

# Similarity flow solutions of a non-Newtonian power-law fluid

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**Abstract:** In this paper we present a mathematical analysis for a steady-state laminar boundary layer flow, governed by the Ostwald-de Wael power-law model of an incompressible non-Newtonian fluid past a semi-infinite power-law stretched flat plate with uniform free stream velocity. A generalization of the usual Blasius similarity transformation is used to find similarity solutions [1]. Under appropriate assumptions, partial differential equations are transformed into an autonomous third-order nonlinear degenerate ordinary differential equation with boundary conditions. Using a shooting method, we establish the existence of an infinite number of global unbounded solutions. The asymptotic behavior is also discussed. Some properties of those solutions depend on the viscosity power-law index.

**Key words:** Boundary-layer, Power-law fluid, Multiple solutions, Similarity transformation

## 1 Introduction

In view of their wide applications in different industrial processes, and also by the interesting mathematical features presented their equations, boundary-layer flows of non-Newtonian fluids have motivated researchers in many branches of engineering in recent years. The most frequently used model in non-Newtonian fluid mechanics is the Ostwald-de Wael model (with a power-law rheology [2, 3, 4, 5, 6]), which the relationship between the shear stress and the strain rate is given as follows

$$\tau_{xy} = k|u_y|^{n-1}u_y, \quad (1)$$

for  $n = 1$  the fluid is called Newtonian with dynamic coefficient of viscosity  $k$ . For  $n > 1$  the behavior of the fluid is dilatant or shear-thickening and for  $0 < n < 1$  the behavior is shear-thinning, in these cases the fluid is non-Newtonian and  $k$  is the fluid consistency. In this work we shall restrict our study to the dilatant fluids, then throughout all the paper, the exponent  $n$  will be taken in the range  $(1, \infty)$ . The problem of laminar flows of power-law non-Newtonian fluids have been studied by several authors. For the sake of brevity, we mention here some examples, Acrivos et al.[7] and Pakdemirli [8] derived the boundary layer equations of power-fluids, Mansutti and Rajagobal [9] investigated the boundary layer flow of dilatant fluids. Adopting the Crocco variable formulation, Nachman and Talliafero [10] established existence and uniqueness of similarity solution for a mass transfer problem. Filipussi et al. [11] obtained similarity solutions and their properties using a phase-plane formalism. Recently numerical solutions have been given by Ece and Büyük in [12] for the steady laminar free convection over a heated flat plate.

More recently Guedda [13] studied the free convection problem of a Newtonian fluid, he showed the existence of an infinite number of solution and studied their asymptotic behavior. In this work we aim to extend the analysis of [13] to the non-Newtonian case, we are interested also in the effect of the power-law index on the existence and the asymptotic behavior of solutions.

The remainder of this work is organized as follows, in the next section, we introduce the mathematical formulation of the problem, section 3 deals with some preliminary tools which will be useful in section 4 and 5 to prove the main results. Finally, we give some concluding remarks in section 6.

## 2 Similarity procedure

The problem is geometrically defined by a semi-infinite power-law stretched rigid plate, over which flows a non-Newtonian fluid obeying to (1). The main hypotheses for the mathematical formulation of this problem are given by:

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- Two-dimensional, incompressible and steady-state laminar flow,
- Physical properties are taken as constants,
- Body force, external gradients pressure and viscous dissipation are neglected.

Under these assumptions, and referred to a Cartesian system of coordinates  $Oxy$ , where  $y = 0$  is the plate, the  $x$ -axis is directed upwards to the plate and the  $y$ -axis is normal to it, the continuity and momentum equations can be simplified, within the range of validity of the Boussinesq approximation [7], to the following equations

$$\begin{cases} uu_x + vu_y = \nu(|u_y|^{n-1}u_y)_y, \\ u_x + v_y = 0, \end{cases} \quad (2)$$

The functions  $u$  and  $v$  are the velocity components in the  $x$ - and  $y$ - directions respectively. The boundary conditions accompanied equation (2) are given by

$$u(x, 0) = U_w(x), \quad v(x, 0) = V_w(x), \quad u(x, y) \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (3)$$

The functions :  $U_w(x) = u_w x^m$  is called the stretching velocity and  $u_w > 0$ , the exponent  $m$  is negative, and  $V_w(x) = v_w x^{\frac{m(2n-1)-n}{n+1}}$  is the suction/injection velocity where  $v_w > 0$  for suction and  $v_w < 0$  for injection.

From the incompressibility of the fluid we introduce the dimensionless stream function  $\psi = \psi(x, y)$  satisfying ( $u = \psi_y$ ,  $v = -\psi_x$ ).

Hence equations (2) are reduced to the single equation

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu(|\psi_{yy}|^{n-1} \psi_{yy})_y. \quad (4)$$

The boundary conditions (3) are transformed into

$$\psi_y(x, 0) = u_w x^m, \quad \psi_x(x, 0) = -v_w x^{\frac{m(2n-1)-n}{n+1}}, \quad \psi_y(x, y) \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (5)$$

Since the broad goal of this paper is to obtain similarity solutions to (4),(5) we introduce the following similarity transformations

$$\psi(x, y) := Ax^\alpha f(t), \quad t := B \frac{y}{x^\beta}. \quad (6)$$

Where  $A, B, \alpha$  and  $\beta$  are real numbers,  $f$  is the transformed dimensionless stream function and  $t$  is the similarity variable. In terms of (6), equation (4) can satisfy the ordinary differential equation of the shape function :  $f$

$$(|f''|^{n-1} f'')' + \alpha f f'' = (\alpha - \beta) f'^2, \quad (7)$$

where the primes denote differentiation with respect to  $t$ , if and only if the following

$$\alpha(2 - n) + \beta(2n - 1) = 1, \quad \text{and} \quad \alpha - \beta = m,$$

holds, and the parameters  $A, B$  and  $\nu$  satisfy

$$\nu A^{n-2} B^{2n-1} = 1, \quad \text{and} \quad m = \alpha - \beta. \quad (8)$$

It follows that

$$\alpha = \frac{1 + m(2n - 1)}{n + 1}, \quad \beta = \frac{1 + m(n - 2)}{n + 1}, \quad \text{and} \quad p = \frac{m(2n - 1) - n}{n + 1}.$$

Consequently, we have

$$\psi(x, y) := \nu^{\frac{1}{n+1}} x^{\frac{1+m(2n-1)}{n+1}} f(t), \quad t := \nu^{-\frac{1}{n+1}} y x^{-\frac{1+m(n-2)}{n+1}}. \quad (9)$$

The corresponding boundary conditions (5) are expressed as

$$f(0) = \frac{-v_w}{A\alpha}, \quad f'(0) = \frac{u_w}{A}, \quad f'(\infty) = \lim_{t \rightarrow \infty} f'(t) = 0. \quad (10)$$

In the remainder, we deal with the following problem

$$\begin{cases} (|f''|^{n-1}f'')' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = \lim_{t \rightarrow \infty} f'(t) = 0. \end{cases} \quad (11)$$

For Newtonian fluids ( $n = 1$ ), problem (11) reads

$$\begin{cases} f''' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 0. \end{cases} \quad (12)$$

We notice that this problem arises from two different contexts in fluid mechanics when looking for similarity solutions. First, in natural convection along a vertical heated flat plate, embedded in a saturated porous medium, where the temperature is a power function with the exponent  $m$ , for more details, we refer the reader to [13, 14, 16] and the references therein. Equation (12) appears also in the study of the boundary layer flow, of a Newtonian fluid, adjacent to a stretching surface with a power-law velocity (see [15, 17]). In [14], the authors proved that (12) with  $a = b = 0$ , has a solution (which is bounded) if  $m \geq -\frac{1}{3}$  and this solution is unique for  $0 \leq m \leq \frac{1}{3}$ . In [15] the author gives a complete study about existence and nonexistence of solutions to (12), where  $b = 1$ .

Recently some new results have been obtained in [13]. The author considered the problem (12), where  $m \in (-\alpha, 0)$ . He showed that, under some assumptions, problem (12) has an infinite number of unbounded solutions and these solutions satisfy  $f(t) \sim t^{\frac{\alpha}{\alpha-m}}$ , as  $t$  goes to infinity.

Based on the results of [13], the interest in this work will be in existence and asymptotic behavior of solutions of problem (11).

### 3 Preliminary results

As it is announced above, the existence of solutions will be established by a shooting method. We replace the boundary condition at infinity by  $f''(0) = d$ , where  $d \neq 0$ . Therefore, we consider the initial value problem

$$\begin{cases} (|f''|^{n-1}f'')' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f''(0) = d. \end{cases} \quad (13)$$

We shall see that for appropriate  $d$  problem (13) has a global unbounded solution and this solution satisfies the boundary condition at infinity.

**Remark 1** We notice that for  $n \neq 1$ , equation (11)<sub>1</sub> can be degenerate or singular at the point  $t_0$  where  $f''(t_0) = 0$ . The existence the  $t_0$  is done for  $d > 0$ . We shall see also that  $f'''$  is not bounded at  $t_0$  (the solution  $f$  is then not classical). By a solution to (11) we will mean a function  $f \in C^2(0, \infty)$  such that  $|f''|^{n-1}f'' \in C^1(0, \infty)$ ,  $f'(\infty) = 0$  and  $f''(\infty) = 0$ . Note also that any solution is classical on any interval where the second derivative does not change the sign.

Consider now the initial value problem (13) with  $n > 1$ ,  $a, d \in \mathbb{R}$ ,  $b \geq 0$  and  $m \in (-\alpha, 0)$ .

By the classical theory of ordinary differential equations the above problem has local (maximal) solutions on some interval  $(0, T_d)$ ,  $T_d \leq \infty$  and they are uniquely determined by  $d$  ( $d \neq 0$ ). Let us denote this such solution by  $f_d$ . Integrating (13)<sub>1</sub> to get the following identity

$$|f_d''|^{n-1}f_d''(t) + \alpha f_d'(t)f_d(t) = |d|^{n-1}d + \alpha ab + (m + \alpha) \int_0^t f_d'(s)^2 ds, \quad \forall t < T_d. \quad (14)$$

which will be used later for proving some results.

A solution  $f_d$  of (11), is of class  $C^2$  on  $[0, T_d)$ , and satisfies  $|f_d''|^{n-1} f_d'' \in C^1([0, T_d))$ . We shall investigate whether  $f_d$  admits an entire extension. First, we give the following result characterizing the existence time  $T_d$ .

**Proposition 2** *Let  $f_d$  be the local solution to (13), if  $T_d$  is finite then the functions  $f_d$ ,  $f_d'$  and  $f_d''$  are unbounded as  $t$  approaches  $T_d$  from below.*

**Proof.** Similar to [15, 18]. ■

Let us note also that if we require a classical solution of (13) (ie.  $f \in C^3(0, \infty)$ ), it is possible that  $f$  ceases to exist at some  $T < \infty$  and such that  $f, f'$  and  $f''$  remain bounded on  $[0, T)$ . More precisely we have the following result.

**Proposition 3** *Let  $f_d$  be the local solution to (13) where  $n > 1$  and  $d \neq 0$ . Assume that there exists  $t_0 \in (0, T_d)$  such that  $f_d''(t_0) = 0$ . Then  $d > 0$ ,  $f_d'' < 0$  on  $(t_0, T_d)$  and  $f_d'''$  is unbounded on  $(0, t_0)$ .*

**Proof.** Assume first that  $d < 0$ . Therefore  $f_d'' < 0$  on  $[0, \varepsilon)$ ,  $\varepsilon$  small, and the following equation

$$n|f_d''|^{n-1} f_d'' + \alpha f f'' - m f'^2 = 0, \quad (15)$$

holds on  $(0, \varepsilon)$ . Hence

$$(f_d'' e^F)' = \frac{m}{n} e^F |f_d''|^{1-n} f_d'^2, \quad \text{on } (0, \varepsilon), \quad (16)$$

where

$$F(t) = \frac{\alpha}{n} \int_0^t f_d |f_d''|^{1-n}(s) ds.$$

Consequently, the function  $t \rightarrow f_d'' e^F(t)$  is decreasing, and then  $f_d''(t)$  remains negative for all  $t \in [0, T_d)$ . A contradiction. Then  $d > 0$ . Actually, we have  $f_d'' > 0$ ,  $f_d' > b$  on  $(0, t_0)$  and equation (15) holds on  $(0, t_0)$ . Assume now that  $f_d'''$  is bounded on  $(0, t_0)$ . Thanks to equation (11)<sub>1</sub> we deduce that  $f_d'(t_0) = 0$  this is contradiction with  $f'(0) > b$ . ■

From the above we can see, in particular, that  $f_d'' < 0$  on  $(0, T_d)$  for any  $d < 0$ . Then  $f_d \in C^\infty([0, T_d))$ . While for the case  $d > 0$  the solution  $f_d$  is not classical.

**Proposition 4** *Let  $f_d$  be the local solution to (13) for  $d \neq 0$  and  $n > 1$ . If  $T_d < \infty$  then  $\lim_{t \rightarrow T_d} f_d(t) = -\infty$ .*

**Proof.** First we show that  $\sup_{[0, T_d)} |f_d(t)| = \infty$ . Suppose not and  $f_d''(t_0) = 0$  holds, for some  $t_0 \in (0, T_d)$ . From (14) we get

$$-(-f_d'')^n(t) + \alpha f_d'(t) f_d(t) = \alpha f_d'(t_0) f_d(t_0) + (m + \alpha) \int_{t_0}^t f_d'(s)^2 ds, \quad \forall t_0 < t < T_d.$$

Hence

$$\frac{\alpha}{2} f_d^2(t) - \alpha f_d'(t_0) f_d(t_0) (t - t_0) - \frac{\alpha}{2} f_d^2(t_0) = (m + \alpha) \int_{t_0}^t \int_{t_0}^\tau f_d'^2(s) ds d\tau + \int_{t_0}^t (-f_d'')^n(s) ds.$$

Since the right-hand side of the above is monotonic increasing with respect to  $t$ , the function  $f_d$  has a finite limit as  $t \rightarrow T_d$ . Consequently the function  $(-f_d'')^n$  is integrable on  $(t_0, T_d)$ . Since  $n > 1$  we deduce that  $f_d''$  is also integrable on  $(t_0, T_d)$ . Therefore the function  $f_d'$  is bounded. Next we use (14) to deduce that  $f_d''$  is also bounded. A contradiction with Proposition 2.

It remains to prove that the hypothesis  $f_d'' > 0$  on  $(0, T_d)$  leads also to a contradiction. Actually, in such situation, we know that  $f_d$  is classical and satisfies (15), which yields to

$$(f_d'')^{n-2} f_d''' \leq -\frac{\alpha}{n} f_d,$$

and then

$$(f_d'')^{n-2} f_d''' \leq \frac{\alpha}{n} \sup_{[0, T_d)} |f_d(t)|.$$

Therefore  $f_d''$  and  $f_d'$  are bounded. A contradiction.

Because  $f_d$  is monotonic on  $(\tau, T_d)$ , for some  $0 < \tau < T_d$ , we deduce that  $|f_d(t)|$  goes to infinity as  $t \rightarrow T_d$ . Finally, to show that  $f_d(t) \rightarrow -\infty$  as  $t \rightarrow T_d$  we assume on the contrary that  $\lim_{t \rightarrow T_d} f_d(t) = \infty$ . Hence the functions  $f_d$  and  $f_d'$  are positive on  $(\tau, T_d)$ . Moreover, using (11)<sub>1</sub>, we can deduce from the Energy-function defined by

$$E(t) = \frac{n}{n+1} |f_d''(t)|^{n+1} - \frac{m}{3} f_d'^3, \quad (17)$$

and satisfies  $E'(t) = -\alpha f_d f_d''^2 \leq 0$ . That  $f_d''$  and  $f_d'$  are bounded. Hence  $f_d$  is also bounded and this is a contradiction with Proposition.2. We conclude that if  $T_d$  is finite the function  $f_d$  goes to minus infinity as  $t \rightarrow \infty$ . ■

## 4 Existence of solutions

In this section we shall obtain a sufficient condition on  $d$  such that the local solution  $f_d$  of (13) is global and satisfies the condition  $f_d'(\infty) = 0$ . We show that, for each  $d$  satisfying  $|d|^{n-1}d > -\alpha ab$ ,  $f_d$  exists on the entire positive axis  $\mathbb{R}^+$  and satisfies  $f_d'(\infty) = 0$ . We begin by a simple observation that: if  $m + \alpha > 0$  and  $|d|^{n-1}d > -\alpha ab$ , (14) yields the important fact that  $f_d$  cannot have a local maximum. Thus we prove the following result.

**Theorem 5** *Let  $a \in \mathbb{R}, b \geq 0$  and  $m \in (-\alpha, 0)$ . For any  $d$  such that  $|d|^{n-1}d > -\alpha ab$ , there exists a unique global solution  $f_d$ , to (13), which goes to infinity with  $t$ , and its first and second derivative tend to 0 as  $t$  approaches infinity.*

For our analysis, we need to distinguish two cases for the parameter  $a = f_d(0)$ ; namely  $a \geq 0$  and  $a < 0$ . First we prove the following lemma.

**Lemma 6** *If  $a \geq 0$  and  $|d|^{n-1}d > -\alpha ab$  the functions  $f_d'$  and  $f_d$  are positive on  $(0, T_d)$  and  $T_d = \infty$ ; that is  $f_d$  is global. Moreover  $f_d'$  and  $f_d''$  are bounded.*

**Proof.** Because  $|d|^{n-1}d + \alpha ab > 0$ , the first assertion of the lemma is immediate from (14). To demonstrate that  $T_d = \infty$  it suffices to show that  $f_d$  remains bounded on any bounded interval  $[0, T]$ . Let us consider the Lyapunov function  $E$  for  $f_d$  defined by (17). Since

$$E'(t) = -\alpha f_d f_d''^2 \leq 0,$$

thanks to (11)<sub>1</sub>, it is seen that

$$\frac{n}{n+1} |f_d''(t)|^{n+1} - \frac{m}{3} f_d'(t)^3 \leq \frac{n}{n+1} |d|^{n+1} - \frac{m}{3} b^3, \quad \forall t < T_d.$$

This in turn implies that  $f_d'', f_d'$  and then  $f_d$  are bounded on  $[0, T]$ . ■

**Lemma 7** *If  $a \geq 0$  and  $|d|^{n-1}d > -\alpha ab$ ,  $f_d(t)$  tends to infinity with  $t$ ,  $f_d'$  and  $f_d''$  tend to zero as  $t \rightarrow \infty$ .*

**Proof.** Since  $f_d'$  is monotonic on  $(t_1, \infty)$ ,  $t_1$ , large enough, and bounded there exists a  $l \geq 0$  such that

$$\lim_{t \rightarrow \infty} f_d'(t) = l.$$

This implies the existence of a sequence  $(t_n)$  tending to infinity with  $n$  satisfying  $\lim_{n \rightarrow +\infty} f_d''(t_n) = 0$  and then  $\lim_{t \rightarrow \infty} f_d''(t) = 0$ , with the help of the energy function  $E$ .

Now we assume that  $f_d$  is bounded, therefore  $l = 0$ . Subsequently

$$|d|^{n-1}d + \alpha ab + (m + \alpha) \int_0^\infty f_d'(t)^2 dt = 0.$$

This is impossible. Therefore  $f_d$  is unbounded and then  $\lim_{t \rightarrow +\infty} f_d(t) = \infty$ . It remains to prove that  $l = 0$ . Assume on the contrary that  $l > 0$ . Together with (14) we get

$$|f_d''|^{n-1} f_d''(t) = -\alpha l^2 t + (m + \alpha) l^2 t + o(t),$$

$$|f_d''|^{n-1} f_d''(t) = m l^2 t + o(t),$$

as  $t$  approaches infinity, that This is only possible if  $m = 0$ . Consequently  $l = 0$ . ■

Next we consider the case  $a < 0$ .

The first simple consequence is that  $f_d(t) < 0$  and  $f_d'(t) > 0$  for small  $t > 0$ . Since  $f_d$  cannot have a local maximum, we have two possibilities

- Either  $f_d(t)$  vanishes at a some point and remains positive after this point.
- Or  $f_d(t) < 0$  for all  $t > 0$ .

Hence the proof of Theorem 5 is completed by the following lemma.

**Lemma 8** Assume  $a < 0$  and  $|d|^{n-1}d > -\alpha ab$ . Then  $f_d$  has exactly one zero, goes to  $\infty$  with  $t$ , and the functions  $f_d', f_d''$  converge to 0 as  $t \rightarrow \infty$ .

**Proof.** Assume that the first assertion holds. Since  $f_d'$  is positive we deduce that  $f_d$  is bounded and then is global. On the other hand, using (14) one sees that  $f_d'' > 0$ . Therefore we get  $\lim_{t \rightarrow \infty} f_d(t) \in (a, 0]$  and  $\lim_{t \rightarrow \infty} f_d'(t) = 0$ , since  $f_d'$  is monotonic. This is absurd since  $f_d'$  is positive and increasing function. Hence  $f_d$  has exactly one zero, say  $t_0$ . To finish the Proof of Lemma 8 and therewith that of Theorem 5 we note that the new function

$$h(t) = f_d(t + t_0)$$

satisfies equation (11)<sub>1</sub> and

$$h(0) \geq 0, \quad h''(0) > -\alpha h(0)h'(0).$$

Therefore we use Lemmas 6 and 7 to conclude. ■

In the next result we complete our analysis on the existence of global solutions by the case  $b < 0$ .

**Theorem 9** Let  $b < 0, a > 0$  and  $m \in (-\alpha, 0)$ . For any  $d > 0$  satisfying

$$ad^n - \frac{1}{2}b^2d^{n-1} + \alpha a^2b > 0. \quad (18)$$

The unique local solution,  $f_d$  to (13) is global unbounded and satisfies  $\lim_{t \rightarrow \infty} f_d'(t) = \lim_{t \rightarrow \infty} f_d''(t) = 0$ .

**Proof.** Since  $a, d > 0$  and  $b < 0$ , there exists a real  $t_0 > 0$  such that  $f_d$  is positive, decreasing and convex on  $(0, t_0)$ . Define

$$T = \sup \{t : f_d(s) > 0, f_d'(s) < 0, f_d''(s) > 0, \text{ for all } s \in (0, t)\}.$$

The real number  $T$  is larger than  $t_0$  and may be infinite.

Assume that  $T = \infty$ . Then the function  $f_d$  has a finite limit at infinity and  $f_d'(t)$  (and  $f_d''$ ) go to zero as  $t \rightarrow \infty$ . Since the function

$$H = f_d |f_d''|^{n-1} f_d'' - \frac{1}{2} f_d'^2 |f_d''|^{n-1} + \alpha f_d^2 f_d',$$

satisfies

$$H' = f(f_d')^2 \left[ m + 2\alpha + \frac{\alpha(n-1)}{2n} \right] - \frac{m(n-1)}{2n} (f_d')^4 (f_d'')^{-1},$$

thanks to (11)<sub>1</sub>, we deduce that  $H$  is increasing on  $(0, \infty)$ . Hence for  $t > 0$  we have

$$H(0) < \lim_{t \rightarrow \infty} H(t) = 0,$$

which yields to

$$ad^n - \frac{1}{2}b^2d^{n-1} + \alpha a^2b < 0.$$



A contradiction. Therefore  $T$  is infinite. Next, we assume that  $f_d(T) = 0$  or  $f_d''(T) = 0$ . Arguing as above we deduce  $H(0) < 0$  and then we get a contradiction. In conclusion if condition (18) holds the function  $f_d$  has a local positive minimum at some  $t_1 > 0$ . We use Theorem 5 to deduce that the new function  $h(t) = f_d(t + t_1)$  is global, unbounded and satisfies  $h'(\infty) = h''(\infty) = 0$ . The proof is finished. ■

**Remark 10** We notice that we can extend the results of Theorem 5 to the case  $(-2\alpha, -\alpha)$  by using the function  $H$  defined above, as in the work by Guedda [13] for the Newtonian case.

## 5 Asymptotic behavior

In this section we shall derive the asymptotic behavior of any possible global unbounded solution to (11) for  $m \in (-2\alpha, 0)$ . First we give the following result

**Lemma 11** *Let  $f$  be a positive solution to (11) for  $m \in (-2\alpha, 0)$ . Then  $f'$  goes to zero at infinity and  $f''$  is negative.*

**Proof.** Since  $f$  is monotonic on  $[t_0, \infty)$ ,  $t_0$  large enough, we get the positivity of  $f'$  and  $f$  on  $(t_0, \infty)$ . In addition we use the Lyapunov function to get the boundedness of  $f'$  and  $f''$ . Arguing as in the previous section we get that  $f' \rightarrow 0$  and  $f'' < 0$  for large  $t$ . ■

**Proposition 12** *Assume that  $n > 1$  and  $m \in (-2\alpha, 0)$ . Let  $f$  be a positive solution to (11). Then*

$$\lim_{t \rightarrow \infty} f_d(t) f_d''(t) = \lim_{t \rightarrow \infty} (|f''|^{n-1} f'')'(t) = 0.$$

**Proof.** Thanks to lemma (11) we have  $f'(t) > 0$ ,  $f''(t) < 0$  for all  $t > t_0$ ,  $t_0$  large enough and  $f'$  and  $f''$  tend to 0 as  $t \rightarrow \infty$ . Then equation (11)<sub>1</sub> can be written as

$$f''' + \frac{\alpha}{n} f f'' |f''|^{1-n} = \frac{m}{n} f'^2 |f''|^{1-n}, \quad \forall t > t_0.$$

By differentiation we have

$$f^{(iv)} + f''' \left[ \frac{\alpha(2-n)}{n} |f''|^{1-n} f - \frac{m(1-n)}{n} |f''|^{-n-1} f'' f'^2 \right] = -\frac{\alpha-2m}{n} f' f'' |f''|^{1-n}. \quad (19)$$

Then the function  $f''' e^G$  is monotonic increasing on  $(t_0, \infty)$ , where

$$G' = \frac{\alpha(2-n)}{n} |f''|^{1-n} f - \frac{m(1-n)}{n} |f''|^{-n-1} f'' f'^2.$$

This indicates that the function  $f'''$  has at most one zero. Because  $f''$  is negative and goes to 0 at infinity, we deduce that  $f'''(t) > 0$  on  $(t_1, \infty)$ , for  $t_1$  large. On the other hand, from (11)<sub>1</sub> we deduce

$$(|f''|^{n-1} f'')' + (\alpha - 2m) f' f'' = -\alpha f f'''.$$

Therefore the function  $t \mapsto (|f''|^{n-1} f'')' + \frac{\alpha-2m}{2} f'^2$  is positive and monotonic decreasing on  $(\inf \{t_0, t_1\}, \infty)$ . Together with the fact that  $f'$  tends to 0 as  $t \rightarrow \infty$  we deduce that

$$\lim_{t \rightarrow +\infty} (|f''|^{n-1} f'')'(t) = 0$$

and then we conclude that  $f f''(t) \rightarrow 0$  as  $t \rightarrow \infty$ , thanks to (11)<sub>1</sub>. ■

**Proposition 13** *Let  $f$  be a solution to (11) where  $n > 1$ ,  $m \in (-2\alpha, 0)$ . Then*

$$\lim_{t \rightarrow +\infty} f f''(t) = \begin{cases} \infty, & \text{if } m + \alpha > 0, \\ L \in (0, \infty), & \text{if } m + \alpha = 0, \\ 0, & \text{if } m + \alpha < 0. \end{cases}$$

**Proof.** Let  $f$  be a global solution to (11). First we claim that there exists  $t_0 \geq 0$  such that

$$|f''|^{n-1} f''(t_0) + \alpha f f'(t_0) > 0.$$

Suppose not; that is

$$|f''|^{n-1} f''(t) + \alpha f f'(t) \leq 0,$$

for all  $t \geq 0$ . Since  $f''(t) \rightarrow 0$  as  $t \rightarrow \infty$  the following

$$f'' + \alpha f f' \leq 0$$

holds on some  $(t_1, \infty)$ ,  $t_1$  large. Consequently the function  $f' + \frac{\alpha}{2} f^2$  is decreasing and goes to infinity with  $t$ , which is absurd. Now we use the identity

$$|f''|^{n-1} f''(t) + \alpha f f'(t) = |f''|^{n-1} f''(t_0) + \alpha f f'(t_0) + (m + \alpha) \int_{t_0}^t f'^2(s) ds,$$

to deduce that  $f f'$  has a limit  $L \in [0, \infty]$  at infinity. This limit is finite for  $\alpha + m = 0$ . Assume that  $\alpha + m \neq 0$ . If  $L \in (0, \infty)$  we get immediately that  $f' \sim \sqrt{\frac{L}{\eta}}$  at infinity which implies that  $f f' \rightarrow \infty$ . A contradiction. Consequently  $L \in \{0, \infty\}$ , we use again the above identity to conclude that  $L = \infty$  if  $m + \alpha > 0$  and  $L = 0$  if  $m + \alpha < 0$ . ■

**Remark 14** We stress that the condition

$$|f''|^{n-1} f''(t_0) + \alpha f f'(t_0) > 0, \quad f'(t_0) \geq 0$$

is necessary and sufficient to obtain a global solution converging to plus infinity with  $t$  in the case  $m \in (-\alpha, 0)$ .

Now we are ready to give the result concerning the large  $t$ -behavior of solutions to (11).

**Theorem 15** Suppose  $n > 1$ ,  $-2\alpha < m < 0$ . Let  $f$  be a solution to (11) such that  $f \rightarrow \infty$ . Then there exists a constant,  $A > 0$ , such that

$$f(t) = t^{\frac{\alpha}{\alpha-m}} (A + o(1)), \quad (20)$$

as  $t \rightarrow \infty$ .

**Proof.** Let  $f$  be a global solution to (11). First we prove the result for the case  $m + \alpha > 0$ .

Let  $t_0$  be a real number such that  $f'' < 0$  and  $f''' > 0$  on  $(t_0, \infty)$ . Dividing equation (11)<sub>1</sub> by  $f f'$  gives

$$\frac{(|f''|^{n-1} f'')'}{f f'} = m \frac{f'}{f} - \alpha \frac{f''}{f'}.$$

Integrating over  $(t_1, t)$ , for  $t_1 > t_0$ , gives

$$\int_{t_1}^t \frac{(|f''|^{n-1} f'')'}{f f'} ds = \log(f^m(t) f'^{-\alpha}(t)) - \log(f^m(t_1) f'^{-\alpha}(t_1)).$$

According to Proposition 13,  $f f'$  goes to infinity with  $t$  and then the left hand side of the above is integrable, consequently  $f^m f'^{-\alpha}$  has a positive finite limit at infinity. The desired asymptotic behavior (20) follows by a simple integration.

Now we deal with the case  $m + \alpha < 0$ . For this sake we define

$$\Psi = \varphi(f) |f''|^{n-1} f'' - \frac{1}{2} \varphi'(f) (f')^2 |f''|^{n-1} + \alpha \varphi(f) f f',$$

where  $\varphi$  is a smooth function. Then it follows from (11)<sub>1</sub>

$$\Psi'(t) = f'^2 [\alpha f \varphi'(f) + (\alpha + m) \varphi] - \frac{1}{2} \varphi''(f) f'^3 |f''|^{n-1} - \frac{n-1}{2} \varphi'(f) f'^2 |f''|^{n-3} f'' f''''.$$



Let the function  $\varphi$  be defined by

$$\varphi(s) = s^{-\frac{m+\alpha}{\alpha}}.$$

It satisfies the following differential equation

$$\alpha s \varphi'(s) + (\alpha + m)\varphi = 0.$$

This implies that

$$\Psi' = -\frac{1}{2}\varphi''(f)f'^3|f''|^{n-1} - \frac{n-1}{2}\varphi'(f)f'^2|f''|^{n-3}f''f''' \geq 0,$$

$$\Psi = \varphi(f) \left[ |f''|^{n-1}f'' - \frac{\alpha+m}{2\alpha f}(f')^2|f''|^{n-1} + \alpha f f' \right],$$

and then

$$|\Psi'(t)| \leq \varepsilon \left[ f^{-\frac{3\alpha+m}{\alpha}}f' + (n-1)(-f'')^{n-2}f''' \right],$$

for all  $t \geq t_0, t_0$  large. Therefore  $\Psi'$  is integrable on  $[0, \infty)$  and then  $\Psi$  has a finite limit at infinity, say  $L$ . Next we show that  $L > 0$ . It will be sufficient to show that  $\Psi(t_1) > 0$  for some  $t_1$  large. Suppose not; that is for any  $t > t_2, t_2$  large we have

$$|f''|^{n-1}f'' - \frac{\alpha+m}{2\alpha f}f'^2|f''|^{n-1} + \alpha f f' \leq 0.$$

Since  $\alpha + m < 0$  then

$$f'' + \alpha f f' \leq 0,$$

from which we deduce, as above, that  $f' + \frac{\alpha}{2}f^2$  is a decreasing function going to infinity with  $t$ . A contradiction. We conclude that  $\lim_{t \rightarrow \infty} f^{-\frac{m}{\alpha}}f' = \frac{L}{\alpha}$ . Finally, a simple integration leads to estimate (20).

To finish, we pay attention to the case  $m = -\alpha$ . Here the identity (14) leads to

$$|f''|^{n-1}f'' + \alpha f f' = |\gamma|^{n-1}\gamma + \alpha ab.$$

According to Theorem 5,  $f$  is global and satisfies  $f \sim t^{\frac{1}{2}}$  at infinity. ■

## 6 Conclusion

The laminar two-dimensional steady boundary layer flow, of a non-Newtonian incompressible fluid, over a stretching surface have been considered. Using the shooting method, existence of global unbounded similarity solutions have been shown, the dependency of those solutions on the power-law index has been investigated, and their asymptotic behavior was also discussed.

Coming back to the original problem (2),(3) we find that, for  $-2\alpha < m < 0$ , the stream function satisfies

$$\psi(x, y) \sim y^{\frac{1+m(2n-1)}{1+m(n-2)}} \quad \text{as} \quad yx^{\frac{(2-n)m-1}{n+1}} \rightarrow \infty.$$

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