t) = \omega_B^A + \frac{1}{2}(g(t) + c) \delta_B^A$, where $\omega_{AB} = -\omega_{BA}$, and c are independent of t . Then (4.13) yields $c = 0$, while (4.16) leaves us with $g''(t) = 0$, and $(C^A)''(t) = 0$, i.e., $g(t) = 2(\kappa t + \lambda)$, and $C^A(t) = \beta^A t + \gamma^A$, where $\kappa, \beta^A, \gamma^A$, and λ are constant coefficients. At last (4.15) implies $X^0 = \kappa t^2 + 2\lambda t + \varepsilon$, with ε a new constant of integration.

We therefore contend that $X \in \mathfrak{sch}(d)$ iff

$$X = (\kappa t^2 + 2\lambda t + \varepsilon) \frac{\partial}{\partial t} + (\omega_B^A x^B + \kappa t x^A + \lambda x^A + \beta^A t + \gamma^A) \frac{\partial}{\partial x^A} \quad (4.19)$$

⁹Let us recall that the Lie derivative of an affine connection along the vector field, X , is given by $L_X \Gamma_{ab}^c = \partial_a \partial_b X^c$ in the *flat* case, in a coordinate system where $\Gamma_{ab}^c = 0$.

where $\omega \in \mathfrak{so}(d)$, $\beta, \gamma \in \mathbb{R}^d$, and $\kappa, \lambda, \varepsilon \in \mathbb{R}$ are respectively infinitesimal rotations, boosts, spatial translations, inversions, dilations, and time translations. We observe that time is dilated twice as much as space [11], a specific property of the Schrödinger Lie algebra (4.19). The Schrödinger *dynamical exponent* is $z = 2$; see, e.g., [32].

Let us recall that the *Galilei Lie algebra* $\mathfrak{gal}(M, \gamma, \theta, \Gamma) \subset \mathfrak{cgal}(M, \gamma, \theta, \Gamma)$ of a NC-structure is plainly defined as its Lie algebra of infinitesimal automorphisms. Thus, $X \in \mathfrak{gal}(M, \gamma, \theta, \Gamma)$ iff [27, 22]

$$L_X \gamma^{ab} = 0, \quad L_X \theta_a = 0 \quad \& \quad L_X \Gamma_{ab}^c = 0 \quad (4.20)$$

for all $a, b, c = 0, 1, \dots, d$, i.e., if (4.11) holds with $f = 0$, and $g = 0$. In the flat case, $\mathfrak{gal}(d) = \mathfrak{gal}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta, \Gamma)$ is clearly spanned by the vector fields (4.19) with $\kappa = 0$, and $\lambda = 0$.

4.3 NC-conformal transformations: lightlike geodesics

So far, we have been dealing with the Galilei-conformal symmetries of the equations of timelike geodesics. What about those of the equations of *lightlike geodesics* (4.3) that model the worldlines of *massless non-relativistic particles* [35]?

Let us now determine the variations (4.7) of the NC-connection, Γ , that preserve the equations of lightlike geodesics, namely Equation (4.1) supplemented by $\dot{t} = 0$.

We thus must have $\delta \Gamma_{ab}^c \dot{x}^a \dot{x}^b = \delta \mu \dot{x}^c$, so that, necessarily,

$$\gamma(df) = 0 \quad (4.21)$$

since $\theta_a \dot{x}^a = 0$; this implies $df = f' \theta$, hence that f is a function of the time axis, T . We furthermore find $\delta \mu = -\dot{f}$. At last, the resulting variation of the NC-connection appears in the new guise

$$\delta \Gamma_{ab}^c = -f' \delta_{(a}^c \theta_{b)} + (f' + g') U^c \theta_a \theta_b + (f + g) \gamma^{ck} \theta_{(a} F_{b)k} \quad (4.22)$$

for all $a, b, c = 0, \dots, d$, where the unit vector field U , i.e., $\theta_a U^a = 1$, and the two-form F are as in (2.2).

We note that the constraint $f + g = 0$ in (4.8), obtained in the “massive” case, does *not* show up, at this stage, in the “massless” case.

Just as in Section 4.2, we will assume that the variations (4.5) of the Galilei structure, and those (4.22) of the NC-connection, are generated by infinitesimal spacetime transformations. We will thus define a new Lie algebra, we call the *Newton-Cartan conformal Lie algebra*, and denote by $\mathfrak{cnc}(M, \gamma, \theta, \Gamma)$, as the Lie algebra of those vector fields that are infinitesimal (i) Galilei-conformal transformation of

(M, γ, θ) , and (ii) transformations which permute *lightlike* geodesics of $(M, \gamma, \theta, \Gamma)$. The *Newton-Cartan conformal Lie algebra*, $\mathbf{cnc}(M, \gamma, \theta, \Gamma)$, is thus spanned by the vector fields, X , of M such that

$$L_X \gamma^{ab} = f \gamma^{ab} \quad (4.23)$$

$$L_X \theta_a = g \theta_a \quad (4.24)$$

$$L_X \Gamma_{ab}^c = -f' \delta_{(a}^c \theta_{b)} + (f' + g') U^c \theta_{(a} \theta_{b)} + (f + g) \gamma^{ck} \theta_{(a} F_{b)k} \quad (4.25)$$

for all $a, b, c = 0, 1, \dots, d$, where f and g are functions of the time axis, T , while U , and F are as in (2.2) and (2.3).

It is worth noticing that the Lie-transport (4.25) of the NC-connection satisfies the very simple condition, viz.,

$$L_X \Gamma^{abc} = 0 \quad (4.26)$$

where $L_X \Gamma^{abc} = (L_X \Gamma_{k\ell}^c) \gamma^{ak} \gamma^{b\ell}$. Interestingly, Equation (4.26) is specific to the so-called *Coriolis* Lie algebra of Galilei-isometries of (M, γ, θ) ; see [22].

We will now determine the Newton-Cartan conformal Lie algebra in the flat case, i.e., the Lie algebra $\mathbf{cnc}(d) = \mathbf{cnc}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta, \Gamma)$ where γ and θ are as in (2.1), as well as $\Gamma_{ab}^c = 0$ — or, equivalently, $U = \partial_0 + u^A \partial_A$, with $u^A = \text{const.}$ for all $A = 1, \dots, d$, and $F = 0$; see (2.2) and (2.3). Let us emphasize that the Galilei (inertial) observer, U , remains here totally arbitrary.

The system (4.23)—(4.25) to solve for $X = X^0 \partial_0 + X^A \partial_A$ reads now

$$\partial_A X_B + \partial_B X_A = -f \delta_{AB} \quad (4.27)$$

$$\partial_A X^0 = 0 \quad (4.28)$$

$$\partial_0 X^0 = g \quad (4.29)$$

$$\partial_0 \partial_0 X^A = (f' + g') u^A \quad (4.30)$$

$$\partial_0 \partial_B X^A = -\frac{1}{2} f' \delta_B^A \quad (4.31)$$

$$\partial_A \partial_B X^C = 0 \quad (4.32)$$

for all $A, B, C = 1, \dots, d$. Much in the same way as in Section 4.2, straightforward computation yields the general solution of that system. More precisely, we find again that $X^A = \omega_B^A x^B - \frac{1}{2} f(t) x^A + C^A(t)$, with the same notation as before. Now, using (4.30), we find $f''(t) = 0$, and $(C^A)'(t) = (f + g) u^A + \tilde{\beta}^A$, where the $\tilde{\beta}^A$ are constant. We thus have

$$f(t) = -2(\kappa t + \lambda). \quad (4.33)$$

with κ, λ new integration constants.

So, we claim with the help of (4.29) that $X \in \mathbf{cnc}(d)$ iff

$$X = h(t) \frac{\partial}{\partial t} + (\omega_B^A x^B + \kappa t x^A + \lambda x^A + (h(t) - \kappa t^2) u^A + \beta^A t + \gamma^A) \frac{\partial}{\partial x^A} \quad (4.34)$$

where $\omega \in \mathfrak{so}(d)$, $\beta, \gamma \in \mathbb{R}^d$, and $\kappa, \lambda \in \mathbb{R}$, the function $X^0 = h$ of T remaining arbitrary, and $u \in \mathbb{R}^d$. We, again, highlight the homomorphism: $\mathbf{cnc}(d) \rightarrow \text{Vect}(\mathbb{R})$ given by $X \mapsto h(t) \partial_t$. Such an homomorphism has already been encountered in the case of the Galilei conformal, $\mathbf{cgal}(d)$, or Schrödinger-Virasoro, $\mathfrak{sv}(d)$, Lie algebras; see (3.1), and (3.11) respectively.

4.4 NC-conformal Lie algebras with dynamical exponent z ; the CGA Lie algebra

The (infinite-dimensional) NC-conformal Lie algebra, $\mathbf{cnc}(M, \gamma, \theta, \Gamma)$, we have just unveiled admits, as we shall see, a whole family of *finite-dimensional* Lie subalgebras.

Much in the same way than in Section 3.2, we will now introduce subalgebras of the latter NC-conformal Lie algebra with prescribed dynamical exponent, z . In contradistinction to the case of generic NC-geodesics, where Condition (4.8) necessarily holds true, viz. $z = 2$, we have a total freedom in the choice of a dynamical exponent, z , in the case of NC-massless geodesics. Indeed, the two rescaling functions, f (see Equation (4.33)), and $g = h'$, remain quite independent in (4.34).

We will therefore define $\mathbf{cnc}_z(M, \gamma, \theta, \Gamma)$ as the Lie subalgebra of $\mathbf{cnc}(M, \gamma, \theta, \Gamma)$ defined by Equation (3.8) — or (3.9); we will call it the *NC-conformal Lie algebra with dynamical exponent z* . Any vector field $X \in \mathbf{cnc}_z(M, \gamma, \theta, \Gamma)$ will thus satisfy Equations (4.23), (4.24), and (4.25), where

$$f + \frac{2}{z}g = 0 \quad (4.35)$$

with $z > 0$, a rational number.

Let us finally establish, in the case of a flat NC-structure (2.1) — again defined by $U = \text{const.}$, and $F = 0$ —, the expression of the generators of the Lie algebra $\mathbf{cnc}_z(d) = \mathbf{cnc}_z(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta, \Gamma)$. Those retain the form (4.34) where, in view of (4.29), (4.33) and (3.9), we now have $(X^0)'(t) = g(t) = -f(t)/q = 2(\kappa t + \lambda)/q$. We will also put $\alpha^A = \alpha u^A$.

At last, we assert that $X \in \mathbf{cnc}_z(d)$ iff

$$X = \left(\frac{1}{2} z \kappa t^2 + z \lambda t + \varepsilon \right) \frac{\partial}{\partial t} + \left(\omega_B^A x^B + \kappa t x^A + \lambda x^A + \frac{(z-2)}{2} t^2 \alpha^A + \beta^A t + \gamma^A \right) \frac{\partial}{\partial x^A} \quad (4.36)$$

where $\omega \in \mathfrak{so}(d)$, $\alpha, \beta, \gamma \in \mathbb{R}^d$, and $\kappa, \lambda, \varepsilon \in \mathbb{R}$.

Apart from the Schrödinger-like generators, we notice the appearance of a new, *acceleration* generator, α , if $z \neq 2$.

Let us emphasize that dealing with *rational* dynamical exponents, z , introduced in (3.10), is clearly allowed by the novel geometric definition (3.8) of Galilei conformal transformations; such a flexibility is furthermore specific to the symmetries of non-relativistic models of massless particles. The values studied in [2] appear for $m = 1$, and $n = N$, i.e., for $z = 2/N$.

Again, just as was the case for the Schrödinger Lie algebra, $\mathfrak{sch}(d)$, we obtain, for each z , an $\mathfrak{so}(2, 1)$ Lie subalgebra, generated by time translations ε , dilations $z\lambda$, and expansions $z\kappa$; see Equation (4.36).

- For $z = 2$, we recover the Schrödinger Lie algebra, i.e.,

$$\mathbf{cnc}_2(d) \cong \mathfrak{sch}(d). \quad (4.37)$$

- For $z = 1$, we get the CGA Lie algebra, namely, the Conformal Galilean Algebra (1.3) of Lukierski, Stichel and Zakrzewski [1].

- In the case $z = \infty$, the Lie algebra $\mathbf{cnc}_\infty(d)$ is the (infinite-dimensional) Lie subalgebra of $\mathbf{cnc}(d)$ defined by $\kappa = 0$, and $\lambda = 0$ in (4.34).¹⁰

5 Conformal Galilean symmetries of physical systems

In order to illustrate our general formalism, we first present a framework, due originally to Souriau [35], which allows us to describe, in particular, both massive and massless Galilean elementary systems in a unified way.

Consider a Hamiltonian system with d degrees of freedom, whose *phase space* is a $2d$ -dimensional symplectic manifold (\mathcal{M}, Ω) , and whose Hamiltonian is a smooth function, H , of \mathcal{M} . The two-form $\Omega = \frac{1}{2}\Omega_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta$ of \mathcal{M} is closed, $d\Omega = 0$, and non-degenerate, $\det(\Omega_{\alpha\beta}) \neq 0$. Using its inverse, $\Omega^{-1} = \frac{1}{2}\Omega^{\alpha\beta} \partial_{\xi^\alpha} \wedge \partial_{\xi^\beta}$, we get the Poisson bracket $\{F, G\} = \Omega^{\alpha\beta} \partial_\alpha F \partial_\beta G$ of two observables F , and G . (The Jacobi identity is equivalent to $d\Omega = 0$.) Then Hamilton's equations read

$$\frac{d\xi^\alpha}{dt} = \{H, \xi^\alpha\} \quad (5.1)$$

where $\alpha = 1, \dots, 2d$, the parameter t being interpreted as “time”.

¹⁰Put $f(t) = 0$ in (4.33).

If $X_H = \{H, \cdot\}$ is the associated Hamiltonian vector field, we see that (5.1) can also be written as¹¹

$$\frac{d\xi}{dt} = X_H \quad \text{where} \quad \Omega(X_H) = -dH \quad (5.2)$$

or, using coordinates, $\Omega_{\alpha\beta} X_H^\alpha = -\partial_\beta H$, for all $\beta = 1, \dots, 2d$.

One can go one step farther [37] and, promoting time as a new coordinate, consider the $(2d + 1)$ -dimensional “evolution space” $\mathcal{V} = \mathcal{M} \times \mathbb{R}$ endowed with the following closed two-form which we write, with some abuse of notation, as

$$\sigma = \Omega - dH \wedge dt. \quad (5.3)$$

The vector fields $Y = \lambda(X_H + \partial/\partial t)$ of \mathcal{V} , with $\lambda \in \mathbb{R}$, clearly satisfy $\sigma(Y) = 0$ since $X_H(H) = 0$, and $Y(t) = \lambda$. Denoting $y = (\xi, t)$ points of \mathcal{V} , we see that the equations of motion (5.1) admit the alternative form

$$\frac{dy}{d\tau} = Y \quad \text{with} \quad \sigma(Y) = 0 \quad (5.4)$$

where τ is now an arbitrary curve-parameter.

Conversely, let us consider a closed two-form, σ , on some general *evolution space*, \mathcal{V} , whose kernel, $K = \ker(\sigma)$, has a (nonzero) constant dimension.¹² Then K (see (5.4)) is an integrable distribution. So, there exists, at each point $y \in \mathcal{V}$, a submanifold whose tangent space is spanned by those vectors in K . Each leaf (or characteristic) of K is a *classical motion*. The set of these motions (which is assumed to be a well-behaved manifold) is Souriau’s *space of motions*, $\mathcal{U} = \mathcal{V}/K$, of the system. The two-form σ passes to the quotient, \mathcal{U} , which becomes, hence, a symplectic manifold. Espousing this point of view, one regards the evolution space as fundamental since it hosts the dynamics in a purely intrinsic way. See also the recent essay [38] supporting this standpoint in the classical and quantum context.

A *symmetry* of the evolution space (\mathcal{V}, σ) is given by a vector field Z which Lie-transportes the two-form σ , namely such that

$$L_Z \sigma = 0. \quad (5.5)$$

A symmetry is called *Hamiltonian* if there exists a function \mathcal{J}_Z of \mathcal{V} such that, globally,

$$\sigma(Z) = -d\mathcal{J}_Z. \quad (5.6)$$

¹¹ The form (5.2) of Hamilton’s equations allows for a variational interpretation; see, e.g., [35, 40, 41].

¹²One says that the two-form is *presymplectic*.

Then, one readily finds that $Y(\mathcal{J}_Z) = 0$ for all $Y \in K$, i.e., that \mathcal{J}_Z (determined by (5.6) up to an overall constant) is a conserved quantity. See [35] for an account on this formulation of Noether's theorem.

Conversely, symplectic manifolds upon which a given group of Hamiltonian symmetries acts transitively can be constructed in a systematic fashion [35]. For example, the homogeneous symplectic manifolds of the Galilei group will represent the spaces of motions of classical, non-relativistic, elementary particles. Skipping the details, here we simply list the results which are important for our purposes.

5.1 Galilean massive particles

Generic elementary systems of the Galilei group (cf. (4.20)) in four-dimensional, flat, NC-spacetime are classified by the *mass*, m , and *spin*, s , invariants. In the “massive” case, $m > 0$, the evolution space of a spinning particle, with $s > 0$, is $\mathcal{V} = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ parametrized by the quadruples $y = (t, \mathbf{x}, \mathbf{v}, \mathbf{u})$, and endowed with the two-form

$$\sigma = mdv_A \wedge (dx^A - v^A dt) - \frac{s}{2} \epsilon_{ABC} u^A du^B \wedge du^C \quad (5.7)$$

where ϵ_{ABC} is the Levi-Civita symbol with $\epsilon_{123} = 1$. Equation (5.7) happens to be of the form (5.3) that unifies the symplectic structure of phase space $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ and the Hamiltonian, namely

$$\sigma = dp_A \wedge dx^A - \frac{1}{2s^2} \epsilon_{ABC} s^A ds^B \wedge ds^C - d\left(\frac{\mathbf{p}^2}{2m}\right) \wedge dt \quad (5.8)$$

where the vector $\mathbf{p} = m\mathbf{v}$ stands for the linear momentum, and $\mathbf{s} = s\mathbf{u}$ for the *classical spin*.¹³ Ordinary phase space has been extended by the sphere S^2 , endowed with its canonical surface element. The evolution space, \mathcal{V} , is therefore 9-dimensional. The two-form (5.7) is closed and has a one-dimensional kernel; the characteristic curves, which are solutions of the free equations of motion (5.4), namely

$$\dot{t} = 1, \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{v}} = 0, \quad \dot{\mathbf{u}} = 0, \quad (5.9)$$

project on spacetime as usual straight worldlines. Those are independent of the spin, which is itself a constant of the motion. These worldlines are, in fact, timelike geodesics since

$$\dot{t} \neq 0. \quad (5.10)$$

¹³At the purely classical level studied here, s is an arbitrary positive number; the “quantizability” [35] requires it to be a half-integer.

As for the symmetries of the model coming from conformal Galilean transformations of flat spacetime, one shows [18, 20] that the only vector fields $X \in \mathfrak{cgal}(3)$ that admit a lift, \tilde{X} , to (\mathcal{V}, σ) verifying $L_{\tilde{X}}\sigma = 0$ (see (5.5)) are necessarily Schrödinger vector fields, $X \in \mathfrak{sch}(3)$. The explicit expression is

$$\begin{aligned} \tilde{X} = & (\kappa t^2 + 2\lambda t + \varepsilon) \frac{\partial}{\partial t} + (\omega_B^A x^B + \kappa t x^A + \lambda x^A + \beta^A t + \gamma^A) \frac{\partial}{\partial x^A} \\ & + (\omega_B^A v^B + \beta^A - \lambda v^A + \kappa(x^A - v^A t)) \frac{\partial}{\partial v_A} \\ & + \omega_B^A u^B \frac{\partial}{\partial u^A} \end{aligned} \quad (5.11)$$

with the notation of (4.11).

We, hence, recover the results of Section 4.2 dealing with the symmetries of the equations of timelike NC-geodesics. Notice that the Schrödinger symmetry still holds in the presence of spin.

The action (5.11) is Hamiltonian, and the conserved quantities, calculated using Noether's theorem (see Equation (5.6)) read¹⁴

$$\mathcal{J}_{\tilde{X}} = \mathbf{J} \cdot \boldsymbol{\omega} - \mathbf{G} \cdot \boldsymbol{\beta} + \mathbf{P} \cdot \boldsymbol{\gamma} - H\varepsilon - K\kappa + D\lambda \quad (5.12)$$

where

$$\begin{aligned} \mathbf{P} &= \mathbf{p} && \text{Linear momentum} \\ \mathbf{G} &= m\mathbf{q} && \text{Galilean boost} \\ \mathbf{J} &= \mathbf{x} \times \mathbf{p} + s\mathbf{u} && \text{Angular momentum} \\ H &= \frac{\mathbf{p}^2}{2m} && \text{Energy} \\ K &= \frac{m\mathbf{q}^2}{2} && \text{Schrödinger expansions} \\ D &= \mathbf{p} \cdot \mathbf{q} && \text{Schrödinger dilations} \end{aligned} \quad (5.13)$$

together with

$$\mathbf{q} = \mathbf{x} - \mathbf{v}t. \quad (5.14)$$

Note that the spin enters the angular momentum only and is, in fact, separately conserved. The space of motions, $\mathcal{U} = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$, therefore inherits from σ the symplectic two-form $\Omega = dp_A \wedge dq^A - (s/2)\epsilon_{ABC} u^A du^B \wedge du^C$. The associated

¹⁴We have put $\omega_A = -\frac{1}{2}\epsilon_{ABC}\omega^{BC}$ for all $A = 1, 2, 3$.

Poisson brackets of the components (5.13) of the *moment map* [35] then realize the *one-parameter central extension* of the Schrödinger group, via

$$\{P_A, G_B\} = m \delta_{AB}. \quad (5.15)$$

Other examples with Schrödinger symmetry involve hydrodynamics [42, 43] and non-relativistic Chern-Simons vortices [44].

5.2 Galilean massless particles

Concerning the second, “Conformal Galilean type” symmetries, the situation is more subtle. The natural candidates are the massless Galilean systems, studied by Souriau forty years ago [35]. In geometrical optics, a classical “light ray” can, in fact, be identified with an *oriented straight line*, \mathcal{D} , in Euclidean space \mathbb{R}^3 . Such a line is characterized by an arbitrary point $\mathbf{x} \in \mathcal{D}$, and its direction, i.e., a unit vector, \mathbf{u} , along \mathcal{D} . The manifold of light rays is readily identified with the (co)tangent bundle TS^2 endowed with its canonical symplectic structure, or a twisted symplectic structure if spin is admitted. This model is based on the Euclidean group.

There exists, indeed, a Galilean version of Euclidean “spinoptics”. The homogeneous symplectic manifolds of the Galilei group to consider are “massless”, i.e., defined by the invariants $m = 0$, $s \neq 0$, and $k > 0$, a new Galilei-invariant [35, 36].

A convenient “evolution space” for these massless models is $\mathcal{V} = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times S^2$ described by the quadruples $y = (t, \mathbf{x}, E, \mathbf{u})$ and endowed with the closed two-form

$$\sigma = k du_A \wedge dx^A - dE \wedge dt - \frac{s}{2} \epsilon_{ABC} u^A du^B \wedge du^C \quad (5.16)$$

where the constants $k > 0$, and s , are the *color* and the *spin*, respectively.¹⁵

The motions of these massless particles, e.g., the classical “light rays”, identified with the characteristic curves of (5.16), project onto spacetime as oriented *lightlike* straight worldlines directed along \mathbf{u} , viz.,

$$\dot{t} = 0, \quad \dot{\mathbf{x}} = \mathbf{u}, \quad \dot{E} = 0, \quad \dot{\mathbf{u}} = 0, \quad (5.17)$$

where we have chosen the parameter τ as the arc-length along the straight line \mathcal{D} . Such a “motion” is *instantaneous*,

$$t = \text{const.} \quad (5.18)$$

¹⁵ For $s = 0$ we get a “Fermat particle”, i.e., “spinless light,” described by the Fermat principle [35, 45]. Quantization requires that s/\hbar be a half-integer, and the color becomes $k = 2\pi\hbar/\lambda$, where λ is the wavelength [35, 36].

and, hence, projects as a lightlike geodesic (4.3) of flat Newton-Cartan spacetime (massless particles have “infinite speed.”)

The Galilei Lie algebra $\mathfrak{gal}(3)$, see (4.20), acts in a Hamiltonian way (5.6) on the evolution space \mathcal{V} according to

$$\tilde{X} = \varepsilon \frac{\partial}{\partial t} + (\omega_B^A x^B + \beta^A t + \gamma^A) \frac{\partial}{\partial x^A} + k u_A \beta^A \frac{\partial}{\partial E} + \omega_B^A u^B \frac{\partial}{\partial u^A}, \quad (5.19)$$

where $\boldsymbol{\omega} \in \mathfrak{so}(3)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$, and $\varepsilon \in \mathbb{R}$.

Using the definition (5.12) of the Hamiltonian, $\mathcal{J}_{\tilde{X}}$, of this action, we find the associated conserved quantities, namely

$$\begin{aligned} \mathbf{P} &= k\mathbf{u} && \text{Linear momentum} \\ \mathbf{G} &= -\mathbf{P}t && \text{Galilean boost} \\ \mathbf{J} &= \mathbf{x} \times \mathbf{P} + s\mathbf{u} && \text{Angular momentum} \\ H &= E && \text{Energy} \end{aligned} \quad (5.20)$$

Let us emphasize that the time-dependent Galilei boost, \mathbf{G} , is a constant of the “motion”, since the latter takes place at constant time, cf. (5.18). Under the Poisson bracket defined by the induced symplectic two-form $\Omega = k du_A \wedge dq^A - dE \wedge dt$, where $\mathbf{q} = -\mathbf{u} \times (\mathbf{u} \times \mathbf{x})$, of the space of motions $\mathcal{U} = TS^2 \times T\mathbb{R}$, the components (5.20) of the moment map close into the *unextended* Galilei group. In particular, translations and Galilean boosts commute: our “photon” is massless. Curiously, the energy, E , remains arbitrary, and determined by the initial conditions.

What about our conformal extensions? Are they symmetries? For the *trajectories*, the answer is positive: the lightlike “instantaneous” geodesics are permuted by construction, see Section 4.3.

Concerning the *dynamics*, the answer is more subtle though: for any finite dynamical exponent z , *none* of the additional geometric symmetries leaves the dynamics invariant. To see this, consider, for example, a z -dilation: while $\mathbf{x} \mapsto e^\lambda \mathbf{x}$, the unit vector, \mathbf{u} , cannot be dilated, $\mathbf{u} \mapsto \mathbf{u}$. Therefore, a “photon” of color k is carried into one with color $ke^{-\lambda}$ (which, in empty space, follows the same trajectories).

There is, however, a way to escape this obstruction: it is enough *... not to dilate \mathbf{x} !* To see this, consider first the spinless “Fermat” case. Then the evolution space can be viewed as the submanifold $\mathcal{V} \subset T^*M$ of the cotangent bundle of spacetime $M = \mathbb{R} \times \mathbb{R}^3$ defined by the equation

$$\gamma^{ab} p_a p_b - k^2 = 0. \quad (5.21)$$

Its presymplectic two-form, given by Equation (5.16) with $s = 0$, is just $\sigma = d\varpi$, where $\varpi = p_a dx^a$ is the restriction to \mathcal{V} of the canonical one-form of T^*M . Recall that a vector field, X , on space-time is *canonically* lifted to T^*M as

$$\tilde{X} = X^a \frac{\partial}{\partial x^a} - p_b \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial p_a}. \quad (5.22)$$

One easily sees that this lift is tangent to the submanifold \mathcal{V} , cf. (5.21), iff one has identically $\tilde{X}(\gamma^{ab} p_a p_b - k^2) = (L_X \gamma)^{ab} p_a p_b = 0$, i.e., iff the vector field X leaves γ invariant, viz.,¹⁶

$$L_X \gamma = 0 \quad (5.23)$$

which is the Galilei-conformal condition (3.8) with $m = 1$ and $n = 0$; the dynamical exponent is therefore $z = \infty$. This corresponds to “dilating” time alone.

In the flat case under study, the general solution of Equation (5.23) is given by (3.14), i.e., by

$$X = h(t) \frac{\partial}{\partial t} + (\omega_B^A(t) x^B + \beta^A(t)) \frac{\partial}{\partial x^A} \quad (5.24)$$

where $h(t)$, $\omega(t) \in \mathfrak{so}(3)$, and $\beta(t)$ depend arbitrarily on time. At last, the maximal Hamiltonian symmetries of the “Fermat” particle model constitute the Lie algebra $\mathfrak{cgal}_\infty(3)$, i.e., an infinite-dimensional conformal extension of the [centrally unextended] Galilei group.

Let us compute the explicit form of the canonical lift, \tilde{X} , of $X \in \mathfrak{cgal}_\infty(3)$ to \mathcal{V} . Using Equation (5.22), we end up with

$$\begin{aligned} \tilde{X} &= h(t) \frac{\partial}{\partial t} + (\omega_B^A(t) x^B + \beta^A(t)) \frac{\partial}{\partial x^A} \\ &\quad + (k(\omega'_{AB}(t) u^A x^B + \beta'_A(t) u^A) - h'(t) E) \frac{\partial}{\partial E} \\ &\quad + \omega_B^A(t) u^B \frac{\partial}{\partial u^A} \end{aligned} \quad (5.25)$$

with the same notation as before. As previously mentioned the $\mathfrak{cgal}_\infty(3)$ -action on \mathcal{V} is, by construction, Hamiltonian if $s = 0$. Using Equation (5.6), one finds the conserved Hamiltonian

$$\mathcal{J}_{\tilde{X}} = (\mathbf{x} \times k\mathbf{u}) \cdot \boldsymbol{\omega} + k\mathbf{u} \cdot \boldsymbol{\beta}(t) - h(t)E. \quad (5.26)$$

What about spin? One easily checks that, in the case $s \neq 0$, the presymplectic two-form (5.16) is no longer $\mathfrak{cgal}_\infty(3)$ -invariant. In fact, elementary calculation

¹⁶This construction is general, and can be extended to the case of any NC-structure [18].

shows that any $X \in \mathfrak{cgal}_\infty(3)$ such that $L_{\tilde{X}}\sigma = 0$ is of the form (5.24) with

$$\boldsymbol{\omega}'(t) = 0. \quad (5.27)$$

Thus, in the general case of massless, spinning Galilean particles, the associated constants of the “motion” retain the final form

$$\mathcal{J}_{\tilde{X}} = (\mathbf{x} \times k\mathbf{u} + s\mathbf{u}) \cdot \boldsymbol{\omega} + k\mathbf{u} \cdot \boldsymbol{\beta}(t) - h(t)E \quad (5.28)$$

with $\boldsymbol{\omega} \in \mathfrak{so}(3)$, $\boldsymbol{\beta}(t)$, and $h(t)$ remaining arbitrary functions of time. Note that the conservation of these quantities is related to the fact that the “motions” are instantaneous (5.18).

In conclusion, Souriau’s “classical photon” admits an infinite-dimensional conformal extension of the unextended Galilei group; see (5.20).

6 Conclusion

In this paper, we have presented a systematic way to derive all types of “non-relativistic conformal transformations” of spacetime M . Due to the degeneracy of the Galilei “metric” (γ, θ) , and to the (relative) independence of Newton-Cartan connections, Γ , there are quite a large number of subcases.

Firstly, the conformal transformations of the “metric” structure alone, (3.1), yield the conformal Galilei Lie algebra $\mathfrak{cgal}(M, \gamma, \theta)$. In the flat case, and in d space dimensions, it is the infinite-dimensional Lie algebra (3.5). Fixing the dynamical exponent, z , via the geometric definition (3.8), yields various (still infinite dimensional) Lie subalgebras, $\mathfrak{cgal}_z(d)$. For $z = 2$, we get the Schrödinger-Virasoro Lie algebra (3.12) of Henkel et al. [32, 2, 3], and for $z = \infty$ we get $\mathfrak{cgal}_\infty(d)$ in (3.14).

The Newton-Cartan structure also involves the choice of a connection, and allows us to consider geodesics, identified with motions of test particles.

- Those Galilei-conformal transformations which preserve the *timelike* geodesics are, in fact, the usual Schrödinger transformations with dynamical exponent $z = 2$, namely $\mathfrak{sch}(d)$ in (4.19).

- Those which preserve the *lightlike* geodesics have a richer structure, though. There is no *a priori* restriction on the dynamical exponent, z . The resulting infinite dimensional algebra, $\mathfrak{cnc}(d)$, is given by (4.34). The latter admits finite-dimensional subalgebras, $\mathfrak{cnc}_z(d)$, for each rational $z < \infty$, see (4.36). For $z = 2$ we find, once again, the Schrödinger algebra, and for $z = 1$, we obtain the CGA (1.3), of Lukierski et al. [1]. The infinite-dimensional Lie algebra $\mathfrak{cnc}_\infty(d)$ completes our classification.

Let us emphasize that our geometric-algebraic framework based on NC-space-time leaves no place to central extensions; the latter only arise when conserved quantities (or (pre)quantization) of concrete physical systems are considered.

All these symmetries were derived by considering as a fundamental the NC-structure of non-relativistic spacetime. What about concrete physical systems? In Section 5 we study two such systems. Souriau’s (pre)symplectic framework [35] allows us to present them in a unified manner.

The dynamics of the concrete physical systems reduce further the geometric symmetries to some of their subgroups. This is understood if we think of free fall: massive particles fall in the same way, independently of their respective masses. The trajectory of a particle can be carried therefore into another one by a geometric transformation. However, such a transformation can change the mass — so it is *not* necessarily a symmetry of the dynamical system.

Firstly, for a massive Galilean particle with spin, we recover the well-known Schrödinger symmetry associated with $\mathfrak{sch}(3)$.

The GCA-type symmetries require no mass [1]. The natural candidates are, therefore, Souriau’s “Galilean photons” [35], which are associated with the coadjoint orbits of the [unextended] Galilei group. These “particles” have an “instantaneous motion” — they have “infinite velocity”. They can carry spin, generalizing “spinless light”, described by the Fermat principle [45].

Can one take such models seriously? The answer is positive since they can be obtained as suitable non-relativistic limits of relativistic massless particles, i.e., those associated with the mass zero coadjoint orbits of the Poincaré group [35]. Even more importantly, the model of “Galilean photon” is a trivial extension of the Euclidean model presented in [45], which has been used to explain the recently observed spin-Hall effect for light [46].

Let us emphasize, in conclusion, that Souriau’s model of massless Galilean particles carry an infinite-dimensional Lie algebra of symmetries, namely $\mathfrak{cgal}_\infty(3)$. Similar results are being found by Stichel and Zakrzewski [48] in a cosmological model.

Acknowledgement. We are indebted to J. Lukierski and M. Henkel for correspondence, and to P. Stichel for informing us about their unpublished paper [48].

References

- [1] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Exotic Galilean conformal symmetry and its dynamical realizations,” *Phys. Lett. A* **357** (2006), 1. [arXiv:hep-th/0511259].
J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Acceleration-Extended Galilean Symmetries with Central Charges and their Dynamical Realizations,” *Phys. Lett. B* **650** (2007), 203. [arXiv:hep-th/0702179].
J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Acceleration-Enlarged Symmetries in Nonrelativistic Space-Time with a Cosmological Constant,” arXiv:0710.3093 [hep-th].
- [2] M. Henkel, “Phenomenology of local scale invariance: from conformal invariance to dynamical scaling.” *Nucl. Phys. B* **641** (2002), 405.
M. Henkel and J. Unterberger, “Schrödinger invariance and space-time symmetries,” *Nucl. Phys. B* **660** (2003) 407. [arXiv:hep-th/0302187].
- [3] M. Henkel, R. Schott, S. Stoimenov, and J. Unterberger, “The Poincare algebra in the context of ageing systems: Lie structure, representations, Appell systems and coherent states,” arXiv:math-ph/0601028. See also in *Quantum probability and infinite-dimensional analysis*, L. Accardi, W. Freudenberg and M. Schürmann (eds), World Scientific (Singapour 2007), pp. 233–240.
- [4] K. Balasubramanian, and J. McGreevy, “Gravity duals for non-relativistic CFTs,” *Phys. Rev. Lett.* **101** (2008), 061601.
D. T. Son, “Toward and AdS/cold atom correspondence: a geometric realization of the Schrödinger symmetry,” *Phys. Rev.* **D78** (2008), 046003.
The number of papers published on the subject is rapidly growing. It is, therefore, impossible to give full credit to all important contributions. For a relation to earlier work and an (incomplete) list of references, see, e.g., C. Duval, M. Hassaine, and P. A. Horvathy, “The geometry of Schrödinger symmetry in non-relativistic CFT,” arXiv:0809.3128 [hep-th]. *Ann. Phys.* **324** (2009), 1158.
- [5] P. C. Stichel, and W. J. Zakrzewski, “Nonrelativistic Dark-Energy Fluid in a Baby Universe,” arXiv:0807.3660 [gr-qc].
- [6] A. V. Galajinsky, “Remark on quantum mechanics with conformal Galilean symmetry,” *Phys. Rev. D* **78** (2008), 087701. arXiv:0808.1553 [hep-th].

- [7] I. Fouxon, and Y. Oz, “CFT Hydrodynamics: symmetries, exact solutions and gravity,” arxiv: 0812.1266 [hep-th].
- [8] A. Bagchi, and R. Gopakumar, “Galilean Conformal Algebras and AdS/CFT,” arXiv:0902.1385 [hep-th].
- [9] M. Alishahiha, A. Davody, and A. Vahedi, “On AdS/CFT of Galilean Conformal Field Theories,” arXiv:0903.3953 [hep-th].
- [10] D. Martelli, and Y. Tachikawa, “Comments on Galilean conformal field theories and their geometric realization,” arXiv:0903.5184 [hep-th].
- [11] R. Jackiw, “Introducing scaling symmetry,” *Phys. Today*, **25** (1972), 23.
 C. R. Hagen, “Scale and conformal transformations in Galilean-covariant field theory,” *Phys. Rev.* **D5** (1972), 377.
 U. Niederer, “The maximal kinematical symmetry group of the free Schrödinger equation,” *Helv. Phys. Acta* **45** (1972), 802.
- [12] G. Burdet, M. Perrin, and P. Sorba, “About the Non-Relativistic Structure of the Conformal Algebra,” *Commun. Math. Phys.* **34** (1973), 85.
- [13] C. Duval, and H.P. Künzle, “Minimal Gravitational Coupling in the Newtonian Theory and the Covariant Schrödinger Equation,” *Gen. Rel. Grav.* **16** (1984), 333.
- [14] J.-M. Lévy-Leblond, “Galilei group and galilean invariance.” In *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972).
 A. Ballesteros, M. Gadella, and M. del Olmo, “Moyal quantization of 2+1 dimensional Galilean systems.” *Journ. Math. Phys.* **33** (1992), 3379.
 Y. Brihaye, C. Gonera, S. Giller, and P. Kosiński, “Galilean invariance in 2 + 1 dimensions,” [hep-th/9503046] (unpublished).
 D. R. Grigore, “Transitive symplectic manifolds in 1 + 2 dimensions.” *Journ. Math. Phys.* **37** (1996), 240.
 D. R. Grigore, “The projective unitary irreducible representations of the Galilei group in 1 + 2 dimensions.” *Journ. Math. Phys.* **37** (1996), 460.
- [15] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, “Galilean-invariant (2 + 1)-dimensional models with a Chern-Simons-like term and $d = 2$ noncommutative geometry.” *Annals of Physics* (N. Y.) **260** (1997), 224.

- [16] C. Duval, and P. A. Horváthy, “The “Peierls substitution” and the exotic Galilei group.” *Phys. Lett.* **B 479** (2000), 284. [hep-th/0002233].
C. Duval, and P. A. Horváthy, “Exotic galilean symmetry in the non-commutative plane, and the Hall effect.” *Journ. Phys.* **A 34** (2001), 10097. [hep-th/0106089].
- [17] P. A. Horváthy, L. Martina, and P. Stichel: “Galilean symmetry in noncommutative field theory,” *Phys. Lett.* **B 564** (2003), 149. [hep-th/0304215].
P. A. Horváthy, and M. S. Plyushchay: “Anyon wave equations and the non-commutative plane.” *Phys. Lett.* **B 595** (2004), 547. [hep-th/0404137].
- [18] C. Duval, “Quelques procédures géométriques en dynamique des particules,” Thèse d’Etat (1982), unpublished.
- [19] G. Burdet, C. Duval, and M. Perrin, “Cartan Structures on Galilean Manifolds: the Chronoprojective Geometry,” *J. Math. Phys.* **24** (1983), 1752.
G. Burdet, C. Duval, and M. Perrin, “Chronoprojective Cartan Structures on Four-Dimensional Manifolds,” *Publ. RIMS, Kyoto Univ.* **19** (1983), 813.
- [20] C. Duval, “Nonrelativistic Conformal Symmetries and Bargmann Structures,” in *Conformal Groups and Related Symmetries. Physical Results and Mathematical Background*, Clausthal 1985, (A.O. Barut & H.D. Doebner Eds), pp 162–182, *Lecture Notes in Physics* **261**, Springer-Verlag (1986).
- [21] B. Carter, and I.M. Khalatnikov, “Canonically covariant formulation of Landau’s Newtonian superfluid dynamics.” *Rev. Math. Phys.* **6** (1994), 277.
- [22] C. Duval, “Galilean isometries,” *Class. Quantum Grav.* **10** (1993), 2217. arXiv:0903.1641.
- [23] C. Duval, G. Burdet, H.P. Künzle, and M. Perrin, “Bargmann Structures And Newton-Cartan Theory,” *Phys. Rev.* **D31**, 1841 (1985).
- [24] C. Duval, G. W. Gibbons, and P. A. Horvathy, “Celestial Mechanics, Conformal Structures and Gravitational Waves,” *Phys. Rev.* **D43**, 3907 (1991).
- [25] J. Gomis, and J.M. Pons, “Poincaré Transformations and Galilei Transformations.” *Phys.Lett.* **A66** (1978), 463.
J. Gomis, A. Poch, and J. M. Pons, “Poincaré wave equations as Fourier transforms of Galilei wave equations.” *J. Math. Phys.* **21** (1980), 2682.

- [26] E. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée,” *Ann. Sci. Ecole Norm. Sup. (4)* **40** (1923), 325.
- [27] A. Trautman, “Sur la théorie newtonienne de la gravitation,” *C.R. Acad. Sci. Paris* **257** (1963), 617; “Comparison of Newtonian and relativistic theories of space time,” pp. 413–425 in *Perspectives in Geometry and Relativity*, (B. Hoffmann, Ed.), Indiana University Press, Bloomington, 1964.
- [28] C. Duval, and H.P. Künzle, “Sur les connexions newtoniennes et l’extension non triviale du groupe de Galilée,” *C.R. Acad. Sci. Paris* **285 A** (1977), 813
- [29] H.P. Künzle, “Galilei and Lorentz structures on spacetime: Comparison of the corresponding geometry and physics,” *Ann. Inst. H. Poincaré. Phys. Théor.* **17** (1972), 337, and references therein.
- [30] C. Duval, and H.P. Künzle, “Dynamics of continua and particles from general covariance of Newtonian gravitation theory,” *Rep. Math. Phys.* **13** (1978), 351.
- [31] H.P. Künzle, “Covariant Newtonian limit of Lorentz space-times,” *General Relativity and Gravitation* **7** (1976), 445.
- [32] M. Henkel, “Schrödinger invariance and strongly anisotropic critical systems,” *J. Stat. Phys.* **75** (1994), 1023.
- [33] F. L. Lin, and S. Y. Wu, “Non-relativistic Holography and Singular Black Hole,” arXiv:0810.0227 [hep-th].
- [34] C. Roger, and J. Unterberger, “The Schrödinger-Virasoro Lie group and algebra: from geometry to representation theory,” *Ann. Henri Poincaré* **7** (2006), 1477.
- [35] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod (1970, ©1969); *Structure of Dynamical Systems. A Symplectic View of Physics*, translated by C.H. Cushman-de Vries (R.H. Cushman and G.M. Tuynman, Translation Editors), Birkhäuser, 1997.
- [36] V. Guillemin, and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge (1984).
- [37] R. Abraham, and J.E. Marsden, *Foundations of Mechanics*, Second Edition, Addison-Wesley Publishing Company, Inc (1987).

- [38] C. Rovelli, “Forget time”, Essay written for the FQXi contest on the Nature of Time, arXiv:0903.3832v3 [gr-qc].
- [39] M. Trümper, “Lagrangian Mechanics and the Geometry of Configuration Space-time,” *Ann. Phys. (N.Y.)* **149** (1983), 203.
- [40] P. A. Horvathy, and L. Úry, “Analogy between statics and dynamics – related to variational mechanics,” *Acta Phys. Hung.* **42** (1977), 251.
P. A. Horvathy, “Variational formalism for spinning particles,” *Journ. Math. Phys.* **20** (1979), 49.
P. A. Horvathy, “The non-commutative Landau problem,” *Ann. Phys. (N.Y.)* **299** (2002), 128. [hep-th/0201007].
- [41] G. Tuynman, “Un principe variationnel pour les variétés symplectiques,” *C. R. Acad. Sci. Paris Sér. I Math.* **326**:3 (1998), 339.
- [42] M. Hassaïne, and P. A. Horvathy, “Field-dependent symmetries of a non-relativistic fluid model,” *Annals Phys.* **282** (2000), 218. [arXiv:math-ph/9904022].
M. Hassaïne, and P. A. Horvathy, “Symmetries of fluid dynamics with polytropic exponent,” *Phys. Lett. A* **279** (2001), 215. [arXiv:hep-th/0009092].
L. O’Raifeartaigh, and V. V. Sreedhar, “The maximal kinematical invariance group of fluid dynamics and explosion-implosion duality,” *Annals Phys.* **293** (2001), 215. [arXiv:hep-th/0007199].
- [43] R. Jackiw, *Lectures on fluid dynamics*. CRM series in mathematical Physics. Springer (2002).
- [44] R. Jackiw, and S-Y. Pi, “Classical and quantum nonrelativistic Chern-Simons Theory.” *Phys. Rev.* **D42** (1990), 3500.
C. Duval, P. Horvathy, and L. Palla: “Conformal symmetry of the coupled Chern-Simons and gauged non-linear Schrödinger equations.” *Phys. Lett.* **B325** (1994), 39. [hep-th/9401065].
C. Duval, P. Horvathy, and L. Palla, “Spinors in non-relativistic Chern-Simons electromagnetism.” *Ann. Phys. (N. Y.)* **249** (1996), 265. [hep-th/9510114].
B. Chakraborty, and A. J. Majumdar, “On Galilean covariance in a nonrelativistic model involving a Chern-Simons term.” *Ann. Phys.* **250** (1996), 112.
R. Banerjee, and P. Mukherjee, “Galilean symmetry in a nonabelian Chern-Simons matter system.” *Ann. Phys.* **264** (1998), 30.

- [45] C. Duval, Z. Horvath, and P. Horvathy, “Geometrical Spinoptics and the Optical Hall Effect,” *J. Geom. Phys.* **57** (2007), 925. [arXiv:math-ph/0509031].
C. Duval, Z. Horvath, and P. A. Horvathy, “Fermat principle for spinning light,” *Phys. Rev. D* **74** (2006), 021701. [arXiv:cond-mat/0509636].
- [46] K. Yu. Bliokh, and Yu. P. Bliokh, “Topological spin transport of photons: the optical Magnus effect and Berry Phase.” *Phys. Lett.* **A333** (2004), 181. [physics/0402110].
M. Onoda, S. Murakami, and N. Nagaosa, “Hall effect for light.” *Phys. Rev. Lett.* **93** (2004), 083901. [cond-mat/0405129].
K. Y. Bliokh, A. Niv, V. Kleiner, and E. Hasman, “Geometrodynamics of Spinning Light,” arXiv:0810.2136 [physics.optics]. *Nature Photon.* **2** (2008), 748.
- [47] K. Y. Bliokh, “Geometrodynamics of polarized light: Berry phase and spin Hall effect in a gradient-index medium,” arXiv:0903.1910 [physics.optics].
- [48] P. C. Stichel, and W. J. Zakrzewski, (to be published).