



# On Hopkins' Picard Group $\text{Pic}_2$ at the prime 3

Nasko Karamanov

## ► To cite this version:

| Nasko Karamanov. On Hopkins' Picard Group  $\text{Pic}_2$  at the prime 3. 2009. hal-00372518

**HAL Id: hal-00372518**

**<https://hal.science/hal-00372518>**

Preprint submitted on 1 Apr 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON HOPKINS' PICARD GROUP $Pic_2$ AT THE PRIME 3

NASKO KARAMANOV

ABSTRACT. In this paper we calculate the algebraic Hopkins' Picard Group  $Pic_2^{alg}$  at the prime  $p = 3$ , which is a subgroup of the group of isomorphism classes of invertible  $K(2)$ -local spectra i.e. of the Hopkins' Picard Group  $Pic_2$ . We use the resolution of the  $K(2)$ -local sphere introduced in [3] and the methods from [5] and [7].

## 1. INTRODUCTION

Let  $\mathcal{C}$  be a symmetric monoidal category with product  $\wedge$  and unit  $I$ . We say that an object  $X$  in  $\mathcal{C}$  is invertible if there exists an object  $Y$  in  $\mathcal{C}$  such that  $X \wedge Y \cong I$ . If the collection of equivalence classes of invertible objects is a set, then the product defines a group structure on it. We denote this group by  $Pic(\mathcal{C})$ , the Picard group of  $\mathcal{C}$ .

For example, by [6] the homomorphism  $\mathbb{Z} \rightarrow Pic(\mathcal{S}) : n \mapsto S^n$  defines an isomorphism between the integers and the Picard group of  $\mathcal{S}$ , the stable homotopy category.

Let  $\mathcal{K}_n$  be the category of  $K(n)$ -local spectra, where  $K(n)$  is the  $n$ -th Morava  $K$ -theory at the prime  $p$ . The unit in  $\mathcal{K}_n$  is given by  $L_{K(n)}S^0$  and the product of two  $K(n)$ -local spectra by  $X \wedge Y := L_{K(n)}(X \wedge Y)$  (as the ordinary smash product of two  $K(n)$ -local spectra need not be  $K(n)$ -local). Hopkins' Picard group is the group  $Pic(\mathcal{K}_n)$  which we denote by  $Pic_n$ . The first account of it appears in [8] and the case  $n = 1$  is treated in details in [6] where also some examples of elements of  $Pic_2$  at the prime  $p = 2$  are given.

In this paper we are interested in  $Pic_2$  at the prime  $p = 3$ .

One way to study  $\mathcal{K}_n$  is through the functor  $E_{n*}X := \pi_* L_{K(n)}(E_n \wedge X)$  where  $E_n$  is the Lubin-Tate (commutative ring) spectrum with coefficients ring  $E_{n*} \cong \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]\langle u, u^{-1} \rangle$ , where the power series ring is over the Witt vectors  $\mathbb{W}\mathbb{F}_{p^n}$ . Recall that  $E_n$  is acted on by the (Big) Morava Stabilizer group  $\mathbb{G}_n = \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$  by  $E_\infty$ -maps (Goerss-Hopkins, Hopkins-Miller). Let  $\mathcal{EG}_n$  be the category of profinite  $E_{n*}[[\mathbb{G}_n]]$ -modules, i.e.  $E_{n*}$ -modules with an  $\mathbb{G}_n$ -action compatible with the action of  $\mathbb{G}_n$  on  $E_{n*}$ . The tensor product (over  $(E_n)_*$ ) gives a monoidal structure on  $\mathcal{EG}_n$ .

**Proposition 1.1.** [6] *Let  $X \in \mathcal{K}_n$ . Then the following conditions are equivalent :*

- a)  $X$  is invertible in  $\mathcal{K}_n$ ;
- b)  $E_{n*}X$  is free  $E_{n*}$ -module of rank 1;
- c)  $E_{n*}X$  is invertible in  $\mathcal{EG}_n$ .

1.1. Let  $Pic_n^{alg} := Pic(\mathcal{EG}_n)$ . By Proposition 1.1 there is a homomorphism :

$$\begin{array}{ccc} \epsilon_n : Pic_n & \rightarrow & Pic_n^{alg} \\ X & \mapsto & E_{n*}X . \end{array}$$

Let  $Pic_n^{alg,0}$  be the subgroup of  $Pic_n^{alg}$  of index 2 of modules concentrated in even degrees. Let  $M \in Pic_n^{alg,0}$  and  $\iota_M$  be a generator of  $M$  in degree 0, as an  $(E_n)_*$ -module. Then for all  $g \in \mathbb{G}_n$  there exists a unique element  $u_g \in (E_n)_0^\times$  such that  $g_*(\iota_M) = u_g \iota_M$ . The map  $\theta_M : g \mapsto u_g$  is a crossed homomorphism and is a well defined element in  $H^1(\mathbb{G}_n; (E_n)_0^\times)$  that does not depend on  $\iota_M$ . Thus we have a homomorphism  $Pic_n^{alg,0} \rightarrow H^1(\mathbb{G}_n; (E_n)_0^\times)$ .

**Proposition 1.2.** [5]  $Pic_n^{alg,0} \cong H^1(\mathbb{G}_n; (E_n)_0^\times)$ .

Not much is known for the kernel  $\kappa_n$  of  $\epsilon_n$  (cf. [8]). When  $n^2 \leq 2(p-1)$  and  $n > 1$  or when  $n = 1$  and  $p > 2$  it is known to be trivial. It is conjectured (Hopkins, [8]) that  $\kappa_n$  is a finite  $p$ -group.

The next theorem is an unpublished result of Goerss, Henn, Mahowald and Rezk.

**Theorem 1.3.** *At the prime  $p = 3$ ,  $\kappa_2 \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ .*

The next two theorems describe some known results for  $Pic_n$ .

**Theorem 1.4.** [6]

$$\begin{aligned} Pic_1 &\cong \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 & p = 2 \\ Pic_1 &\cong \mathbb{Z}_p \times \mathbb{Z}/2(p-1) & p > 2. \end{aligned}$$

The spectrum  $S^1$  is a generator of  $Pic_1$  in the case  $p > 2$ . In an unpublished result and using Shimomura's calculations of  $\pi_* L_2 S$  at primes  $p > 3$  Hopkins shows

**Theorem 1.5.** *For primes  $p > 3$*

$$Pic_2 \cong \mathbb{Z}_p^2 \times \mathbb{Z}/2(p^2 - 1).$$

The main result of this paper is the following theorem.

**Theorem 1.6.** *At the prime 3*

$$Pic_2^{alg} \cong \mathbb{Z}_3^2 \times \mathbb{Z}/16$$

*generated by  $(E_2)_* S^1$  and  $(E_2)_* S^0[det]$ , where  $det$  is a suitable character of  $\mathbb{G}_2$ .*

Theorem 1.3 and Theorem 1.6 imply the following theorem.

**Theorem 1.7.** *At the prime 3*

$$Pic_2 \cong \mathbb{Z}_3^2 \times \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/16.$$

1.2. This paper is organized as follows. In section 2 we recall the basic properties of the Morava stabilizer group  $\mathbb{G}_n$  and describe some important subgroups in the case  $n = 2$  and  $p = 3$ . We also recall the GHMR resolution of [3] and the spectral sequence of [5] and [7] that we use for the most difficult part of our calculation. In section 3 we define two elements of  $Pic_n^{alg}$  that turn out to be generators in the case of  $Pic_2^{alg}$ . In section 4 we present three short exact sequences that we use to simplify the calculations. In Section 5 we describe the part of the first page of the spectral sequence that is needed for the calculations. The final calculations for  $Pic_2^{alg,0}$  are done in section 6, and  $Pic_2^{alg}$  is treated in the last section.

The author would like to thank Hans-Werner Henn for many usefull discussions and for sharing his knowledge on the subject.

## 2. ON THE MORAVA STABILIZER GROUP AND THE GHMR RESOLUTION

In this section we recall some basic properties of the Morava stabilizer group and some important finite subgroups in the case  $n = 2$  and  $p = 3$ . We also describe the main tool of this work, that is the algebraic GHMR resolution of the  $K(2)$ -local sphere constructed in [3]. This resolution is used in [5] and [7] to determine the homotopy of the mod-3 Moore spectrum localized at  $K(2)$ . We will use some of the calculations of [5] and [7] and the spectral sequence used there. For more details the reader is referred to the corresponding papers.

2.1. Recall that  $\mathbb{S}_n$  is the group of automorphisms of the Honda formal group law  $\Gamma_n$  with  $p$ -series  $[p]_{\Gamma_n}(x) = x^{p^n}$ , that is, the group of units in the endomorphism ring  $\text{End}(\Gamma_n)$ . Let  $\mathcal{O}_n$  be the non-commutative ring extension of  $\mathbb{W}\mathbb{F}_{p^n}$  (the Witt vectors over  $\mathbb{F}_{p^n}$ , that we denote by  $\mathbb{W}$  from now on) generated by an element  $S$  satisfying  $S^n = p$  and  $Sw = w^\sigma S$  where  $w \in \mathbb{W}$  and  $\sigma$  is the lift of the Frobenius automorphism of  $\mathbb{F}_{p^n}$ . Then  $\text{End}(\Gamma_n)$  can be identified with  $\mathcal{O}_n$ . For example, in the case  $n = 2$  and  $p = 3$  each element  $g$  of  $\mathbb{S}_2$  can be written as  $g = g_1 + g_2 S$  with  $g_1 \in \mathbb{W}^\times$  and  $g_2 \in \mathbb{W}$ .

2.2. Right multiplication of  $\mathbb{S}_n$  on  $\text{End}(\Gamma_n)$  defines a homomorphism  $\mathbb{S}_n \rightarrow GL(\mathbb{W})$ . Composition with the determinant can be extended to  $\mathbb{G}_n$  to obtain a homomorphism  $\mathbb{G}_n \rightarrow \mathbb{W}^\times \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$  and it is easy to check that this lands in  $\mathbb{Z}_p^\times \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ . The quotient of  $\mathbb{Z}_p^\times$  with its torsion subgroup, isomorphic to  $\mathbb{Z}/(p-1)$ , can be identified with  $\mathbb{Z}_p$  and we get a homomorphism called *reduced determinant* or *reduced norm*:

$$\mathbb{G}_n \rightarrow \mathbb{Z}_p.$$

The kernel of this homomorphism is denoted by  $\mathbb{G}_n^1$  and in the case when  $p-1$  divides  $n$  we have  $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$ .

2.3. The element  $S$  generates a two sided maximal ideal  $\mathfrak{m}$  in  $\mathcal{O}_n$  with quotient  $\mathcal{O}_n/\mathfrak{m} \cong \mathbb{F}_{p^n}$ . The strict Morava stabilizer group  $S_n$  is the kernel of the homomorphism  $\mathcal{O}_n^\times \rightarrow \mathbb{F}_{p^n}^\times$  induced by reduction modulo  $\mathfrak{m}$ . We denote its intersection with  $\mathbb{G}_n^1$  by  $S_n^1$ .

2.4. Let  $n = 2$  and  $p = 3$  from now on. Let  $\omega$  be a primitive eighth root of unity,  $\phi \in Gal(\mathbb{F}_9/\mathbb{F}_3)$  the generator,  $t := \omega^2$ ,  $\psi := \omega\phi$  and  $a := \frac{1}{2}(1 + \omega S)$ . It is easy to verify that  $a$  is an element of order 3. These elements satisfy  $\psi a = a\psi$ ,  $t\psi = \psi t^3$ ,  $ta = a^2 t$  and  $\psi^2 = t^2$ . Then  $a$ ,  $\psi$  and  $t$  generate a subgroup of order 24, denoted  $G_{24}$ ,  $\omega$  and  $\phi$  a subgroup isomorphic to the semi-dihedral group of order 16, denoted  $SD_{16}$ . The elements  $\omega^2$  and  $\phi$  generate a subgroup of  $SD_{16}$  isomorphic to the quaternion group of order 8, denoted  $Q_8$  and we have

$$(1) \quad SD_{16} \cong Q_8 \rtimes Gal(\mathbb{F}_9/\mathbb{F}_3).$$

2.5. The action of the element  $a$  on  $\mathbb{F}_9[[u_1]][u, u^{-1}]$  is described in [5, Cor. 4.7]. For our purposes we only need the following formulae :

$$\begin{aligned} a_* u &\equiv (1 + (1 + \omega^2)u_1)u && \text{mod } (u_1^3) \\ a_* u_1 &\equiv u_1 - (1 + \omega^2)u_1^2 && \text{mod } (u_1^3). \end{aligned}$$

The (integral) action of  $\omega$  is given by

$$(2) \quad \omega_* u_1 = \omega^2 u_1 \text{ and } \omega_* u = \omega u$$

and the Frobenius  $\phi$  acts  $\mathbb{Z}_3$ -linearly by extending the action of the Frobenius on  $\mathbb{W}$  via

$$(3) \quad \phi_* u_1 = u_1 \text{ and } \phi_* u = u.$$

**2.6. The GHMR resolution.** In [3] a resolution of the trivial  $\mathbb{G}_2^1$ -module  $\mathbb{Z}_3$  is constructed that has the following form

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \mathbb{Z}_3 \longrightarrow 0$$

where  $C_0 = C_3 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[G_{24}]} \mathbb{Z}_3$  and  $C_1 = C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi$  and  $\chi$  is the non trivial character of  $SD_{16}$  defined over  $\mathbb{Z}_3$ , on which  $\omega$  and  $\phi$  act by multiplication by  $-1$ . The complete ring  $\mathbb{Z}_p[[G]]$  is by definition  $\lim_{U,n} \mathbb{Z}_p/p^n[G/U]$  where  $U$  runs through the open subgroups of  $G$ . Then we have the following lemma (cf. [5, Lemma 6.1]).

**Lemma 2.1.** *Let  $M$  be a left  $\mathbb{G}_2^1$ -module. Then there is a first quadrant cohomological spectral sequence  $E_r^{*,*}$ ,  $r \geq 1$  with*

$$(4) \quad E_1^{s,t} = \text{Ext}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}^t(C_s; M) \implies H^{s+t}(\mathbb{G}_2^1; M)$$

in which  $E_1^{s,t} = 0$  for  $0 < s < 3$  and  $t > 0$ , and for  $s \geq 0$  and  $t > 3$ , and also

$$E_1^{0,t} \cong E_1^{3,t} \cong H^t(G_{24}; M) \text{ and } E_1^{1,0} \cong E_1^{2,0} \cong \text{Hom}_{SD_{16}}(\chi; M).$$

Note that  $\text{Hom}_{SD_{16}}(\chi; M) \cong \{m \in M \mid \omega_* m = \phi_* m = -m\}$ .

**2.7.** Let  $N_0$  be the kernel of  $\partial_0$  and  $j : N_0 \rightarrow C_0$  the inclusion. As explained in the remark after [5, Lemma 6.1] the differentials in the spectral sequence can be evaluated if we know projective resolutions  $Q_\bullet$  of  $N_0$  and  $P_\bullet$  of  $C_0$  as well as a chain map  $\phi : Q_\bullet \rightarrow P_\bullet$  covering  $j$ . These data can be assembled in a double complex  $T_{\bullet\bullet}$  with  $T_{\bullet 0} = P_\bullet$ ,  $T_{\bullet 1} = Q_\bullet$ , vertical differentials  $\delta_P$  and  $\delta_Q$  and horizontal differentials  $(-1)^n \phi_n : Q_n \rightarrow P_n$ . The filtration of the spectral sequence of this double complex agrees (up to reindexing) with that of the spectral sequence of the lemma. Hence extension problems in the spectral sequence (4) can be studied by using the double complex. As in [5] we obtain a resolution  $P_\bullet := \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[G_{24}]} P'_\bullet$  induced from an explicit resolution of the trivial  $G_{24}$ -module  $\mathbb{Z}_3$ .

**Lemma 2.2.** [5, Lemma 6.2] *Let  $\bar{\chi}$  be the  $\mathbb{Z}_3[Q_8]$ -module whose underlying  $\mathbb{Z}_3$ -module is  $\mathbb{Z}_3$  and on which  $t$  acts by multiplication by  $-1$  and  $\psi$  by the identity. Then the trivial  $\mathbb{Z}_3[G_{24}]$ -module  $\mathbb{Z}_3$  admits a projective resolution  $P'_\bullet$  of period 4 of the following form*

$$\xrightarrow{a^2-a} \bar{\chi} \uparrow_{Q_8}^{G_{24} e+a+a^2} \bar{\chi} \uparrow_{Q_8}^{G_{24} a^2-a} \bar{\chi} \uparrow_{Q_8}^{G_{24} e+a+a^2} \bar{\chi} \uparrow_{Q_8}^{G_{24} a^2-a} 1 \uparrow_{Q_8}^{G_{24}} \longrightarrow \mathbb{Z}_3.$$

We obtain  $Q_\bullet$  from splicing the exact complex  $0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow N_0 \rightarrow 0$  with the projective resolution  $P_\bullet$  of  $C_3 = C_0$  (as  $C_1$  and  $C_2$  are projective). If we denote by  $e$  the unit of  $\mathbb{G}_2^1$ , by  $e_i$  the generators  $e \otimes 1$  of  $C_i$  and by  $\tilde{e}_i$  the generators  $e \otimes 1$  of  $P_i$ , then by [5, Lemma 6.3] there is a chain map  $\phi : Q_\bullet \rightarrow P_\bullet$  covering the homomorphism  $j$  such that  $\phi_0 : Q_0 = C_1 \rightarrow P_0$  sends  $e_1$  to  $(e - \omega)\tilde{e}_0$ .

2.8. We denote by  $E$  the spectral sequence for  $M = E_{2*}/(3) \cong \mathbb{F}_9[[u_1]][u, u^{-1}]$ . The structure of the  $E_1$ -page is well known (cf. [4] for the group  $\mathbb{S}_n$ , the case of  $\mathbb{G}_n$  can be deduced in the same way).

**Proposition 2.3.** *As  $\mathbb{F}_3[\beta, v_1]$ -modules ( $\beta$  acting trivially on  $E_1^{s,*,*}$  if  $s = 1, 2$ )*

$$E_1^{s,*,*} \cong \begin{cases} \mathbb{F}_3[[v_1^6 \Delta^{-1}]][\Delta^{\pm 1}, v_1, \alpha, \tilde{\alpha}, \beta]/(\alpha^2, \tilde{\alpha}^2, v_1 \alpha, v_1 \tilde{\alpha}, \alpha \tilde{\alpha} + v_1 \beta) e_s & s = 0, 3 \\ \omega^2 u^4 \mathbb{F}_3[[u_1^4]][v_1, u^{\pm 8}] e_s & s = 1, 2 \\ 0 & s > 3 \end{cases}$$

where

$$\begin{aligned} |e_s| &= (s, 0, 0) & |v_1| &= (0, 0, 4) & |\Delta| &= (0, 0, 24) \\ |\beta| &= (0, 2, 12) & |\alpha| &= (0, 1, 4) & |\tilde{\alpha}| &= (0, 1, 12). \end{aligned}$$

Recall that  $v_1 = u_1 u^{-2}$  is invariant modulo 3 with respect to the action of  $\mathbb{G}_2$  and therefore all the differentials in the spectral sequence are  $v_1$ -linear. The element  $\alpha \in H^1(G_{24}; (E_2)_4)$  is defined as the modulo 3 reduction of  $\delta^0(v_1)$ , where  $\delta^0$  is the Bockstein with respect to the short exact sequence

$$0 \longrightarrow E_{2*} \longrightarrow E_{2*} \longrightarrow E_{2*}/(3) \longrightarrow 0.$$

The element  $\tilde{\alpha} \in H^1(G_{24}; (E_2)_{12})$  is defined as  $\delta^1(v_2)$ , where  $v_2 = u^{-8}$  and  $\delta^1$  is the Bockstein with respect to the short exact sequence

$$0 \longrightarrow E_{2*}/(3) \longrightarrow E_{2*}/(3) \longrightarrow E_{2*}/(3, u_1) \longrightarrow 0$$

and  $\beta \in H^2(G_{24}; (E_2)_{12})$  is the modulo 3 reduction of  $\delta^0 \delta^1(v_2)$ . The definition of  $\Delta$  is more complicated and we have the following formula (cf. [5, Prop. 5.1]).

$$(5) \quad \Delta \equiv \omega^2(1 - \omega^2 u_1^2 + u_1^4) u^{-12} \pmod{(u_1^6)}.$$

One of the main results in [7] (see also [5, Thm. 1.2]) is the following theorem.

**Theorem 2.4.** *There are elements*

$$\Delta_k \in E_1^{0,0,24k} \quad b_{2k+1} \in E_1^{1,0,8(2k+1)} \quad \bar{b}_{2k+1} \in E_1^{2,0,8(2k+1)}$$

for each  $k \in \mathbb{Z}$  satisfying

$$\Delta_k \equiv \Delta^k e_0 \quad b_{2k+1} \equiv \omega^2 u^{-4(2k+1)} e_1 \quad \bar{b}_{2k+1} \equiv \omega^2 u^{-4(2k+1)} e_2$$

(where the first congruence is modulo  $(u_1^2)$  and the last two modulo  $(u_1^4)$ ) such that

$$\begin{aligned} d_1(\Delta_k) &= \begin{cases} b_{2(3m+1)+1} \equiv (-1)^{m+1} \omega^2 (1 + u_1^4) u^{-12k} & k = 2m + 1 \\ (-1)^{m+1} m v_1^{4 \cdot 3^n - 2} b_{2 \cdot 3^n (3m-1)+1} & k = 2 \cdot 3^n m, m \not\equiv 0 \pmod{3} \end{cases} \\ d_1(b_{2k+1}) &= \begin{cases} (-1)^n v_1^{6 \cdot 3^n + 2} \bar{b}_{3^{n+1}(6m+1)} & k = 3^{n+1}(3m+1) \\ (-1)^n v_1^{10 \cdot 3^n + 2} \bar{b}_{3^n(18m+11)} & k = 3^n(9m+8) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3. TWO ELEMENTS OF  $Pic_n^{alg,0}$ 

3.1. We have two distinguished elements in  $Pic_n^{alg,0} \cong H^1(\mathbb{G}_n; (E_n)_0^\times)$ . In the case  $n = 2$  and  $p = 3$  these will generate the first cohomology. The first one is given by the crossed homomorphism :

$$\begin{aligned} \eta : \mathbb{G}_n &\rightarrow (E_n)_0^\times \\ g &\mapsto \frac{g_* u}{u}. \end{aligned}$$

The second one is given as the composition of the reduced norm and the canonical inclusion :

$$det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times \rightarrow (E_n)_0^\times.$$

We denote the corresponding elements of  $H^1(\mathbb{G}_n; (E_n)_0^\times)$  again by  $\eta$  and  $det$ . Note that by the isomorphism of Proposition 1.2 the element  $(E_n)_* S^2 \in Pic_n^{alg,0}$  is sent to  $\eta$  (as  $u \in (E_n)_{-2}$  gives rise to a generator of  $(E_n)_0 S^2$ ).

3.2. The reduction  $\mathbb{W}[[u_1]]^\times \rightarrow \mathbb{W}^\times$  is equivariant with respect to the inclusion  $\mathbb{W}^\times \rightarrow \mathbb{G}_n$ . Taking into account the Galois group we obtain a homomorphism :

$$red : H^1(\mathbb{G}_n; (E_n)_0^\times) \rightarrow H^1(\mathbb{W}^\times \rtimes Gal; \mathbb{W}^\times) \rightarrow H^1(\mathbb{W}^\times; \mathbb{W}^\times)^{Gal}$$

and the last homomorphism is induced by the short exact sequence  $1 \rightarrow \mathbb{W}^\times \rightarrow \mathbb{W}^\times \rtimes Gal \rightarrow Gal \rightarrow 1$  and the corresponding spectral sequence.

**Proposition 3.1.** *Let  $n = 2$  and  $p > 2$ . Then the image of the homomorphism  $red$  is (topologically) generated by the images of  $\eta$  and  $det$ .*

*Proof.* Recall that when  $p > 2$  then  $\mathbb{W}^\times \cong \mathbb{W} \times \mathbb{F}_{p^n}$  with the obvious Galois action. Thus

$$H^1(\mathbb{W}^\times; \mathbb{W}^\times)^{Gal} \cong \text{End}(\mathbb{W}^\times)^{Gal} \cong \mathbb{Z}_p^n \times \mathbb{Z}/(p^n - 1).$$

The image of  $det$  is given by the composition

$$\mathbb{W}^\times \rightarrow \mathbb{G}_n \xrightarrow{det} \mathbb{Z}_p^\times \rightarrow (E_n)_0^\times \rightarrow \mathbb{W}^\times.$$

If  $g = g_0 + g_1 S + \dots + g_{n-1} S^{n-1}$  with  $g_i \in \mathbb{W}$  and  $w \in \mathbb{W}$  then

$$gw = (g_0 + g_1 S + \dots + g_{n-1} S^{n-1})w = g_0 w + g_1 w^\phi S + \dots + g_{n-1} w^{\phi^{n-1}} S^{n-1}$$

and the composition above sends  $w$  to  $w w^\phi \dots w^{\phi^{n-1}}$ . The image of  $\eta$  is given by the composition

$$\mathbb{W}^\times \rightarrow \mathbb{G}_n \xrightarrow{\eta} (E_n)_0^\times \rightarrow \mathbb{W}^\times$$

and this is easily verified to be the identity.  $\square$

## 4. REDUCTIONS

In this short section we present three short exact sequences that we use in our calculations. The last two were also used by Hopkins in the case  $n = 2$  and  $p > 3$ .

4.1. The first one

$$(6) \quad 1 \rightarrow \mathbb{G}_2^1 \rightarrow \mathbb{G}_2 \rightarrow \mathbb{Z}_3 \rightarrow 1$$

was described in section 2. We use the Lyndon-Hochschild-Serre spectral sequence associated to (6) to calculate  $H^1(\mathbb{G}_2; \mathbb{W}[[u_1]]^\times)$ . The main difficulty is computing  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)$ . This is done in Theorem 6.4.

4.2. The reduction modulo 3 gives a short exact sequence

$$(7) \quad 0 \rightarrow \mathbb{W}[[u_1]] \xrightarrow{\exp(p^-)} \mathbb{W}[[u_1]]^\times \rightarrow \mathbb{F}_9[[u_1]]^\times \rightarrow 0$$

We will use the long exact sequence associated to (7) to calculate  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]])$ . The difficult part is  $H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]^\times)$  (Corollary 6.2).

4.3. We have another short exact sequence coming from the reduction modulo  $u_1$

$$(8) \quad 1 \rightarrow U_1 \rightarrow \mathbb{F}_9[[u_1]]^\times \rightarrow \mathbb{F}_9^\times \rightarrow 1$$

where  $U_1 := \{h \in \mathbb{F}_9[[u_1]]^\times \mid h \equiv 1 \pmod{(u_1)}\}$ . The hard part is to calculate  $H^1(\mathbb{G}_2^1; U_1)$ . This is by far the hardest part of this work (cf. Theorem 5.10). Note that the group  $U_1$  is 3-profinite.

## 5. THE SPECTRAL SEQUENCE

We use the spectral sequence (4) with  $M = U_1$  and denote it by  $\overline{E}$  to distinguish it from the (additive) case  $M = E_{2*}/(3)$  that we also make use of. We start with the  $\overline{E}_1$ -page. As we only need to calculate the first cohomology, it is sufficient to determine  $\overline{E}_1^{0,0}$ ,  $\overline{E}_1^{0,1}$  and  $\overline{E}_1^{1,0} \cong \overline{E}_1^{2,0}$  and the corresponding differentials and extension problems.

5.1. **The term  $\overline{E}_1^{0,1}$ .**

**Proposition 5.1.**  $H^1(G_{24}; \mathbb{F}_9[[u_1]]^\times) \cong \mathbb{Z}/6$ .

*Proof.* Let  $\mathbb{F}_9((u_1))^\times$  be the multiplicative group of the field of fractions of  $\mathbb{F}_9[[u_1]]$ . Each element of  $\mathbb{F}_9((u_1))^\times$  is of the form  $u_1^n f$  with  $f \in \mathbb{F}_9[[u_1]]^\times$  and  $n \in \mathbb{Z}$ . The map

$$\mathbb{F}_9((u_1))^\times \rightarrow \mathbb{Z} : u_1^n f \mapsto n$$

is a group homomorphism with kernel  $\mathbb{F}_9[[u_1]]^\times$ . Thus we have a short exact sequence of  $G_{24}$ -modules

$$(9) \quad 1 \rightarrow \mathbb{F}_9[[u_1]]^\times \rightarrow \mathbb{F}_9((u_1))^\times \rightarrow \mathbb{Z} \rightarrow 1$$

where  $G_{24}$  acts trivially on  $\mathbb{Z}$ .

By Hilbert 90, the multiplicative version, we have  $H^1(G_{24}; \mathbb{F}_9((u_1))^\times) = 0$  and thus the long exact sequence induced by (9) yields

$$H^0(G_{24}; \mathbb{F}_9[[u_1]]^\times) \rightarrow H^0(G_{24}; \mathbb{F}_9((u_1))^\times) \rightarrow H^0(G_{24}; \mathbb{Z}) \rightarrow H^1(G_{24}; \mathbb{F}_9[[u_1]]^\times).$$

By Proposition 2.3 we have  $H^0(G_{24}; \mathbb{F}_9[[u_1]]^\times) \cong \mathbb{F}_3[[v_1^6 \Delta^{-1}]]^\times$  and by a similar argument we conclude  $H^0(G_{24}; \mathbb{F}_9((u_1))^\times) \cong \mathbb{F}_3((v_1^6 \Delta^{-1}))^\times$ . By (5) we have  $v_1^6 \Delta^{-1} \equiv u_1^6 \pmod{(u_1^8)}$  so the image of the homomorphism

$$H^0(G_{24}; \mathbb{F}_9((u_1))^\times) \rightarrow H^0(G_{24}; \mathbb{Z}) \cong \mathbb{Z}$$

is  $6\mathbb{Z}$ , and the result follows.  $\square$

Note that  $\eta$  is not defined on  $U_1$  (as for example  $\omega_* u/u = \omega u \notin U_1$ ), but  $8\eta$  is well defined.

**Proposition 5.2.**  $\overline{E}_1^{0,1} \cong H^1(G_{24}; U_1) \cong \mathbb{Z}/3$  generated by the restriction of  $8\eta$ .

*Proof.* The short exact sequence (8) induces a long exact sequence

$$\rightarrow H^1(G_{24}; U_1) \rightarrow H^1(G_{24}; \mathbb{F}_9[[u_1]]^\times) \rightarrow H^1(G_{24}; \mathbb{F}_9^\times) \rightarrow$$

The group in the middle is isomorphic to  $\mathbb{Z}/6$  by Proposition 5.1 and the group on the right is 2-torsion. The group  $U_1$  is 3-profinite, so there is no 2-torsion in  $H^*(G_{24}; U_1)$ . As  $H^0(G_{24}; \mathbb{F}_9^\times)$  is 2-torsion, the first morphism above is injective.

For the second part of the proposition we use the resolution  $P'_\bullet$  of  $G_{24}$  constructed in Lemma 2.2, to show that the cocycle of  $8\eta$  can not be a cobord. Recall that  $\eta$  was defined as a crossed homomorphism and thus it can be easily described using the standard (bar) resolution (cf. [1, III.3]). A cocycle representing the image of  $8\eta$  in the standard resolution  $B_\bullet$  of  $G_{24}$  is given by  $B_1 \rightarrow U_1 : [g] \mapsto g_* u^8 / u^8$ . By comparing these resolutions we find a representing cocycle in  $P'_\bullet$ . A homomorphism  $\phi_\bullet : P'_\bullet \rightarrow B_\bullet$  over the identity of  $\mathbb{Z}_3$  is given by

$$\begin{aligned} \phi_0 : P'_0 &\rightarrow B_0 & \phi_1 : P'_1 &\rightarrow B_1 \\ e'_0 &\mapsto \frac{1}{8} \sum_{g \in Q_8} g & e'_1 &\mapsto \frac{1}{8} \left( \sum_{g \in Q_8} \bar{\chi}(g^{-1}) g \right) a[a] \end{aligned}$$

where  $e'_i$  are the generators  $e \otimes 1$  of  $P'_i$  and  $\{[g]\}_{g \in G_{24}}$  is a  $G_{24}$ -basis of  $B_1$  (cf. [1, I.5]). Thus the composition

$$P'_1 \rightarrow B_1 \rightarrow U_1 : e'_1 \mapsto \frac{a_*^2 u^8}{a_* u^8}$$

is the desired cocycle. Using the formula from 2.5 we obtain

$$\frac{a_*^2 u^8}{a_* u^8} \equiv 1 - (1 + \omega^2) u_1 \pmod{(u_1^2)}.$$

Now we will show that this cocycle can not be a cobord. A morphism from  $P'_0 \rightarrow U_1$  sends  $e'_0$  to a  $Q_8$ -invariant element  $h$  of  $U_1$ . Thus by Proposition 5.3 we have  $h \equiv 1 \pmod{(u_1^2)}$  and thus the composition  $P'_1 \rightarrow P'_0 \rightarrow U_1$  sends  $e'_1$  to an element congruent to 1 modulo  $u_1^2$  which is not the case for  $8\eta$ .  $\square$

**5.1. The 0-th line.** In the following proposition we give the structure of the 0-th line of the first page of our spectral sequence. We end up with a nice description of the corresponding groups as products of copies of the 3-adics.

Recall that  $v_1 = u_1 u^{-2}$  is in degree 4 and  $\Delta$  is in degree 24. Thus  $v_1^6 \Delta^{-1}$  is in degree 0.

**Proposition 5.3.**

- a)  $\overline{E}_1^{0,0} \cong \{g \in ((E_2)_0/3)^{G_{24}} \cong \mathbb{F}_3[[v_1^6 \Delta^{-1}]]^\times \mid g \equiv 1 \pmod{(u_1)}\}.$
- b) Let  $g_k \in \overline{E}_1^{0,0}$  be such that  $g_k \equiv 1 + v_1^{6k} \Delta^{-k} \pmod{(u_1^{6k+2})}$ . Then

$$\overline{E}_1^{0,0} \cong \prod_{\substack{k \geq 1 \\ k \not\equiv 0 \pmod{3}}} \mathbb{Z}_3 \{g_k\}$$

- c)  $\overline{E}_1^{1,0} = \{h \in U_1 \mid \exists k \in (E_2/3)_0^{Q_8} \cong \mathbb{F}_3[[\omega^2 u_1^2]], h = \omega_* k/k\}$
- d) Let  $h_k \in \overline{E}_1^{1,0}$  be such that  $h_k \equiv 1 + \omega^2 u_1^{4k+2} \pmod{(u_1^{4k+4})}$ . Then

$$\overline{E}_1^{1,0} \cong \overline{E}_1^{2,0} \cong \prod_{\substack{k \geq 0 \\ k \not\equiv 1 \pmod{3}}} \mathbb{Z}_3 \{h_k\}$$

*Proof.* Proposition 2.3 implies (a). By (a) each element  $g \in \overline{E}_1^{0,0}$  can be written as a product  $\prod_{k \geq 1} g_k^{\lambda_k}$  with  $\lambda_k \in \{-1, 0, 1\}$ . As  $g_{3k} \equiv g_k^3 \pmod{(u_1^{18k+2})}$  we obtain the result. To get the action of  $Q_8$  on  $(E_2/3)_0$  we use formulae (2) and (3) and then (c) follows. As  $\omega^2 = -\omega^6$  we have  $h_{3k+1}^{-1} \equiv 1 + \omega^6 u_1^{4(3k+1)+2} \equiv (1 + \omega^2 u_1^{4k+2})^3 \equiv h_k^3 \pmod{(u_1^{12k+8})}$  and we obtain (d).  $\square$

5.2. The goal of what follows is to construct families of generators  $\{g_k\}_{k \geq 1, k \not\equiv 0 \pmod{3}}$  and  $\{h_k\}_{k \geq 0, k \not\equiv 1 \pmod{3}}$  as in the previous proposition on which  $\overline{d}_1$  is easy to describe.

We start with a particular element  $m \in \overline{E}_1^{1,0}$  that is related to  $8\eta$  and plays the same role as the element  $b_1$  in [5].

**Proposition 5.4.** *There exists  $m \in \overline{E}_1^{1,0}$  such that*

- a)  $m \equiv 1 - \omega^2 u_1^2 \pmod{(u_1^4)}$
- b)  $\overline{d}_1(m) = 1$
- c)  $24\eta = m$  in  $H^1(\mathbb{G}_2^1; U_1)$ .

*Proof.* We imitate the proof of [5, Prop. 5.5] and use paragraph 2.7. By definition  $8\eta$  is a permanent cycle so there are cochains  $c \in \text{Hom}_{\mathbb{Z}_2[[\mathbb{G}_2^1]]}(P_1, U_1)$  and  $d \in \text{Hom}_{\mathbb{Z}_2[[\mathbb{G}_2^1]]}(Q_0, U_1)$  such that  $c + d$  is a cocycle in the total complex of the double complex  $\text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{\bullet\bullet}, U_1)$  and such that  $c$  represents the restriction of  $8\eta$  in  $H^1(G_{24}; U_1)$ . From the proof of the Proposition 5.2 we have an explicit cocycle  $c_1$  for  $8\eta$  in the resolution  $P_\bullet$ , so there exists  $h_c \in \text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0; U_1)$  such that  $c = c_1 + \delta_P(h_c)$ . As  $24\eta = 1$  in  $H^1(G_{24}; U_1)$  (Proposition 5.2) there exists  $h \in \text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0; U_1)$  such that  $\delta_P(h) = 3c_1$ . In the double complex the cochain  $3c + 3d$  is cohomologous to  $3d - \delta_Q(h + 3h_c)$ . One can easily check that  $h = u^{24}/(u^8(a_* u^8)(a_*^2 u^8))$ . Then  $3d - (e - \omega)_*(h + 3h_c)$  is a cocycle concentrated in  $\text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{1,0}, U_1)$  representing  $24\eta$ . The formula for  $m$  can be deduced using the formulae for the action of  $a$  and  $\omega$  given in paragraph 2.5.  $\square$

The next lemma is elementary but crucial as it relates the differentials of  $\overline{E}$  and  $E$ , and thus suggests that we could use the generators from Theorem 2.4 to construct convenient generators  $g_k$  and  $h_k$ .

**Lemma 5.5.** *Let  $f \in \mathbb{F}_9[[u_1]]$  be such that  $f \equiv 0 \pmod{(u_1^k)}$ . Then*

$$\frac{1}{1+f} \equiv 1 - f \pmod{(u_1^{2k})}.$$

**Proposition 5.6.** *Let  $g_k := 1 + v_1^{6k} \Delta_{-k}$  for  $k \geq 1, k \not\equiv 0 \pmod{3}$ . Then*

$$\overline{d}_1(g_k) \equiv \begin{cases} 1 + (-1)^{m+1} \omega^2 (u_1^{12m+6} + u_1^{12m+10}) \pmod{(u_1^{12m+12})} & k = 2m + 1 \\ 1 + (-1)^{m+1} m \omega^2 u_1^{12m+2} \pmod{(u_1^{12m+4})} & k = 2m. \end{cases}$$

*Proof.* By Lemma 5.5 we have  $\overline{d}_1(1 + v_1^{6k} \Delta_{-k}) \equiv 1 + v_1^{6k} d_1(\Delta_{-k}) \pmod{(u_1^{12k})}$  where we have used the  $v_1$ -linearity of  $d_1$ . Then the result follows from the formulae from Theorem 2.4.  $\square$

5.3. As in [5] we will use the image of  $\bar{d}_1$  to define  $h_k$ . We allow the generators  $h_k$  to be congruent to  $1 - \omega^2 u_1^{4k+2}$  modulo higher powers of  $u_1$ . Note that  $12m + 2 = 4 \cdot 3m + 2$  thus for  $m \geq 1$  and  $m \not\equiv 0 \pmod{3}$  we can define

$$h_{3m} := \bar{d}_1(g_{2m})$$

and then by definition these elements are in the kernel of  $\bar{d}_1$ .

For the other case, note that  $1 + \omega^2(u_1^{12m+6} + u_1^{12m+10}) \equiv (1 + \omega^2 u_1^{12m+6})(1 + \omega^2 u_1^{12m+10}) \equiv (1 - \omega^2 u_1^{4m+2})^3(1 + \omega^2 u_1^{12m+10}) \pmod{u_1^{12m+12}}$ . Thus if  $h_m$  is already defined we could use  $\bar{d}_1(g_{1+2m})$  to define  $h_{3m+2}$  by

$$h_{3m+2} := \bar{d}_1(g_{1+2m})h_m^3.$$

As  $h_m \equiv 1 \pm \omega^2 u_1^{4m+2} \equiv 1 \pm v_1^{4m+2} b_{-1-2m} \pmod{u_1^{4m+6}}$  we can use Lemma 5.5 to calculate  $\bar{d}_1$ . In order to do that we will use the partition of  $\mathbb{N}$  introduced in Theorem 2.4. If  $-1 - 2m = 1 + 2(3l + 1)$  then  $1 + 2m$  is divisible by 3. The other cases are of relevance : if  $-1 - 2m = 1 + 2 \cdot 3^n(3l - 1)$  then  $m = 3^n(-3l + 1) - 1$  with  $l \not\equiv 0 \pmod{3}$ , if  $-1 - 2m = 1 + 2 \cdot 3^n(9l + 8)$  then  $m = 3^n(-9l - 8) - 1$  with  $l \leq -1$  and the last one, if  $-1 - 2m = 1 + 2 \cdot 3^{n+1}(3l + 1)$  then  $m = 3^{n+1}(-3l - 1) - 1$  with  $l \leq -1$ . In each of these cases, if  $h_m$  is defined then  $h_{3m+2}$  belongs to the same family with  $n$  augmented by one. This allows to define recursively, for  $l, n \geq 0$  :

$$\begin{aligned} h_{3^{n+1}(3l+1)-1} &:= h_{3^n(3l+1)-1}^3 \bar{d}_1(g_{1+2 \cdot 3^n(3l+1)}) & l \not\equiv 0 \pmod{3} \\ h_{3^{n+1}(9l+1)-1} &:= h_{3^n(9l+1)-1}^3 \bar{d}_1(g_{1+2 \cdot 3^n(9l+1)}) \\ h_{3^{n+2}(3l+2)-1} &:= h_{3^{n+1}(3l+2)-1}^3 \bar{d}_1(g_{1+2 \cdot 3^{n+1}(3l+2)}). \end{aligned}$$

For this to work we need to define the corresponding element of each family for  $n = 0$ . In the first case  $h_{3l}$  is already defined. In the other cases we define:

$$\begin{aligned} h_0 &:= m \\ h_{9l} &:= \omega_*(1 + v_1^{36l+2} b_{-1-18l}) / (1 + v_1^{36l+2} b_{-1-18l}) & l > 0 \\ h_{9l+5} &:= \omega_*(1 + v_1^{36l+22} b_{-11-18l}) / (1 + v_1^{36l+22} b_{-11-18l}) & l \geq 0. \end{aligned}$$

Note that  $b_{-1-18l}$  and  $b_{-11-18l}$  belong to the two families of Theorem 2.4 that have non trivial image under  $d_1$ . In both of these cases we can apply Lemma 5.4. This would not have been the case if we would have defined  $h_0$  as  $1 + v_1^2 b_{-1}$  as then Lemma 5.5 does not give the enough precision. The following proposition and the recursive definition of the generators describe  $\bar{d}_1 : \bar{E}_1^{1,0} \rightarrow \bar{E}_1^{2,0}$ . The proof uses Theorem 2.4 and Lemma 5.5.

**Proposition 5.7.**

$$\begin{aligned} z_{1,l} := \bar{d}_1(h_{9l}) &\equiv 1 + \omega^2 u_1^{36l+14} \pmod{u_1^{36l+18}} \\ z_{2,l} := \bar{d}_1(h_{9l+5}) &\equiv 1 + \omega^2 u_1^{36l+30} \pmod{u_1^{36l+34}}. \end{aligned}$$

A more complete description of  $\bar{d}_1$  is given with the following proposition.

**Proposition 5.8.** *The following complexes are exact :*

$$\begin{aligned} \prod_{n \geq 0} \mathbb{Z}_3\{g_{1+2 \cdot 3^n(3l+1)}\} &\rightarrow \prod_{n \geq 0} \mathbb{Z}_3\{h_{3^n(3l+1)-1}\} \rightarrow 1 & \text{for } l \notin 3\mathbb{N} \\ \prod_{n \geq 0} \mathbb{Z}_3\{g_{1+2 \cdot 3^n(9l+1)}\} &\rightarrow \prod_{n \geq 0} \mathbb{Z}_3\{h_{3^n(9l+1)-1}\} \rightarrow \mathbb{Z}_3\{z_{1,l}\} & \text{for } l > 0 \\ \prod_{n \geq 0} \mathbb{Z}_3\{g_{1+2 \cdot 3^{n+1}(3l+2)}\} &\rightarrow \prod_{n \geq 0} \mathbb{Z}_3\{h_{3^{n+1}(3l+2)-1}\} \rightarrow \mathbb{Z}_3\{z_{2,l}\} & \text{for } l \geq 0. \end{aligned}$$

The first homology of the following complex

$$\prod_{n \geq 0} \mathbb{Z}_3\{g_{1+2 \cdot 3^n}\} \rightarrow \prod_{n \geq 0} \mathbb{Z}_3\{h_{3^n-1}\} \rightarrow 1$$

is isomorphic to  $\mathbb{Z}_3\{h_0\}$ . In each complex the infinite matrix of the first homomorphism has the following form  $\begin{pmatrix} -3 & & \\ 1 & -3 & \\ & 1 & -3 \\ & & \dots \end{pmatrix}$  and when non trivial the infinite matrix of the second morphism has the following form  $(1 \ 3 \ 9 \ 27 \ \dots)$ .

*Proof.* The proof is a consequence of Proposition 5.3, Proposition 5.4, the definitions of the generators in paragraph 5.3 and Proposition 5.8.  $\square$

**Corollary 5.9.**  $\overline{E}_2^{1,0} \cong \mathbb{Z}_3$ .

*Proof.* This is consequence of the previous proposition. Indeed, the relevant part of the 0-th line of the first page of the spectral sequence is the product over  $l$  of the four complexes of the previous proposition together with the exact complex

$$\prod_{m > 0} \mathbb{Z}_3\{g_{2m}\} \rightarrow \prod_{m > 0} \mathbb{Z}_3\{h_{3m}\} \rightarrow 1.$$

$\square$

**Theorem 5.10.**  $H^1(\mathbb{G}_2^1; U_1) \cong \mathbb{Z}_3$  is generated by  $8\eta$ .

*Proof.* We only need to resolve the extension problem

$$0 \rightarrow \overline{E}_2^{1,0} \cong \mathbb{Z}_3 \rightarrow H^1(\mathbb{G}_2^1; U_1) \rightarrow \mathbb{Z}/3\mathbb{Z} \cong \overline{E}_2^{0,1} \rightarrow 0.$$

But this is immediate due to Proposition 5.4  $\square$

## 6. GETTING TO $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)$

In this section we prove the main theorem by using the short exact sequences from Section 4. The element  $\eta$  again plays an important role in the proof.

**Proposition 6.1.**  $H^1(\mathbb{G}_2^1; \mathbb{F}_9^\times) \cong \mathbb{F}_9^\times$ .

*Proof.* There is a short exact sequence

$$1 \rightarrow S_2^1 \rightarrow \mathbb{G}_2^1 \rightarrow SD_{16} \rightarrow 1$$

that gives a spectral sequence and as  $S_2^1$  acts trivially on  $\mathbb{F}_9^\times$  we have

$$H^*(\mathbb{G}_2^1; \mathbb{F}_9^\times) \cong H^*(SD_{16}; \mathbb{F}_9^\times).$$

There is another short exact sequence

$$1 \rightarrow \langle \omega \rangle \rightarrow SD_{16} \rightarrow \text{Gal} \rightarrow 1$$

(where  $\text{Gal} := \text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ ) and thus a spectral sequence

$$H^*(\text{Gal}; H^*(\langle \omega \rangle; \mathbb{F}_9^\times)) \Rightarrow H^*(SD_{16}; \mathbb{F}_9^\times).$$

By using the standard resolution it is easily seen that the group  $H^*(\langle \omega \rangle; \mathbb{F}_9^\times)$  is isomorphic to  $\mathbb{F}_9^\times$  in each degree as  $\omega$  acts trivially on  $\mathbb{F}_9^\times$ .

The group  $H^1(\langle \omega \rangle; \mathbb{F}_9^\times)$  is generated by the identity which is Galois invariant thus

$$H^0(\text{Gal}; H^1(\langle \omega \rangle; \mathbb{F}_9^\times)) \cong \mathbb{F}_9^\times$$

and by Hilbert 90

$$H^1(\text{Gal}; H^0(<\omega>; \mathbb{F}_9^\times)) = H^1(\text{Gal}; \mathbb{F}_9^\times) = 0.$$

As the image of  $\eta$  in  $H^1(\mathbb{G}_2^1; \mathbb{F}_9^\times) \cong H^1(SD_{16}; \mathbb{F}_9^\times)$  reduces to the identity in the group  $H^1(<\omega>; \mathbb{F}_9^\times)$ , the differential

$$d_2 : H^0(\text{Gal}; H^2(<\omega>; \mathbb{F}_9^\times)) \rightarrow H^1(\text{Gal}; H^0(<\omega>; \mathbb{F}_9^\times))$$

has to be trivial.  $\square$

**Corollary 6.2.**  $H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]^\times) \cong \mathbb{Z}_3 \times \mathbb{F}_9^\times$  generated by  $\eta$ .

*Proof.* The short exact sequence (8) induces a long exact sequence

$$\rightarrow H^0(\mathbb{G}_2^1; \mathbb{F}_9^\times) \rightarrow H^1(\mathbb{G}_2^1; U_1) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]^\times) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{F}_9^\times) \rightarrow H^2(\mathbb{G}_2^1; U_1).$$

By Theorem 5.10 we have  $H^1(\mathbb{G}_2^1; U_1) \cong \mathbb{Z}_3$  and by Proposition 6.1 we have  $H^1(\mathbb{G}_2^1; \mathbb{F}_9^\times) \cong \mathbb{F}_9^\times$ . As  $U_1$  is a 3-profinite group, the first and the last homomorphisms are trivial.  $\square$

**Proposition 6.3.**

- a)  $H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]) = 0$
- b) The group  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]])$  is 3-profinite.
- c)  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]) = 0$ .

*Proof.* (a) is direct consequence of [5, Thm. 1.6]. The group  $\mathbb{W}[[u_1]]$  is 3-profinite and there is a resolution of finite type (Lazard) of the trivial  $\mathbb{G}_2^1$ -module  $\mathbb{Z}_3$  and (b) follows. Multiplication by 3 induces a short exact sequence

$$\mathbb{W}[[u_1]] \xrightarrow{\times 3} \mathbb{W}[[u_1]] \rightarrow \mathbb{F}_9[[u_1]]$$

which induces a long exact sequence

$$\rightarrow H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]) \rightarrow$$

From (a) and the long exact sequence above it follows that the homomorphism

$$H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]])$$

is surjective i.e. the group  $G := H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]])$  is 3-divisible. As  $G$  is 3-profinite, it is the limit of finite 3-groups  $G/I_n$ . Thus the homomorphism  $G/I_n \rightarrow G/I_n$  induced by the multiplication by 3 is surjective and therefore  $G$  is trivial.  $\square$

**Theorem 6.4.**  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times) \cong \mathbb{Z}_3 \times \mathbb{F}_9^\times$  generated by  $\eta$ .

*Proof.* We use the long exact sequence in  $H^1(\mathbb{G}_2^1; -)$  induced from the short exact sequence (7). The homomorphism  $H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times) \rightarrow H^1(\mathbb{G}_2^1; \mathbb{F}_9[[u_1]]^\times)$  is injective by Proposition 6.3 (c) and also surjective as the image of  $\eta \in H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)$  is a generator (by Corollary 6.2).  $\square$

Finally we get to the proof of 1.6(a).

**Theorem 6.5.**  $H^1(\mathbb{G}_2; \mathbb{W}[[u_1]]^\times) \cong \mathbb{Z}_3^\times \times \mathbb{F}_9^\times$  generated by  $\eta$  and  $\det$ .

*Proof.* We use the short exact sequence (6). We have

$$H^1(\mathbb{Z}_3; H^0(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)) \cong H^1(\mathbb{Z}_3; \mathbb{Z}_3) \cong \mathbb{Z}_3$$

and

$$H^0(\mathbb{Z}_3; H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)) \cong H^0(\mathbb{Z}_3; \mathbb{Z}_3 \times \mathbb{F}_9^\times) \cong \mathbb{Z}_3 \times \mathbb{F}_9^\times.$$

The theorem follows from

$$1 \rightarrow H^1(\mathbb{Z}_3; H^0(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)) \rightarrow H^1(\mathbb{G}_2; \mathbb{W}[[u_1]]^\times) \rightarrow H^0(\mathbb{Z}_3; H^1(\mathbb{G}_2^1; \mathbb{W}[[u_1]]^\times)) \rightarrow 1$$

□

Note the  $det$  is the image of the identity and the image of  $\eta$  is a generator.

## 7. $Pic_2^{alg}$

In this short section we prove of Theorem 1.6 i.e. we calculate  $Pic_2^{alg}$ .

We are left with the short exact sequence

$$0 \rightarrow Pic_2^{alg,0} \rightarrow Pic_2^{alg} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

that comes from the definition of  $Pic_2^{alg,0}$  (cf. paragraph 1.1). Note that the isomorphism of Proposition 1.2 sends  $(E_2)_*S^2$  to  $\eta$  (cf. paragraph 3.1). Thus  $(E_2)_*S^2$  is an element of  $Pic_1^{alg,0}$  that generates  $\mathbb{Z}_3 \times \mathbb{Z}/8$  in  $Pic_2^{alg}$ . But  $(E_2)_*S^1$  is not an element of  $Pic_2^{alg,0}$ , therefore its image in the above sequence is a generator of  $\mathbb{Z}/2$ . Thus  $(E_2)_*S^1$  itself must generate  $\mathbb{Z}_3 \times \mathbb{Z}/16$ .

## REFERENCES

1. Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR MR1324339 (96a:20072)
2. Ethan S. Devinatz and Michael J. Hopkins, *The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts*, Amer. J. Math. **117** (1995), no. 3, 669–710. MR MR1333942 (97a:55007)
3. P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the  $K(2)$ -local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822. MR MR2183282 (2006j:55016)
4. Paul Goerss, Hans-Werner Henn, and Mark Mahowald, *The homotopy of  $L_2V(1)$  for the prime 3*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 125–151. MR MR2039763 (2004k:55010)
5. H.-W. Henn, N. Karamanov, and M. Mahowald, *The homotopy of the  $K(2)$ -local Moore spectrum at the prime 3 revisited*, available at <http://arxiv.org/abs/0811.0235>.
6. Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky, *Constructions of elements in Picard groups*, Topology and representation theory (Evanston, IL, 1992), Contemp. Math., vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 89–126. MR MR1263713 (95a:55020)
7. N. Karamanov, *À propos de la cohomologie du deuxième groupe stabilisateur de Morava; application aux calculs de  $\pi_*L_{K(2)}V(0)$  et du groupe  $pic_2$  de Hopkins*, <http://130.79.4.117:8080/Record.htm?idlist=1record=19385121124911033039>.
8. N. P. Strickland, *On the  $p$ -adic interpolation of stable homotopy groups*, Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), London Math. Soc. Lecture Note Ser., vol. 176, Cambridge Univ. Press, Cambridge, 1992, pp. 45–54. MR MR1232198 (94i:55018)