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Abstract. We consider the Euler equations on the Lie algebra $\mathrm{so}(4,\mathbb{C})$ with a diagonal quadratic Hamiltonian. It is known that this system always admits three functionally independent polynomial first integrals. We prove that if the system has a rational first integral functionally independent of the known three ones (so called fourth integral), then it has a polynomial fourth first integral. This is a consequence of a more general fact that for these systems the existence of a Darboux polynomial with non vanishing cofactor implies the existence of a polynomial fourth integral.

Key words: Euler equations; Rational and polynomial first integrals; Darboux polynomials

1. Introduction

For a given system of (polynomial) ordinary differential equations depending on parameters, a question is how to identify those values of the parameters for which the equations have (rational or polynomial) first integrals? Except for some simple cases, this problem is very hard and there are no general methods to solve it.

In this paper we present a partial result concerning the integrability problem for the so-called *Euler equations on Lie algebras* [1–4, 13, 14], for which the problem is also largely open.

Let us recall some relevant definition. Let $(L, [\cdot,\cdot])$ be a finite dimensional (real or complex) Lie algebra and L^* its dual. For $f, g \in C^\infty(L^*)$ their *Lie-Poisson bracket* $\{f, g\}$ is defined by

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle,$$

where $x \in L^*$, $df(x)$, $dg(x) \in (L^*)^* = L^{**} \approx L$, and for $x \in L^*$ and $y \in L$, we denote $\langle x, y\rangle = x(y)$.

Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis to a fixed basis $\{e_1, \ldots, e_n\}$ of L. An element $x \in L^*$ can be written as $x = \sum_{i=1}^n x_i e_i^*$, where the coefficients x_i (real or complex) depend smoothly on x thus defining the functions $\in C^\infty(L^*)$, for $1 \le i \le n$.

For a given function $H \in C^\infty(L^*)$, the system of differential equations

$$\frac{dx_i}{dt} = \{x_i, H\}, \quad 1 \le i \le n, \tag{1.1}$$

is called *Euler equations on the Lie algebra* L with the *Hamiltonian* H.

Recall that a function $F \in C^\infty(L^*)$ is a *Casimir function* of the Lie algebra L if $\{f, F\} = 0$, for every $f \in C^\infty(L^*)$. It is clear (e.g., [13]) that a function F defined on L^* is a first integral of system (1.1) if and only if $\{F, H\} = 0$. In particular, the Hamiltonian H and any Casimir function of the Lie algebra L are first integrals of system (1.1).

Only for Hamiltonians *H* that are functionally independent of the Casimir functions, the right sides of system (1.1) does not vanish identically. That is why we will always suppose that the Hamiltonian H under considerations is functionally independent of the Casimir functions.

From now on we will concentrate only on the complex six dimensional Lie algebra $\mathrm{so}(4, \mathbb{C})$, the Lie algebra of the complex Lie group SO(4,$\mathbb{C}$), and study one of the simplest Euler equations on it, namely the Euler equations corresponding to the so called *diagonal quadratic Hamiltonian*.

In an appropriate basis of the Lie algebra $\mathrm{so}(4, \mathbb{C})$ (see [1]), the Euler equations corresponding to the diagonal quadratic Hamiltonian $\frac{1}{2}\sum_{i=1}^6 \lambda_i x_i^2$, take the following elegant form:

$$\begin{aligned}
\frac{dx_1}{dt} &= (\lambda_3-\lambda_2)x_2x_3+(\lambda_6-\lambda_5)x_5x_6,\\
\frac{dx_2}{dt} &= (\lambda_1-\lambda_3)x_1x_3+(\lambda_4-\lambda_6)x_4x_6,\\
\frac{dx_3}{dt} &= (\lambda_2-\lambda_1)x_1x_2+(\lambda_5-\lambda_4)x_4x_5,\\
\frac{dx_4}{dt} &= (\lambda_3-\lambda_5)x_3x_5+(\lambda_6-\lambda_2)x_2x_6,\\
\frac{dx_5}{dt} &= (\lambda_4-\lambda_3)x_3x_4+(\lambda_1-\lambda_6)x_1x_6,\\
\frac{dx_6}{dt} &= (\lambda_2-\lambda_4)x_2x_4+(\lambda_5-\lambda_1)x_1x_5,
\end{aligned} \tag{1.2}$$

where $\lambda := (\lambda_1,\dots,\lambda_6) \in \mathbb{C}^6$. Exactly the same construction holds for the Lie algebra $\mathrm{so}(4,\mathbb{R})$, where $\lambda := (\lambda_1,\dots,\lambda_6) \in \mathbb{R}^6$ and the form of equations (1.2) remains unchanged.

The Lie algebra $\mathrm{so}(4,\mathbb{C})$ admits two functionally independent polynomial Casimir functions. Thus any system of Euler equations on it always admits three functionally independent first integrals. Indeed, the above system possesses three first integrals:

$$H_1 = x_1x_4+x_2x_5+x_3x_6, \quad H_2=\sum_{i=1}^{6}x_i^2, \quad H_3=\sum_{i=1}^{6}\lambda_i x_i^2, \tag{1.3}$$

where the first integrals H_1 and H_2 are Casimir functions of the Lie algebra $\mathrm{so}(4,\mathbb{C})$ and H_3 is the Hamiltonian. These three first integrals are functionally independent unless all the $\lambda_i,\ 1\le i\le 6$, are equal, in which case the right hand sides of system (1.2) vanish.

For the Lie algebra $\mathrm{so}(4,\mathbb{C})$, on the constant level manifolds of two functionally independent Casimir functions, any Euler system, at least locally, can be reduced to the standard Hamiltonian equations with two degrees of freedom (see Secs. 6.1-6.2 and Theorem 6.22 from [13]). Therefore whatever the chosen notion of integrability, the system (1.2) in order to be integrable needs a supplementary first integral H_4, functionally independent of $H_1,\ H_2$ and H_3, called shortly a *fourth integral*. The only known cases when a fourth integral exists are the *Manakov case*, defined by the condition

$$\begin{aligned}
M = {}&\lambda_1\lambda_4(\lambda_2+\lambda_5-\lambda_3-\lambda_6)+\lambda_2\lambda_5(\lambda_3+\lambda_6-\lambda_1-\lambda_4)\\
&+\lambda_3\lambda_6(\lambda_1+\lambda_4-\lambda_2-\lambda_5)=0,
\end{aligned}$$

and the *product case*, defined by the conditions

$$\lambda_1=\lambda_4, \quad \lambda_2=\lambda_5, \quad \lambda_3=\lambda_6.$$

In both cases the fourth integral can be found among the polynomials of degree 2 at most (see [1, 9, 15]).

In our study, we will concentrate only on the existence of a *fourth rational integral*. In fact, it is well known, its absence implies the absence of an algebraic fourth integral [7, 17, 18] as well as the absence of a meromorphic fourth integral defined on some neighbourhood of 0 of $\mathbb{C}^6$ [19].

In [15], two of us proved the following theorem:

Theorem 1.1. *If for some $\lambda \in \mathbb{C}^6$, the Euler equations (1.2) admit a rational fourth integral, then they admit a polynomial fourth integral.*

Let us note that from the validity of Theorem 1.1 in complex setting, its validity in real one follows immediately.

The proof of Theorem 1.1 presented in [15], based on *Holomorphic Rectification Theorem* (see Theorem 1.18 in [5]) is elementary but quite long and quite involved. The aim of this note is to present a simpler proof based on the more powerful *Holomorphic Frobenius Integrability Theorem* (see Theorem 2.9 in [5]). The present paper is self-contained and independent of [15]. To put it in such a form some overlaps with [15] were unavoidable.

The paper is organized as follows. In Sec. 2 we collect various facts needed for the proof. In Sec. 3, Theorem 1.1 is obtained with the help of more general Theorem 3.1. Actually, the proof of Theorem 1.1 is based on the study of so called *Darboux polynomials* (see Sec. 2.1) for the Euler equations (1.2) and the rich symmetry properties of these equations and Theorem 1.1 is a direct consequence of Theorem 3.1 concerning Darboux polynomials. Let us stress that all proofs are completely elementary.

Finally let us note that in [8] exact counterparts of Theorems 1.1 and 3.1 are proved for so called natural polynomial hamiltonian systems of an arbitrary degree of freedom.

2. Preliminaries

2.1. Darboux polynomials

Consider a polynomial system of ordinary differential equations defined in $\mathbb{C}^n$

$$\frac{dx_j}{dt} = V_j(x_1, \ldots, x_n), \qquad 1 \leq j \leq n. \tag{2.1}$$

For a holomorphic function F defined on some open subset of $\mathbb{C}^n$, let us define

$$d(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} V_i.$$

The operator d is called the *derivation* associated with system of differential equations (2.1).

A polynomial $P \in \mathbb{C}[x_1, \ldots, x_n] \setminus \mathbb{C}$ is called a *Darboux polynomial* of system (2.1) if for some polynomial $S \in \mathbb{C}[x_1, \ldots, x_n]$ we have

$$d(P) = SP. \tag{2.2}$$

The polynomial S is called a *cofactor* of the Darboux polynomial P. When $S \neq 0$, P is called a *proper* Darboux polynomial. When $S = 0$, P is nothing but a first integral of system (2.1).

Here we mention some properties of the Darboux polynomials that we will need throughout:

(D1) Let P_1 and P_2 be non-zero relatively prime polynomials that are not first integrals of system (2.1). Then the rational function P_1/P_2 is a first integral of system (2.1) if and only if P_1 and P_2 are its proper Darboux polynomials with the same cofactor.
(D2) All factors of a Darboux polynomial of system (2.1) are also its Darboux polynomials.
(D3) If P_1 and P_2 are two Darboux polynomials of system (2.1) with cofactors S_1 and S_2, respectively, then P_1P_2 is also its Darboux polynomial with cofactor $S_1 + S_2$.
(D4) Let us suppose that the right-hand sides of system (2.1) are homogeneous polynomials of the same degree. Let P be a Darboux polynomial of system (2.1). Then its cofactor S is homogeneous and all homogeneous components of P are also Darboux polynomials of system (2.1).

See [11] for more details.

2.2. Permutational symmetries

The Euler equations (1.2) possess an invariance property, called *permutational symmetry*. In general, permutational symmetries can be described as follows. Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let $V(x, \lambda) = (V_1(x, \lambda), \ldots, V_n(x, \lambda))$ depends holomorphically on $(x, \lambda) \in \mathbb{C}^{2n}$. Let us consider the following system of differential equations

$$\frac{dx}{dt} = V(x, \lambda). \tag{2.3}$$

Let σ be an element of the symmetric group S_n, i.e., the group of all permutations of $\{1, \ldots, n\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, we will denote $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

A permutation $\sigma \in \mathrm{S}_n$ will be called a *permutational symmetry* of system (2.3) if for all $(x, \lambda) \in \mathbb{C}^{2n}$, we have

$$V_k(\sigma(x), \sigma(\lambda)) = \varepsilon V_{\sigma(k)}(x, \lambda), \qquad 1 \leq k \leq n,$$

where $\varepsilon = \pm 1$ is a constant independent of k. Clearly, all permutational symmetries of system (2.3) form a group.

Theorem 2.1. *Let σ be a permutational symmetry of system (2.3).*

(a) If $F = F(x)$ is a first integral of system (2.3), then the function $\tilde{F} = F \circ \sigma^{-1}$ is a first integral of the system

$$\frac{dx}{dt} = V(x, \sigma(\lambda)). \tag{2.4}$$

(b) If $P = P(x)$ is a Darboux polynomial of system (2.3) (see (2.2)), then

$$\tilde{d}(\tilde{P}) = \tilde{S}\tilde{P},$$

where $\tilde{P} = P \circ \sigma^{-1}$, $\tilde{S} = S \circ \sigma^{-1}$, and $\tilde{d}$ is the derivation associated with system (2.4).

For the proof of (a) see Sec. II of [9]. The proof of (b) follows the same line.

The group of permutational symmetries of the Euler equations (1.2) consists of 24 elements. Among others, it contains the following five permutations:

$$\begin{aligned}
\tau_2(1,2,3,4,5,6) &= (2,1,3,5,4,6),\\
\tau_3(1,2,3,4,5,6) &= (3,2,1,6,5,4),\\
\tau_4(1,2,3,4,5,6) &= (4,2,6,1,5,3),\\
\tau_5(1,2,3,4,5,6) &= (5,4,3,2,1,6),\\
\tau_6(1,2,3,4,5,6) &= (6,2,4,3,5,1).
\end{aligned} \tag{2.5}$$

For more details see Sec. II of [9] where, in its notations, $\tau_2 = \sigma_1$, $\tau_3 = \sigma_3$, $\tau_4 = \sigma_7$, $\tau_5 = \sigma_8 \circ \sigma_1$ and $\tau_6 = \sigma_7 \circ \sigma_3$.

Let P be a proper Darboux polynomial of system (1.2), that is $d(P) = SP$, where d is the corresponding derivation and $S \in \mathbb{C}[x_1, \dots, x_6] \backslash \{0\}$. Since the right-hand sides of system (1.2) are homogeneous of the same degree, it follows from (D4) that the cofactor is a homogeneous linear form, i.e.,

$$S = \sum_{i=1}^{6} \alpha_i x_1,$$

where $\alpha_i \in \mathbb{C},\ 1 \leq i \leq 6$, are some complex constants and at least one of them is non-zero, say $\alpha_{i_0} \neq 0$.

According to (2.5) $\tau_{i_0}(i_0) = 1$. Now, Theorem 2.1b implies that without any loss of generality, we can always assume that $\alpha_1 \neq 0$. This fact will be used in the proof of Theorem 1.1.

From now on, d will always denote the derivation associated with the Euler equations (1.2).

2.3. Another invariance property

Beside permutational symmetries, the Euler equations (1.2) possess also another invariance property related to the change of signs of the couples of variables (x_1, x_4), (x_2, x_5), and (x_3, x_6) respectively. More precisely, let us denote:

$$\begin{aligned}\tau_{14}(x_1,x_2,x_3,x_4,x_5,x_6) &= (-x_1,x_2,x_3,-x_4,x_5,x_6),\\ \tau_{25}(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1,-x_2,x_3,x_4,-x_5,x_6),\\ \tau_{36}(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1,x_2,-x_3,x_4,x_5,-x_6).\end{aligned} \tag{2.6}$$

It is easy to see that for $(ij) = (14)$, $(ij) = (25)$, and $(ij) = (36)$,

$$\tau_{ij}{}^{-1} \circ d \circ \tau_{ij} = -d,$$

which means that under any of the transformations of (2.6), the right side of equations (1.2) changes the sign.

For the polynomial $T \in \mathbb{C}[x_1, \ldots, x_6]$, let us denote $T_{(ij)} := T \circ \tau_{ij}$. Thus if T is a first integral of the system (1.2), then $T_{(14)}$, $T_{(25)}$, and $T_{(36)}$ also are first integrals of this system.

Moreover, if P is its Darboux polynomial, that is, $d(P) = SP$, then $d(P_{(ij)}) = -S_{(ij)}P_{(ij)}$. In particular, if

$$d(P)(x) = (\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + \alpha_5x_5 + \alpha_6x_6)P(x), \tag{2.7}$$

then

$$d(P_{(14)})(x) = (\alpha_1x_1 - \alpha_2x_2 - \alpha_3x_3 + \alpha_4x_4 - \alpha_5x_5 - \alpha_6x_6)P_{(14)}(x) \tag{2.8}$$

and

$$d(P_{(25)})(x) = (-\alpha_1x_1 + \alpha_2x_2 - \alpha_3x_3 - \alpha_4x_4 + \alpha_5x_5 - \alpha_6x_6)P_{(25)}(x). \tag{2.9}$$

2.4. Explicit form of some linear differential operators

On $\mathbb{C}^6$, equipped with coordinates $(x_1, \ldots, x_6)$, we consider the linear differential operator X_{ij}, for $1 \le i < j \le 6$, defined by the formula

$$X_{ij}(G) = \det \frac{\partial(H_1, H_2, H_3, G)}{\partial(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_6)}$$

where G is a holomorphic function on $\mathbb{C}^6$, the functions H_1, H_2, H_3 are given by (1.3), and $\hat{x}_r$ means the absence of x_r.

Two such operators X_{25} and X_{36} will play a crucial role in the proof of Theorem 1.1 and we need an explicit formula of them. To simplify the notations, we write: $\lambda_{ij} = \lambda_i - \lambda_j$ for $i \neq j$, $1 \leq i, j \leq 6$. A direct calculation gives the formulae:

$$\begin{aligned}
X_{25} = & \left(\lambda_{63}x_1x_3x_6 + \lambda_{34}x_3^2x_4 + \lambda_{46}x_4x_6^2\right)\frac{\partial}{\partial x_1} \\
& + \left(\lambda_{16}x_1^2x_6 + \lambda_{41}x_1x_3x_4 + \lambda_{64}x_4^2x_6\right)\frac{\partial}{\partial x_3} \\
& + \left(\lambda_{13}x_1x_3^2 + \lambda_{61}x_1x_6^2 + \lambda_{36}x_3x_4x_6\right)\frac{\partial}{\partial x_4} \\
& + \left(\lambda_{31}x_1^2x_3 + \lambda_{14}x_1x_4x_6 + \lambda_{43}x_3x_4^2\right)\frac{\partial}{\partial x_6}, \\
X_{36} = & \left(\lambda_{52}x_1x_2x_5 + \lambda_{24}x_2^2x_4 + \lambda_{45}x_4x_5^2\right)\frac{\partial}{\partial x_1} \\
& + \left(\lambda_{15}x_1^2x_5 + \lambda_{41}x_1x_2x_4 + \lambda_{54}x_4^2x_5\right)\frac{\partial}{\partial x_2} \\
& + \left(\lambda_{12}x_1x_2^2 + \lambda_{51}x_1x_5^2 + \lambda_{25}x_2x_4x_5\right)\frac{\partial}{\partial x_4} \\
& + \left(\lambda_{21}x_1^2x_2 + \lambda_{14}x_1x_4x_5 + \lambda_{42}x_2x_4^2\right)\frac{\partial}{\partial x_5}.
\end{aligned}$$

It is easy to see that outside of some very special subcases of the Manakov case, these two differential operators are not identically zero. Note that $X_{25}(H_r) = X_{36}(H_r) = 0, 1 \leq r \leq 3$.

2.5. Linear partial differential equations

Let us consider the following system of k, where $1 \leq k \leq n-1$, linear partial differential equations

$$\sum_{i=1}^{n} a_{ij}(x)\frac{\partial F}{\partial x_i} = 0, \quad 1 \leq j \leq k, \tag{2.10}$$

where a_{ij}, $1 \leq i \leq n$, are holomorphic functions defined on some open subset $\mathcal{U} \subset \mathbb{C}^n$.

Theorem 2.2. *Let $x_0 \in \mathcal{U}$ be such that* $\operatorname{rank} A(x_0) = k$*, where the matrix $A(x) = (a_{ij}(x))_{1\leq i\leq n, 1\leq j\leq k}$. Let us suppose that F and $F_1, \ldots, F_{n-k}$ are holomorphic on $\mathcal{U}$ solutions of system (2.10) such that $F_1, \ldots, F_{n-k}$ are functionally independent at x_0, that means that the vectors* $(\operatorname{grad} F_r)(x_0)$*, $1 \leq r \leq n-k$, are linearly independent. Then there exists a neighbourhood $\mathcal{V}$ of the point $(F_1(x_0), \ldots, F_{n-k}(x_0))$ and a holomorphic function Ω defined on $\mathcal{V}$, such that for any x in some neighbourhood of x_0 we have*

$$F(x) = \Omega(F_1(x), \ldots, F_{n-k}(x)). \tag{2.11}$$

This result is a direct consequence of *Holomorphic Frobenius Integrability Theorem* (see Sec. 2.11.20 in [12], Theorem 2.9 in [5] and Appendix 8 in [6]).

Throughout, $\mathcal{U}$ denotes a subset of $\mathbb{C}^6$ defined by the condition that for all $1 \le i < j \le 6$, and any point $z \in \mathcal{U}$, the vectors $(\mathrm{grad}\, H_1)(z)$, $(\mathrm{grad}\, H_2)(z)$, $(\mathrm{grad}\, H_3)(z)$, $(\mathrm{grad}\, x_i)(z)$, $(\mathrm{grad}\, x_j)(z)$ are linearly independent. Unless all λ_i, $1 \le i \le 6$, are equal, $\mathcal{U}$ is always an open dense subset of $\mathbb{C}^6$. Below, when saying that identity (2.11) is locally fulfilled, we will understand that this is so on a neighbourhood of some point from $\mathcal{U}$.

3. Proof of Theorem 1.1.

Let us suppose that the irreducible rational fraction P_1/P_2, where $P_1, P_2 \in \mathbb{C}[x_1,\ldots,x_6]$, is a first integral of system (1.2) and that P_1 (and thus also P_2) is not its first integral. Then (D1) from Sec. 2.1 implies that P_1 and P_2 are proper Darboux polynomials of system (1.2). Since the right-hand sides of system (1.2) are homogeneous of the same degree, it follows from (D2) and (D4) that system (1.2) admits also an irreducible homogeneous proper Darboux polynomial *P* and its cofactor is a homogeneous linear form, i.e.,

$$S = \sum_{i=1}^{6} \alpha_i x_1,$$

where $\alpha_i,\ 1 \le i \le 6$, are some constants. Since $S \neq 0$, at least one of its coefficients is not zero. As explained in Sec. 2.2, without any loss of generality we can assume that $\alpha_1 \neq 0$.

Theorem 1.1 is now a direct consequence of

Theorem 3.1. *If for some $\lambda \in \mathbb{C}^6$, the Euler equations (1.2) have a proper Darboux polynomial, then they have a polynomial fourth integral.*

Proof. Let P be a proper Darboux polynomial of the Euler equations (1.2). Without any lost of generality we can suppose that P is an irreducible and homogeneous polynomial. The proof is naturally divided into three almost independent parts.

Part 1. Construction of a polynomial first integral.

From (2.7) and (2.8) it immediately follows that $R = PP_{(14)}$ is a Darboux polynomial of system (1.2) with cofactor $2(\alpha_1 x_1 + \alpha_4 x_4)$, i.e.,

$$d(R)(x) = 2(\alpha_1 x_1 + \alpha_4 x_4)R(x), \tag{3.1}$$

Thus from (2.9) one deduces that the polynomial $U = R_{(25)}$ satisfies

$$d(U)(x) = -2(\alpha_1 x_1 + \alpha_4 x_4)U(x),$$

and finally (see (D3) in Sec. 2.1) that

$$d(V) = 0,$$

where

$$V := RU = RR_{(25)} = (PP_{(14)})(PP_{(14)})_{(25)} = PP_{(14)}P_{(25)}P_{(14)(25)}.$$

This means that V is a polynomial first integral of the Euler equations (1.2).

The main difficulty is to decide whether V is a fourth integral. We will prove that this is always the case outside of some very special subcases of the Manakov case. This is proved in *Part 2* if the polynomials R and U are relatively prime and in *Part 3* if this is not the case. As in the Manakov case the polynomial fourth integral always exists, this will prove Theorem 3.1.

Part 2. R and U are relatively prime polynomials.

We have to decide when the first integrals H_1, H_2, H_3 (see (1.3)) and V are functionally independent. Let us suppose that they are functionally dependent.

Then for all $\alpha_i,\ 1 \le i \le 6$

$$X_{ij}(V) = X_{ij}(R)U + X_{ij}(U)R = 0. \tag{3.2}$$

We will prove that outside of very special subcases of the Manakov case this contradicts $\alpha_1 \neq 0$.

We have supposed that the polynomials R and U are relatively prime and thus (3.2) shows that either R divides $X_{ij}(R)$, i.e.,

$$X_{ij}(R) = f_{ij}R, \tag{3.3}$$

where f_{ij} is a homogeneous polynomial of second degree, or $X_{ij}(R) = X_{ij}(U) = 0$. For the first possibility, according to (3.2) and (3.3), we have that

$$X_{ij}(U) = -f_{ij}U. \tag{3.4}$$

In particular $X_{25}(R) = f_{25}R$ and $X_{25}(U) = -f_{25}U$. Applying to the first identity the change of variables τ_{25} (see Sec. 2.3), we conclude that $X_{25}(U) = (f_{25} \circ \tau_{25})U$ and finally that $f_{25} = -f_{25} \circ \tau_{25}$. But this is impossible because f_{25} cannot depend on x_2 and x_5. Indeed, the maximal powers of x_2 and of x_5 in $X_{25}(R)$ are never greater than their respective maximal powers in R. Thus $f_{25} = 0$ and consequently $X_{25}(R) = X_{25}(U) = 0$.

We have thus proved that R satisfies the equation

$$X_{25}(R) = \det \frac{\partial(H_1, H_2, H_3, R)}{\partial(x_1, x_3, x_4, x_6)} = 0. \tag{3.5}$$

Let us note that not only $U = R \circ \tau_{25}$ but also $U = R \circ \tau_{36}$. This is so because R is a homogeneous polynomial of even degree and contains monomials that have only an even sum of powers of x_1 and x_4. Thus the monomials of R containing an even sum of powers of x_2 and x_5 contain also an even sum of powers of x_3 and x_6 and respectively, the monomials of R containing an odd sum of powers of x_2 and x_5 contain an odd sum of powers of x_3 and x_6.

As $U = R \circ \tau_{36}$, exactly in the same way as (3.5), we prove that $f_{36} = 0$ or, equivalently, that

$$X_{36}(R) = \det \frac{\partial(H_1, H_2, H_3, R)}{\partial(x_1, x_2, x_4, x_5)} = 0. \tag{3.6}$$

Equations (3.5) and (3.6) represent a system of linear homogeneous partial differential equations for R. This system has four solutions: H_1, H_2, H_3 and R. Consequently, H_1, H_2, H_3, and R are also solutions of the linear homogeneous partial differential equation

$$Y(\Phi) = 0, \tag{3.7}$$

where $Y = [X_{25}, X_{36}]$ is the commutator (Lie bracket) of the vector fields X_{25} and X_{36}.

The three vector fields X_{25}, X_{36} and Y are necessarily linearly dependent on $\mathbb{C}^6$. Indeed, let us suppose that they are linearly independent. System (3.5)-(3.7) has three functionally independent solutions H_1, H_2 and H_3. Then, according to Theorem 2.2, the fourth solution R of this system is represented locally as a function of H_1, H_2, and H_3, i.e., R is a first integral of system (1.2). But this is a contradiction because R is a proper Darboux polynomial of (1.2).

We compute the vector field Y (using Maple) and obtain

$$\begin{aligned}
Y = \Big[&\lambda_{25}x_2x_5(\lambda_{63}x_1x_3x_6 + \lambda_{34}x_3^2x_4 + \lambda_{46}x_4x_6^2)\\
&+ \lambda_{63}x_3x_6(\lambda_{52}x_1x_2x_5 + \lambda_{24}x_2^2x_4 + \lambda_{45}x_4x_5^2)\\
&+ (\lambda_{42}x_2^2 + \lambda_{54}x_5^2)(\lambda_{13}x_1x_3^2 + \lambda_{61}x_1x_6^2 + \lambda_{36}x_3x_4x_6)\\
&+ (\lambda_{34}x_3^2 + \lambda_{46}x_6^2)(\lambda_{12}x_1x_2^2 + \lambda_{51}x_1x_5^2 + \lambda_{25}x_2x_4x_5)\Big]\frac{\partial}{\partial x_1}\\
&+ \Big[(\lambda_{14}x_1x_2 + 2\lambda_{45}x_4x_5)(\lambda_{13}x_1x_3^2 + \lambda_{61}x_1x_6^2 + \lambda_{36}x_3x_4x_6)\\
&+ (2\lambda_{51}x_1x_5 + \lambda_{14}x_2x_4)(\lambda_{63}x_1x_3x_6 + \lambda_{34}x_3^2x_4 + \lambda_{46}x_4x_6^2)\Big]\frac{\partial}{\partial x_2}\\
&+ \Big[(\lambda_{41}x_1x_3 + 2\lambda_{64}x_4x_6)(\lambda_{12}x_1x_2^2 + \lambda_{51}x_1x_5^2 + \lambda_{25}x_2x_4x_5)\\
&+ (2\lambda_{16}x_1x_6 + \lambda_{41}x_3x_4)(\lambda_{52}x_1x_2x_5 + \lambda_{24}x_2^2x_4 + \lambda_{45}x_4x_5^2)\Big]\frac{\partial}{\partial x_3}\\
&+ \Big[\lambda_{52}x_2x_5(\lambda_{13}x_1x_3^2 + \lambda_{61}x_1x_6^2 + \lambda_{36}x_3x_4x_6)\\
&+ \lambda_{36}x_3x_6(\lambda_{12}x_1x_2^2 + \lambda_{51}x_1x_5^2 + \lambda_{25}x_2x_4x_5)\\
&+ (\lambda_{13}x_3^2 + \lambda_{61}x_6^2)(\lambda_{52}x_1x_2x_5 + \lambda_{24}x_2^2x_4 + \lambda_{45}x_4x_5^2)\\
&+ (\lambda_{21}x_2^2 + \lambda_{15}x_5^2)(\lambda_{63}x_1x_3x_6 + \lambda_{34}x_3^2x_4 + \lambda_{46}x_4x_6^2)\Big]\frac{\partial}{\partial x_4}
\end{aligned}$$

$$+\Big[(2\lambda_{12}x_1x_2+\lambda_{41}x_4x_5)(\lambda_{63}x_1x_3x_6+\lambda_{34}x_3^2x_4+\lambda_{46}x_4x_6^2)$$
$$+(\lambda_{41}x_1x_5+2\lambda_{24}x_2x_4)(\lambda_{13}x_1x_3^2+\lambda_{61}x_1x_6^2+\lambda_{36}x_3x_4x_6)\Big]\frac{\partial}{\partial x_5}$$
$$+\Big[(2\lambda_{31}x_1x_3+\lambda_{14}x_4x_6)(\lambda_{52}x_1x_2x_5+\lambda_{24}x_2^2x_4+\lambda_{45}x_4x_5^2)$$
$$+(\lambda_{14}x_1x_6+2\lambda_{43}x_3x_4)(\lambda_{12}x_1x_2^2+\lambda_{51}x_1x_5^2+\lambda_{25}x_2x_4x_5)\Big]\frac{\partial}{\partial x_6}.$$

Let N be the 3×6 matrix composed by the coefficients of vector fields X_{25}, X_{36} and Y. The linear dependence of those vector fields means that the condition

$$\text{rank}N\leq 2 \tag{3.8}$$

should be fulfilled. We consider two cases:

1. $\lambda_1\neq\lambda_4$,
2. $\lambda_1=\lambda_4$.

Case 1. Let N_{123} be the minor of N that consists of first, second and third column. In particular, condition (3.8) implies

$$\delta=\det N_{123}\equiv 0.$$

The Maple computation of δ gives

$$\delta=\lambda_{14}\delta_1\delta_2\delta_3,$$

where

$$\delta_1=\lambda_{31}x_1^2x_3^2+\lambda_{16}x_1^2x_6^2+2\lambda_{63}x_1x_3x_4x_6+\lambda_{34}x_3^2x_4^2+\lambda_{46}x_4^2x_6^2,$$
$$\delta_2=\lambda_{56}x_1x_5x_6+\lambda_{64}x_2x_4x_6+\lambda_{45}x_3x_4x_5,$$
$$\delta_3=\lambda_{12}x_1^2x_2^2+\lambda_{51}x_1^2x_5^2+2\lambda_{25}x_1x_2x_4x_5+\lambda_{42}x_2^2x_4^2+\lambda_{54}x_4^2x_5^2.$$

As we consider now the case $\lambda_1\neq\lambda_4$, then the condition $\delta\equiv 0$ implies that either $\delta_1\equiv 0$ or $\delta_2\equiv 0$ or $\delta_3\equiv 0$. From $\delta_1\equiv 0$ we obtain

$$\lambda_1=\lambda_3=\lambda_4=\lambda_6$$

that is a subcase of the Manakov case. From $\delta_2\equiv 0$ we obtain

$$\lambda_4=\lambda_5=\lambda_6$$

that is a subcase of the Manakov case. From $\delta_3\equiv 0$ we obtain

$$\lambda_1=\lambda_2=\lambda_4=\lambda_5$$

that is a subcase of the Manakov case.

In this way we conclude that in Case 1 condition (3.8) is eventually fulfilled in some subcases of the Manakov case only.

Case 2. In this case, condition (3.8) is fulfilled. Equations (3.5) and (3.6) have four functionally independent solutions. We know three of them explicitly: they are H_1, H_2 and H_3. As it is easily seen, equations (3.5) and (3.6) have also the solution

$$\overline{R} = \lambda_{42}x_2^2 + \lambda_{45}x_5^2.$$

We prove that out of the Manakov case, $\overline{R}$ is functionally independent of H_1, H_2 and H_3. Indeed, if $\overline{R}$ were functionally dependent on H_1, H_2, and H_3, then $\overline{R}$ would be a first integral of the Euler equations (1.2). We compute $d(\overline{R})$ and obtain

$$d(\overline{R}) = 2(\lambda_{24}\lambda_{34}x_1x_2x_3 + \lambda_{54}\lambda_{64}x_1x_5x_6 + \lambda_{24}\lambda_{64}x_2x_4x_6 \\ + \lambda_{54}\lambda_{34}x_3x_4x_5).$$

A necessary condition for the equation $d(\overline{R}) \equiv 0$ to hold is that either $\lambda_2 = \lambda_4$ or $\lambda_3 = \lambda_4$. Let us first suppose that $\lambda_2 = \lambda_4$. Then

$$d(\overline{R}) = 2\lambda_{54}x_5(\lambda_{64}x_1x_6 + \lambda_{34}x_3x_4)$$

and therefore either $\lambda_5 = \lambda_4$ or $\lambda_3 = \lambda_4 = \lambda_6$. Both possibilities together with the condition of Case 2, i.e., $\lambda_1 = \lambda_4$, lead to subcases of the Manakov case.

Let us suppose now that $\lambda_3 = \lambda_4$. Then

$$d(\overline{R}) = 2\lambda_{64}x_6(\lambda_{54}x_1x_5 + \lambda_{24}x_2x_4)$$

and therefore either $\lambda_6 = \lambda_4$ or $\lambda_2 = \lambda_4 = \lambda_5$. As above, both possibilities lead to subcases of the Manakov case.

So far we have proved that outside of the Manakov case the functions H_1, H_2, H_3 and $\overline{R}$ are four functionally independent solutions of equations (3.5) and (3.6). By Theorem 2.2 we know that locally any other holomorphic solution of those two equations is a function of H_1, H_2, H_3 and $\overline{R}$. Thus our Darboux polynomial R, being a solution of equations (3.5) and (3.6), is represented locally in the following way

$$R = W(H_1, H_2, H_3, \overline{R}),$$

where W is some holomorphic function.

The polynomial R satisfies equation (3.1) and therefore

$$d(R) = \frac{\partial W}{\partial \overline{R}} d(\overline{R}) = 2(\alpha_1 x_1 + \alpha_4 x_4)W,$$

which shows that the functions H_1, H_2, H_3, $\overline{R}$, $d(\overline{R})$ and $\alpha_1x_1 + \alpha_4x_4$ are functionally dependent. Thus the determinant $\varDelta$ of their Jacobi matrix is identically zero, i.e.

$$\Delta = 16 \begin{vmatrix} x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \lambda_4 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & \lambda_4 x_4 & \lambda_5 x_5 & \lambda_6 x_6 \\ 0 & \lambda_{42} x_2 & 0 & 0 & \lambda_{45} x_5 & 0 \\ \frac{\partial d(\overline{R})}{\partial x_1} & \frac{\partial d(\overline{R})}{\partial x_2} & \frac{\partial d(\overline{R})}{\partial x_3} & \frac{\partial d(\overline{R})}{\partial x_4} & \frac{\partial d(\overline{R})}{\partial x_5} & \frac{\partial d(\overline{R})}{\partial x_6} \\ \alpha_1 & 0 & 0 & \alpha_4 & 0 & 0 \end{vmatrix} \equiv 0.$$

It is easy to compute Δ (using Maple) but this is not necessary. As Δ equals zero identically, we can compute it only for some fixed particular values of certain variables x_i, $1 \leq i \leq 6$. For $x_1 = 0$, $x_5 = 0$, and $x_6 = 0$ we obtain easily (and without any computer algebra tools) that

$$\Delta\Big|_{x_1=x_5=x_6=0} = 16\alpha_1 x_2 x_3 x_4^2 \lambda_{34}\lambda_{42}(\lambda_{64}\lambda_{42}x_2^2 + \lambda_{34}\lambda_{45}x_3^2) \equiv 0. \quad (3.9)$$

As $\alpha_1 \neq 0$, identity (3.9) is fulfilled in three cases only:

A. $\lambda_2 = \lambda_4$,
B. $\lambda_3 = \lambda_4$,
C. $\lambda_4 = \lambda_5 = \lambda_6$.

Case A. We compute Δ for $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ and obtain

$$\Delta\Big|_{x_1=x_2=x_3=0,\lambda_2=\lambda_4} = 16\alpha_1 \lambda_{45}^2 \lambda_{43}\lambda_{64} x_4^2 x_5^3 x_6 \equiv 0. \quad (3.10)$$

As $\alpha_1 \neq 0$ identity (3.10) is fulfilled if and only if either $\lambda_5 = \lambda_4$ or $\lambda_3 = \lambda_4$ or $\lambda_6 = \lambda_4$. Each of these possibilities together with the conditions $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_4$ leads to a subcase of the Manakov case.

Case B. We compute Δ also for $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ and obtain

$$\Delta\Big|_{x_1=x_2=x_3=0,\lambda_3=\lambda_4} = 16\alpha_1 \lambda_{54}\lambda_{46}^2 \lambda_{42} x_4^2 x_5 x_6^3 \equiv 0. \quad (3.11)$$

As $\alpha_1 \neq 0$, identity (3.11) is fulfilled if and only if either $\lambda_5 = \lambda_4$ or $\lambda_6 = \lambda_4$ or $\lambda_2 = \lambda_4$. Each of these possibilities together with the conditions $\lambda_1 = \lambda_4$ and $\lambda_3 = \lambda_4$ leads to a subcase of the Manakov case.

Case C. This case is directly a subcase of the Manakov case.

Thus the assumption that H_1, H_2, H_3 and V are functionally dependent when R and U are relatively prime can eventually be true in some very special subcases of the Manakov case only.

Remark. We have to note here that there are, indeed, some subcases of the Manakov case when our procedure does not lead to a fourth integral. For example, when $\lambda_1 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ and $\lambda_2 = -\lambda_3$ (subcase of the case C), the polynomial $P = x_2 + x_3$ is a proper Darboux polynomial of the Euler equations (1.2). Applying our procedure on *P*, one obtains, however, a polynomial first integral that is functionally dependent on H_3.

But we know that in the Manakov case there always exists a polynomial fourth integral. That is why we do not exclude the Manakov case from the condition of the theorem.

Part 3. R and U are not relatively prime polynomials.

We have for R and U

$$R = PP_{(14)} \quad \text{and} \quad U = P_{(25)}P_{(14)(25)}.$$

Since the polynomial P is irreducible, the polynomials $P_{(14)}$, $P_{(25)}$ and $P_{(14)(25)}$ are also irreducible. Thus polynomials R and U are not relatively prime in the following 8 cases only:

1. $P = P_{(25)}$;
2. $P = -P_{(25)}$;
3. $P = P_{(14)(25)}$;
4. $P = -P_{(14)(25)}$;
5. $P_{(14)} = P_{(25)}$, that is, equivalent to case 3;
6. $P_{(14)} = -P_{(25)}$, that is, equivalent to case 4;
7. $P_{(14)} = P_{(14)(25)}$, that is, equivalent to case 1;
8. $P_{(14)} = -P_{(14)(25)}$, that is, equivalent to case 2.

Let us examine case 1. The cofactor of P is

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6.$$

According to (2.9) the cofactor of $P_{(25)}$ is

$$-\alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 x_3 - \alpha_4 x_4 + \alpha_5 x_5 - \alpha_6 x_6.$$

Since P and $P_{(25)}$ are equal in the case under consideration, comparing the two cofactors we find

$$\alpha_1 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0,\ \alpha_6 = 0.$$

This contradicts, however our assumption that $\alpha_1 \neq 0$. In the same way, cases 2, 3 and 4 also lead to $\alpha_1 = 0$. □

As an example of an application of the constructing procedure of the fourth integral described in the above proof, let us consider the product case when $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq \lambda_3$. One can easily see that in this case the polynomial

$$P = \frac{\lambda_{21}}{c} x_2 + x_3 + \frac{\lambda_{21}}{c} x_5 + x_6,$$

where $c = \sqrt{\lambda_{13}\lambda_{21}}$, is a proper Darboux polynomial of system (1.2) with cofactor $c(x_1 + x_4)$. Here $P = P_{(14)}$ and thus $R = PP_{(14)} = P^2$ and $U = (P^2)_{(25)} = P^2_{(25)}$. Finally, the polynomial

$$V = RU = \left(PP_{(25)}\right)^2 = \left[-\frac{\lambda_{21}}{\lambda_{13}}(x_2 + x_5)^2 + (x_3 + x_6)^2\right]^2$$

is a fourth integral of (1.2). In fact, in this example, already $PP_{(25)}$ is a fourth integral.

The explicit form of the polynomial fourth integral when $\lambda_2 \neq \lambda_1$ and $\lambda_2 \neq \lambda_3$ or when $\lambda_3 \neq \lambda_1$ and $\lambda_3 \neq \lambda_2$ follows now from Theorem 2.1b applied to the permutational symmetries $\tau = \tau_2 \circ \tau_3$ and τ^2 respectively.

Remark. When comparing our system (1.2) with its "twin brother", which are the Euler-Poisson equations of heavy rigid body motion (see [2,3,14,16, 17]), we conclude from [20] (see also [10]) that for these equations the exact counterpart of Theorem 1.1 holds. Nevertheless, the exact counterpart of Theorem 3.1 for the Euler-Poisson equations fails. Indeed, in the non-integrable so-called Hess-Appelrot case, proper Darboux polynomial exists.

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