A new proof of the uniqueness of the flow for ordinary differential equations with BV vector fields

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Abstract We provide in this article a new proof of the uniqueness of the flow solution to ordinary differential equations with BV vector-fields that have divergence in L^∞ (or in L^1) and that are nearly incompressible (see the text for the definition of this term). The novelty of the proof lies in the fact it does not use the associated transport equation.

1 Introduction and statement of our main result

In 1989, P.-L. Lions and R. DiPerna showed in [DL89] the existence and the uniqueness of the almost everywhere defined flow solution to an ordinary differential equation of the type:

$$\dot{y}(t) = b(t, y(t))\,, \tag{1}$$

for $W^{1,1}$ vector fields b with $L^1_{loc}(\mathbb{R}_t, L^\infty_y)$ divergence (along with some technical assumptions). For such 'singular' vector fields, the only possibility is to solve the equation *almost everywhere* on the space Ω of initial conditions. In that case, one defines a flow $X(t,x) : \mathbb{R} \times \Omega \to \Omega$ satisfying:

$$\begin{cases} \dot{X}(t,x) = b(t, X(t,x)) & \text{for all } t, \\ X(0,x) = x \end{cases} . \tag{2}$$

for almost all $x \in \Omega$. An initial time $s \neq 0$ may of course be chosen, and the flow then depends parametrically on this initial time s. With a view to simplifying the presentation, we will assume henceforth and throughout this article that the field b is time-independent. Our arguments may be modified to cover the time-dependent case.

In the present article, we also adopt a notion of *almost everywhere flow solution* similar to that of DiPerna and Lions. We denote by $(X(t,\cdot)_\#\lambda)(E) = \lambda(X(-t,E))$ the pushforward of the Lebesgue measure λ. In the sequel, the vector-field b will always be assumed at least L^1_{loc}.

Definition 1 (Almost everywhere flows) *An almost everywhere flow solution to* (2) *is a measurable function $X(t,x) : \mathbb{R} \times \Omega \to \Omega$ satisfying the following conditions:*

(i) For almost all $x \in \Omega$, the map $t \mapsto X(t,x)$ is a continuous solution to $\dot{\gamma} = b(\gamma)$ satisfying $\gamma(0) = x$:

$$\text{for almost all } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad X(t,x) = x + \int_0^t b(X(s,x))\,ds$$

(ii) For all t, the measure $X(t,\cdot)_\#\lambda$ is absolutely continuous with respect to λ, and there exist a time-dependent function $C(t) > 0$ such that:

$$\forall t \in \mathbb{R}_+,\ \forall |\tau| \leq t, \quad \frac{1}{C(t)}\lambda \leq X(\tau,\cdot)_\#\lambda \leq C(t)\lambda\,,$$

(iii) X is a one-parameter transformation group, i.e. satisfies:

$$X(t, X(s,x)) = X(s+t, x), \quad \text{for almost all } x \in \Omega, \forall s,\ t$$

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Remark 1 *Di Perna and Lions originally define in [DL89] a flow solution with condition (i) replaced by* $X \in \mathcal{C}(\mathbb{R}, L^1)$ satisfies the ordinary differential equation (2) in the sense of distribution. *Their definition is equivalent to ours. It is indeed shown in [DL89] that the original definition implies* (i)*, and it can be shown that conditions* (ii) *and* (iii) *together imply that* $X \in \mathcal{C}(\mathbb{R}, L^1)$*.*

Remark 2 *Condition* (ii) *is in particular satisfied if* b *is sufficiently smooth, and* $\operatorname{div}(b) \in L^\infty$*. In that case* $C(t) = e^{\|\operatorname{div}(b)\|_\infty |t|}$ *is convenient. Besides this case, for a class of ordinary differential equations coming from some particular types of hyperbolic equations, such as the Keyfitz-Krantzer system,* $\operatorname{div}(b)$ *is only* L^1*, but an estimate of the form* (ii) *may be established using a maximal principle. See the work [Bre03] by Bressan for more details on these systems and that by L. Ambrosio, F. Bouchut and C. De Lellis [ABDL04] for a discussion on the relevance of condition* (ii)*.*

Before stating our result, we give a brief state-of-the-art survey on the theory of ordinary differential equations with vector fields of low regularity. The seminal work [DL89] by DiPerna and Lions has been complemented and extended notably by L. Ambrosio in [Amb04]. Several other authors have made important contributions. We would like to specifically cite the work [Ler04] by N. Lerner which has inspired our own, present work. To date, the minimal conditions that are known to guarantee the existence and the uniqueness of the flow are the BV regularity of the vector field, a L^1 bound on the divergence together with a near-incompressibility condition (or more classicaly a bounded divergence condition) of the type (ii). The classical proofs of such results are based upon the consideration of the associated transport equation, written either in the conservative form:

$$\frac{\partial u}{\partial t} + \operatorname{div}(b(x)u) = 0, \tag{3}$$

or in the non-conservative form:

$$\frac{\partial u}{\partial t} + b(x) \cdot \nabla_x u = 0, \tag{4}$$

both with the initial condition $u(0,x) = u_0(x)$. Remark that, for divergence-free fields, the two equations coincide. When the existence and the uniqueness of the solution to the transport equation is established, for any given initial condition, one deduces the same result for the a.e. flow solution to the ordinary differential equation. The key ingredient for the resolution of the transport equation is a commutation lemma (first stated in [DL89]), which says that:

$$\rho_\varepsilon * \operatorname{div}(bu) - \operatorname{div}((b * \rho_\varepsilon)u) \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ in } L^1.$$

C. De Lellis and G. Crippa have recently given in [CDL08] a new proof of the existence and uniqueness of the flow solution of (2), not using the the associated transport equation. Their very interesting approach provides regularity estimates for $W^{1,p}$ vector-fields with $p > 1$ but seemingly fails for $W^{1,1}$ vector-fields, unfortunately.

1.1 Main result

The purpose of this article is to give a new and direct proof of the uniqueness of the a.e. flow solution to (2) for BV vector fields, *without* arguing on the associated transport equation. We adopt the approach already used in [HLBL07] for $W^{1,1}$ vector fields. Basically, the commutation lemma instrumental in the proof contained in this prior publication is replaced by another strategy of proof, namely the introduction of a second variable. This is explained in details in the next paragraph.

Our result is the following:

Theorem 1 *Let* b *be a* BV *vector field on the* N*-dimensional torus* $\mathbb{T}^N$*. If* $\operatorname{div}(b) \in L^1$*, then there exists at most one a.e. flow solution to* (2)*, in the sense of Definition 1.*

Remark 3 *In [CDP96], B. Perthame and I. Capuzzo Dolcetta remarked that the assumption "*$b \in W^{1,1}$*" of the original work by DiPerna and Lions could be replaced by the weaker assumption "the symmetric part of* Db *is a matrix-valued* L^1 *function". This observation seems to not be valid for the present strategy of proof, and more generally in the* BV *case. The reason is, their argument is based on the use of radially symmetric regularization kernels, while the regularization kernels we use here for the* BV *case are typically anisotropic.*

1.2 Main idea of the proof

To start with, we outline here the proof performed in details in the next section. As already said, the proof uses a technique introduced in [HLBL07]. In that work, a smooth convolution kernel ρ, with normalized integral, is considered. It is then proved that for any two a.e. flows X and Y solutions to the ordinary differential equation with $W^{1,1}$ coefficients,

$$\lim_{\varepsilon\to 0}\frac{d}{dt}\left(\int\int |X(t,x)-Y(t,y)|\frac{1}{\varepsilon^d}\rho(\frac{x-y}{\varepsilon})\,dx\,dy\right)=0.$$

Now, the limit of the integral is

$$\frac{d}{dt}\left(\int |X(t,x)-Y(t,x)|\,dx\right)=0.$$

This shows that, for all t,

$$\int |X(t,x)-Y(t,x)|\,dx=0,$$

since this quantity vanishes at initial time. The uniqueness of the solution follows. Remark that the introduction of the extra-variable y allows to perform the calculation without using the transport equation.

Our aim is to now modify the above approach and treat BV vector fields. For this purpose, we use a convolution kernel well adapted to the geometry of the flow and the possible singularities of the BV vector field under consideration. In short, we consider the regularization kernel

$$\frac{1}{\varepsilon^d}\rho(x,\frac{x-y}{\varepsilon}) \text{ with } \rho(x,z)=F_0(|U(x)z|^2)\det U(x)\,,\ \text{and } U(x)=Id+\gamma\eta(x)\otimes\eta(x)\,.$$

Here, F_0 is a smooth function, γ is a constant that will be sent to infinity, and η is an approximation of the direction normal to the jumps of the measure Db. The purpose of such a construction is to have a regularization that decreases faster in the direction normal to the jumps. The idea of a direction-dependent regularization was first introduced by P.L. Lions in [Lio98]. N. Lerner introduced the specific position-dependent regularization used here in [Ler04] with a view to simplifying the proof of uniqueness originally given by L. Ambrosio for the BV case. His argument, however, is still based upon the equivalence with the transport equation. In the present paper, we combine his argument with the approach consisting in introducing a second variable, already employed in [HLBL07] for $W^{1,1}$ vector fields.

2 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We denote by $\mu_1(t,\cdot)$ (resp. $\mu_2(t,\cdot)$) the L^∞ density of the measure $X(-t,\cdot)_\#\lambda$ (resp. $Y(-t,\cdot)_\#\lambda$) with respect to λ.

Consider now the kernel

$$\frac{1}{\varepsilon^d}\rho(x,\frac{x-y}{\varepsilon}),$$

where ρ is a smooth, compactly supported, function, from $\mathbb{T}^n\times\mathbb{T}^n$ to $\mathbb{R}^+$ which we will make precise below. Assume in addition ρ satisfies $\int\rho(x,z)\,dz=1$ for all x. Our aim is to estimate

$$I_\varepsilon(t)=\frac{d}{dt}\left(\int\int |X(t,x)-Y(t,y)|\,\frac{1}{\varepsilon^d}\rho(x,\frac{x-y}{\varepsilon})\mu_1(t,x)\mu_2(t,y)\,dx\,dy\right). \tag{5}$$

where X and Y are two flow solutions to (2). In the sense of distributions,

$$\lim_{\varepsilon\to 0} I_\varepsilon(t)=\frac{d}{dt}\left(\int |X(t,x)-Y(t,x)|\mu_1(t,x)\mu_2(t,x)\,dx\right). \tag{6}$$

This is established using the Lebesgue continuity of the functions Y and μ_2 at almost every point, along with the L^∞ bound on μ_1. Remark that the Lebesgue continuity may be used if the support of $\rho(x,\cdot)$ is not exceedingly stretched in one direction (more specifically, we should have some constant $c>0$ such that $\forall x\in\mathbb{T}^n,\ B(0,c^{-1})\subset \mathrm{Supp}\rho(x,\cdot)\subset B(0,c)$, See [Ste70] for more details). The kernel we shall use satisfies such a condition for all $\varepsilon>0$, even though in the limit of a vanishing ε, it is infinitely stretched. Our purpose is to show that the limit (6) is

$$\lim_{\varepsilon\to 0} I_\varepsilon(t)=-\int |X(t,x)-Y(t,x)|\,\mathrm{div}(b)(x)\mu_1(t,x)\mu_2(t,x)\,dx.$$

This will eventually prove the uniqueness of the flow solution to (2) using the bounds from below on μ_1 and μ_2 inferred from (ii). To this end, we first perform the change of variable $(x,y) \to (X(t,x), Y(t,y))$ in $I_\varepsilon(t)$, and then differentiate under the integral:

$$I_\varepsilon(t) = \frac{d}{dt}\left(\int\int |x-y| \, \frac{1}{\varepsilon^d} \rho(X(-t,x), \frac{X(-t,x)-Y(-t,y)}{\varepsilon}) \, dx\, dy \right),$$

which we write $I_\varepsilon(t) = I^1_\varepsilon(t) + I^2_\varepsilon(t)$, with:

$$\begin{aligned} I^1_\varepsilon(t) &= -\int\int |x-y| \, \frac{1}{\varepsilon^d} \partial_1\rho(X(-t,x), \frac{X(-t,x)-Y(-t,y)}{\varepsilon}) \cdot b(X(-t,x)) \, dx\, dy \\ I^2_\varepsilon(t) &= -\int\int |x-y| \, \frac{1}{\varepsilon^{d+1}} \partial_2\rho(X(-t,x), \frac{X(-t,x)-Y(-t,y)}{\varepsilon}) \cdot (b(X(-t,x)) - b(Y(-t,y))) \, dx\, dy. \end{aligned}$$

Then, we return to the original variables (x,y):

$$\begin{aligned} I^1_\varepsilon(t) &= -\int\int |X(t,x)-Y(t,y)| \, \frac{1}{\varepsilon^d} \partial_1\rho(x, \frac{x-y}{\varepsilon}) \cdot b(x)\mu_1(t,x)\mu_2(t,y) \, dx\, dy \\ I^2_\varepsilon(t) &= -\int\int |X(t,x)-Y(t,y)| \, \frac{1}{\varepsilon^{d+1}} \partial_2\rho(x, \frac{x-y}{\varepsilon}) \cdot (b(x) - b(y))\mu_1(t,x)\mu_2(t,y) \, dx\, dy, \end{aligned}$$

and next use the change of variable $z = (y-x)/\varepsilon$:

$$I^1_\varepsilon(t) = -\int\int |X(t,x)-Y(t,x+\varepsilon z)| \, \partial_1\rho(x,z) \cdot b(x)\mu_1(t,x)\mu_2(t,x+\varepsilon z) \, dx\, dz \tag{7}$$

$$I^2_\varepsilon(t) = -\int\int |X(t,x)-Y(t,x+\varepsilon z)| \, \partial_2\rho(x,z) \cdot \frac{(b(x)-b(x+\varepsilon z))}{\varepsilon} \mu_1(t,x)\mu_2(t,x+\varepsilon z) \, dx\, dz. \tag{8}$$

We now need to estimate these two terms when ε goes to zero. We begin with the easiest of the two, namely I^1_ε.

Step 1: Limit of I^1_ε

Because ρ is smooth, $b \in L^1$, and almost all points are Lesbesgue points for the two functions Y and μ_2, we can use the Lebesgue dominated convergence theorem and obtain:

$$\lim_{\varepsilon\to 0} I^1_\varepsilon(t) = -\int |X(t,x)-Y(t,x)| \left(\int \partial_1\rho(x,z) \, dz \right) \cdot b(x)\mu_1(t,x)\mu_2(t,x) \, dx.$$

Now

$$\int \partial_1\rho(x,z) \, dz = \frac{d}{dx}\left(\int \rho(x,z) \, dz \right) = 0\,,$$

since $\int \rho(x,z)\, dz = 1$, for all x. Thus,

$$\lim_{\varepsilon\to 0} I^1_\varepsilon(t) = 0. \tag{9}$$

The treatment for I^2_ε is more elaborate and will necessitate several steps.

Step 2: Bound for I^2_ε

We now wish to pass to the limit $\varepsilon \to 0$ in (8). If b were $W^{1,1}$, the limit could easily be identified. It would suffice to replace $(b(x+\varepsilon z) - b(x))/\varepsilon$ by $\int_0^1 Db(x+\theta\varepsilon z)\cdot z \, d\theta$ in (8), and next use the Lebesgue dominated convergence theorem. All this does not require making specific the convolution kernel ρ (See below and [HLBL07]). Owing to the presence of the singular part of Db, we have to argue more carefully.

To proceed further, we recall the following result.

Proposition 1 [from [AFP00, Theorem 1.28, Corollary 1.29]] *Let b be a BV vector-field on $\mathbb{T}^n$.*
(i) The Radon-Nikodym decomposition of its derivative Db writes

$$Db = D^a b + D^s b, \quad \textit{with} \quad D^a b << \mathcal{L}^d,\ D^s b \perp \mathcal{L}^d,$$

where the superscript a stand for "absolute continuous part", and s stand for "singular" respectively. As $D^a b$ is absolutely continuous with respect to the Lebesgue measure, we write it:

$$D^a b = \partial^a b \, dx\,,$$

where O^s is a L^1 matrix-valued fonction.
(ii) In addition, the polar decomposition of the singular part $D^s b$ of the measure Db writes:

$$D^s b = M^s \, |D^s b| \, ,$$

where $|D^s b|$ is the total variation of the matrix-valued measure $D^s b$, and M^s a matrix-valued fonction, such that $|M^s(x)| = 1$, $|D^s b|$-a.e (the norm used for M is the norm induced on matrices by the Euclidian norm of $\mathbb{R}^n$).

In view of the above decomposition, we now claim that

$$\begin{aligned} \limsup_{\varepsilon \longrightarrow 0} \int\int\int_0^1 |X(t,x) - Y(t,x+\varepsilon z)| \left| \partial_2 \rho(x,z) \cdot \left(\frac{b(x+\varepsilon z) - b(x)}{\varepsilon} - \partial^a b(x+\varepsilon\theta z) \cdot z \right) \right| \\ \mu_1(t,x) \mu_2(t,x+\varepsilon z) \, d\theta \, dx \, dz \\ \leq 2C(t)^2 \int\int |\partial_2 \rho(x,z) \cdot M^s(x) \cdot z| \; d|Db|(x) \, dz. \end{aligned} \tag{10}$$

For convenience, we denote by

$$I^2_{\varepsilon,a} = - \int\int\int_0^1 |X(t,x) - Y(t,x+\varepsilon z)| \, \partial_2 \rho(x,z) \cdot \partial^a b(x+\varepsilon\theta z) \cdot z \, d\theta \, dx \, dz,$$

in the left-hand side, and

$$\bar{I}^2_s(t) = \int\int |\partial_2 \rho(x,z) \cdot M^s(x) \cdot z| \; d|Db|(x) \, dz,$$

in the right-hand side.

To prove our claim, we regularize X, Y, μ_1 and μ_2, using some smooth X^α, Y^α, μ_1^α and μ_2^α. Next, we replace $(b(x+\varepsilon z) - b(x))/\varepsilon$ by $\int_0^1 Db(x+\theta\varepsilon z) \cdot z \, d\theta$ (an equality true for almost all (x,z)) and perform the change of variable $x' = x + \varepsilon\theta z$ (we use it even for the measure Db because this is a linear change of variable). We obtain:

$$\begin{aligned} I^{2,\alpha}_\varepsilon(t) &:= - \int\int |X^\alpha(t,x) - Y^\alpha(t,x+\varepsilon z)| \, \partial_2 \rho(x,z) \cdot \frac{(b(x+\varepsilon z) - b(x))}{\varepsilon} \mu_1^\alpha(t,x) \mu_2^\alpha(t,x+\varepsilon z) \, dx \, dz \\ &= - \int\int\int_0^1 |X^\alpha(t,x) - Y^\alpha(t,x+\varepsilon z)| \, \partial_2 \rho(x,z) \cdot Db(x+\theta\varepsilon z) \cdot z \mu_1^\alpha(t,x) \mu_2^\alpha(t,x+\varepsilon z) \, dx \, dz \, d\theta \\ &= - \int\int\int_0^1 |X^\alpha(t,x-\varepsilon\theta z) - Y^\alpha(t,x+\varepsilon(1-\theta)z)| \\ &\qquad \partial_2 \rho(x-\varepsilon\theta z,z) \cdot Db(x) \cdot z \mu_1^\alpha(t,x-\varepsilon\theta z) \mu_2^\alpha(t,x+\varepsilon(1-\theta)z) \, dx \, dz \, d\theta \, . \end{aligned} \tag{11}$$

Let us decompose $I^{2,\alpha}_\varepsilon$ in two parts, according to the above Proposition 1,

$$I^{2,\alpha}_\varepsilon(t) = I^{2,\alpha}_{\varepsilon,a}(t) + I^{2,\alpha}_{\varepsilon,s}(t),$$

where:

$$I^{2,\alpha}_{\varepsilon,a}(t) = - \int\int\int_0^1 |X^\alpha(t,x-\varepsilon\theta z) - Y^\alpha(t,x+\varepsilon(1-\theta)z)| \tag{12}$$

$$\partial_2 \rho(x-\varepsilon\theta z,z) \cdot \partial^a b(x) \cdot z \, \mu_1^\alpha(t,x-\varepsilon\theta z) \mu_2^\alpha(t,x+\varepsilon(1-\theta)z) \, dx \, dz \, d\theta \tag{13}$$

$$|I^{2,\alpha}_{\varepsilon,s}(t)| \leq 2C(t)^2 \int\int\int_0^1 \partial_2 \rho(x-\varepsilon\theta z,z) \cdot M^s(x) \cdot z \, |D^s b|(x) \, dz \, d\theta, \tag{14}$$

where we have used that $|X^\alpha - Y^\alpha| \leq 2$ (as we work on the torus). And letting ε going to zero, we obtain (10) for X^α, Y^α and the μ_i^α. Then (10) is obtained letting X^α, Y^α, μ_1^α and μ_2^α approximate X, Y, μ_1 and μ_2, respectively.

The majoration (10) being established, we proceed as follows. Arguing as above for I_ε, that is using the smoothness of ρ and the fact that almost every point is a Lebesgue point for Y, μ_1 and μ_2, we obtain:

$$\begin{aligned}\lim_{\varepsilon\to 0} I^2_{\varepsilon,a}(t) &= \lim_{\varepsilon\to 0} -\int\int\int_0^1 |X(t,x-\varepsilon\theta z)-Y(t,x+\varepsilon(1-\theta)z)| \\ &\qquad \partial_2\rho(x-\varepsilon\theta z,z)\cdot\partial^a b(x)\cdot z\,\mu_1(t,x-\varepsilon\theta)\mu_2(t,x+\varepsilon(1-\theta)z)\,d\theta\,dx\,dz \\ &= -\int\int |X(t,x)-Y(t,x)|\partial_2\rho(x,z)\cdot\partial^a b(x)\cdot z\,\mu_1(t,x)\mu_2(t,x)\,dx\,dz \\ &= -\int |X(t,x)-Y(t,x)|R_a(x)\mu_1(t,x)\mu_2(t,x)\,dx,\end{aligned}$$

with $R_a(x)=\int\partial_2\rho(x,z)\cdot\partial^a b(x)\cdot z\,dz$. To calculate this term, we integrate by parts and use the property $\forall x\in\mathbb{T}^n,\ \int\rho(x,z)\,dz=1$:

$$\begin{aligned}R_a(x) &= \sum_{i,j}\int\frac{\partial\rho}{\partial z_i}(x,z)\frac{\partial^a b_i}{\partial x_j}(x)z_j\,dz=\sum_{i,j}\frac{\partial^a b_i}{\partial x_i}(x)\int-\rho(x,z)\frac{\partial z_j}{\partial z_i}\,dz=-\sum_i\frac{\partial^a b_i}{\partial z_i}(x)\int\rho(x,z)\,dz \\ &= -\operatorname{div}^a b. \qquad (15)\end{aligned}$$

So we have obtained:

$$\lim_{\varepsilon\to 0} I^2_{\varepsilon,a}=\int|X(t,x)-Y(t,x)|\operatorname{div}^a b(x)\mu_1(t,x)\mu_2(t,x)\,dx. \qquad (16)$$

The next step consists in proving that the right-hand side of (10) may be chosen arbitrarily small.

Step 3: A bound on the singular part

In order to estimate the right hand side of (10), we now use a geometric information, namely, the special form of $M^s(x)$, proved by G. Alberti [Alb93].

Theorem 2 [Alberti's rank one Theorem, [AFP00, Theorem 3.94]] *Let b be a BV vector-field defined on $\mathbb{T}^d$, and write $Db=D^sb+D^ab$ the Radon-Nikodym decomposition of its gradient. Consider $D^sb=M\,|D^sb|$ the polar decomposition of the singular part as in Proposition 1. Then, M is of rank one $|D^sb|$-almost everywhere, that is, there exists two vector-valued functions ξ_b and η_b $|D^sb|$-measurables, such that ξ_b and η_b are unit vectors $|D^sb|$-a.e. and satisfy:*

$$M(x)=\xi_b(x)\otimes\eta_b(x),\quad |Db^s|-\textit{almost everywhere},$$

where $\xi_b\otimes\eta_b$ denotes the linear map $x\mapsto\langle\eta_b,x\rangle\xi_b$.

Corollary 1 *As a consequence, the singular part of the divergence is $\operatorname{div}^s b=\langle\xi,\eta\rangle|D^sb|$. If we assume that the divergence of b belongs to L^1, it follows that*

$$\langle\xi,\eta\rangle=0,\quad |Db^s|-a.\ e.,$$

a property that will be crucial in the sequel.

Using the decomposition provided by Theorem 2, we rewrite our bound in (10), which we denote $\bar{I}^2_s(t)$ in the sequel:

$$\bar{I}^2_s(t)\le 2C(t)^2\int\int|\langle\partial_2\rho(x,z),\xi_b(x)\rangle||\langle\eta_b(x),z\rangle|\,d|Db^s|(x)\,dz.$$

In order to render the right-hand side arbitrarily small, we now make specific our convolution kernel ρ. We choose:

$$\rho(x,z)=F_0(|U(x)z|^2)\det(U(x)),$$

where F_0 is a smooth, compactly supported, non negative function such that $\int_{\mathbb{R}^d}F_0(|z|^2)\,dz=1$, and U is a smooth, matrix-valued function, such that $U(x)$ is an orientation preserving matrix for all x. Note that owing to the presence of the determinant, the integral of $\rho(x,\cdot)$ remains equals to one independently of x. The dilation matrix $U(x)$ is set to $U(x)=Id+\gamma\eta(x)\otimes\eta(x)$ (with the notation $a\otimes b$ for the endomorphism $x\to\langle b,x\rangle a$), where η is a smooth vector-valued function. On the jump of the measure Db, η will be chosen later as an approximation of the direction normal to the jump set. The factor γ will be chosen as large as possible. It may possibly depend upon x and be large only on a neighbourhood of the singular set of the measure Db, but we for simplicity of the calculation we will not use that not essential possibility here.

The partial derivative of ρ writes:

$$\partial_2\rho(x,z) = 2F_0'(|U(x)z|^2)\langle U(x)z, U(x)\cdot\rangle \det(U(x)).$$

We use this in the bound on $\bar{I}_s^2(t)$ to obtain:

$$\bar{I}_s^2(t) \le C\int\int |F_0'(|U(x)z|^2)||\langle U(x)z, U(x)\xi_b(x)\rangle||\langle \eta_b(x), z\rangle| \det(U(x))\, d|Db^s|(x)\, dz,$$

where here and below C denotes various irrelevant constants. To simplify this term, we perform the change of variable $z \to U(x)z$, and obtain:

$$\bar{I}_s^2(t) \le C\int\int |F_0'(|z|^2)||\langle z, U(x)\xi_b(x)\rangle|\, |\langle \eta_b(x), U^{-1}z\rangle|\, d|Db^s|(x)\, dz. \tag{17}$$

We next intend to use the special form $U(x) = Id + \gamma(x)\eta(x)\otimes\eta(x)$, to bound from above the two scalar products. Let us first formally illustrate our argument, performing our calculation with $\eta = \eta_b$, as if η_b were smooth. In this case,

$$|\langle z, U(x)\xi_b(x)\rangle| = |\langle z, \xi_b(x)\rangle| \le |z|,$$

because $\langle \eta_b, \xi_b\rangle = 0$ and ξ_b has unit norm. For the second scalar product,

$$|\langle \eta_b(x), U^{-1}(x)z\rangle| = \frac{1}{1+\gamma}|\langle \eta_b(x), z\rangle| \le \frac{1}{1+\gamma},$$

because $U^{-1} = \mathrm{Id} - \dfrac{\gamma}{1+\gamma}\eta_b\otimes\eta_b$.

Inserting these bounds in (17), we obtain:

$$\bar{I}_s^2(t) \le \frac{C}{1+\gamma}\int\int |F_0'(|z|^2)|\, |Db^s|(x)\, dz \le \frac{C(F_0,b)}{1+\gamma},$$

where the constant $C(F_0,b)$ depends only of F_0 and b. It remains then to let γ to infinity to obtain $\bar{I}_s^2(t) = 0$ and conclude our (formal) proof.

We now modify the above formal argument using an approximation η of η_b, instead of η_b itself. First, we remark

$$\begin{aligned}
|\langle z, U(x)\xi_b\rangle| &= |\langle z, \xi_b + \gamma\langle \xi_b, \eta\rangle\eta\rangle| \\
&\le (1+\gamma|\langle \xi_b, \eta\rangle|)\, |z| \\
&\le (1+\gamma|\langle \xi_b, \eta-\eta_b\rangle|)\, |z| \\
&\le (1+\gamma|\eta-\eta_b|)\, |z|,
\end{aligned} \tag{18}$$

where we have used $\langle \xi_b, \eta_b\rangle = 0$, $|Db^s|$-a.e (from Corollary 1), and that ξ_b, η_b are unit vectors. To bound the scalar product $|\langle \eta_b(x), U^{-1}z\rangle|$, we decompose z in $z = z_\eta + z_\perp$, where z_η is the projection of z on $\mathbb{R}\cdot\eta$:

$$\begin{aligned}
|\langle \eta_b, U^{-1}z\rangle| &= \left|\left\langle \eta_b, z - \frac{\gamma}{1+\gamma}\langle \eta, z\rangle\eta\right\rangle\right| \\
&= \left|\left\langle \eta_b, z_\perp + \frac{1}{1+\gamma}z_\eta\right\rangle\right| \\
&\le \left|\left\langle \eta_b - \eta, z_\perp + \frac{1}{1+\gamma}z_\eta\right\rangle\right| + \left|\left\langle \eta, z_\perp + \frac{1}{1+\gamma}z_\eta\right\rangle\right| \\
&\le \left(|\eta_b - \eta| + \frac{1}{1+\gamma}\right)|z|.
\end{aligned} \tag{19}$$

From (18) and (19) we deduce:

$$|\langle z, U(x)\xi_b\rangle|\, |\langle \eta_b, U^{-1}z\rangle| \le \left(2|\eta-\eta_b| + \frac{1}{1+\gamma} + \gamma|\eta-\eta_b|^2\right)|z|^2.$$

We insert this bound in (17) and obtain:

$$\begin{aligned}
\bar{I}_s^2(t) &\le C\int \left(|\eta-\eta_b| + \frac{1}{1+\gamma} + \gamma|\eta-\eta_b|^2\right)\left(\int F_0'(|z|^2)|z|^2\, dz.\right)|Db^s|(x) \\
&\le C(F_0)\left(\frac{1}{1+\gamma} + (1+2\gamma)\int |\eta-\eta_b|\, |Db^s|(x)\right),
\end{aligned} \tag{20}$$

because the integral $\int r_0(|z|)|z|\,dz$ is fixed, and both η_b and η are unit vectors.

We finally show that

$$\inf_{\gamma>0,\eta \text{ smooth}} \left(\frac{1}{1+\gamma} + (1+2\gamma) \int |\eta - \eta_b|\, |Db^s|(x) \right) = 0.$$

To this end, we first choose γ such that $1/(1+\gamma)$ is small, and then construct a smooth function η, sufficiently close to η_b on the support of $D^s b$ so that $(1+2\gamma)\int |\eta - \eta_b|\, |Db^s|(x)$ is also small (use for that classical approximation theorem with respect to the Radon measure $D^s b$). Note that η can be arbitrarily extended to the whole torus as its value outside the support of $D^s b$ is irrelevant. This concludes the proof of the convergence of the right-hand side of (10) to zero.

Step 4: Conclusion Collecting all the previous results, we obtain

$$\frac{d}{dt} \int |X(t,x) - Y(t,x)| \mu_1(t,x)\mu_2(t,x)\, dx = - \int |X(t,x) - Y(t,x)|\, \mathrm{div}(b)(x) \mu_1(t,x)\mu_2(t,x)\, dx. \tag{21}$$

where we have replaced $\mathrm{div}^a(b)$ by $\mathrm{div}(b)$, since we are dealing with vector fields b having at least, divergence in L^1. If $\mathrm{div}(b) \in L^\infty$, then

$$\frac{d}{dt} \int |X(t,x) - Y(t,x)| \mu_1(t,x)\mu_2(t,x)\, dx \le C \int |X(t,x) - Y(t,x)| \mu_1(t,x)\mu_2(t,x)\, dx\,.$$

Since the integral in the right hand side vanishes initially, we conclude that

$$\int |X(t,x) - Y(t,x)| \mu_1(t,x)\mu_2(t,x)\, dx = 0$$

and finally that $X(t,\cdot) = Y(t,\cdot)$ a.e. in x since the μ_i are bounded away from 0. Note that, as usual, if only the solution at positive times if of interest, an assumption on the negative part $\mathrm{div}(b)^-$ of the divergence suffices to conclude.

When only the weaker hypothesis $\mathrm{div}(b) \in L^1$ holds, we have to slightly adapt the above argument. We choose a smooth compactly supported function $\phi(x)$, insert a factor $\phi(X(t,x))$ in the integral (5) defining I_ε. We now estimate

$$I_\varepsilon^\phi(t) = \frac{d}{dt} \left(\int \int \phi(X(t,x)) |X(t,x) - Y(t,y)|\, \frac{1}{\varepsilon^d} \rho(x, \frac{x-y}{\varepsilon}) \mu_1(t,x)\mu_2(t,y)\, dx\, dy \right).$$

The above argument carries over to the present case. An equality similar to (21) is obtained:

$$\begin{aligned} \frac{d}{dt} \int \phi(X(t,x)) |X(t,x) - Y(t,x)| \mu_1(t,x)\mu_2(t,x)\, dx \\ = - \int \phi(X(t,x)) |X(t,x) - Y(t,x)|\, \mathrm{div}(b)(x) \mu_1(t,x)\mu_2(t,x)\, dx \end{aligned} \tag{22}$$

which can also be written (using the change of variable $x = X(t,x)$)

$$\begin{aligned} \frac{d}{dt} \left(\int \phi(x) |x - Y(t, X(-t,x))| \mu_2(t, X(-t,x))\, dx \right) \\ = - \int \phi(x) |x - Y(t, X(-t,x))|\, \mathrm{div}(b)(X(-t,x)) \mu_2(t, X(-t,x))\, dx. \end{aligned} \tag{23}$$

We next define $u(t,x) = |x - Y(t, X(-t,x))| \mu_2(t, X(-t,x))$. Equation (23) holding for all ϕ, it follows that

$$\frac{\partial u}{\partial t} + \mathrm{div}(b)(X(-t,x))\, u = 0, \tag{24}$$

in the distributional sense. There is no derivative of u with respect to x in the equation, so that the variable x is only a parameter. Since $\mathrm{div}(b) \in L^1$ and condition (ii) holds, we have

$$\int_x \int_0^T |\, \mathrm{div}(b)(X(-t,x))|\, dt dx < +\infty$$

for all time T. So that, for almost all x, $\int_0^T |\, \mathrm{div}(b)(X(-t,x))|\, dt < +\infty$. Therefore equation (24) is well-posed for almost all x, and since by construction its solution u vanishes at initial time, it vanishes for all time: $u(t,x) = 0$ for all t, a.e. in x. This concludes the proof: $X \equiv Y$.

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